LOCAL RETURN RATES IN SUBSTITUTIVE SUBSHIFTS

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ABSTRACT. Local lower and upper return rates express the asymptotic growth of the Poincaré return time of cylinders around a given point of a dynamical system. We show that in substitutive subshifts the lower (upper) local return time assumes almost everywhere its minimum (maximum) value and give an algorithm which yields these two values.

1. INTRODUCTION

In a topologically transitive dynamical system (X, F), for every neighbourhood U of a point $x \in X$ there exists $k > 0$ with $F^k(U) \cap U \neq \emptyset$. The least k with this property is the Poincaré return time $\tau(U) = \min\{k > 0 : F^k(U) \cap U \neq \emptyset\}$ of U. As U shrinks, $\tau(U)$ grows (except when x is a periodic point). This dependence is expressed by the local return rates introduced by Hirata et al [4]. The lower and upper local return rates are function $\underline{R}_\xi, \overline{R}_\xi : X \to [0, \infty]$ defined for a given dynamical system (X, F) and a measurable partition ξ of X. If $\Sigma \subseteq A^{\mathbb{N}}$ is a subshift, and $\xi = \{[a] : a \in A\}$ is the canonical clopen partition, then

$$\underline{R}(y) = \liminf_{k\to\infty} \frac{\tau([y_{[0,k)}])}{k}, \quad \overline{R}(y) = \limsup_{k\to\infty} \frac{\tau([y_{[0,k)}])}{k}$$

Here $y \in \Sigma$ and $[y_{[0,k)}] = \{z \in \Sigma : z_{[0,k)} = y_{[0,k)}\}$ is the cylinder of the prefix of y of length k.

Hirata et al [4] show that both $\underline{R}$ and $\overline{R}$ are subinvariant, i.e., $\underline{R}(\sigma(y)) \leq \underline{R}(y)$ and $\overline{R}(\sigma(y)) \leq \overline{R}(y)$. Moreover if μ is an invariant measure and (Σ, σ, μ) is ergodic, then both $\underline{R}$ and $\overline{R}$ are μ-almost everywhere constant, so there exist constants $0 \leq \mathbf{r}_0 \leq \mathbf{r}_1 \leq \infty$, such that $\underline{R}(y) = \mathbf{r}_0$ a.e. and $\overline{R}(y) = \mathbf{r}_1$ a.e.

Saussol et al. [9] show that if (Σ, σ, μ) is ergodic with positive entropy, then $\underline{R}(y) \geq 1$ almost everywhere. This does not hold in systems with zero entropy. Cassaigne et al. show that $\mathbf{r}_0 = \frac{3-\sqrt{5}}{2} < 1$ holds for the Fibonacci subshift, which is the Sturmian subshift of the golden angle rotation. Afraimovich et al [1] show examples of irrational rotations with unbounded continued fractions, where $\mathbf{r}_0 = 0$. These results are generalized in Kupsa [5] who treats the general case of irrational rotations and their corresponding Sturmian subshifts. In the present paper we present another generalization of Cassaigne et al [2]. We show that in substitutive subshifts $\mathbf{r}_0$ is the minimum of the range $\underline{R}(\Sigma)$ while $\mathbf{r}_1$ is the maximum of the range $\overline{R}(\Sigma)$. Moreover we describe an algorithm which computes $\mathbf{r}_0$ and $\mathbf{r}_1$ for a given substitution.

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2. Subshifts

For an alphabet A denote by A^* the set of finite words and by $A^{\mathbb{N}}$ the space of one-sided infinite words with the product topology. Denote by $|u|$ the length of a word $u \in A^*$ and by $|u|_a$ the number of occurrences of a letter a in u. The empty word is denoted by λ and $A^+ = A^* \setminus \{\lambda\}$ is the set of nonempty words. We write $v \sqsubseteq u$, if $v = u_{[i,j)} = u_i \cdots u_{j-1}$ is a subword of u for some $0 \leq i \leq j \leq |u|$.

The shift map $\sigma : A^{\mathbb{N}} \to A^{\mathbb{N}}$ is defined by $\sigma(x)_i = x_{i+1}$. A subshift is any subset $\Sigma \subseteq A^{\mathbb{N}}$ which is closed and σ-invariant, i.e., $\sigma(\Sigma) \subseteq \Sigma$. A subshift is determined by its language $\mathcal{L}(\Sigma) = \{u \in A^* : \exists x \in \Sigma, u \sqsubseteq x\}$. The cylinder set of a word $u \in \mathcal{L}(\Sigma)$ is $[u] = \{x \in \Sigma : x_{[0,|u|)} = u\}$.

Given $y \in \Sigma$ we define the sequence of free positions $s = (s_k)_{k\geq 0}$ in y by induction. Set $s_0 = 0$ and if s_{k-1} has been already defined, then $s_k > s_{k-1}$ is the largest integer, such that for all n,

$$s_{k-1} < n \leq s_k \;\Rightarrow\; [y_{[0,n)}] = [y_{[0,s_k)}].$$

If we set $\tau_k = \tau([y_{[0,s_k)}])$, then for $s_{k-1} < n \leq s_k$ we have $\tau([y_{[0,n)}]) = \tau_k$ and

$$\begin{aligned}
\underline{R}(y) &= \liminf_{k\to\infty} \frac{\tau([y_{[0,k)}])}{k} = \liminf_{k\to\infty} \frac{\tau_k}{s_k} = 1/\limsup_{k\to\infty} \frac{s_k}{\tau_k} \\
\overline{R}(y) &= \limsup_{k\to\infty} \frac{\tau([y_{[0,k)}])}{k} = \limsup_{k\to\infty} \frac{\tau_k}{s_{k-1}} = 1/\liminf_{k\to\infty} \frac{s_k}{\tau_{k+1}}
\end{aligned}$$

3. Substitutive subshifts

A subshift is substitutive, if it is the orbit closure of an aperiodic fixed fixed point of a primitive substitution (see e.g., Durand et all [3] or Kůrka [6]). Recall that a substitution over an alphabet A is a map $\vartheta : A \to A^+$. It extends to a monoid morphism $\vartheta : A^* \to A^*$ and to a map $\vartheta : A^{\mathbb{N}} \to A^{\mathbb{N}}$ by concatenation. A substitution is primitive, if its matrix $M_{ab} = |\vartheta(a)|_b$ is primitive. The matrix M has then spectral radius $\alpha > 1$ and corresponding left and right positive eigenvectors μ, ν which are normalized to satisfy

$$\mu M = \alpha\mu, \quad M\nu = \alpha\nu, \quad \sum_{a\in A} \mu_a = 1, \quad \sum_{a\in A} \mu_a \nu_a = 1.$$

By the Perron-Frobenius theorem we have

$$\lim_{k\to\infty} \frac{|\vartheta^k(a)|_b}{\alpha^k} = \nu_a \mu_b, \quad \lim_{k\to\infty} \frac{|\vartheta^k(a)|}{\alpha^k} = \nu_a$$

If ϑ is a primitive substitution, then there exists a ϑ-periodic point $x \in A^{\mathbb{N}}$ and we assume that x is not σ-periodic. By passing to a power of ϑ, we can assume that x is a fixed point, $\vartheta(x) = x$ and moreover, the lower norm $|\vartheta| = \min\{|\vartheta(a)| : a \in A\}$ is at least 2. The corresponding subshift is the orbit closure

$$\Sigma_\vartheta = \overline{\mathcal{O}(x)} = \{y \in A^{\mathbb{N}} : \forall n, \exists k, y_{[0,n)} = x_{[k,k+n)}\}$$

and does not depend on the choice of the fixed point x. The subshift Σ_ϑ is minimal and uniquely ergodic. In particular, for every $y \in \Sigma_\vartheta$,

$$\lim_{n\to\infty} \#\{i < n : y_i = a\}/n = \mu_a.$$

We use the same symbol μ for the measure $\mu(W)$ of a set $W \subseteq \Sigma_\vartheta$. The complexity function $P(n) = \#\mathcal{L}^n(\Sigma_\vartheta) = \#\{u \in \mathcal{L}(\Sigma_\vartheta) : |u| = n\}$ is sublinear, i.e., there exist

$0 < a < b$ such that $an \le P(n) \le bn$ for each n. The return times of cylinders are sublinear too. If $u \in \mathcal{L}^n(\Sigma_\vartheta)$, then $an \le \tau([u]) \le bn$. We show now that in substitutive subshifts $\mathbf{r}_0 < \mathbf{r}_1$.

Proposition 1. *If Σ is a substitutive subshift, then there exists $y \in \Sigma$ such that $\overline{R}(y) > \underline{R}(y)$.*

Proof. Let $0 < a < b$ be constants which satisfy $an \le P(n) \le bn$ and $an \le \tau([u]) \le bn$ for each $u \in \mathcal{L}^n(\Sigma_\vartheta)$. Fix a real number $0 < c < 1$ and assume that for all $y \in \Sigma$ and for all k, $s_{k+1} \le (c+1)s_k$. Then $s_k \le (c+1)^{k-1}$ and

$$2^k \le P(s_k) \le bs_k \le b(c+1)^k$$

and this is a contradiction. Thus there exists $y \in \Sigma$ and an increasing sequence $k_1 < k_2 < \cdots$, such that $s_{k_i+1} - s_{k_i} \ge cs_{k_i}$. It follows

$$\frac{\tau_{k_i+1}}{s_{k_i}} - \frac{\tau_{k_i+1}}{s_{k_i+1}} \ge \frac{\tau_{k_i+1} \cdot c \cdot s_{k_i}}{s_{k_i} s_{k_i+1}} \ge ac$$

so $\overline{R}(y) - \underline{R}(y) \ge ac$. □

We shall use frequently the following "decoding" theorem.

Theorem 2 (Mossé [8])**.** *Let ϑ be a primitive substitution with an aperiodic fixed point x. Define a function $h : \mathbb{N} \to \mathbb{N}$ by $h(n) = |\vartheta(x_{[0,n)})|$. Then there exists a context length $m > 0$ such that for every $u \in \mathcal{L}(\Sigma_\vartheta)$ of length at least $2m$ there exist $i, j \in \mathbb{N}$ with $0 \le i \le m$, $|u| - m \le j \le |u|$ and unique word $v \in \mathcal{L}(\Sigma)$ with $u_{[i,j)} = \vartheta(v)$, such that whenever $x_{[n,n+|u|)} = u$ then there exist i', j' with $n + i = h(i')$, $n + j = h(j')$ and $x_{[i',j')} = v$.*

As an auxiliary construction we consider also the two-sided subshift $\Theta_\vartheta \subseteq A^{\mathbb{Z}}$ with the same language $\mathcal{L}(\Theta_\vartheta) = \mathcal{L}(\Sigma_\vartheta) = \mathcal{L}(x)$. The cylinder of a word $u \in \mathcal{L}(x)$ positioned at $n \in \mathbb{Z}$ is the set $[u]_n = \{y \in \Theta_\vartheta : y_{[n,n+|u|)} = u\}$. The cylinder of the empty word is the full space $[\lambda] = [\lambda]_0 = \Theta_\vartheta$. We extend the substitution to a map $\vartheta : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ by

$$\vartheta(\ldots u_{-2}u_{-1}.u_0u_1\ldots) = \ldots\vartheta(u_{-2})\vartheta(u_{-1}).\vartheta(u_0)\vartheta(u_1)\ldots$$

where the dot is placed immediately before the zero coordinate. As a consequence of Theorem 2 we have

Proposition 3.
1. $\vartheta(\Theta_\vartheta) \subseteq \Theta_\vartheta$.
2. $\vartheta : \Theta_\vartheta \to \Theta_\vartheta$ is one-to-one and open.
3. If $u \in \mathcal{L}(\Sigma_\vartheta)$, then $\vartheta([u]_0) = [\vartheta(u)]_0$ in Θ_ϑ.
4. For every $y \in \Theta_\vartheta$ there exists a unique $z \in \Theta_\vartheta$ and unique $i < |\vartheta(z_0)|$, such that $y = \sigma^i(\vartheta(z))$.

Definition 4. *For a clopen (closed and open) set $W \subseteq \Theta_\vartheta$, we set*

$$\begin{aligned}
l(W) &= \max\{l \le 0 : \forall y \in W, \forall z \in A^{\mathbb{Z}}, (z_{[l,\infty)} = y_{[l,\infty)} \Rightarrow z \in W)\} \\
p(W) &= \min\{n \le 0 : \forall y, z \in W, y_{[n,0)} = z_{[n,0)}\} \\
q(W) &= \max\{n \ge 0 : \forall y, z \in W, y_{[0,n)} = z_{[0,n)}\} \\
r(W) &= \min\{l \ge 0 : \forall y \in W, \forall z \in A^{\mathbb{Z}}, (z_{(-\infty,l)} = y_{(-\infty,l)} \Rightarrow z \in W)\}
\end{aligned}$$

Denote by $|W| = r(W) - l(W)$ the length of W and by $c(W) \in A^{q(W)-p(W)}$ the common central part of W, such that for all $y \in W$, $y_{[p(W),q(W))} = c(W)$.

Then $l(W) \leq p(W) \leq q(W) \leq r(W)$ and W is a union of cylinders of length $|W|$ positioned at $l(W)$. All these cylinders coincide at $[p(W), q(W))$. For the full set $W = [\lambda]$ we have $l(W) = p(W) = q(W) = r(W) = 0$.

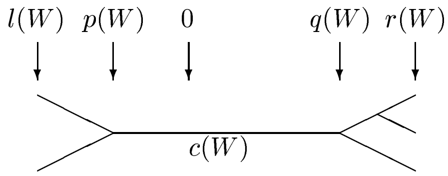

FIGURE 1. A clopen set

If $W \subseteq \Theta_\vartheta$ is a clopen set, then $\vartheta(W)$ is a clopen set too. We investigate the properties of the iterates $\vartheta^k(W)$.

Proposition 5. *There exists an algorithm which, given a clopen set W, computes the limit*

$$\chi(W) = \lim_{k\to\infty} q(\vartheta^k(W)) \cdot \alpha^{-k}.$$

Proof. Let $f : A \to A$ be a finite dynamical system given by $f(a) = \vartheta(a)_0$ and set $A_0 = \{a \in A : [a] \cap W \neq \emptyset\}$. If for all $k \geq 0$ $f^k(A_0)$ contains at least two elements, then $q(\vartheta^k(W)) = 0$ and $\chi(W) = 0$. Assume that for some $j > 0$, $f^j(A_0)$ is a singleton, so $q(\vartheta^j(W)) > 0$. Let $v_k = \vartheta^k(W)_{[0,q(\vartheta^k(W)))}$, so $q(\vartheta^k(W)) = |v_k|$. Since $|\vartheta| \geq 2$, $|v_{k+1}| \geq 2|v_k|$ and $|v_k|$ tend to infinity. Set

$$m_1 = \left\lceil \frac{2m}{|\vartheta|} \right\rceil, \quad m_2 = \left\lceil \frac{m}{|\vartheta| - 1} \right\rceil, \quad q_j = q(\vartheta^j(W))$$

where m is the context length from Theorem 2. Let $j_0 \geq 0$ be the first integer for which $q_{j_0} \geq m_1$. For $j \geq j_0$ set

$$V_j = \{y_{[q_j - m_1, q_j + m_2)} : y \in \vartheta^j(W)\}$$

By Theorem 2, for every $y \in \vartheta^j(W)$ we have $q_{j+1} \leq |\vartheta(y_{[0,q_j)})| + m$ and therefore

$$|\vartheta(y_{[0,q_j+m_2)})| - q_{j+1} \geq |\vartheta(y_{[q_j,q_j+m_2)})| - m \geq m_2 \cdot |\vartheta| - m \geq m_2$$

Thus $\vartheta(y)_{[q_{j+1}-m_1, q_{j+1}+m_2)}$ is a subword of $\vartheta(y_{[q_j-m_1,q_j+m_2)})$ and V_{j+1} is determined by V_j. Since V_j are finite (and bounded), there exist $j_0 \leq j < j + r$ such that $V_{j+r+i} = V_{j+i}$ for all $i \geq 0$. There exist $b, c \in \mathcal{L}(\Sigma_\vartheta)$ such that

$$\begin{aligned} y \in \vartheta^j(W) &\Rightarrow y_{[0,q_j)} = b \\ y \in \vartheta^{j+r}(W) &\Rightarrow y_{[0,q_{j+r})} = \vartheta^r(b)c \\ y \in \vartheta^{j+lr}(W) &\Rightarrow y_{[0,q_{j+lr})} = \vartheta^{lr}(b)\vartheta^{(l-1)r}(c)\cdots\vartheta^r(c)c \end{aligned}$$

It follows

$$\begin{aligned} \chi(W) &= \lim_{l\to\infty} \frac{|\vartheta^{lr}(b)\vartheta^{(l-1)r}(c)\cdots\vartheta^r(c)c|}{\alpha^{j+lr}} \\ &= \alpha^{-j}\sum_{i<|b|}\nu_{b_i} + \left(\alpha^{-j-r} + \alpha^{-j-2r} + \cdots\right)\sum_{i<|c|}\nu_{c_i} \\ &= \alpha^{-j}\sum_{i<|b|}\nu_{b_i} + \frac{\alpha^{-j}}{\alpha^r - 1}\sum_{i<|c|}\nu_{c_i} \end{aligned}$$

$\square$

Proposition 6. *There exists an algorithm which, given a clopen set W, yields the limit*

$$\varrho(W) = \lim_{k\to\infty} \tau(\vartheta^k(W)) \cdot \alpha^{-k} > 0.$$

Proof. Set $b = r(W) - l(W)$. Let U be the set of all words $u \in \mathcal{L}(x)$ such that

$$[u_{[0,b)}]_{l(W)} \subseteq W, \quad [u_{[a,a+b)}]_{l(W)} \subseteq W$$

for some $a > 0$ (Figure 2). Let $a_u = a$ be the least integer with this property, so $|u| = a_u + b$. Assume that $k \geq 0$ and let $w \in \vartheta^k(W) \cap \sigma^{-\tau(\vartheta^k(W))}(\vartheta^k(W))$. There exist $z, v \in W$ such that $w = \vartheta^k(z)$, $\sigma^{\tau(\vartheta^k(W))}(w) = \vartheta^k(v)$. By Theorem 2 there exists $a > 0$ with $z = \sigma^a(v)$ Then $u = z_{[l(W),l(W)+a+b)} \in U$ and $a_u = a$, so $\tau(\vartheta^k(W)) = |\vartheta^k(u_{[0,a_u)})|$. For every $u \in U$ there exists a limit

$$t_u = \lim_{k\to\infty} |\vartheta^k(u_{[0,a_u)})| \cdot \alpha^{-k} = \sum_{i<a_u} \nu_{u_i}.$$

Since U is a finite set, we get $\varrho(W) = \min\{t_u : u \in U\} > 0$. $\square$

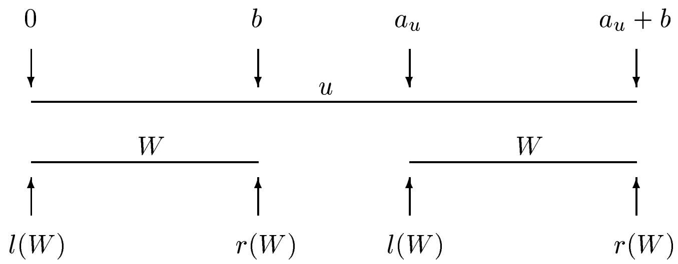

FIGURE 2. Return time

Definition 7. *We say that a clopen set $W \subseteq \Theta_\vartheta$ is decodable, if for some $i \in \mathbb{Z}$, $\sigma^{-i}(W) \subseteq \vartheta(\Theta_\vartheta)$. If $i \geq 0$ is the least integer with this property, we write, by an abuse of notation,*

$$\vartheta^{-1}(W) = \vartheta^{-1}(\sigma^{-i}(W)) = \{z \in \Theta_\vartheta : \sigma^i(\vartheta(z)) \in W\}$$

We say that a clopen set $W \subseteq \Theta_\vartheta$ is short, if both $p(W) - l(W)$ and $r(W) - q(W)$ are less than $(m+1)|\vartheta|/(|\vartheta|-1)$, where m is the context length from Theorem 2.

If W is decodable, then clearly $\vartheta(\vartheta^{-1}(W)) = \sigma^{-i}(W)$.

Proposition 8. *If W is a clopen set with $|c(W)| = q(W) - p(W) \geq 2m$, where m is the context length, then W is decodable, and*

$$\begin{aligned}
r(\vartheta^{-1}(W)) - q(\vartheta^{-1}(W)) &\leq \frac{r(W) - q(W) + m}{|\vartheta|} + 1 \\
q(\vartheta^{-1}(W)) - p(\vartheta^{-1}(W)) &\leq \frac{q(W) - p(W)}{|\vartheta|} + 1 \\
p(\vartheta^{-1}(W)) - l(\vartheta^{-1}(W)) &\leq \frac{p(W) - l(W) + m}{|\vartheta|} + 1
\end{aligned}$$

If W is also short, then so is $\vartheta^{-1}(W)$.

Proof. By Theorem 2 there exist i,j such that $p(W) \le i \le p(W)+m$, $q(W)-m \le j \le q(W)$ and unique v such that for each $y \in W$, $y_{[i,j)} = \vartheta(v)$. Moreover, there exists $z \in \Theta_\vartheta$ with $\vartheta(z) = \sigma^i(y)$ and $z \in [v]_0$, so W is decodable. We have

$$r(\vartheta^{-1}(W)) - q(\vartheta^{-1}(W)) \le \frac{r(W)-j}{|\vartheta|} + 1 \le \frac{r(W)-q(W)+m}{|\vartheta|} + 1$$

Similarly we obtain the inequality for $p(\vartheta^{-1}(W)) - l(\vartheta^{-1}(W))$, while the inequality for $q(\vartheta^{-1}(W)) - p(\vartheta^{-1}(W))$ is obvious. If W is short, then

$$r(\vartheta^{-1}(W)) - q(\vartheta^{-1}(W)) \le \frac{\frac{(m+1)|\vartheta|}{|\vartheta|-1} + m}{|\vartheta|} + 1 \le \frac{(m+1)|\vartheta|}{|\vartheta|-1}$$

so $\vartheta^{-1}(W)$ is short too. □

Definition 9. *Let $V \subset W$ be clopen sets. We say that V is a maximal clopen subset of W, if $\chi(V) > \chi(W)$ and there is no clopen set U with $V \subset U \subset W$ and $\chi(U) > \chi(W)$.*

Lemma 1. *Let U, V be maximal clopen subsets of W. If $U \cap V \ne \emptyset$, then $U = V$.*

Proof. Assume that $w \in U \cap V$ and set $c = \min\{\chi(U), \chi(V)\} > \chi(W)$. For $c_k = \min\{q(\vartheta^k(U)), q(\vartheta^k(V))\}$ we have $\lim_{k\to\infty} c_k \alpha^{-k} = c$. If $u, v \in U \cup V$, then

$$\vartheta^k(u)_{[0,c_k)} = \vartheta^k(w)_{[0,c_k)} = \vartheta^k(v)_{[0,c_k)}$$

so $q(\vartheta^k(U \cup V)) \ge c_k$ and $\chi(U \cup V) \ge c > \chi(W)$. Since U, V are maximal, we get $U = U \cup V = V$. □

We construct now a finite graph associated to a substitution. Denote by $\mathcal{W}$ the set of all clopen sets $W \subseteq \Theta_\vartheta$ which are short and not decodable. By Proposition 8, $\mathcal{W}$ is finite. We say that a pair $e = (W_0, W)$ is an edge, if $W_0 \in \mathcal{W}$ and W is a maximal clopen subset of W_0. Denote by $\mathcal{E}$ the set of edges. We have the source and target maps $s, t : \mathcal{E} \to \mathcal{W}$ defined as follows. If $e = (W_0, W) \in \mathcal{E}$ is an edge, then $s(e) = W_0$. Its target is $t(e) = W_1 = \vartheta^{-L(e)}(W)$, where $L(e) \ge 0$ is the least integer such that W_1 is not decodable. Proposition 8 implies that W_1 is short, so $W_1 \in \mathcal{W}$. The offset of an edge $e = (W_0, W)$ is $\chi(e) = \chi(W) - \chi(W_0) > 0$ and its probability is $P(e) = \mu(W)/\mu(W_0)$. Let $\mathcal{G}_0 = (\mathcal{W}_0, \mathcal{E}_0, s, t)$ be the subgraph of $\mathcal{G} = (\mathcal{W}, \mathcal{E}, s, t)$ of those vertices which are reachable from the initial vertex $[\lambda] = \Theta_\vartheta$. Given a vertex $W \in \mathcal{W}_0$ the outgoing edges determine a clopen partition of W and the sum of their probabilities is 1.

Lemma 2. *For every measurable set $W \subseteq \Theta_\vartheta$ we have*

$$\mu(\vartheta(W)) = \frac{\mu(W)}{\sum_{a\in A} \mu_a |\vartheta(a)|}$$

Proof. For $y \in \Theta_\vartheta$ and $n > 0$ set $k_n = |\vartheta(y_{[0,n)})|$. If $u \in \mathcal{L}(\Theta_\vartheta)$, then $\vartheta(u)$ occurs in $\vartheta(y_{[m,k_n-m)})$ only at positions $|\vartheta(y_{[0,j)})|$, such that $y_{[j,j+|u|)} = u$. It follows

$$\begin{aligned}\mu(\vartheta([u]_0) &= \lim_{n\to\infty} \frac{\#\{i < k_n : \vartheta(y)_{[0,|\vartheta(u)|)} = \vartheta(u)\}}{k_n} \\ &= \lim_{n\to\infty} \frac{\#\{i < n : y_{[0,u)} = u\}}{n} \cdot \frac{n}{k_n} = \frac{\mu([u]_0)}{\sum_{a\in A} \mu_a |\vartheta(a)|}\end{aligned}$$

□

Proposition 10. *For every $y \in \Sigma_\vartheta$ there exists a path $(e_k : W_k \to W_{k+1})_{k\geq 0}$ in $\mathcal{G}_0$ from the initial vertex $W_0 = [\lambda]$ and integers $(l_k)_{k\geq 0}$ such that $l_{k+1} - l_k = L(e_k)$, and $W_k = \vartheta^{-l_k}([y_{[0,s_k)}])$. Conversely any infinite path in $\mathcal{G}_0$ which starts in W_0 yields a unique point $y \in \Sigma_\vartheta$ with this property. Moreover,*

$$\mu([y_{[0,s_k)}]) = P(e_0)\cdots P(e_{k-1})$$

Proof. For a fixed k set $U_n = \vartheta^{-n}(y_{[0,s_k)})) \in \mathcal{W}$, where $0 \leq n \leq l_k$ and $l_k \geq 0$ is the first integer for which U_{l_k} is not decodable. Then $c(U_{l_k}) < 2m$ and by induction we get that U_{l_k} is short. Thus $W_k = U_{l_k} \in \mathcal{W}$. Set $V_k = \vartheta^{-l_k}(y_{[0,s_{k+1})})$. Since $[y_{[0,s_{k+1})}]$ is a maximal clopen subset of $[y_{[0,s_k)}]$, $e = (W_k, V_k)$ is an edge and for $W_{k+1} = t(e)$ (target) we get that $y_{[0,s_{k+1})} = \vartheta^{l_{k+1}}(W_{k+1})$. We have $\mu([y_{[0,s_0)}]) = \mu([\lambda]) = 1$ and

$$\frac{\mu([y_{[0,s_{k+1})}]}{\mu([y_{[0,s_k)}]} = \frac{\mu(\vartheta^{l_k}(V_k))}{\mu(\vartheta^{l_k}(W_k))} = \frac{\mu(V_k)}{\mu(W_k)} = P(e_k)$$

□

Proposition 11. *For an edge $e = (W_0, W) : W_0 \to W_1$ consider a linear function*

$$f_e(z) = a_e z + b_e = \frac{\varrho(W_0)z + \chi(e)}{\varrho(W_1)\alpha^{L(e)}}$$

Given $y \in \Sigma_\vartheta$, let l_k be the sequence from Proposition 10 and let k_i be the sequence of times whose transitions pass through e, i.e., $W_{k_i} = W_0$ and $W_{k_i+1} = W_1$. Then

$$\lim_{i\to\infty} \frac{s_{k_i+1}}{\tau_{k_i+1}} - f_e\left(\frac{s_{k_i}}{\tau_{k_i}}\right) = 0$$

The coefficents a_e and b_e satisfy $a_e \leq 1$ and $b_e > 0$. Moreover, the product of slopes a_e along a cycle of the graph is strictly smaller than 1.

Proof. Since $\tau_{k_i} = \tau([y_{[0,s_{k_i})}]) = \tau(\vartheta^{l_{k_i}}(W_0))$, and

$$\lim_{i\to\infty} \frac{s_{k_i+1} - s_{k_i}}{\alpha^{k_i}} = \lim_{i\to\infty} \frac{q(\vartheta^{l_{k_i}}(W)) - q(\vartheta^{l_{k_i}}(W_0))}{\alpha^{k_i}} = \chi(W) - \chi(W_0) = \chi(e),$$

we get

$$\begin{aligned} &\frac{s_{k_i+1}}{\tau_{k_i+1}} - f_e\left(\frac{s_{k_i}}{\tau_{k_i}}\right) \\ &= \frac{s_{k_i+1} - s_{k_i}}{\alpha^{l_{k_i}} \cdot \alpha^{L(e)}} \cdot \frac{\alpha^{l_{k_i+1}}}{\tau_{k_i+1}} + \frac{s_{k_i}}{\tau_{k_i}}\left(\frac{\tau_{k_i}}{\tau_{k_i+1}} - \frac{\varrho(W_0)}{\varrho(W_1)\alpha^{L(e)}}\right) - \frac{\chi(e)}{\varrho(W_1)\alpha^{L(e)}} \\ &\to \frac{\chi(e)}{\tau(W_1)\alpha^{L(e)}} + \frac{s_{k_i}}{\tau_{k_i}} \cdot 0 - \frac{\chi(e)}{\tau(W_1)\alpha^{L(e)}} = 0 \end{aligned}$$

Since $W \subset W_0$,

$$\frac{\tau(\vartheta^k(W_0))}{\alpha^k} \leq \frac{\tau(\vartheta^k(W))}{\alpha^k} = \frac{\tau(\vartheta^{k+L(e)}(W_1))}{\alpha^{k+L(e)}} \cdot \alpha^{L(e)}$$

so $\varrho(W_0) \leq \varrho(W_1)\alpha^{L(e)}$. If $e = e_0, \ldots, e_{k-1} : W_0 \to W_1 \to \cdots \to W_k = W_0$ is a cycle in $\mathcal{G}$, then $a_e = a_{e_0} \cdots a_{e_{k-1}} = \alpha^{-L(e_0)-\cdots-L(e_{k-1})} < 1$. □

For the sequence s_k/τ_{k+1} we consider the graph $\mathcal{G}_2$ whose vertices are $\mathcal{E}_0$ and whose edges are $\mathcal{E}_2 = \{(d,e) \in \mathcal{E}^2 : t(d) = s(e)\}$. The source and target maps and probabilities are $s(d,e) = d$, $t(d,e) = e$, $P(d,e) = P(e)$. The paths in $\mathcal{G}_0$ are in one-to-one correspondence with those paths in $\mathcal{G}_2$ whose initial vertex $e \in \mathcal{E}_0$ satisfies $s(e) = \lambda$ in $\mathcal{G}_0$.

Proposition 12. *For a pair of edges $W_0 \xrightarrow{d} W_1 \xrightarrow{e} W_2$ consider a linear function*

$$g_{de}(z) = \frac{\varrho(W_1)z + \chi(d)\alpha^{-L(d)}}{\varrho(W_2)\alpha^{L(e)}}$$

Given $y \in \Sigma_\vartheta$, let l_k be the sequence from Proposition 10 and let k_i be the sequence of times whose transitions pass through d, e, i.e., $W_{k_i} = W_0$, $W_{k_i+1} = W_1$ and $W_{k_i+2} = W_2$. Then

$$\lim_{i\to\infty} \frac{s_{k_i+1}}{\tau_{k_i+2}} - g_{de}\left(\frac{s_{k_i}}{\tau_{k_i+1}}\right) = 0$$

Proof. We have

$$\begin{aligned}
&\frac{s_{k_i+1}}{\tau_{k_i+2}} - g_{de}\left(\frac{s_{k_i}}{\tau_{k_i+1}}\right) \\
&= \quad \frac{s_{k_i+1} - s_{k_i}}{\alpha^{l_{k_i}} \cdot \alpha^{L(d)+L(e)}} \cdot \frac{\alpha^{l_{k_i+2}}}{\tau_{k_i+2}} + \frac{s_{k_i}}{\tau_{k_i+1}} \left(\frac{\tau_{k_i+1}}{\tau_{k_i+2}} - \frac{\varrho(W_1)}{\varrho(W_2)\alpha^{L(e)}}\right) - \frac{\chi(d)}{\varrho(W_2)\alpha^{L(d)+L(e)}} \\
&\to \quad \frac{\chi(d)}{\varrho(W_2)\alpha^{L(d)+L(e)}} + \frac{s_{k_i}}{\tau_{k_i+1}} \cdot 0 - \frac{\chi(d)}{\varrho(W_2)\alpha^{L(d)+L(e)}} = 0
\end{aligned}$$

□

Theorem 13. *Let $\vartheta : A \to A^+$ be a primitive substitution with an aperiodic fixed point $x \in A^{\mathbb{N}}$. Set*

$$\mathbf{r}_0 = \min \underline{R}(\Sigma_\vartheta), \quad \mathbf{r}_1 = \max \overline{R}(\Sigma_\vartheta)$$

Then $0 < \mathbf{r}_0 < \mathbf{r}_1 < \infty$, $\underline{R}(y) = \mathbf{r}_0$ a.e., and $\overline{R}(y) = \mathbf{r}_1$ a.e.

Proof. Say that $C \subseteq \mathcal{W}_0$ is a final irreducible component of $\mathcal{G}_0$, if for every $W \in C$ and $W' \in \mathcal{W}_0$ we have $W' \in C$ iff there exists a path from W to W'. Denote by $C_1, \ldots, C_p$ the final irreducible components of $\mathcal{G}_0$. The set $Y_i \subseteq \Sigma_\vartheta$ of those y which ultimately attain C_i is open, has positive measure, and $Y = Y_1 \cup \cdots \cup Y_p$ has measure 1. Say that a path $e = e_0, \ldots, e_{j-1}, e_j, \ldots, e_{k-1}$ in C_i is simple, if $e_0, \ldots, e_{j-1}$ is a cycle, i.e., $t(e_{j-1}) = s(e_0)$, $e_0, \ldots, e_{j-1}$ are pairwise distinct, and $e_j, \ldots, e_{k-1}$ are pairwise distinct. The composition $f_{e_{j-1}} \cdots f_{e_0}$ has a unique fixed point z and we set $z_e = f_{e_{k-1}} \cdots f_{e_j}(z)$. The set of simple paths is finite. Denote by $c_i > 0$ the minimum of all $1/z_e$ over all simple paths in C_i. Then for almost all $y \in Y_i$, $\underline{R}(y) = c_i$. Consider now two different final irreducible components C_i, C_j. Since Y_i, Y_j are open and $(\Sigma_\vartheta, \sigma)$ is minimal, there exists $k > 0$ such that $Y_{ij} = Y_i \cap \sigma^{-k}(Y_j)$ is nonempty and has positive measure. For almost all $y \in Y_{ij}$ we have $\underline{R}(y) = c_i$ and $c_i \geq \underline{R}(\sigma^k(y)) \geq c_j$. Thus all c_i are equal $c_1 = \cdots = c_p = \mathbf{r}_0 > 0$ and for allmost all $y \in \Sigma_\vartheta$ we have $\underline{R}(y) = \mathbf{r}_0$. If $y \in \Sigma_\vartheta \setminus Y$, then for some $k \geq 0$, $\sigma^k(y) \in Y$, so $\underline{R}(y) \geq \underline{R}(\sigma^k(y)) \geq \mathbf{r}_0$, so $\mathbf{r}_0 = \min \underline{R}(\Sigma_\vartheta)$.

Similarly denote by $D_1, \ldots D_p$ all final irreducible components of $\mathcal{G}_2$, $Y_i \subseteq \Sigma_\vartheta$ the set of those points which ultimately attain D_i. if $e = e_0 \ldots, e_{j-1}, e_j, \ldots, e_{k-1}$ is a simple path in $\mathcal{G}_2$, then the composition $g_{e_{j-1}} \cdots g_{e_0}$ has a single fixed point z and we set $z_e = g_{e_{k-1}} \cdots g_{e_j}(z)$. Since all coefficients of all functions g_{e_j} are positive, we have $z_e > 0$. Denote by $d_i < \infty$ the maximum of all $1/z_e$ over all simple paths in D_i. Then for almost all $y \in Y_i$, $\overline{R}(y) = d_i$. Consider now two different final irreducible components D_i, D_j. Since Y_i, Y_j are open and $(\Sigma_\vartheta, \sigma)$ is minimal, there exists $k > 0$ such that $Y_{ij} = Y_i \cap \sigma^{-k}(Y_j)$ is nonempty and has positive measure. The set $\sigma^k(Y_{ij}) \subseteq Y_j$ has a positive measure too, so for allmost all $y \in \sigma^k(Y_{ij})$, $\overline{R}(y) = d_i$. If $y = \sigma^k(z)$ with $z \in Y_{ij}$, then $d_j = \overline{R}(y) \leq \overline{R}(z) \leq d_i$. So all d_i are equal, $d_1 = \cdots = d_p = \mathbf{r}_1$, and $\overline{R}(y) = \mathbf{r}_1$ for allmost all $y \in Y$. If $y \in \Sigma_\vartheta \setminus Y$, then there exists $k > 0$ and $z \in Y$ with $y = \sigma^k(z)$, so $\overline{R}(y) \leq \overline{R}(z) \leq \mathbf{r}_1$. Thus $\mathbf{r}_1 = \max \overline{R}(\Sigma_\vartheta)$. By Proposition 1, $\mathbf{r}_0 < \mathbf{r}_1$. □

Corollary 14. *There exists an algorithm which computes the values $\mathbf{r}_0$ and $\mathbf{r}_1$ of a given substitution.*

4. The Feigenbaum subshift

The Feigenbaum subshift is generated by the substitution

$$\vartheta = \begin{cases} 0 & \rightarrow & 11 \\ 1 & \rightarrow & 10 \end{cases}$$

with fixed point $x = \vartheta^\infty(1) = 1011\ 1010\ 1011\ 1011\ 1011\ 1010\ 1011\ 1010\ldots$ The context length is $m = 2$, the spectral radius is $\alpha = 2$, and the normalized eigenvectors are $\mu = (\frac{1}{3}, \frac{2}{3})$, $\nu = (1,1)$. We show that we get the graph with vertices $W_0 = [\lambda]$, $W_1 = [1]_0$, $W_2 = [11]_0$. By Propositiion 6 we get $\varrho(W_1) = \varrho(W_2) = 1$. Denote by C_k, the common prefix of $\vartheta^k(0)$ and $\vartheta^k(1)$, so $\vartheta^k(W_0) = [C_k]_0$. We have $C_1 = 1$, $C_2 = 101$, $C_3 = 1011101, \ldots$ and $|C_k| = 2^k - 1$. If $u \in \mathcal{L}(\Sigma_\vartheta)$, then $c(\vartheta^k([u]_0)) = \vartheta^k(u)C_k$, so $q(\vartheta^k([u]_0)) = (|u|+1)2^k - 1$ and

$$\chi([u]_0) = \lim_{k\to\infty} \frac{(|u|+1)2^k - 1}{2^k} = |u| + 1$$

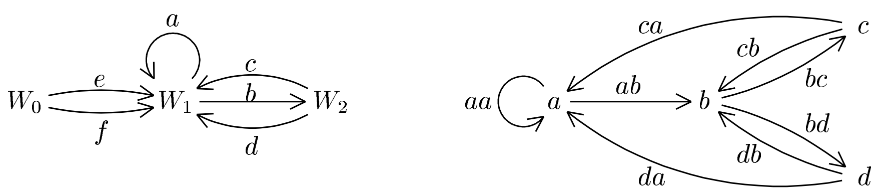

FIGURE 3. Graphs of the Feigenbaum subshift

In the graph there are two edges leading from the initial vertex $W_0 = [\lambda]$: $e = (W_0, [1]_0) : W_0 \to W_1$ with $L(e) = 0$ and $f = (W_0, [0]_0)$. Since $[0]_0 = [01]_0$ and $\tau^{-1}([01]_0) = [1]_0$, we get $f : W_0 \to W_1$ with $L(f) = 1$. Continuing in this way we get edges (Figure 3)

$$\begin{array}{llll}
e = ([\lambda],[1]) : & W_0 \to W_1, & L(e) = 0, & \chi(e) = 1 \\
f = ([\lambda],[01]) : & W_0 \to W_1, & L(f) = 1, & \chi(f) = 2 \\
a = ([1],[101]) : & W_1 \to W_1, & L(a) = 1, & \chi(a) = 2, \quad f_a(z) = \frac{z+2}{2} \\
b = ([1],[11]) : & W_1 \to W_2, & L(b) = 0, & \chi(b) = 1, \quad f_b(z) = z+1 \\
c = ([11],[1101]) : & W_2 \to W_1, & L(c) = 2, & \chi(c) = 2, \quad f_c(z) = \frac{z+2}{4} \\
d = ([11],[11101]) : & W_2 \to W_1, & L(d) = 2, & \chi(d) = 3, \quad f_d(z) = \frac{z+3}{4}
\end{array}$$

For any $z \in \mathbb{R}$ we have $\lim_{n\to\infty} f_{a^n}(z) = 2$, and 2 is the fixed point of f_a. The maximum of iterations of functions f_a, f_b, f_c and f_d is attained by $f_b(2) = 3$. The

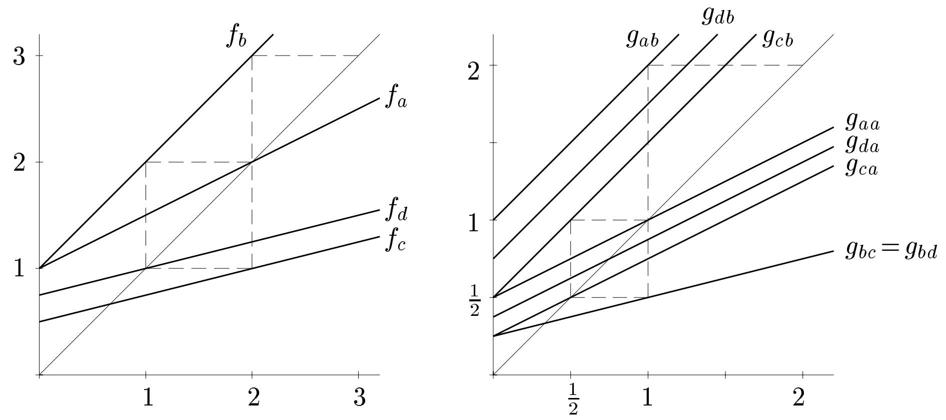

FIGURE 4. Functions of the Feigenbaum subshift

minimum is attained by the iterations of the function $f_{bc}(z) = f_c(f_b(z)) = (z+3)/4$ whose fixed point is 1. Thus we get

$$1 \leq \liminf_{k\to\infty} \frac{s_k}{\tau_k} \leq \limsup_{k\to\infty} \frac{s_k}{\tau_k} \leq 3, \quad \mathbf{r}_0 = \frac{1}{3}$$

By Proposition 12 we get

$$\begin{array}{llllllll} g_{aa}(z) & = & \frac{z+1}{2}, & g_{ab}(z) & = & z+1 & g_{bc}(z) & = \frac{z+1}{4}, \quad g_{bd}(z) = \frac{z+1}{4} \\ g_{ca}(z) & = & \frac{2z+1}{4}, & g_{cb}(z) & = & z+\frac{1}{2} & g_{da}(z) & = \frac{4z+3}{8}, \quad g_{db}(z) = \frac{4z+3}{4} \end{array}$$

The maximum of iterations of these functions is attained from the fixed point 1 of g_{aa} by $g_{ab}(1) = 2$. The minimum is attained at the fixed point of the function $g_{cbc}(z) = g_{bc}(g_{cb}(z)) = \frac{2z+3}{8}$ which is $z = \frac{1}{2}$, so $\mathbf{r}_1 = 2$.

$$\frac{1}{2} \leq \liminf_{k\to\infty} \frac{s_k}{\tau_{k+1}} \leq \limsup_{k\to\infty} \frac{s_k}{\tau_{k+1}} \leq 2$$

Corollary 15.

$$\frac{1}{3} \leq \underline{R}(y) \leq 1, \quad \frac{1}{2} \leq \overline{R}(y) \leq 2, \quad \underline{R}(y) = \frac{1}{3} \quad a.e., \quad \overline{R}(y) = 2 \quad a.e.$$

5. The Fibonacci subshift

The Fibonacci subshift is generated by the substitution

$$\vartheta = \begin{cases} 0 & \to & 1 \\ 1 & \to & 10 \end{cases}$$

with fixed point $x = \vartheta^\infty(1) = 10110\ 101\ 10110\ 10110101\ 1011010110110\ldots$ The context length is $m = 1$. The spectral radius $\alpha = \frac{\sqrt{5}+1}{2}$ satisfies $\alpha^2 = \alpha + 1$. The normalized eigenvectors are

$$\mu = \left(\frac{3-\sqrt{5}}{2}, \frac{\sqrt{5}-1}{2}\right), \quad \nu = \left(\frac{\sqrt{5}+1}{2\sqrt{5}}, \frac{3+\sqrt{5}}{2\sqrt{5}}\right)$$

The Fibonacci numbers $F_k = (\alpha^{k+1} - (-\alpha)^{-k-1})/\sqrt{5}$ are $F_0 = F_1 = 1$, $F_2 = 2$, $F_3 = 3$, $F_4 = 5, \ldots$ We have $|\vartheta^k(0)| = F_k$, $|\vartheta^k(1)| = F_{k+1}$. We show that the vertices of the graph are $W_0 = [\lambda]$ and $W_1 = [1]$ (Figure 5).

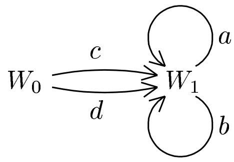

FIGURE 5. Graphs of the Fibonacci subshift

Set $C_k = \vartheta^{k-1}(1)\cdots\vartheta(1)1$, so $C_1 = 1$, $C_2 = 101$, $C_3 = 101101\ldots$. Then

$$\vartheta^k(W_0) = [C_k]_0, \quad \vartheta^k(W_1) = [\vartheta^k(1)C_k]_0 = [C_{k+1}]_0$$

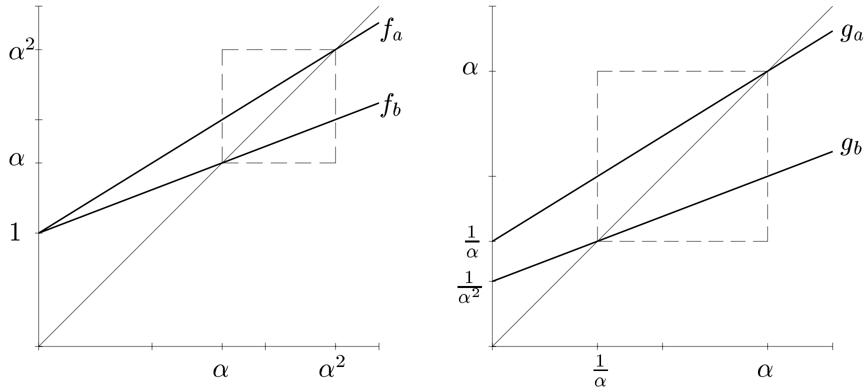

FIGURE 6. Functions of the Fibonacci subshift

We have edges

$$\begin{array}{llll}
c = ([\lambda],[1]): & W_0 \to W_1, & L(c) = 0, & \\
d = ([\lambda],[01]): & W_0 \to W_1, & L(d) = 1, & \\
a = ([1],[101]): & W_1 \to W_1, & L(a) = 1, & \chi(a) = \alpha^3/\sqrt{5}, \quad f_a(z) = \frac{z}{\alpha} + 1 \\
b = ([1],[1101]): & W_1 \to W_1, & L(b) = 2, & \chi(b) = \alpha^4/\sqrt{5}, \quad f_b(z) = \frac{z}{\alpha^2} + 1
\end{array}$$

Indeed $\varrho(W_1) = \nu_1 = \alpha^2/\sqrt{5}$ and

$$\begin{aligned}
\chi(a) &= \lim_{k\to\infty} \frac{|\vartheta^k(01)|}{\alpha^k} = \lim_{k\to\infty} \frac{F_{k+2}}{\alpha^k} = \frac{\alpha^3}{\sqrt{5}} \\
\chi(b) &= \lim_{k\to\infty} \frac{|\vartheta^k(101)|}{\alpha^k} = \lim_{k\to\infty} \frac{F_{k+3}}{\alpha^k} = \frac{\alpha^4}{\sqrt{5}}
\end{aligned}$$

The bounds are fixed points $f_a(\alpha^2) = \alpha^2$, $f_b(\alpha) = \alpha$, so

$$\alpha = \frac{\alpha^2}{\alpha^2 - 1} \le \frac{s_k}{\tau_k} \le \frac{\alpha}{\alpha - 1} = \alpha^2, \quad \mathbf{r}_0 = \alpha^{-2}$$

For $s_k/\tau_{k+1} = s_k/F_{l_{k+1}}$ we get functions

$$g_{aa}(z) = g_{ba}(z) = g_a(z) = \frac{z+1}{\alpha}, \quad g_{ab}(z) = g_{bb}(z) = g_b(z) = \frac{z+1}{\alpha^2}$$

with fixed points $g_a(\alpha) = \alpha$, $g_b(\frac{1}{\alpha}) = \frac{1}{\alpha}$, so

$$\frac{1}{\alpha} = \frac{1}{\alpha^2 - 1} \leq \frac{s_k}{\tau_{k+1}} \leq \frac{1}{\alpha - 1} = \alpha, \quad \mathbf{r}_1 = \alpha$$

Corollary 16.

$$\frac{1}{\alpha^2} \leq \underline{R}(x) \leq \frac{1}{\alpha} \leq \overline{R}(x) \leq \alpha$$

with $\underline{R}(x) = \alpha^{-2}$, $\overline{R}(x) = \alpha$ *almost everywhere.*

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