Dirichlet forms and symmetric Markovian semigroups on $\mathbb{Z}_2$-graded von Neumann algebras

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Abstract

We extend the construction of Dirichlet forms and symmetric Markovian semigroups on standard forms of von Neumann algebras given in [Par] to the case of $\mathbb{Z}_2$-graded von Neumann algebras. We apply the extension to construct symmetric Markovian semigroups on CAR algebras with respect to gauge invariant quasi-free states and also investigate detailed properties such as ergodicity of the semigroups.

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1 Introduction

In [Par], the author has used the theory of Dirichlet forms and Markovian semigroups on standard forms of von Neumann algebras developed by Cipriani[Cip1] to construct Dirichlet forms which generate symmetric Markovian semigroups. The result has been employed to construct Dirichlet forms and associated symmetric Markovian semigroups on CCR algebras with respect to quasi-free states[BKP] and quantum mechanical systems[BK]. The purpose of this paper is to extend the construction method of Dirichlet forms in [Par] to the case of $\mathbb{Z}_2$-graded algebras by using the notion of superderivations[DL2]. We then apply the result to construct symmetric Markovian semigroups on CAR algebras with respect to gauge invariant quasi-free states and also investigate detailed properties such as ergodicity of the semigroups. This application is similar to our previous work on CCR algebras[BKP].

The need to construct Markovian semigroups on von Neumann algebras, which are symmetric with respect to a non-tracial state, is clear for various applications to open systems[Dav], quantum statistical mechanics[BR], and quantum probability theory[Acc, AFL, Part]. Although on the abstract level we have quite well-developed theory[Cip1, GL1, GL2, GL3], the progress in concrete applications is very slow. We would like to mention a few works in this direction. The completely positive Hamiltonian semigroup for quantum spin chains in the ground state representation has been considered in [Mat]. In [MZ1, MZ2], the generalized conditional expectation is used to construct generators of spin-flip type dynamics for quantum spin systems. As we mentioned above, one of authors gave a general construction method of Dirichlet forms on standard forms of von Neumann algebras and applied the method to construct translation invariant Markovian semigroups for quantum spin systems[Par]. The method of [Par] has been extended to construct symmetric Markovian semigroups on the CCR with respect to quasi-free states[BKP], and on quantum mechanical systems[BK]. In [CFL, KP], quantum Ornstein-Uhlenbeck semigroups were constructed by means of noncommutative Dirichlet forms.

Let us describe the content of this paper briefly. For a $\mathbb{Z}_2$-graded von Neumann algebra $\mathcal{M}$ acting on a Hilbert space $\mathcal{H}$, let ξ_0 be a cyclic and separating vector for $\mathcal{M}$ which is invariant under the $\mathbb{Z}_2$-grading (see (2.6)). We consider a construction of Dirichlet forms on the natural standard form $(\mathcal{M}, \mathcal{H}, \mathcal{P}, J)$ associated with the pair $(\mathcal{M}, \xi_0)$. For any admissible function f (Definition 2.1) and any odd analytic element x in $\mathcal{M}$, we construct a bounded Dirichlet form which generates a Markovian semigroup on $\mathcal{H}$ (Theorem 2.1).

Applying this result, we construct Dirichlet forms and associated symmetric Markovian semigroups on CAR algebras with respect to gauge invariant quasi-free states. More precisely, let $\mathcal{A}(\mathfrak{h}_0)$ be the CAR algebra over a complex separable pre-Hilbert space $\mathfrak{h}_0$ and let ω be a gauge invariant quasi-free state on $\mathcal{A}(\mathfrak{h}_0)$. For any normalized admissible function f and complete orthonormal system (CONS) $\{g_n\} \subset \mathfrak{h}_0$, we construct a Dirichlet form and corresponding symmetric Markovian semigroup on the natural standard form associated to the GNS representation of the pair $(\mathcal{A}(\mathfrak{h}_0), \omega)$. It turns out that the form we constructed is independent of the admissible function f and the CONS $\{g_n\}$ chosen (Theorem 3.1). By establishing a (chaos) decomposition of the quasi-free Hilbert space (see Section 4) and analyzing the spectrum of the generator (Dirichlet operator) of the semigroup, we prove that the semigroup is ergodic in the sense of [Cip2] and tends to the equilibrium exponentially fast (Theorem 3.2).

We organize the paper as follows. In section 2, we introduce the some terminologies in the theory of noncommutative Dirichlet forms in the sense of Cipriani[Cip1] and give a brief review on $\mathbb{Z}_2$-graded von Neumann algebras and the notion of superderivations[DL2]. We then extend the construction of Dirichlet forms given in [Par] to the case of $\mathbb{Z}_2$-graded von Neumann algebras. In Section 3, as an application of the results in previous section, we give an explicit expression of a Dirichlet form on the natural standard form associated to CAR algebra with a quasi-free state. In Section 4, we make a chaos decomposition of $\mathcal{H}$ which is essentially same as that in [AW], and then prove the ergodicity of the semigroup and the existence of a spectral gap.

2 Construction of Dirichlet forms on $\mathbb{Z}_2$-graded von Neumann algebras

In this section, we first introduce necessary terminologies in the theory of noncommutative Dirichlet forms in the sense of Cipriani [Cip1] and the notion of superderivations [DL2], and then by employing the method developed in [Par], we construct Dirichlet forms on standard forms of $\mathbb{Z}_2$-graded von Neumann algebras which generate symmetric Markovian semigroups.

Let $\mathcal{M}$ be a σ-finite von Neumann algebra acting on a complex Hilbert space $\mathcal{H}$ and let ξ_0 be a cyclic and separating vector for $\mathcal{M}$. We use Δ and J to denote respectively, the modular operator and the modular conjugation associated to the pair $(\mathcal{M},\, \xi_0)$[BR]. The associated modular automorphism group is denoted by σ_t: $\sigma_t(A) = \Delta^{it} A \Delta^{-it}, \forall A \in \mathcal{M},\, t \in \mathbb{R}$. $j : \mathcal{M} \to \mathcal{M}'$ is the antilinear $*$-isomorphism defined by $j(A) = JAJ$, $A \in \mathcal{M}$.

The natural positive cone $\mathcal{P}$ associated with the pair $(\mathcal{M},\, \xi_0)$ is the closure of the set

$$\{Aj(A)\xi_0 : A \in \mathcal{M}\}.$$

By a general result, the closed convex cone $\mathcal{P}$ can be obtained by the closure of the set

$$\{\Delta^{1/4} A A^* \xi_0 : A \in \mathcal{M}\}$$

and this cone $\mathcal{P}$ is self-dual in the sense that

$$\Big\{\xi \in \mathcal{H} : \langle \xi, \eta \rangle \geq 0,\ \forall \eta \in \mathcal{P}\Big\} = \mathcal{P}.$$

Then the form $(\mathcal{M}, \mathcal{H}, \mathcal{P}, J)$ is the standard form associated with the pair $(\mathcal{M}, \xi_0)$. For details we refer to [Ara] and Section 2.5 of [BR]. We shall use the fact that $\mathcal{H}$ is the complexification of the real subspace $\mathcal{H}^J = \Big\{\xi \in \mathcal{H} : \langle \xi, \eta \rangle \in \mathbb{R},\ \forall \eta \in \mathcal{P}\Big\}$, whose elements are called *J-real*: $\mathcal{H} = \mathcal{H}^J \bigoplus i\mathcal{H}^J$. The cone $\mathcal{P}$ gives rise to a structure of ordered Hilbert space on $\mathcal{H}^J$ (denoted by $\leq$) and to an anti-unitary involution J on $\mathcal{H}$, which preserves $\mathcal{P}$ and $\mathcal{H}^J$: $J(\xi + i\eta) := \xi - i\eta$, $\forall \xi, \eta \in \mathcal{H}^J$. Also note that any J-real element $\xi \in \mathcal{H}^J$ can be decomposed uniquely as a difference $\xi = \xi_+ - \xi_-$, of two orthogonal, positive elements, called the positive and the negative part of ξ : ξ_+, $\xi_- \in \mathcal{P}$ and $\langle \xi_+, \xi_- \rangle = 0$. The order interval $\{\eta \in \mathcal{H} : 0 \leq \eta \leq \xi_0\}$ will be denoted by $[0, \xi_0]$. This is a closed convex subset of $\mathcal{H}$, and we shall denote the nearest point projection onto $[0, \xi_0]$ by $\eta \mapsto \eta_I$.

A bounded operator A on $\mathcal{H}$ is called *J-real* if $AJ = JA$ and *positive preserving* if $A\mathcal{P} \subset \mathcal{P}$. The semigroup $\{T_t\}_{t\geq 0}$ is said to be *J-real* if T_t is J-real for any $t \geq 0$ and it is called *positive preserving* if T_t is positive preserving for any $t \geq 0$. A bounded operator $A : \mathcal{H} \to \mathcal{H}$ is called *sub-Markovian* (with respect to ξ_0) if $0 \leq \xi \leq \xi_0$ implies $0 \leq A\xi \leq \xi_0$. A is called *Markovian* if it is sub-Markovian and also $A\xi_0 = \xi_0$. A semigroup $\{T_t\}_{t\geq 0}$ is said to be *sub-Markovian* (with respect to ξ_0) if T_t is sub-Markovian for every $t \geq 0$. The semigroup $\{T_t\}_{t\geq 0}$ is called *Markovian* if T_t is Markovian for every $t \geq 0$.

Next, we consider a sesquilinear form on some linear manifold of $\mathcal{H}$: $\mathcal{E}(\cdot, \cdot) : D(\mathcal{E}) \times D(\mathcal{E}) \to \mathbb{C}$. We also consider the associated quadratic form: $\mathcal{E}[\cdot] : D(\mathcal{E}) \to \mathbb{C}$, $\mathcal{E}[\xi] := \mathcal{E}(\xi, \xi)$. A real valued quadratic form $\mathcal{E}[\cdot]$ is said to be *semi-bounded* if $\inf\{\mathcal{E}[\xi] : \xi \in D(\mathcal{E}),\ ||\xi|| = 1\} = -b > -\infty$. A quadratic form $(\mathcal{E}, D(\mathcal{E}))$ is said to be *J-real* if $JD(\mathcal{E}) \subset D(\mathcal{E})$ and $\mathcal{E}[J\xi] = \overline{\mathcal{E}[\xi]}$ for any $\xi \in D(\mathcal{E})$. For a given semi-bounded quadratic form $\mathcal{E}$, one considers the inner product given by $\langle \xi, \eta \rangle_\lambda := \mathcal{E}(\xi, \eta) + \lambda \langle \xi, \eta \rangle$, for $\lambda > b$. The form $\mathcal{E}$ is *closed* if $D(\mathcal{E})$ is a Hilbert space for some of the above norms. The form $\mathcal{E}$ is called *closable* if it admits a closed extension.

Associated to a semi-bounded closed form $\mathcal{E}$, there are a self-adjoint operator $(H, D(H))$ and a strongly continuous, symmetric semigroup $\{T_t\}_{t\geq 0}$. Each of the above objects determines uniquely the others according to well known relations (see Section 3.1 of [BR]).

From now on we will consider only J-real, real-valued, semi-bounded, densely defined quadratic forms. It is easy to check that these forms satisfy the relation: $\mathcal{E}[\xi + i\eta] = \mathcal{E}[\xi] + \mathcal{E}[\eta]$ for all $\xi + i\eta \in D(\mathcal{E})^J + iD(\mathcal{E})^J = D(\mathcal{E})$ where $D(\mathcal{E})^J := D(\mathcal{E}) \cap H^J$.

A J-real, real-valued, densely defined quadratic form $(\mathcal{E}, D(\mathcal{E}))$ is called *Markovian* with respect to $\xi_0 \in \mathcal{P}$ if

$$\eta \in D(\mathcal{E})^J \text{ implies } \eta_I \in D(\mathcal{E}) \text{ and } \mathcal{E}[\eta_I] \leq \mathcal{E}[\eta]. \tag{2.1}$$

A closed Markovian form is called a *Dirichlet form*.

Next, we collect main results of [Cip1]. Let $(\mathcal{E}, D(\mathcal{E}))$ be a J-real, real valued, densely defined closed form. Assume that the following properties hold:

$$
\begin{aligned}
&(a) \quad \xi_0 \in D(\mathcal{E}), \\
&(b) \quad \mathcal{E}(\xi,\xi) \geq 0 \text{ for } \xi \in D(\mathcal{E}), \\
&(c) \quad \xi \in D(\mathcal{E})^J \text{ implies } \xi_\pm \in D(\mathcal{E}) \text{ and } \mathcal{E}(\xi_+,\xi_-) \leq 0.
\end{aligned}
\tag{2.2}
$$

Then $\mathcal{E}$ is a Dirichlet form if and only if $\mathcal{E}(\xi,\xi_0) \geq 0$ for all $\xi \in D(\mathcal{E}) \cap \mathcal{P}$. The above result follows from Proposition 4.5 (b) and Proposition 4.10 (ii) of [Cip1].

The following is Theorem 4.11 of [Cip1] : Let $\{T_t\}_{t\geq 0}$ be a J-real, strongly continuous, symmetric semigroup on $\mathcal{H}$ and let $(\mathcal{E}, D(\mathcal{E}))$ be the associated densely defined J-real, real valued quadratic form. Then the followings are equivalent.

$$
\begin{aligned}
&(a) \quad \{T_t\}_{t\geq 0} \text{ is sub-Markovian.} \\
&(b) \quad (\mathcal{E}, D(\mathcal{E})) \text{ is a Dirichlet form .}
\end{aligned}
\tag{2.3}
$$

We refer the reader to [Cip1] for the details.

Let $\gamma : \mathcal{M} \to \mathcal{M}$ be a $\mathbb{Z}_2$-grading of $\mathcal{M}$, so that γ is a $*$-automorphism of $\mathcal{M}$ which is involutive: $\gamma^2 = \mathrm{id}$. Thus $\mathcal{M}$ is a direct sum of $\mathcal{M}_e := \{A \in \mathcal{M} : \gamma(A) = A\}$ and $\mathcal{M}_o := \{A \in \mathcal{M} : \gamma(A) = -A\}$. The elements of $\mathcal{M}_e$ are called *even* and those of $\mathcal{M}_o$ *odd*. The identity element $\mathbf{1} \in \mathcal{M}$ is even. A *superderivation* δ on a $\mathbb{Z}_2$-graded von Neumann algebra $(\mathcal{M}, \gamma)$ is a linear map satisfying that for all $A, B \in \mathcal{M}$,

$$
\begin{aligned}
&(a)\, \delta(AB) = \delta(A)B + \gamma(A)\delta(B), \\
&(b)\, \delta(\gamma(A)) = -\gamma(\delta(A)).
\end{aligned}
\tag{2.4}
$$

A superderivation δ is called *inner* if there is $x \in \mathcal{M}$ such that

$$
\delta(A) = xA - \gamma(A)x,\ \forall\, A \in \mathcal{M}. \tag{2.5}
$$

The element x which defines an inner superderivation as in (2.5) is odd. See Lemma 1.1 of [DL2].

In this paper we assume that ξ_0 is invariant under γ in the sense that

$$
\langle \xi_0, A\xi_0 \rangle = \langle \xi_0, \gamma(A)\xi_0 \rangle \quad A \in \mathcal{M}. \tag{2.6}
$$

Then there exists a unitary operator U_γ on $\mathcal{H}$ such that

$$
\begin{aligned}
\gamma(A) &= U_\gamma A U_\gamma^{-1},\ A \in \mathcal{M}, \\
U_\gamma \mathcal{P} &\subset \mathcal{P}, \\
U_\gamma \xi_0 &= \xi_0, \\
U_\gamma^2 &= \mathbf{1}.
\end{aligned}
\tag{2.7}
$$

In fact $U_\gamma A\xi_0 = \gamma(A)\xi_0$, where $A \in \mathcal{M}$. See the proof of Corollary 2.5.32 of [BR]. Here we have used the property $\gamma^2 = \mathrm{id}$ for the last relation. In this case, it can be checked that U_γ commutes with Δ and J, which implies for any $t \in \mathbb{R}$, $A \in \mathcal{M}$,

$$\begin{aligned} \gamma(\sigma_t(A)) &= \sigma_t(\gamma(A)), \\ \gamma(j(A)) &= j(\gamma(A)). \end{aligned} \tag{2.8}$$

In order to express Dirichlet forms, let us introduce the notion of admissible functions in [Par].

Definition 2.1 *An analytic function $f : D \to \mathbb{C}$ on a domain D containing the strip $Im\ z \in [-1/4, 1/4]$ is said to be admissible if the following properties hold:*

(a) $f(t) \geq 0$ *for* $\forall t \in \mathbb{R}$,

(b) $f(t + i/4) + f(t - i/4) \geq 0$ *for* $\forall t \in \mathbb{R}$,

(c) there exist $M > 0$ and $p > 1$ such that the bound

$$|f(t + is)| \leq M(1 + |t|)^{-p}$$

holds uniformly in $s \in [-1/4, 1/4]$.

We remark that there exists an admissible function satisfying Definition 2.1. See Lemma 3.1 of [Par].

We are ready to give a construction of Dirichlet forms on the standard form $(\mathcal{M}, \mathcal{H}, \mathcal{P}, J)$ associated with pair $(\mathcal{M}, \xi_0)$, where $\mathcal{M}$ is a $\mathbb{Z}_2$-graded von Neumann algebra by γ. Denote by $\mathcal{M}_{an}$ the dense subset of $\mathcal{M}$ consisting of σ_t-analytic elements on a domain containing the strip $I_{1/2} := \{z : |\mathrm{Im} z| \leq \frac{1}{2}\}$[BR]. By Proposition 2.5.21 of [BR], any element $A \in \mathcal{M}_{an}$ is strongly analytic. For a given admissible function f and odd element $x \in \mathcal{M}_o \cap \mathcal{M}_{an}$, define a sesquilinear form $\mathcal{E} : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{C}$ by

$$\begin{aligned} &\mathcal{E}(\eta, \xi) \qquad\qquad (2.9)\\ &= \int \Big\langle \big(\sigma_{t-i/4}(x) - j(\sigma_{t-i/4}(x^*))U_\gamma\big)\eta, \big(\sigma_{t-i/4}(x) - j(\sigma_{t-i/4}(x^*))U_\gamma\big)\xi \Big\rangle f(t)dt \\ &\quad + \int \Big\langle \big(\sigma_{t-i/4}(x^*) - j(\sigma_{t-i/4}(x))U_\gamma\big)\eta, \big(\sigma_{t-i/4}(x^*) - j(\sigma_{t-i/4}(x))U_\gamma\big)\xi \Big\rangle f(t)dt \\ &\equiv \mathcal{E}^{(1)}(\eta, \xi) + \mathcal{E}^{(2)}(\eta, \xi). \end{aligned}$$

Here U_γ is the unitary operator satisfying $\gamma(A) = U_\gamma A U_\gamma^{-1}$, $A \in \mathcal{M}$. The associated quadratic form $\mathcal{E}[\cdot]$ is given by

$$\begin{aligned} \mathcal{E}[\xi] &= \int \|\big(\sigma_{t-i/4}(x) - j(\sigma_{t-i/4}(x^*))U_\gamma\big)\xi\|^2 f(t)dt \\ &\quad + \int \|\big(\sigma_{t-i/4}(x^*) - j(\sigma_{t-i/4}(x))U_\gamma\big)\xi\|^2 f(t)dt. \end{aligned} \tag{2.10}$$

The form given in the above is a bounded form since $\|\sigma_{t-i/4}(x)\| = \|\sigma_{-i/4}(x)\|$ for any $x \in \mathcal{M}_{an}$.

Recall that $\mathcal{M}\xi_0$ is a dense subset in $\mathcal{H}$ and x an odd element. Using the fact that $JB\xi_0 = \Delta^{1/2}B^*\xi_0$, $B \in \mathcal{M}$, one obtains that for any $A \in \mathcal{M}$,

$$\begin{aligned}
&\big(\sigma_{t-i/4}(x^\#) - j(\sigma_{t-i/4}((x^\#)^*))U_\gamma\big)A\xi_0 \\
&\quad = \sigma_{t-i/4}(x^\#)A\xi_0 - \gamma(A)j(\sigma_{t-i/4}((x^\#)^*))\xi_0 \\
&\quad = \big(\sigma_{t-i/4}(x^\#)A - \gamma(A)\sigma_{t-i/4}(x^\#)\big)\xi_0 \qquad (2.11)
\end{aligned}$$

where $x^\#$ stands for x or x^*. Thus we have used inner superderivations in (2.5) to define the sesquilinear form in (2.9).

The following is the main result which corresponds to Theorem 3.1 of [Par].

Theorem 2.1 *For given the admissible function f and odd element $x \in \mathcal{M}_o \cap \mathcal{M}_{an}$, let $(\mathcal{E}, \mathcal{H})$ be defined as in (2.9). Let H be the self-adjoint operator associated with $(\mathcal{E}, \mathcal{H})$, i.e., $\mathcal{E}[\xi] = \langle \xi, H\xi \rangle$, $\forall \xi \in \mathcal{H}$. Assume that there exists a constant $M > 0$ such that the bound*

$$\sup_{s \in [-1/4, 1/4]} \|\sigma_{t+is}(x)\| \leq M \qquad (2.12)$$

holds uniformly in $t \in \mathbb{R}$. Then the following properties hold:
(a) $H\xi_0 = 0$,
(b) $\mathcal{E}[J\xi] = \mathcal{E}[\xi]$, $\forall \xi \in \mathcal{H}$, (J-real)
(c) $\mathcal{E}(\xi_+, \xi_-) \leq 0$ for $\forall \xi \in \mathcal{H}^J$.
Furthermore $(\mathcal{E}, \mathcal{H})$ is a Dirichlet form.

We will produce the proof of the Theorem 2.1 at the end of this section. The following is a consequence of Theorem 2.1:

Theorem 2.2 *Let H be the self-adjoint operator associated with $(\mathcal{E}, \mathcal{H})$ defined as in (2.9) and let $T_t = e^{-tH}$, $t \geq 0$. Then $\{T_t\}_{t \geq 0}$ is a J-real, strongly continuous, symmetric Markovian semigroup.*

Proof: Clearly $\mathcal{E}[\cdot] \geq 0$. Thus Theorem 2.1 and (2.3) imply that $\{T_t\}_{t\geq 0}$ is J-real, strongly continuous, sub-Markovian. It follows from Theorem 2.1 (a) that $T_t\xi_0 = \xi_0$ for any $t \geq 0$. Thus $\{T_t\}_{t\geq 0}$ is Markovian. $\square$

Remark 2.1 *Consider the symmetric embedding:*

$$\begin{aligned}
i_0 : \mathcal{M} &\longrightarrow \mathcal{H} \\
i_0(A) &= \Delta^{1/4}A\xi_0
\end{aligned}$$

Define the maps S_t on $\mathcal{M}$ by

$$S_t : \mathcal{M} \to \mathcal{M}, \quad i_0 \circ S_t \equiv T_t \circ i_0.$$

It follows from Theorem 2.12 of [Cip1] that $\{S_t\}_{t\geq 0}$ is a weak continuous, Markovian semigroup on $\mathcal{M}$. Notice that S_t preserves parity : $S_t(\gamma(A)) = \gamma(S_t(A))$, where $A \in \mathcal{M}$. This can be checked from the fact $U_\gamma H U_\gamma^{-1} = H$ where H is defined in Theorem 2.2.*

We now produce the proof of Theorem 2.1. Let us mention that we modify the proof of Theorem 3.1 of [Par].

Proof of Theorem 2.1 (a) Notice that $U_\gamma \xi_0 = \xi_0$, $JA\xi_0 = \Delta^{\frac{1}{2}} A^* \xi_0$ for any $A \in \mathcal{M}$. We obtain that

$$\begin{aligned} \big(\sigma_{t-i/4}(x) - j(\sigma_{t-i/4}(x^*))U_\gamma\big)\xi_0 &= \Delta^{\frac{1}{4}}\sigma_t(x)\xi_0 - J\Delta^{\frac{1}{4}}\sigma_t(x^*)\Delta^{-\frac{1}{4}}\xi_0 \\ &= \Delta^{\frac{1}{4}}\sigma_t(x)\xi_0 - \Delta^{\frac{1}{4}}\sigma_t(x)\xi_0 \\ &= 0, \end{aligned}$$

and also $\big(\sigma_{t-i/4}(x^*) - j(\sigma_{t-i/4}(x))U_\gamma\big)\xi_0 = 0$. Thus (a) follows from (2.9) and the above facts.

(b) Notice that $U_\gamma x U_\gamma^{-1} = -x$ for any $x \in \mathcal{M}_o$. The first relation in (2.8) implies that

$$\sigma_{t-i/4}(x) = -U_\gamma \sigma_{t-i/4}(x) U_\gamma^{-1}. \tag{2.13}$$

Thus a direct calculation shows that for any $\xi \in \mathcal{H}$,

$$\begin{aligned} &\|\big(\sigma_{t-i/4}(x) - j(\sigma_{t-i/4}(x^*))U_\gamma\big)J\xi\|^2 \\ &\quad = \|U_\gamma J\big(-j(\sigma_{t-i/4}(x))U_\gamma + \sigma_{t-i/4}(x^*)\big)\xi\|^2 \\ &\quad = \|\big(\sigma_{t-i/4}(x^*) - j(\sigma_{t-i/4}(x))U_\gamma\big)\xi\|^2. \end{aligned}$$

Here we have used $U_\gamma^{-1} = U_\gamma$ and U_γ commutes with J. Thus we have $\mathcal{E}^{(1)}[J\xi] = \mathcal{E}^{(2)}[\xi]$. The method used in the above also implies that $\mathcal{E}^{(2)}[J\xi] = \mathcal{E}^{(1)}[\xi]$.

(c) By the expression of $\mathcal{E}(\xi,\eta)$ in (2.9), $\mathcal{E}(\xi_+,\xi_-)$ can be written as

$$\begin{aligned} \mathcal{E}(\xi_+,\xi_-) &= \mathcal{E}^{(1)}(\xi_+,\xi_-) + \mathcal{E}^{(2)}(\xi_+,\xi_-) \\ &= (\mathrm{I}^{(1)} + \mathrm{II}^{(1)}) + (\mathrm{I}^{(2)} + \mathrm{II}^{(2)}) \end{aligned} \tag{2.14}$$

where

$$\begin{aligned} \mathrm{I}^{(1)} = \int \Big(&\langle \sigma_{t-i/4}(x)\xi_+, \sigma_{t-i/4}(x)\xi_- \rangle \\ &+ \langle \sigma_{t-i/4}(x^*)\xi_-, \sigma_{t-i/4}(x^*)\xi_+ \rangle\Big) f(t)\, dt, \end{aligned} \tag{2.15}$$

$$\begin{aligned} \mathrm{II}^{(1)} &= -\int \Big(\langle \sigma_{t-i/4}(x)\xi_+, j\,(\sigma_{t-i/4}(x^*))U_\gamma \xi_- \rangle \\ &\qquad + \langle j\,(\sigma_{t-i/4}(x^*))U_\gamma \xi_+, \sigma_{t-i/4}(x)\xi_- \rangle \Big) f(t)\,dt. \end{aligned}$$

and $\mathrm{I}^{(2)}$ and $\mathrm{II}^{(2)}$ are obtained from $\mathrm{I}^{(1)}$ and $\mathrm{II}^{(1)}$ respectively by replacing x by x^* in the above. Here we have used the facts that $U_\gamma = U_\gamma^{-1}$, the antilinearity of J and (2.13) to obtain the second term of $\mathrm{I}^{(1)}$ in (2.15). As a consequence of Theorem 4(7) of [Ara], $\mathcal{M}\xi_+ \perp \mathcal{M}\xi_-$, which implies $\mathrm{I}^{(1)} = 0$. Similarly $\mathrm{I}^{(2)} = 0$. See also proof of Proposition 5.3(ii) of [Cip1].

Next, we consider $\mathrm{II}^{(1)}$. Since $\sigma_{t-i/4}(x)^* = \sigma_{t+i/4}(x^*)$, it follows from (2.7) and (2.13) that

$$\begin{aligned} &\langle \sigma_{t-i/4}(x)\xi_+\,,\, j\,(\sigma_{t-i/4}(x^*))U_\gamma \xi_- \rangle \\ &\qquad = \langle\, \xi_+\,,\, \sigma_{t+i/4}(x^*) j(\sigma_{t+i/4}(x)^*)U_\gamma \xi_- \rangle \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} &\langle j\,(\sigma_{t-i/4}(x^*))U_\gamma \xi_+\,,\, \sigma_{t-i/4}(x)\xi_- \rangle \\ &\qquad = \langle -U_\gamma j\,(\sigma_{t-i/4}(x^*))\xi_+\,,\, -U_\gamma \sigma_{t-i/4}(x)U_\gamma \xi_- \rangle \\ &\qquad = \langle \xi_+\,,\, \sigma_{t-i/4}(x) j(\sigma_{t-i/4}(x^*)^*)U_\gamma \xi_- \rangle. \end{aligned} \tag{2.17}$$

Substituting (2.16) and (2.17) into the second expression of (2.15), we get that

$$\begin{aligned} \mathrm{II}^{(1)} &= -\int \langle\, \xi_+, \sigma_{t+i/4}(x^*) j(\sigma_{t+i/4}(x)^*)U_\gamma \xi_- \rangle\, f(t)\,dt \\ &\qquad - \int \langle \xi_+, \sigma_{t-i/4}(x) j(\sigma_{t-i/4}(x^*)^*)U_\gamma \xi_- \,\rangle f(t)\,dt. \end{aligned}$$

Notice that for any $x \in \mathcal{M}_{an}$ and $\xi \in \mathcal{H}$, the map

$$z \mapsto j(\sigma_z(x)^*)U_\gamma \xi$$

is analytic on a domain containing the strip $I_{1/2}$. In fact, the analyticity follows from the facts that $\langle \eta, j\big(\sigma_z(x)^*\big)U_\gamma \xi \rangle = \langle \sigma_z(x)^* J U_\gamma \xi, J\eta \rangle = \langle JU_\gamma \xi, \sigma_z(x) J\eta \rangle$ for any $\eta, \xi \in \mathcal{H}$, and that weak analyticity implies strong analyticity [RS, Theorem VI.4].

Using the Cauchy integral theorem, the assumption in the theorem, the property (c) in Definition 2.1 and $\sigma_t(x)^* = \sigma_t(x^*)$, we obtain that

$$\begin{aligned} \mathrm{II}^{(1)} = &\quad -\int \langle \xi_+, \sigma_t(x^*) j(\sigma_t(x^*))U_\gamma \xi_- \rangle f(t - i/4)\,dt \\ &\quad - \int \langle \xi_+, \sigma_t(x) j(\sigma_t(x))U_\gamma \xi_- \rangle f(t + i/4)\,dt. \end{aligned}$$

Replacing x by x^* in the above, we obtain the expression of $\mathrm{II}^{(2)}$. Thus we get

$$\begin{aligned} \mathrm{II} &= -\int \langle \xi_+, [\sigma_t(x) j(\sigma_t(x)) + \sigma_t(x^*) j(\sigma_t(x^*))] U_\gamma \xi_- \rangle \\ &\quad \cdot (f(t - i/4) + f(t + i/4))\, dt. \end{aligned}$$

Recall that $U_\gamma \mathcal{P} \subset \mathcal{P}$ in (2.7). Since $\sigma_t(x^\#) j(\sigma_t(x^\#)) U_\gamma \xi_- \in \mathcal{P}$, $\langle \xi_+, \sigma_t(x^\#) j(\sigma_t(x^\#)) U_\gamma \xi_- \rangle \geq 0$ for $\forall\, t \in \mathbb{R}$, where $x^\#$ is either x or x^*. By the property (b) in Definition 2.1, we conclude that $\mathrm{II} \leq 0$. This proved the part (c) of the theorem.

Clearly $\mathcal{E}[\cdot] \geq 0$. Note that $\mathcal{E}(\xi, \xi_0) = 0$ for all $\xi \in \mathcal{H}$. Theorem 2.1 (b),(c) imply that the form $(\mathcal{E}, \mathcal{H})$ satisfies the conditions in (2.2). Thus $(\mathcal{E}, \mathcal{H})$ is a Dirichlet form. □

3 Markovian semigroups on CAR algebras with quasi-free states

In this section, we apply the results in Section 2 to construct Dirichlet forms and associated Markovian semigroups on CAR algebras with respect to quasi-free states.

We first review the notion of CAR algebras. For the details we refer to Section 5.2.2 of [BR]. Let $\mathfrak{h}_0$ be a separable pre-Hilbert space with inner product $(\cdot, \cdot)$ and $\mathfrak{h}$ the completion of $\mathfrak{h}_0$. Let $\mathcal{A}(\mathfrak{h})$ be the C^*-algebra generated by the identity $\mathbf{1}$ and elements $a(f)$, $f \in \mathfrak{h}$, satisfying

$$\begin{aligned} &(a)\, f \mapsto a(f) \text{ is antilinear}, \\ &(b)\, \{a(f), a(g)\} = 0, \\ &(c)\, \{a(f), a(g)^*\} = (f, g)\mathbf{1} \end{aligned} \tag{3.1}$$

for all f, $g \in \mathfrak{h}$, where $\{A, B\} := AB + BA$. Notice that

$$\|a(f)\| = \|f\| \quad \forall f \in \mathfrak{h}, \tag{3.2}$$

and so $\mathcal{A}(\mathfrak{h}_0) = \mathcal{A}(\mathfrak{h})$. Let $\gamma : \mathcal{A}(\mathfrak{h}) \to \mathcal{A}(\mathfrak{h})$ be the $*$-automorphism defined by

$$\gamma(a(f)) = -a(f), \tag{3.3}$$

for all $f \in \mathfrak{h}$. Then $(\mathcal{A}(\mathfrak{h}), \gamma)$ is a $\mathbb{Z}_2$-graded C^*-algebra.

Next, we describe quasi-free states on $\mathcal{A}(\mathfrak{h})$. Let A be a bounded and non-negative operator on $\mathfrak{h}$. Recall that a vector $g \in \mathfrak{h}$ is called an analytic vector for an operator B on $\mathfrak{h}$ if for each $n \in \mathbb{N}$, $g \in D(B^n)$ and for some $t > 0$,

$$\sum_{n=0}^{\infty} \frac{\|B^n g\|}{n!} t^n < \infty.$$

In the rest of this paper, we assume that A satisfies the following properties:

Assumption 3.1 *(a) There exists* $\alpha > 0$ *such that* $0 < A \leq \alpha\mathbf{1}$.
(b) The inverse A^{-1} *of* A *exists as a (unbounded) self-adjoint and positive operator on* $\mathfrak{h}$.
(c) For any $z \in \mathbb{C}$, A^z *leaves* $\mathfrak{h}_0$ *invariant, i.e.,* $A^z\mathfrak{h}_0 \subset \mathfrak{h}_0$. *Moreover,* $z \mapsto A^z f$ *is entire analytic for any* $f \in \mathfrak{h}_0$.
(d) Any $g \in \mathfrak{h}_0$ *is an analytic vector for* $A^{-1/2}$.

We remark that a dense subspace $\mathfrak{h}_0$ of $\mathfrak{h}$ satisfying Assumption 3.1 exists by the spectral theorem.

Example 3.1 (Ideal Fermi gases) *Let* $\mathfrak{h}$ *be the space* $L^2(\mathbb{R}^d, dx)$ *and* Δ *the Laplacian operator on* $\mathfrak{h}$. *Let* A *be given by*

$$A = e^{-\beta(-\Delta - \mu\mathbf{1})},$$

where $\beta > 0, \mu \in \mathbb{R}$. *The subspace* $\mathfrak{h}_0$ *is given by* $\mathfrak{h}_0 = \{f \in \mathfrak{h} : \widehat{f} \in C_c(\mathbb{R}^d)\}$, *where* $\widehat{f}$ *denotes the Fourier transform of* f. *Then Assumption 3.1 satisfied with* $\alpha = e^{\beta\mu}$.

For given bounded operator A on $\mathfrak{h}$ satisfying Assumption 3.1, the gauge invariant quasi-free state ω on $\mathcal{A}(\mathfrak{h})$ is defined by

$$\omega(a(f_m)^* \cdots a(f_1)^* a(g_1) \cdots a(g_n)) = \delta_{nm} \det((g_i, A(\mathbf{1} + A)^{-1} f_j)) \tag{3.4}$$

for any $f_1, \cdots, f_m, g_1, \cdots, g_n \in \mathfrak{h}$. Let $\sigma_t : \mathcal{A}(\mathfrak{h}) \to \mathcal{A}(\mathfrak{h})$ be the $*$-automorphisms defined by

$$\sigma_t(a(f)) = a(A^{it} f) \tag{3.5}$$

for any $f \in \mathfrak{h}$, $t \in \mathbb{R}$. Then one can check KMS conditions

$$\omega(B\sigma_{-i}(C)) = \omega(CB)$$

for any analytic elements $B, C \in \mathcal{A}(\mathfrak{h})$[BR].

Let $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ be the GNS representation of $(\mathcal{A}(\mathfrak{h}), \omega)$. Define $\mathcal{M}_\omega := \pi_\omega(\mathcal{A}(\mathfrak{h}))''$, $a(f)_\omega := \pi_\omega(a(f)), a(f)^*_\omega := \pi_\omega(a(f)^*), f \in \mathfrak{h}$, $\sigma_t^\omega := \pi_\omega \sigma_t \pi_\omega^{-1}$ and $\gamma_\omega := \pi_\omega \gamma \pi_\omega^{-1}$. From now on we suppress ω from the notations, e.g., $\mathcal{M} := \mathcal{M}_\omega$, $\mathcal{H} := \mathcal{H}_\omega$, $a(f) := a_\omega(f), a(f)^* := a_\omega(f)^*$, $\sigma_t := \sigma_t^\omega$ and $\gamma := \gamma_\omega$ etc. By continuity, σ_t and γ extend to $\mathcal{M}$. We also write $\xi_0 = \Omega_\omega$. Notice that ω satisfies the σ_t-KMS conditions [BR]. Thus, by the uniqueness of the modular automorphism, σ_t is the modular automorphism on $\mathcal{M}$. See Theorem 5.3.10 of [BR]. Let Δ and J denote the modular operator and the modular conjugation associated with the pair $(\mathcal{M}, \xi_0)$ respectively. Thus $\sigma_t(B) = \Delta^{it} B \Delta^{-it}, \forall B \in \mathcal{M}$. It follows from (3.3) and (3.4) that

$$\omega(\gamma(B)) = \omega(B), \ \ \forall B \in \mathcal{M}.$$

Since ω is γ-invariant, that is, $\langle\xi_0, \gamma(B)\xi_0\rangle = \langle\xi_0, B\xi_0\rangle$, $\forall B \in \mathcal{M}$, there exists a unitary operator U_γ on $\mathcal{H}$ satisfying the properties in (2.7).

For any $g \in \mathfrak{h}$, define an odd element $B(g) \in \mathcal{M}_o$ by

$$B(g) := \frac{1}{\sqrt{2}}(a(g) + a(g)^*). \tag{3.6}$$

Using the antilinearity of $a(g)$ and (3.6), we get that for any $g \in \mathfrak{h}$,

$$\begin{aligned} a(g) &= \frac{1}{\sqrt{2}}(B(g) + iB(ig)), \\ a(g)^* &= \frac{1}{\sqrt{2}}(B(g) - iB(ig)). \end{aligned} \tag{3.7}$$

From the CAR relations (3.1), we obtain that for any $f, g \in \mathfrak{h}$,

$$\begin{aligned} \{a(f), B(g)\} &= \frac{1}{\sqrt{2}}(f, g)\mathbf{1}, \\ \{a(f)^*, B(g)\} &= \frac{1}{\sqrt{2}}(g, f)\mathbf{1}. \end{aligned} \tag{3.8}$$

Denote by $\mathcal{M}_{fin}$ the algebra generated by $\mathbf{1}$ and $a(f)^\#$, $f \in \mathfrak{h}_0$, where $a(f)^\#$ is either $a(f)$ or $a(f)^*$ for $f \in \mathfrak{h}_0$. Because of (3.7), $\mathcal{M}_{fin}$ is equal to the algebra generated by $\mathbf{1}$ and $B(f)$, $f \in \mathfrak{h}_0$. Since $\mathcal{M}_{fin}$ is norm dense in $\pi_\omega(\mathcal{A}(\mathfrak{h}))$, $\mathcal{M}_{fin}\xi_0$ is also dense in $\mathcal{H}$. For any $f \in \mathfrak{h}_0$, $z \in \mathbb{C}$, we write

$$\begin{aligned} \sigma_z(a(f)) &:= a(A^{i\bar{z}}f), \\ \sigma_z(a(f)^*) &:= a(A^{iz}f)^*. \end{aligned} \tag{3.9}$$

In fact, using (3.4) and Assumption 3.1(c), one can check that for any $f \in \mathfrak{h}_0$ and $\xi \in \mathcal{M}_{fin}\xi_0$, the map $z \mapsto \sigma_z(a(f)^\#)\xi$ is the analytic extension on $\mathbb{C}$ of the map $t \mapsto \sigma_t(a(f)^\#)\xi$ (where $a(f)^\#$ is either $a(f)$ or $a(f)^*$). Thus for any $g \in \mathfrak{h}_0$, $a(g)$, $a(g)^*$ and $B(g)$ are σ_t-entire analytic odd elements of $\mathcal{M}$.

Let us turn to construction of a Dirichlet form which generates the symmetric Markovian semigroup on CAR algebra $\mathcal{A}(\mathfrak{h})$ with respect to the quasi-free state ω. An admissible function f is said to be *normalized* if $\int f(t)\,dt = 1$. For given normalized function f and a complete orthonormal system(CONS) $\{g_j\}_{j=1}^\infty \subset \mathfrak{h}_0$ for $\mathfrak{h}$, define a sesquilinear form $\mathcal{E} : D(\mathcal{E}) \times D(\mathcal{E}) \to \mathbb{C}$ as follows:

$$\begin{aligned} D(\mathcal{E}) &= \{\xi \in \mathcal{H} : \sum_{j=1}^\infty \mathcal{E}_j(\xi, \xi) < \infty\}, \\ \mathcal{E}(\eta, \xi) &= \sum_{j=1}^\infty \mathcal{E}_j(\eta, \xi), \quad \eta, \xi \in D(\mathcal{E}) \end{aligned} \tag{3.10}$$

where for each $j \in \mathbb{N}$,

$$\begin{aligned} D(\mathcal{E}_j) &= \mathcal{H}, \\ \mathcal{E}_j(\eta,\xi) &= \int \Big\langle \big(\sigma_{t-i/4}(a(g_j)) - j(\sigma_{t-i/4}(a(g_j)^*))U_\gamma\big)\eta, \\ &\qquad\qquad \big(\sigma_{t-i/4}(a(g_j)) - j(\sigma_{t-i/4}(a(g_j)^*))U_\gamma\big)\xi\Big\rangle f(t)dt \\ &\quad + \int \Big\langle \big(\sigma_{t-i/4}(a(g_j)^*) - j(\sigma_{t-i/4}(a(g_j)))U_\gamma\big)\eta, \\ &\qquad\qquad \big(\sigma_{t-i/4}(a(g_j)^*) - j(\sigma_{t-i/4}(a(g_j)))U_\gamma\big)\xi\Big\rangle f(t)dt. \end{aligned} \tag{3.11}$$

We also define the associated quadratic forms by

$$\begin{aligned} \mathcal{E}_j[\xi] &= \mathcal{E}_j(\xi,\xi), \quad \xi \in \mathcal{H},\, j \in \mathbb{N}, \\ \mathcal{E}[\xi] &= \sum_{j=1}^{\infty} \mathcal{E}_j[\xi], \quad \xi \in D(\mathcal{E}). \end{aligned} \tag{3.12}$$

We remark that the expression $\mathcal{E}_j(\eta,\xi)$ in (3.11) can be obtained from $\mathcal{E}(\eta,\xi)$ in (2.9) by replacing x by $a(g_j)$.

The followings are the main results in this section. It turns out that the form defined as in (3.10) and (3.11) is independent of the CONS $\{g_j\} \subset \mathfrak{h}_0$ for $\mathfrak{h}$ and the normalized admissible function f we have chosen:

Theorem 3.1 *Let $(\mathcal{E}, D(\mathcal{E}))$ be defined as in (3.10) and (3.11). Then the form $(\mathcal{E}, D(\mathcal{E}))$ is a densely defined Dirichlet form, and independent of the CONS $\{g_j\} \subset \mathfrak{h}_0$ for $\mathfrak{h}$ and the normalized admissible function f we have chosen. Moreover, let $\{T_t\}_{t\ge 0}$ be the semigroup associated to the form $(\mathcal{E}, D(\mathcal{E}))$. Then the semigroup $\{T_t\}_{t\ge 0}$ is J-real, strongly continuous, symmetric and Markovian.*

Theorem 3.2 *Let $\{T_t\}_{t\ge 0}$ be the symmetric Markovian semigroup associated to the form $(\mathcal{E}, D(\mathcal{E}))$ in Theorem 3.1 and H the Dirichlet operator, i.e., $T_t = e^{-tH}, t \ge 0$. Then the following results hold:*

(a) H is essentially self-adjoint on $\mathcal{M}_{fin}\xi_0$.

(b) The zero is a simple eigenvalue of H with eigenvector ξ_0. Moreover $(0,2)\cap\sigma(H) = \emptyset$.

By the spectral theorem, Theorem 3.2 (b) implies that for any $\xi \in \mathcal{H}$ and $t \ge 0$,

$$\|T_t\xi - \langle\xi_0,\xi\rangle\xi_0\|_{\mathcal{H}} \le e^{-2t}\|\xi - \langle\xi_0,\xi\rangle\xi_0\|_{\mathcal{H}}.$$

Thus $\{T_t\}_{t\ge 0}$ converges to the equilibrium exponentially fast. Note that Theorem 3.2 (b) implies that the vector ξ_0 is a simple, strictly positive ground state for the generator H. In view of [Cip2], T_t satisfies the indecomposability and the ergodicity (for each positive ξ, η, there exists $t > 0$ such that $\langle\xi, T_t\eta\rangle > 0$). See Theorem 4.3 of [Cip2].

Remark 3.1 *The main results in this section can be generalized in several ways. For an instance, if one replace g_j by $B^\lambda g_j$ (for the definition of B see the equation (4.1)), $j \in \mathbb{N}$, for some $\lambda \in \mathbb{R}$ in the definition of $\mathcal{E}(\eta, \xi)$ in (3.10) and (3.11), and modifies Assumption 3.1 (d) appropriately, then all of the results in this section still hold with a modified spectral gap in Theorem 3.2 (b), i.e., $(0, 2^{1+2\lambda}) \cap \sigma(H) = \emptyset$.*

In the rest of this section, we produce the proof of Theorem 3.1. The proof of Theorem 3.2 will be postponed to the next section.

Proof of Theorem 3.1 Notice that $a(g_j) \in \mathcal{M}_o \cap \mathcal{M}_{an}$, $\forall j \in \mathbb{N}$ (see below of the equation (3.9)). Since $\|\sigma_{t+is}(a(g_j))\| = \|A^s g_j\|$ (see (3.9) and (3.2)), the assumption (2.12) is satisfied. Thus Theorem 2.1 impies that each $\mathcal{E}_j$, $j \in \mathbb{N}$, is a Dirichlet form. Thus if one can show that $D(\mathcal{E})$ is dense in $\mathcal{H}$, then it follows from Theorem 5.2(ii) of [Cip1] that $(\mathcal{E}, D(\mathcal{E}))$ is a Dirichlet form and it generates a J-real, strongly continuous, symmetric and sub-Markovian semigroup $T_t = e^{-tH}$, $t \geq 0$. Using the calculations used in the proof of Theorem 2.1 (a), we get that for each $j \in \mathbb{N}$,

$$\big(\sigma_{t-i/4}(a(g_j)^\#) - j(\sigma_{t-i/4}((a(g_j)^\#)^*))U_\gamma\big)\xi_0 = 0.$$

Thus $H\xi_0 = 0$ and so $\{T_t\}_{t\geq 0}$ is Markovian.

Let us show that $D(\mathcal{E})$ is dense. Since $\mathcal{M}_{fin}\xi_0$ is dense in $\mathcal{H}$, it is sufficient to show that $\mathcal{M}_{fin}\xi_0$ is contained in $D(\mathcal{E})$. For any $B \in \mathcal{M}$, it follows from (2.11) and (3.9) that

$$\begin{aligned}
&\big(\sigma_{t-i/4}(a(g_j)) - j(\sigma_{t-i/4}(a(g_j)^*))U_\gamma\big)B\xi_0 \\
&= a(A^{it-1/4}g_j)B\xi_0 - \gamma(B)a(A^{it-1/4}g_j)\xi_0 \\
&= \begin{cases} [a(A^{it-1/4}g_j), B]\xi_0 & \text{if} \quad B \in \mathcal{M}_e \\ \{a(A^{it-1/4}g_j), B\}\xi_0 & \text{if} \quad B \in \mathcal{M}_o \end{cases}
\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}
&\big(\sigma_{t-i/4}(a(g_j)^*) - j(\sigma_{t-i/4}(a(g_j)))U_\gamma\big)B\xi_0 \\
&= \begin{cases} [a(A^{it+1/4}g_j)^*, B]\xi_0 & \text{if} \quad B \in \mathcal{M}_e \\ \{a(A^{it+1/4}g_j)^*, B\}\xi_0 & \text{if} \quad B \in \mathcal{M}_o. \end{cases}
\end{aligned} \tag{3.14}$$

Let $B \in \mathcal{M}_{fin}$ be any element of the form $B = B(f_1)\cdots B(f_n) \in \mathcal{M}_{fin}$, $f_1, \cdots, f_n \in \mathfrak{h}_0$, $n \in \mathbb{N}$ (see (3.6) for the definition of $B(f_i)$). Consider the case for $n = 2m$, that is, $B \in \mathcal{M}_e$. Using the CARs (3.1) and (3.8), we obtain that

$$\begin{aligned}
[a(A^{it-1/4}g_j), B] &= \sum_{k=1}^n \frac{(-1)^{k+1}}{\sqrt{2}}(A^{it-1/4}g_j, f_k)B(f_1)\cdots\widehat{B}(f_k)\cdots B(f_n), \quad (3.15)\\
[a^*(A^{it+1/4}g_j), B] &= \sum_{k=1}^n \frac{(-1)^{k+1}}{\sqrt{2}}(f_k, A^{it+1/4}g_j)B(f_1)\cdots\widehat{B}(f_k)\cdots B(f_n)
\end{aligned}$$

where $\widehat{B}(f)$, $f \in \mathfrak{h}_0$, denotes that $B(f)$ is omitted. By the Schwarz inequaity and the Bessel inequality it is easy to see that for any $h_1, h_2 \in \mathfrak{h}_0, t \in \mathbb{R}$,

$$\sum_{j=1}^{\infty} |(A^{it}h_1, g_j)\,(g_j, A^{it}h_2)| \leq ||h_1||\,||h_2||.$$

Using the above inequality and that $||B(h)|| \leq ||h||$ for any $h \in \mathfrak{h}$, we get from (3.13), (3.14) and (3.15) that

$$\sum_{j=1}^{\infty} \mathcal{E}_j[B\xi_0] < \infty.$$

Thus $B\xi_0 \in D(\mathcal{E})$. The result same as that in the above holds for odd n. Hence $\mathcal{M}_{fin}\xi_0$ is contained in $D(\mathcal{E})$ and so $D(\mathcal{E})$ is dense.

Next we will show that $(\mathcal{E}, D(\mathcal{E}))$ is independent of the choice of the CONS $\{g_j\} \subset \mathfrak{h}_0$ for $\mathfrak{h}$ and the normalized admissible function f. Using the Parseval relations, we get that for all $t \in \mathbb{R}$, $k, k' \in \mathbb{N}$,

$$\begin{aligned}&\sum_{j=1}^{\infty}(f_k, A^{it-1/4}g_j)(A^{it-1/4}g_j, f_{k'})\\&= (f_k, A^{-1/2}f_{k'})\end{aligned}$$

and

$$\begin{aligned}&\sum_{j=1}^{\infty}(A^{it+1/4}g_j, f_k)(f_{k'}, A^{it+1/4}g_j)\\&= (f_{k'}, A^{1/2}f_k).\end{aligned}$$

It follows from (3.12), (3.13), (3.14), (3.15), the above relations and Dominated convergence theorem that

$$\begin{aligned}\mathcal{E}[B\xi_0] = \sum_{k=1}^{n}\sum_{k'=1}^{n} &\frac{(-1)^{k+k'}}{2}\{(f_k, A^{-1/2}f_{k'}) + (f_{k'}, A^{1/2}f_k)\} \\ &\cdot\langle B(f_1)\cdots\widehat{B}(f_k)\cdots B(f_n)\xi_0, B(f_1)\cdots\widehat{B}(f_{k'})\cdots B(f_n)\xi_0\rangle,\end{aligned} \tag{3.16}$$

where $B\xi_0 = B(f_1)\cdots B(f_n)\xi_0 \in \mathcal{M}_{fin}\xi_0$. It turns out that $\mathcal{M}_{fin}\xi_0$ is a core for the Dirichlet operator H associated to the form $(\mathcal{E}, D(\mathcal{E}))$ (Theorem 3.2 (a)). Thus H is independent of normalized admissible function f and CONS $\{g_j\} \subset \mathfrak{h}_0$ and so is $(\mathcal{E}, D(\mathcal{E}))$. $\square$

4 Decomposition of quasi-free Hilbert space : Proof of Theorem 3.2

As in [BKP], we will decompose the Hilbert space $\mathcal{H} = \mathcal{H}_\omega$ into direct sum of $\mathcal{H}^{(m,n)}$, $m, n \in \mathbb{N} \cup \{0\}$, where $\mathcal{H}^{(m,n)}$ is the Hilbert space of m quasi-particles and n anti quasi-particles.

We then use the results to prove Theorem 3.2. The decomposition method we use is essentially same as that in [AW]. See also Example 5.2.20 in [BR].

Denote by B the operator given by

$$B := A^{-\frac{1}{2}} + A^{\frac{1}{2}}. \tag{4.1}$$

In this section, $a(g)^{\#}$, $g \in \mathfrak{h}_0$ stands for $a(g)$ or $a(g)^*$. And we write

$$\begin{aligned} \delta(a(g)^{\#}) &:= a(g)^{\#} - j(\sigma_{-i/2}((a(g)^{\#})^*))U_\gamma \\ &= a(g)^{\#} + U_\gamma j(\sigma_{-i/2}((a(g)^{\#})^*)), \end{aligned} \tag{4.2}$$

as a bounded operator on $\mathcal{H}$. For $g \in \mathfrak{h}_0$, $C \in \mathcal{M}$, we have

$$\delta(a(g)^{\#})C\xi_0 = a(g)^{\#}C\xi_0 - \gamma(C)a(g)^{\#}\xi_0 \tag{4.3}$$

(see the argument of (2.11)). Recall (2.13), $U_\gamma^{-1} = U_\gamma$ and the fact that U_γ commutes with J and Δ. It follows from (3.9) and (4.2) that for any $g \in \mathfrak{h}_0$

$$\begin{aligned} \delta(a(B^{-\frac{1}{2}}A^{-\frac{1}{4}}g)) &= a(B^{-\frac{1}{2}}A^{-\frac{1}{4}}g) - j(\sigma_{-i/2}(a(B^{-\frac{1}{2}}A^{-\frac{1}{4}}g)^*))U_\gamma \\ &= a(B^{-\frac{1}{2}}A^{-\frac{1}{4}}g) + U_\gamma j(a(B^{-\frac{1}{2}}A^{\frac{1}{4}}g)^*), \end{aligned} \tag{4.4}$$

$$\begin{aligned} \delta(a(B^{-\frac{1}{2}}A^{\frac{1}{4}}g)^*) &= a(B^{-\frac{1}{2}}A^{\frac{1}{4}}g)^* - j(\sigma_{-i/2}(a(B^{-\frac{1}{2}}A^{\frac{1}{4}}g)))U_\gamma \\ &= a(B^{-\frac{1}{2}}A^{\frac{1}{4}}g)^* + U_\gamma j(a(B^{-\frac{1}{2}}A^{-\frac{1}{4}}g)). \end{aligned}$$

Since $(j(a(g)))^* = j(a(g)^*)$ and $U_\gamma = U_\gamma^{-1} = U_\gamma^*$, a computation shows that

$$\begin{aligned} (\delta(a(B^{-\frac{1}{2}}A^{-\frac{1}{4}}g)))^* &= a(B^{-\frac{1}{2}}A^{-\frac{1}{4}}g)^* + j(a(B^{-\frac{1}{2}}A^{\frac{1}{4}}g))U_\gamma \\ &= a(B^{-\frac{1}{2}}A^{-\frac{1}{4}}g)^* - U_\gamma j(a(B^{-\frac{1}{2}}A^{\frac{1}{4}}g)) \\ &= a(B^{\frac{1}{2}}A^{\frac{1}{4}}g)^* - \delta(a(B^{-\frac{1}{2}}A^{\frac{3}{4}}g)^*). \end{aligned} \tag{4.5}$$

Here we have used the fact that $B^{-\frac{1}{2}}(A^{-\frac{1}{4}} + A^{\frac{3}{4}}) = B^{\frac{1}{2}}A^{\frac{1}{4}}$. Using the method similar to that used in the above, we get that

$$(\delta(a(B^{-\frac{1}{2}}A^{\frac{1}{4}}g)^*))^* = a(B^{\frac{1}{2}}A^{-\frac{1}{4}}g) - \delta(a(B^{-\frac{1}{2}}A^{-\frac{3}{4}}g)). \tag{4.6}$$

From notational brevity, we write that for any $g \in \mathfrak{h}_0$

$$\begin{aligned} D_1(g) &:= \delta(a(B^{-\frac{1}{2}}A^{-\frac{1}{4}}g)), \\ D_2(g) &:= \delta(a(B^{-\frac{1}{2}}A^{\frac{1}{4}}g)^*). \end{aligned} \tag{4.7}$$

Then it follows from (4.5) - (4.7) that

$$\begin{aligned} D_1(g)^* &= a(B^{\frac{1}{2}}A^{\frac{1}{4}}g)^* - D_2(A^{\frac{1}{2}}g), \\ D_2(g)^* &= a(B^{\frac{1}{2}}A^{-\frac{1}{4}}g) - D_1(A^{-\frac{1}{2}}g). \end{aligned} \tag{4.8}$$

We first collect some properties of $D_i(g)$ for $g \in \mathfrak{h}_0$ and $i = 1, 2$.

Lemma 4.1 *$D_i(g)\xi_0 = 0$ for any $g \in \mathfrak{h}_0$ and $i = 1, 2$.*

Proof: This follows from (4.3) and (4.7). □

Lemma 4.2 *The following relations hold for any g, $h \in \mathfrak{h}_0$:*

(a) $\{D_i(g), D_j(h)\} = 0,\ i = 1, 2,\ j = 1, 2,$

(b) $\{D_1(g), a(h)\} = 0,$

(c) $\{D_2(g), a(h)^*\} = 0,$

(d) $\{D_1(g), a(B^{\frac{1}{2}} A^{\frac{1}{4}} h)^*\} = (g, h)\mathbf{1},$

(e) $\{D_2(g), a(B^{\frac{1}{2}} A^{-\frac{1}{4}} h)\} = (h, g)\mathbf{1}.$

Proof: Notice that $a(h)^{\#} U_\gamma = -U_\gamma a(h)^{\#}$ for $h \in \mathfrak{h}_0$. The proofs follows from direct computations using the definitions in (4.7) and the CARs in (3.1). □

Proposition 4.1 *The following canonical anti-commutation relations (CARs) hold for any g, $h \in \mathfrak{h}_0$:*

(a) $\{D_1(g), D_1(h)^*\} = (g, h)\mathbf{1},$
$\{D_1(g), D_1(h)\} = 0, \quad \{D_1(g)^*, D_1(h)^*\} = 0,$

(b) $\{D_2(g), D_2(h)^*\} = (h, g)\mathbf{1},$
$\{D_2(g), D_2(h)\} = 0, \quad \{D_2(g)^*, D_2(h)^*\} = 0,$

(c) $\{D_1(g), D_2(h)\} = 0, \quad \{D_1(g), D_2(h)^*\} = 0,$
$\{D_1(g)^*, D_2(h)\} = 0, \quad \{D_1(g)^*, D_2(g)^*\} = 0.$

Proof: The anti-commutation relations follow from (4.7), (4.8), (3.1) and Lemma 4.2. □

Now we are ready to decompose the Hilbert space $\mathcal{H} = \mathcal{H}_\omega$, called quasi-free Hilbert space. According to Lemma 4.1 and the CARs in Proposition 4.1, $D_i(g)$ and $D_i(h)^*$, $i = 1, 2$, $g, h \in \mathfrak{h}_0$, can be thought as annihilation and creation operators respectively. We remark that $h \mapsto D_1(h)^*$ is linear, but $g \mapsto D_2(g)^*$ is conjugate linear. With an abuse of terminology, we call $D_1(h)^*$ and $D_2(h)^*$ the creation operators for quasi-particles and anti quasi-particles respectively for $h \in \mathfrak{h}_0$. The following is the decomposition of $\mathcal{H}$:

Theorem 4.1 *The following decomposition holds:*

$$\mathcal{H} = \bigoplus_{m,n=0}^{\infty} \mathcal{H}^{(m,n)}$$

where for each $m, n \in \mathbb{N} \cup \{0\}$, $\mathcal{H}^{(m,n)}$ is the closure of the subspace spanned by the vectors of the form

$$(\prod_{j=1}^{m} D_1(g_j)^*)(\prod_{l=1}^{n} D_2(h_l)^*)\xi_0,\ g_j,\ h_l \in \mathfrak{h}_0.$$

In the case in which $m = 0\,(n = 0)$, we replace the operator in the first(second) parenthesis in the above by identity.

Proof: Remark that $\mathcal{M}_{fin}$ is the algebra generated by 1, $B(f)$, $f \in \mathfrak{h}_0$ is dense in $\mathcal{M}$. It follows from (4.8) and (3.6) that any $B(g)$, $g \in \mathfrak{h}_0$ can be written as linear sum of four $D_i(h)^{\#}, h \in \mathfrak{h}_0$, $i = 1, 2$. Thus any $(\prod_{l=1}^{m} B(g_l))\xi_0$, $g_l \in \mathfrak{h}_0$, $l = 1, ..., m$ can be expressed as a finite linear combination of the vectors of the form

$$(\prod_{j=1}^{p} D_1(g_j)^{\#})(\prod_{l=1}^{q} D_2(h_l)^{\#})\xi_0,\ g_j,\ h_l \in \mathfrak{h}_0,$$

where $D(g)^{\#}$ is either $D(g)$ or $D(g)^*$. Using Lemma 4.1 and the CARs in Proposition 4.1, the above vector can be expressed as a finite linear combination of the vectors of the form

$$(\prod_{j=1}^{p'} D_1(g_j)^*)(\prod_{l=1}^{q'} D_2(h_l)^*)\xi_0,\ g_j,\ h_l \in \mathfrak{h}_0,\ p', q' \in \mathbb{N} \cup \{0\}.$$

The set of finite linear combinations of the vectors of the above form is dense in $\mathcal{H}$. Thus the decomposition follows from Lemma 4.1 and the CARs in Proposition 4.1. □

Recall that $\mathfrak{h}_0$ is a dense subspace of a complex Hilbert space $\mathfrak{h}$. Let $\mathcal{F} = \mathcal{F}(\mathfrak{h})$ be the anti-symmetric Fock space over $\mathfrak{h}$, and $a(g)$ and $a(g)^*$, $g \in \mathfrak{h}$, the annihilation and creation operator on $\mathcal{F}$ respectively. Denote by Ω the vacuum vector in $\mathcal{F}$. Let $C : \mathfrak{h} \to \mathfrak{h}$ be an anti-unitary operator. If $\mathfrak{h}$ is a L^2-space, one may consider that C is the complex conjugation. Denote by $\Gamma(C)$ the second quantization of C. See Section 5.2.1 of [BR]. Let $\mathcal{F}_1$, Ω_1, $a_1(g)$ and $a_1(g)^*$, $g \in \mathfrak{h}$ be the identical copies of $\mathcal{F}$, Ω, $a(g)$ and $a(g)^*$, $g \in \mathfrak{h}$, respectively. Notice that $\Gamma(C)a(g)^{\#}\Gamma(C)^{-1} = a(Cg)^{\#}$. We write that $\mathcal{F}_2 = \Gamma(C)\mathcal{F}(= \mathcal{F})$, $\Omega_2 = \Omega$, $a_2(g) = a(Cg)$, and $a_2(g)^* = a(Cg)^*$, $g \in \mathfrak{h}$. Then the following anti-commutation relations hold: for any g, $h \in \mathfrak{h}$

$$\begin{aligned} &\{a_2(g), a_2(h)^*\} = (h, g)\mathbf{1}, \\ &\{a_2(g), a_2(h)\} = 0. \end{aligned} \tag{4.9}$$

Proposition 4.2 *Let U be the operator defined by*

$$U : \mathcal{H} \to \mathcal{F}_1 \otimes \mathcal{F}_2$$
$$(\prod_{j=1}^{m} D_1(g_j)^*)(\prod_{l=1}^{n} D_2(h_l)^*)\xi_0 \mapsto (\prod_{j=1}^{m}(a_1(g_j)^* \otimes \mathbf{1}))(\prod_{l=1}^{n}(\theta \otimes a_2(h_l)^*))\Omega_1 \otimes \Omega_2$$

for $g_j,\, h_l \in \mathfrak{h}_0$, $j = 1, ..., m$, $l = 1, ..., n$, $m, n \in \mathbb{N} \cup \{0\}$, where θ is an operator which anti-commutes with $a(g)$, $a(h)^*$ for $g, h \in \mathfrak{h}_0$ and satisfies $\theta\Omega_1 = \Omega_1$. Then U is unitary.

Proof: Notice that $\theta^2 = \mathbf{1}$ and for g, h in $\mathfrak{h}_0$

$$(a_1(g)^* \otimes \mathbf{1})(\theta \otimes a_2(h)^*)\Omega_1 \otimes \Omega_2 = -(\theta \otimes a_2(h)^*)(a_1(g)^* \otimes \mathbf{1})\Omega_1 \otimes \Omega_2. \tag{4.10}$$

Since $D_1(g)^\#$ and $a_1(g)^\#$, and $D_2(g)^\#$ and $a_2(g)^\#$ for $g \in \mathfrak{h}_0$ satisfy the same anti-commutation relations respectively by Proposition 4.1 and (4.9), the unitarity of U follows from Lemma 4.1, (4.10) and $a_i(g)\Omega_i = 0$, $i = 1, 2$ for any $g \in \mathfrak{h}_0$. □

We next turn to the spectral analysis of H, where H is the generator of the symmetric Markovian semigroup $\{T_t\}_{t\geq 0}$ associated to the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$.

Let us first describe the basic idea of the proof of Theorem 3.2. Recall that the vectors in $\mathcal{M}_{fin}\xi_0$ can be expressed as finite linear combination of the vectors of the form

$$(\prod_{j=1}^{p} D_1(f_j)^*)(\prod_{l=1}^{q} D_2(h_l)^*)\xi_0,\ f_j,\, h_l \in \mathfrak{h}_0,\ p, q \in \mathbb{N} \cup \{0\}.$$

Let $\{g_j\}_{j=1}^{\infty} \subset \mathfrak{h}_0$ be a CONS for $\mathfrak{h}$. In the proof of Theorem 3.1, we showed that the form $(\mathcal{E}, \mathcal{M}_{fin}\xi_0)$ is independent of the CONS $\{g_j\} \in \mathfrak{h}_0$ for $\mathfrak{h}$. Thus one can show that for any $\xi \in \mathcal{M}_{fin}\xi_0$

$$\begin{aligned}\mathcal{E}[\xi] &= \sum_{j=1}^{\infty} \int \left(\|D_1(A^{it}B^{1/2}g_j)\xi\|^2 + \|D_2(A^{it}B^{1/2}g_j)\xi\|^2\right) f(t)dt \\ &= \sum_{j=1}^{\infty} \left(\|D_1(B^{1/2}g_j)\xi\|^2 + \|D_2(B^{1/2}g_j)\xi\|^2\right) \\ &= \langle \xi, \widehat{H}\xi \rangle,\end{aligned} \tag{4.11}$$

where

$$\widehat{H} = \sum_{j=1}^{\infty} \{D_1(B^{1/2}g_j)^* D_1(B^{1/2}g_j) + D_2(B^{1/2}g_j)^* D_2(B^{1/2}g_j)\}$$

as a bilinear form on $\mathcal{M}_{fin}\xi_0 \times \mathcal{M}_{fin}\xi_0$. If one can show that $\mathcal{M}_{fin}\xi_0 \subset D(H)$, $H = \widehat{H}$ on $\mathcal{M}_{fin}\xi_0$, and that $\mathcal{M}_{fin}\xi_0$ is a core for H, then one expects that the spectrum of H can be analyzed completely. The following is one of main results in this section:

Theorem 4.2 *(a) Let $\tilde{H} : \mathcal{M}_{fin}\xi_0 \to \mathcal{H}$ be the operator defined by*

$$
\begin{aligned}
&\tilde{H}\big(\prod_{p=1}^{m} D_1(f_p)^*\big)\big(\prod_{q=1}^{n} D_2(h_q)^*\big)\xi_0 \qquad (4.12)\\
= \;& \sum_{k=1}^{m}\big(\prod_{p=1}^{k-1} D_1(f_p)^*\big)D_1(Bf_k)^*\big(\prod_{p=k+1}^{m} D_1(f_p)^*\big)\big(\prod_{q=1}^{n} D_2(h_q)^*\big)\xi_0 \\
&+\sum_{k=1}^{n}\big(\prod_{p=1}^{m} D_1(f_p)^*\big)\big(\prod_{q=1}^{k-1} D_2(h_q)^*\big)D_2(Bh_k)^*\big(\prod_{q=k+1}^{n} D_2(h_q)^*\big)\xi_0
\end{aligned}
$$

for any $m, n \in \mathbb{N} \cup \{0\}$ and $f_p, h_q \in \mathfrak{h}_0$, $p = 1, ..., m, q = 1, ..., n$. Then the relation

$$\mathcal{E}(\eta, \xi) = \langle \eta, \tilde{H}\xi \rangle$$

holds for any $\eta, \xi \in \mathcal{M}_{fin}\xi_0$, where $\mathcal{E}$ is defined as in (3.10) and (3.11).
(b) $\tilde{H}$ is essentially self-adjoint and the self-adjoint extension denoted by $\tilde{H}$ again is equal to the Dirichlet operator H.

Proof: (a) Let $(\mathcal{E}_1, \mathcal{M}_{fin}\xi_0)$ be the form given by

$$\mathcal{E}_1[\xi] = \sum_{j=1}^{\infty} \|D_1(B^{\frac{1}{2}}g_j)\xi\|^2,$$

where $\{g_j\}$, $g_j \in \mathfrak{h}_0$ is a CONS for $\mathfrak{h}$. We write that $\tilde{H} = \tilde{H}_1 + \tilde{H}_2$, where the image under $\tilde{H}_1$(resp. $\tilde{H}_2$) is defined by the first (resp. second) vector in the right hand side in (4.12). The CARs in Proposition 4.1 (a) and Lemma 4.1 imply that

$$
\begin{aligned}
&\mathcal{E}_1[(\prod_{p=1}^{m} D_1(f_p)^*)\xi_0]\\
&= \sum_{j=1}^{\infty}\sum_{p=1}^{m}\sum_{q=1}^{m}(-1)^{p+q}(B^{\frac{1}{2}}f_p, g_j)(g_j, B^{\frac{1}{2}}f_q)G(f_1, ..., f_m; p, q)
\end{aligned}
$$

where

$$
\begin{aligned}
&G(f_1, ..., f_m; p, q)\\
:=\;& \langle\Big(\prod_{\tau=1}^{p-1} D_1(f_\tau)^*\Big)\Big(\prod_{\tau=p+1}^{m} D_1(f_\tau)^*\Big)\xi_0, \Big(\prod_{\tau=1}^{q-1} D_1(f_\tau)^*\Big)\Big(\prod_{\tau=q+1}^{m} D_1(f_\tau)^*\Big)\xi_0\rangle.
\end{aligned}
$$

Using the Parserval relations and the fact that

$$\begin{aligned}
&\sum_{p=1}^{m}(-1)^{p+q}(f_p, Bf_q)G(f_1, ..., f_m; p, q) \\
&= (-1)^{q-1}\langle D_1(Bf_q)\big(\prod_{p=1}^{m} D_1(f_p)^*\big)\xi_0, \Big(\prod_{\tau=1}^{q-1} D_1(f_\tau)^*\Big)\Big(\prod_{\tau=q+1}^{m} D_1(f_\tau)^*\Big)\xi_0\rangle \\
&= \langle\big(\prod_{p=1}^{m} D_1(f_p)^*\big)\xi_0, \Big(\prod_{\tau=1}^{q-1} D_1(f_\tau)^*\Big)D_1(Bf_q)^*\Big(\prod_{\tau=q+1}^{m} D_1(f_\tau)^*\Big)\xi_0\rangle,
\end{aligned}$$

we have that

$$\begin{aligned}
&\mathcal{E}_1[\big(\prod_{p=1}^{m} D_1(f_p)^*\big)\xi_0] \\
&= \sum_{q=1}^{m}\langle\big(\prod_{p=1}^{m} D_1(f_p)^*\big)\xi_0, \Big(\prod_{\tau=1}^{q-1} D_1(f_\tau)^*\Big)D_1(Bf_q)^*\Big(\prod_{\tau=q+1}^{m} D_1(f_\tau)^*\Big)\xi_0\rangle \\
&= \langle\big(\prod_{p=1}^{m} D_1(f_p)^*\big)\xi_0, \tilde{H}_1\big(\prod_{p=1}^{m} D_1(f_p)^*\big)\xi_0\rangle.
\end{aligned}$$

Notice that $\tilde{H}_2 = \tilde{H} - \tilde{H}_1$ commutes with $D_1(f)^*$ for any $f \in \mathfrak{h}_0$ by (4.12). Thus by the polarization identity, we proved that

$$\mathcal{E}_1(\eta, \xi) = \langle\eta, \tilde{H}_1\xi\rangle$$

for any η, $\xi \in \mathcal{M}_{fin}\xi_0$. The method similar to that used in the above implies that

$$\mathcal{E}_2(\eta, \xi) = \langle\eta, \tilde{H}_2\xi\rangle$$

for any η, $\xi \in \mathcal{M}_{fin}\xi_0$. This proved the part (a) of the theorem.

(b) By Proposition 4.2, we have that

$$U\tilde{H}U^{-1} = d\Gamma_1(B) \otimes \mathbf{1} + \mathbf{1} \otimes d\Gamma_2(B), \tag{4.13}$$

where each i, $i = 1, 2$, $d\Gamma_i(B)$ is the second quantization of B on $\mathcal{F}_i$. We remark that $d\Gamma_2(B)$ is anti-unitary equivalent to $d\Gamma_1(B)$. By Assumption 3.1 (d), any $f \in \mathfrak{h}_0$ is an analytic vector for B, and so it is easy to check that $(\prod_{p=1}^{m} a(f_p)^*)\Omega$ is an analytic vector for $d\Gamma(B)$ for any $f_p \in \mathfrak{h}_0$, $p = 1, ..., m$. Thus it follows that any $\xi \in \mathcal{M}_{fin}\xi_0$ is an analytic vector for $\tilde{H}$. Since $\tilde{H}\mathcal{M}_{fin}\xi_0 \subset \mathcal{M}_{fin}\xi_0$ by (4.12) and $\tilde{H} = H$ on $\mathcal{M}_{fin}\xi_0$ by the part (a) of the theorem, it follows from Corollary 2 of Theorem X. 39 of [RS] that $\tilde{H}$ and H are essentially self-adjoint on $\mathcal{M}_{fin}\xi_0$, and so $\tilde{H} = H$. □

Finally we can produce the proof of Theorem 3.2.

Proof of Theorem 3.2 (a) follows from Theorem 4.2.
(b) Recall $0 < A \leq \alpha \mathbf{1}$. Let

$$m = \begin{cases} \alpha^{-1/2} + \alpha^{1/2} & \alpha < 1 \\ 2 & \alpha \geq 1 \end{cases}$$

Since $B = A^{-\frac{1}{2}} + A^{\frac{1}{2}}$,

$$\inf \sigma(B) \geq m.$$

It follows from the above lower bound and (4.13) that zero is a simple eigenvalue with eigenvector ξ_0 and

$$\inf(\sigma(H) - \{0\}) \geq m \geq 2.$$

This completes the proof of the theorem. $\square$

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