Eigenvalue asymptotics for the Schrödinger operator with a δ-interaction on a punctured surface

P. Exner[a,b] and K. Yoshitomi[c]

a) Department of Theoretical Physics, Nuclear Physics Institute, Academy of Sciences, 25068 Řež, Czech Republic
b) Doppler Institute, Czech Technical University, Břehová 7, 11519 Prague, Czech Republic
c) Department of Mathematics, Tokyo Metropolitan University, Minami-Ohsawa 1-1, Hachioji-shi, Tokyo 192-0397, Japan
exner@ujf.cas.cz, yositomi@comp.metro-u.ac.jp

Abstract. Given $n \geq 2$, we put $r = \min\{\, i \in \mathbb{N};\ i > n/2 \,\}$. Let Σ be a compact, C^r-smooth surface in $\mathbb{R}^n$ which contains the origin. Let further $\{S_\epsilon\}_{0 \leq \epsilon < \eta}$ be a family of measurable subsets of Σ such that $\sup_{x \in S_\epsilon} |x| = \mathcal{O}(\epsilon)$ as $\epsilon \to 0$. We derive an asymptotic expansion for the discrete spectrum of the Schrödinger operator $-\Delta - \beta\delta(\cdot - \Sigma \setminus S_\epsilon)$ in $L^2(\mathbb{R}^n)$, where β is a positive constant, as $\epsilon \to 0$. An analogous result is given also for geometrically induced bound states due to a δ interaction supported by an infinite planar curve.

1 Introduction

Schrödinger operators with δ-interactions supported by subsets of a lower dimension in the configuration space have been studied by numerous authors – see, e.g., [1]–[4] and references therein. Recently such systems attracted a new attention as models of "leaky" quantum wires and similar structures; new results have been derived about a curvature-induced discrete spectrum [6, 7] and the strong-coupling asymptotics [5, 8, 9, 10, 11].

The purpose of this paper is to discuss another question, namely how the discrete spectrum of such operators behave with respect to a perturbation of the interaction support. Since the argument we are going to use can be formulated in any dimension, we consider here generally n-dimensional Schrödinger operators, $n \geq 2$, with a δ-interaction supported by a punctured surface. On the other hand, we restrict our attention to the situation when the surface codimension is one and the Schrödinger operator in question is defined naturally by means of the appropriate quadratic form.

Formally speaking, our result says that up to an error term the eigenvalue shift resulting from removing an ϵ-neighbourhood of a surface point is the same as that of adding a repulsive δ interaction at this point with the coupling constant proportional to the puncture "area". We will formulate this claim precisely in Theorem 1 below for any sufficiently smooth compact surface in $\mathbb{R}^n$ and prove it in Section 3. Furthermore, the compactness requirement is not essential in the argument; in Section 4 we will derive an analogous asymptotic formula for an infinite planar curve which is not a straight line but it is asymptotically straight in a suitable sense.

2 The main result

Put $r := \min\{\, i \in \mathbb{N};\ i > n/2 \,\}$. Let Σ be a compact, C^r-smooth surface in $\mathbb{R}^n$ which contains the origin, $0 \in \Sigma$. Let further $\{S_\epsilon\}_{0 \leq \epsilon < \eta}$ be a family of subsets of Σ which obeys the following hypotheses:

(H.1) The set S_ϵ is measurable with respect to the $(n-1)$-dimensional Lebesgue measure on Σ for any $\epsilon \in [0, \eta)$.

(H.2) $\sup_{x \in S_\epsilon} |x| = \mathcal{O}(\epsilon)$ as $\epsilon \to 0$.

Next we fix $\beta > 0$ and define for $0 \leq \epsilon < \eta$ the quadratic form q_ϵ by

$$q_\epsilon[u, v] := (\nabla u, \nabla v)_{L^2(\mathbb{R}^n)} - \beta \int_{\Sigma \setminus S_\epsilon} u(x)\overline{v(x)}\, dS\,, \quad u, v \in H^1(\mathbf{R}^n)\,;$$

it is easily seen to be closed and bounded from below. Let H_ϵ be the self-adjoint operator associated with q_ϵ. Since $\Sigma \setminus S_\epsilon$ is bounded, we have

$$\sigma_{\mathrm{ess}}(H_\epsilon) = [0, \infty) \quad \text{and} \quad \sharp\sigma_{\mathrm{disc}}(H_0) < \infty\,.$$

By the min-max principle, there exists a unique $\beta^* \geq 0$ such that $\sigma_{\rm disc}(H_0)$ is non-empty if $\beta > \beta^*$ while $\sigma_{\rm disc}(H_0) = \emptyset$ for $\beta \leq \beta^*$. The critical coupling is dimension-dependent: we have $\beta^* = 0$ when $n = 2$ and $\beta^* > 0$ for $n \geq 3$, see [4, Thm 4.2]. Since our aim is to derive asymptotic properties of the discrete spectrum, we will assume throughout that

(H.3) $\beta > \beta^*$.

Let N be the number of negative eigenvalues of H_0. Since

$$0 \leq q_\epsilon[u,u] - q_0[u,u] \to 0 \quad \text{as} \quad \epsilon \to 0 \quad \text{for} \quad u \in H^1(\mathbb{R}^2)\,,$$

there exists $\eta' \in (0, \eta)$ such that for $\epsilon \in (0, \eta')$ the operator H_ϵ has exactly N negative eigenvalues denoted by $\lambda_1(\epsilon) < \lambda_2(\epsilon) \leq \cdots \leq \lambda_N(\epsilon)$, and moreover

$$\lambda_j(\epsilon) \to \lambda_j(0) \quad \text{as} \quad \epsilon \to 0 \quad \text{for} \quad 1 \leq j \leq N$$

(see [13, Chap. VIII, Thm 3.15]). Let $\{\varphi_j(x)\}_{j=1}^N$ be an orthonormal system of eigenfunctions of H_0 such that $H_0\varphi_j = \lambda_j(0)\varphi_j$ for $1 \leq j \leq N$. Pick a sufficiently small $a > 0$ so that the set $\{\, x \in \mathbb{R}^n : \ |x| < a \,\} \setminus \Sigma$ consists of two connected components, which we denote by $B_\pm$. By the elliptic regularity theorem we have $\varphi_j \in H^r(B_\pm)$. Since $r > n/2$ by assumption, the Sobolev trace theorem implies that the function φ_j is continuous on a Σ-neighbourhood of the origin. We also note that one can suppose without loss of generality that $\varphi_1(x) > 0$ in $\mathbb{R}^n$. For a given $\mu \in \sigma_{\rm disc}(H_0)$ we define

$$\begin{aligned} m(\mu) &:= \min\{1 \leq j \leq N;\ \mu = \lambda_j(0)\}\,, \\ n(\mu) &:= \max\{1 \leq j \leq N;\ \mu = \lambda_j(0)\}\,, \\ C(\mu) &:= \Big(\varphi_i(0)\overline{\varphi_j(0)} \Big)_{m(\mu) \leq i,j \leq n(\mu)}\,. \end{aligned}$$

Let $s_{m(\mu)} \leq s_{m(\mu)+1} \leq \cdots \leq s_{n(\mu)}$ be the eigenvalues of the matrix $C(\mu)$. In particular, if $\mu = \lambda_j(0)$ is a simple eigenvalue of H_0, we have $m(\mu) = n(\mu) = j$ and $s_j = |\varphi_j(0)|^2$. Our main result can be then stated as follows.

Theorem 1 *Adopt the assumptions (H.1)–(H.3). Let $\mu \in \sigma_{\rm disc}(H_0)$, then the asymptotic formula*

$$\lambda_j(\epsilon) = \mu + \beta\, \mathrm{meas}_\Sigma(S_\epsilon) s_j + o(\epsilon^{n-1}) \quad \textit{as} \quad \epsilon \to 0$$

holds for $m(\mu) \leq j \leq n(\mu)$, where $\mathrm{meas}_\Sigma(\cdot)$ stands for the $(n$–$1)$-dimensional Lebesgue measure on Σ.

It should be stressed that our problem involves a singular perturbation and thus it cannot be reduced to the general asymptotic perturbation theory of quadratic forms described in [13, Sec. VIII.4]. Indeed, we have

$$q_\epsilon[u,u] = q_0[u,u] + \beta\, \mathrm{meas}_\Sigma(S_\epsilon)|u(0)|^2 + \mathcal{O}(\epsilon^n) \quad \text{as} \quad \epsilon \to 0$$

for $u \in C_0^\infty(\mathbb{R}^n)$ and the quadratic form $C_0^\infty(\mathbb{R}^n) \ni u \mapsto |u(0)|^2 \in \mathbb{R}$ does not extend to a bounded form on $H^1(\mathbb{R}^n)$, because the set

$$\{\, u \in C_0^\infty(\mathbb{R}^n);\ u = 0 \quad \text{in a neighbourhood of the origin}\,\}$$

is dense in $H^1(\mathbb{R}^n)$. We eliminate this difficulty by using the compactness of the map $H^1(\mathbb{R}^n) \ni f \mapsto f|_\Sigma \in L^2(\Sigma)$, which will enable us to prove Theorem 1 along the lines of the asymptotic-perturbation theorem proof.

Let us remark that our functional-analytic argument has a distinctive advantage over another technique employed in such situations, usually called the matching of asymptotic expansions – see [12] for a thorough review – since the latter typically requires a sort of self-similarity for the perturbation domains. Our technique needs no assumption of this type.

3 Proof of Theorem 1

We denote $R(\zeta,\epsilon) = (H_\epsilon - \zeta)^{-1}$ for $\zeta \in \rho(H_\epsilon)$ and $R(\zeta) = (H_0 - \zeta)^{-1}$ for $\zeta \in \rho(H_0)$. Put $\kappa := \frac{1}{2}\,\mathrm{dist}(\{\mu\}, \sigma(H_0) \setminus \{\mu\})$. Since $\lambda_j(\cdot)$ is continuous at the origin for $1 \le j \le N$, there is an $\eta_0 \in (0,\eta')$ such that

$$\begin{aligned}\sigma(H_\epsilon) \cap [\mu-\kappa, \mu+\kappa] &= \sigma(H_\epsilon) \cap (\mu - \kappa/2, \mu+\kappa/2)\\ &= \{\lambda_{m(\mu)}(\epsilon), \lambda_{m(\mu)+1}(\epsilon), \dots, \lambda_{n(\mu)}(\epsilon)\}\end{aligned}$$

holds if $0 < \epsilon \le \eta_0$. Choosing the circle $C := \{\, z \in \mathbb{C};\ |z-\mu| = \frac{3}{4}\kappa \,\}$ we put

$$w_j(\zeta,\epsilon) := R(\zeta,\epsilon)\varphi_j - R(\zeta)\varphi_j \quad \text{for} \quad 0 < \epsilon \le \eta_0\,,\ \zeta \in C\,.$$

Our first aim is to check that

$$\|w_j(\zeta,\epsilon)\|_{H^1(\mathbb{R}^n)} = \mathcal{O}(\epsilon^{(n-1)/2}) \quad \text{as} \quad \epsilon \to 0 \tag{1}$$

holds uniformly with respect to $\zeta \in C$ for $m(\mu) \le j \le n(\mu)$. We observe that the function $\lambda \mapsto (1+|\lambda|)(\lambda-\zeta)^{-1}$ is bounded on $\sigma(H_\epsilon)$, uniformly in $\epsilon \in (0,\eta_0)$, so by spectral theorem there exists a $K_1 > 0$ such that

$$\|u\|^2_{H^1(\mathbb{R}^n)} \le K_1 |(q_\epsilon - \zeta)[u,u]|$$

for $\zeta \in C$, $0 < \epsilon < \eta_0$, and $u \in H^1(\mathbb{R}^n)$. Moreover, by the Sobolev imbedding theorem, there exists a constant $K_2 > 0$ such that

$$\|u\|_{L^2(\Sigma)} \leq K_2 \|u\|_{H^1(\mathbb{R}^n)} \quad \text{for} \quad u \in H^1(\mathbb{R}^n)\,.$$

Combining these two estimates we get

$$\begin{aligned}
\|w_j(\zeta,\epsilon)\|^2_{H^1(\mathbb{R}^n)} &\leq K_1\, |(q_\epsilon - \zeta)[w_j(\zeta,\epsilon), w_j(\zeta,\epsilon)]| \\
&= K_1 \left| -\beta \int_{S_\epsilon} R(\zeta)\varphi_j \overline{w_j(\zeta,\epsilon)}\, dS \right| \\
&\leq K_1\beta\, \|R(\zeta)\varphi_j\|_{L^2(S_\epsilon)} \|w_j(\zeta,\epsilon)\|_{L^2(S_\epsilon)} \\
&\leq \frac{4K_1K_2\beta}{3\kappa}\, \|\varphi_j\|_{L^2(S_\epsilon)} \|w_j(\zeta,\epsilon)\|_{H^1(\mathbb{R}^n)}\,. \qquad (2)
\end{aligned}$$

Since $\|\varphi_j\|_{L^2(S_\epsilon)} = \mathcal{O}(\epsilon^{(n-1)/2})$ as $\epsilon \to 0$ by the assumptions (H.1), (H.2) and the continuity of $\varphi_j|_\Sigma$ at the origin, we arrive at the relation (1).

In the next step we are going to demonstrate that the convergence is in fact slightly faster, namely

$$\sup_{\zeta\in C} \|w_j(\zeta,\epsilon)\|_{H^1(\mathbb{R}^n)} = o(\epsilon^{(n-1)/2}) \quad \text{as} \quad \epsilon \to 0\,. \qquad (3)$$

We will proceed by contradiction. Suppose that (3) does not hold; then there would exist a constant $\delta > 0$, a sequence $\{\epsilon_i\}_{i=1}^\infty \subset (0,\eta_0)$ which tends to zero, and $\{\zeta_i\}_{i=1}^\infty \subset C$ such that

$$\epsilon_i^{-(n-1)/2} \|w_j(\zeta_i,\epsilon_i)\|_{H^1(\mathbb{R}^n)} \geq \delta \quad \text{for all} \quad i \in \mathbb{N}\,. \qquad (4)$$

Notice that the map $H^1(\mathbb{R}^n) \ni f \mapsto f|_\Sigma \in L^2(\Sigma)$ is compact due to the boundedness of the map $H^1(\mathbb{R}^n) \ni g \mapsto g|_\Sigma \in H^{1/2}(\Sigma)$ and the compactness of the imbedding $H^{1/2}(\Sigma) \ni h \mapsto h \in L^2(\Sigma)$ – cf. [14, Chap. 1, Thms 8.3 and 16.1]. Since the sequence $\left\{\epsilon_i^{-(n-1)/2} w_j(\zeta_i,\epsilon_i)\right\}_{i=1}^\infty$ is bounded in $H^1(\mathbb{R}^n)$, there is a subsequence $\{i(k)\}_{k=1}^\infty$ of $\{i\}_{i=1}^\infty$ such that $\left\{\epsilon_{i(k)}^{-(n-1)/2} w_j(\zeta_{i(k)},\epsilon_{i(k)})\right\}_{k=1}^\infty$ converges in $L^2(\Sigma)$. Let us denote

$$g := \lim_{k\to\infty} \epsilon_{i(k)}^{-(n-1)/2} w_j(\zeta_{i(k)},\epsilon_{i(k)}) \in L^2(\Sigma)\,;$$

then we have

$$\left\| \epsilon_{i(k)}^{-(n-1)/2} w_j(\zeta_{i(k)}, \epsilon_{i(k)}) \right\|_{L^2(S_\epsilon)}$$
$$\leq \left\| \epsilon_{i(k)}^{-(n-1)/2} w_j(\zeta_{i(k)}, \epsilon_{i(k)}) - g \right\|_{L^2(\Sigma)} + \left(\int_{S_{\epsilon_{i(k)}}} |g(x)|^2 \, dS \right)^{1/2} \to 0$$

as $k \to \infty$. Combining this result with the inequalities (2) we infer that

$$\epsilon_{i(k)}^{-(n-1)/2} \| w_j(\zeta_{i(k)}, \epsilon_{i(k)}) \|_{H^1(\mathbb{R}^n)} \to 0 \quad \text{as} \quad k \to \infty \,,$$

which violates the relation (4); in this way we have proved (3).

Now we denote by P_ϵ the spectral projection of H_ϵ associated with the interval $(\mu - 3\kappa/4, \mu + 3\kappa/4)$. It follows from (3) that

$$\begin{aligned} P_\epsilon \varphi_j - \varphi_j &= \frac{\sqrt{-1}}{2\pi} \oint_{|\zeta - \mu| = 3\kappa/4} w_j(\zeta, \epsilon) \, d\zeta \\ &= o(\epsilon^{(n-1)/2}) \quad \text{in} \quad H^1(\mathbb{R}^n) \quad \text{as} \quad \epsilon \to 0 \end{aligned}$$

holds for $m(\mu) \leq j \leq n(\mu)$. Consequently, we have

$$\begin{aligned} &(H_\epsilon P_\epsilon \varphi_i, P_\epsilon \varphi_j)_{L^2(\mathbb{R}^n)} - \mu \delta_{i,j} - \beta \varphi_i(0) \overline{\varphi_j(0)} \, \mathrm{meas}_\Sigma(S_\epsilon) \\ &\quad = q_\epsilon[P_\epsilon \varphi_i, P_\epsilon \varphi_j] - q_0[\varphi_i, \varphi_j] - \beta \varphi_i(0) \overline{\varphi_j(0)} \, \mathrm{meas}_\Sigma(S_\epsilon) \\ &\quad = q_\epsilon[\varphi_i, \varphi_j] - q_0[\varphi_i, \varphi_j] - q_\epsilon[(I - P_\epsilon)\varphi_i, (I - P_\epsilon)\varphi_j] \\ &\qquad - \beta \varphi_i(0) \overline{\varphi_j(0)} \, \mathrm{meas}_\Sigma(S_\epsilon) \\ &\quad = -q_\epsilon[(I - P_\epsilon)\varphi_i, (I - P_\epsilon)\varphi_j] + \beta \int_{S_\epsilon} \varphi_i(x) \overline{\varphi_j(x)} \, dS \\ &\qquad - \beta \varphi_i(0) \overline{\varphi_j(0)} \, \mathrm{meas}_\Sigma(S_\epsilon) \\ &\quad = o(\epsilon^{n-1}) \end{aligned} \tag{5}$$

and

$$(P_\epsilon \varphi_i, P_\epsilon \varphi_j)_{L^2(\mathbb{R}^n)} = \delta_{i,j} + o(\epsilon^{n-1}) \tag{6}$$

as $\epsilon \to 0$ for $m(\mu) \leq i, j \leq n(\mu)$, where we have used, in the last step of (5), the assumptions (H.1), (H.2), and the continuity of the restrictions $\varphi_i|_\Sigma$ and $\varphi_j|_\Sigma$ at the origin. Let us now introduce the matrices

$$\begin{aligned} L(\epsilon) &:= ((H_\epsilon P_\epsilon \varphi_i, P_\epsilon \varphi_j)_{L^2(\mathbb{R}^n)})_{m(\mu) \leq i,j \leq n(\mu)} \,, \\ M(\epsilon) &:= ((P_\epsilon \varphi_i, P_\epsilon \varphi_j)_{L^2(\mathbb{R}^n)})_{m(\mu) \leq i,j \leq n(\mu)} \,. \end{aligned}$$

It is clear that $\lambda_{m(\mu)}(\epsilon), \lambda_{m(\mu)+1}(\epsilon), \ldots, \lambda_{n(\mu)}(\epsilon)$ are the eigenvalues of the matrix $L(\epsilon)M(\epsilon)^{-1}$, which by (5), (6) is equal to

$$L(\epsilon)M(\epsilon)^{-1} = \mu I + \beta \, \mathrm{meas}_\Sigma(S_\epsilon) \, C(\mu) + o(\epsilon^{n-1})$$

on $\mathbb{R}^d$, $d := (n(\mu) - m(\mu))^2$, as $\epsilon \to 0$, where I stands for the identity matrix. This concludes the argument. ∎

4 Perturbation of an infinite curve

As we have mentioned, the compactness of Σ did not play an essential role in the above argument, and we can use the same technique for punctured non-compact manifolds of unit codimension as well, as long as the corresponding Hamiltonian has a discrete spectrum. At present this is known to be true (without restriction to the coupling constant β) in the case $n = 2$.

We shall thus consider "puncture" perturbations of infinite asymptotically straight curves. Let $\Lambda : \mathbb{R} \to \mathbb{R}^2$ be a C^2-smooth curve parameterized by its arc length. Fix $\beta > 0$ and assume that $\Lambda(0) = 0$. Given $\epsilon \geq 0$, we define

$$t_\epsilon[u, v] := (\nabla u, \nabla v)_{L^2(\mathbb{R}^2)} - \beta \int_{\Lambda(\mathbb{R} \setminus (-\epsilon, \epsilon))} u(x) \overline{v(x)} \, dS \,, \quad u, v \in H^1(\mathbb{R}^2) \,.$$

Let T_ϵ be the self-adjoint operator associated with the quadratic form t_ϵ. We adopt the following assumptions about the curve Λ.

(H.4) The curve Λ is not a straight line.

(H.5) There exists $c \in (0, 1)$ such that $|\Lambda(s) - \Lambda(t)| \geq c|t - s|$ for $s, t \in \mathbb{R}$.

(H.6) There exist $d > 0$, $\rho > 1/2$, and $w \in (0, 1)$ such that the inequality

$$1 - \frac{|\Lambda(s) - \Lambda(s')|}{|s - s'|} \leq d \left[1 + |s + s'|^{2\rho}\right]^{-1/2}$$

holds in the sector $\left\{ (s, s') \in \mathbb{R}^2 ;\ w < \frac{s}{s'} < w^{-1} \right\}$.

¿From [6, Prop 5.1 and Thm 5.2] we know that under these conditions

$$\sigma_{\mathrm{ess}}(T_0) = [-\beta^2/4, \infty) \quad \text{and} \quad 1 \leq \sharp \sigma_{\mathrm{disc}}(T_0) \leq \infty .$$

Let $K := \{ j \in \mathbb{N};\ j \leq \sharp \sigma_{\mathrm{disc}}(T_0) \}$. For $j \in K$, we denote by $\kappa_j(\epsilon)$ the j-th eigenvalue of T_ϵ counted with multiplicity. The function $\kappa_j(\cdot)$ is monotone non-decreasing, continuous function in a neighbourhood of the origin.

Let $\{\psi_j(x)\}_{j\in K}$ be an orthonormal system of eigenfunctions of T_0 such that $T_0\psi_j = \kappa_j(0)\psi_j$ for $j \in K$. Each function ψ_j is continuous on Λ. For $\mu \in \sigma_{\rm disc}(T_0)$, we define

$$\begin{aligned} p(\mu) &:= \min\{\, j \in K;\ \mu = \kappa_j(0)\,\}\,, \\ r(\mu) &:= \max\{\, j \in K;\ \mu = \kappa_j(0)\,\}\,, \\ D(\mu) &:= \Big(\psi_i(0)\overline{\psi_j(0)}\Big)_{p(\mu)\le i,j\le r(\mu)}\,. \end{aligned}$$

Let $e_{p(\mu)} \le e_{p(\mu)+1} \le \cdots \le e_{r(\mu)}$ be the eigenvalues of the matrix $D(\mu)$. As in the compact case, if $\mu = \kappa_j(0)$ is a simple eigenvalue of H_0, we have $p(\mu) = r(\mu) = j$ and $e_j = |\psi_j(0)|^2$. The asymptotic behaviour now looks as follows.

Theorem 2 *Assume that (H.4)–(H.6) and take $\mu \in \sigma_{\rm disc}(T_0)$. Then*

$$\kappa_j(\epsilon) = \mu + 2\beta e_j\epsilon + o(\epsilon) \quad as \quad \epsilon \to 0$$

holds for $p(\mu) \le j \le r(\mu)$.

Proof is analogous to that of Theorem 1. ∎

Let us mention in conclusion that the results derived here raise some interesting questions, for instance, what is the following term in the expansion, how the asymptotic behaviour looks like for non-smooth surfaces, and whether similar formulae are valid in the case of $\operatorname{codim}\Sigma = 2, 3$ when the corresponding generalized Schrödinger operator has to be defined by means of appropriate boundary conditions.

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