ON CONTACT ANOSOV FLOWS

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Abstract. Exponential decay of correlations for $\mathcal{C}^4$ Contact Anosov flows is established. This implies, in particular, exponential decay of correlations for all smooth geodesic flows in strictly negative curvature.

1. Introduction

The study of decay of correlations for hyperbolic systems goes back to the work of Sinai [36] and Ruelle [32]. While a manifold of results were obtained thru the years for maps, some positive results have been established for Anosov flows only recently. Notwithstanding the proof of ergodicity, and mixing, for geodesic flows on manifolds of negative curvature [15, 1, 35] the first quantitative results consisted in the proof of exponential decay of correlations for geodesic flows on manifolds of constant negative curvature in two [4, 23, 30] and three [26] dimensions. The proof there is group theoretical in nature and therefore ill suited to generalizations to the non constant curvature case.[1] The conjecture that all Axiom A mixing flows exhibit exponential decay of correlations had already been proven false by Ruelle [34, 27] who produced piecewise constant ceiling suspensions with arbitrarily slow rate of decay.

The next advance was due to Chernov [3] who put forward the first *dynamical* proof showing sub-exponential decay of correlations for geodesic flows on surfaces of variable negative curvature. The basic idea was to construct a suitable stochastic approximation of the flow (see also [20] for a generalization of such a point of view).

The last substantial advance in the field is due to the work of Dolgopyat [7, 8, 9]. He was able to use the thermodynamics formalism [36, 33, 28] and elaborate the necessary estimate on the Perron-Frobenius operator to control the Laplace transform of the correlation function. As a consequence he established exponential decay of correlations for all Anosov flows with $\mathcal{C}^1$ strong stable and unstable foliations. He also gave conditions for fast decay of correlations (for $\mathcal{C}^\infty$ observable) in more general cases.

Unfortunately, $\mathcal{C}^1$ strong stable and unstable foliations seem to be a quite rare phenomenon for higher dimensional Anosov flows [29, 10, 37]. One is therefore led to think that, unless some further geometrical structure is present, Anosov flows decay typically slower than exponentially.

The simplest geometrical structure that can be considered is certainly a contact structure, geodesic flows in particular. In this case an explicit formula by Katok and Burns [16] provides an approximation to the *temporal function* which is the real quantity on which some smoothness is required. An improvement on the error term for the above formula, than can be found in this paper (Appendix B, Lemma B.7), shows that, for a Contact Anosov flow, if the strong foliations are τ-Hölder, with $\tau > \sqrt{3}-1$, then the temporal functions is likely to be $\mathcal{C}^1$ (see Remark B.8). On the other hand, geodesic flows that are a-pinched[2] have foliations that are $\mathcal{C}^{2\sqrt{a}}$ ([18] and Appendix

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[1] Although some partial results for slowly varying curvature were obtained by perturbative techniques [4].

[2] That is, such that there exists $C > 0$ for which $-C \le$ sectional curvatures $< -aC$, clearly it must be $a \in (0, 1)$. Recall that here we are considering higher dimensional manifolds, geodesic flows on surfaces always have $\mathcal{C}^1$ foliations.

B; see also [13, 11] for more complete results on such an issue). Dolgopyat result would then, at best, imply that any geodesic flow in negative curvature which is a-pinched, with $a > 1 - \frac{\sqrt{3}}{2}$, enjoys exponential decay of correlations.

Given the fact that the above numbers do not look particularly inspiring it is then natural to guess that all Anosov Contact flows exhibit exponential decay of correlations. This is exactly what it is proved in the present paper (Theorem 2.4).

To obtain such a result I built on Dolgopyat's work and on the results in [2] where it is introduced a functional space over which the Perron-Frobenius operator can be studied directly, without any coding, contrary to the previous approaches by Dolgopyat, Chernov and Pollicott.

Over such a space all the thermodynamics quantities studied by Dolgopyat have a particularly simple analogous with a specially transparent interpretation. It is then possible to establish a spectral gap for the generator of the flow and this, in turn, implies exponential decay of correlations.

The simplification of the approach is considerable as is testified by the length of the (self-contained) proof. In addition, the transparency of the relevant quantities allows to recognize that in certain cases the results of Dolgopyat can be dramatically improved. To keep the exposition as simple as possible I have chosen to restrict it to the main case in which new results can be obtained: spectral properties of Contact Anosov flows with respect to the Contact volume. This allows to choose a function space simpler than the one needed in the general case (see [2] for a more general choice of the Banach space that would accommodate any Anosov flow with respect to any equilibrium measure).

The plan of the paper is as follows. Section two starts by describing the type of flows under consideration and the key objects used in the proof. Then the main result is stated precisely (Theorem 2.4). After that a proof of the result is presented. The proof is complete provided one assumes Lemma 2.7, Lemma 2.9 and Proposition 2.12. Lemma 2.7 is proven in section four. Lemma 2.9 is proven in section four. Section five contains the proof of Proposition 2.12 modulo and inequality, Lemma 5.2, which is proven in section six.

Finally, for the reader convenience, the paper contains three appendices. Appendix A contains a collection of needed–but already well established–facts on Anosov flows. Appendix B is devoted to the discussion of known–and less known–properties of Contact flows. Appendix C contains few technical facts about averages that will certainly not surprise the experts but needed to be proven somewhere.

2. Statements and results

We will consider a $\mathcal{C}^4$, $2d+1$ dimensional, connected compact Riemannian manifold $\mathcal{M}$ and a $\mathcal{C}^4$ flow[3] $T_t : \mathcal{M} \to \mathcal{M}$ defined on it which satisfies the following conditions.

Condition 1. *At each point $x \in \mathcal{M}$ there exists a splitting of the tangent space $\mathcal{T}_x\mathcal{M} = E^s(x) \oplus E^c(x) \oplus E^u(x)$. The splitting is invariant with respect to T_t, E^c is one dimensional and coincides with the flow direction, in addition there exists $A, \mu > 0$ such that*

$$\|dT_t v\| \leq Ae^{-\mu t}\|v\| \quad \text{for each } v \in E^s \text{ and } t \geq 0$$
$$\|dT_t v\| \geq Ae^{\mu t}\|v\| \quad \text{for each } v \in E^u \text{ and } t \leq 0.$$

That is, the flow is Anosov.

Condition 2. *There exists a $\mathcal{C}^2$ one form α on $\mathcal{M}$, such that $\alpha \wedge (d\alpha)^d$ is nowhere zero, which is left invariant by T_t (that is $\alpha(dT_t v) = \alpha(v)$ for each $t \in \mathbb{R}$ and tangent vector $v \in \mathcal{T}\mathcal{M}$). In other words T_t is a* Contact Flow.

Remark 2.1. *From now on I will assume $\mathcal{M}$ to be a Riemannian manifold with the Riemannian volume being the same as the contact volume $\alpha \wedge (d\alpha)^d$. This is not really necessary, yet it is convenient and can be done without loss of generality.*

With a slight abuse of notation let us define on $\mathcal{C}^1(\mathcal{M}, \mathbb{C})$ the following group of operators

$$T_t\varphi := \varphi \circ T_t \; ; \quad \mathcal{L}_t f := f \circ T_{-t} \tag{2.1}$$

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[3] That is $T_0 = \mathbf{Id}$ and $T_{t+s} = T_t \circ T_s$ for each $t, s \in \mathbb{R}$.

The operator $\mathcal{L}_t$ specifies the evolution of the densities and therefore should determine the statistical properties of the system. Unfortunately, the spectral properties of $\mathcal{L}_t$ on $\mathcal{C}^1(\mathcal{M},\mathbb{C})$ are not well connected to the statistical properties of the map. To establish such a connection it is necessary to enlarge the space. In order to do so we must define weaker norms. Clearly such norms will need to have a relation with the dynamical properties of the system.

The simplest way to embed the dynamics of a system into the topology is to introduce a *dynamical distance*. In our case several natural possibilities are available: for each $\sigma \in \mathbb{R}$ let

$$d_\sigma^+(x,y) := \int_0^\infty e^{\sigma t} d(T_t x, T_t y)\,dt; \quad d_\sigma^-(x,y) := \int_{-\infty}^0 e^{-\sigma t} d(T_t x, T_t y)\,dt, \tag{2.2}$$

where $d(\cdot,\cdot)$ is the Riemannian metric of $\mathcal{M}$.

Remark 2.2. *Note that d_σ^+ and d_σ^- are distances only if σ is sufficiently small (that is, negative and larger, in absolute value, than the absolute values of all the Lyapunov exponents), otherwise they are only* pseudo–distances.[4]

In the present article we are interested only to the special cases of (2.2) considered in the following Lemma (the trivial proof is left to the reader).

Lemma 2.3. *Choose $\lambda \in (0,\mu)$ and let $d_s := d_\lambda^+$ and $d_u := d_\lambda^-$. Then d_u is a pseudo-distance on $\mathcal{M}$ and $d_u(T_{-t}x, T_{-t}y) \leq e^{-\lambda t} d_u(x,y)$. In addition, d_u, restricted to any strong-unstable manifold, is a smooth function and it is equivalent to the restriction of the Riemannian metric, while points belonging to different unstable manifolds are at an infinite distance. The analogous properties hold for d_s.*

We can now start to describe the spaces on which we will consider the operators T_t and $\mathcal{L}_t$. First of all let us fix $\delta > 0$ that will need to be sufficiently small (how small will be specified later in the paper) and define

$$H_{s,\beta}(\varphi) := \sup_{d_s(x,y)\leq\delta} \frac{|\varphi(x)-\varphi(y)|}{d_s(x,y)^\beta}; \quad |\varphi|_{s,\beta} := |\varphi|_\infty + H_{s,\beta}(\varphi). \tag{2.3}$$

Definition 1. *In the following by the Banach space $\mathcal{C}_s^\beta(\mathcal{M},\mathbb{C}) \subset \mathcal{C}^0(\mathcal{M},\mathbb{C})$ we will mean the closure of $\mathcal{C}^1(\mathcal{M},\mathbb{C})$ with respect to the norm $|\cdot|_{s,\beta}$. Similar definitions hold with respect to the metric d_u and the Riemannian metric d (giving the space of Hölder function $\mathcal{C}^\beta$).*

Let us also define the unit ball $\mathcal{D}_\beta := \{\varphi \in \mathcal{C}_s^\beta(\mathcal{M},\mathbb{C}) \mid |\varphi|_{s,\beta} \leq 1\}$. For a given $\beta < 1$, and $f \in \mathcal{C}^1(\mathcal{M},\mathbb{C})$, let

$$\begin{aligned} &\|f\|_w := \sup_{\varphi\in\mathcal{D}_1} \int_{\mathcal{M}} \varphi f \\ &\|f\| := \|f\|_s + \|f\|_u\,; \quad \|f\|_s := \sup_{\varphi\in\mathcal{D}_\beta} \int_{\mathcal{M}} \varphi f\,; \quad \|f\|_u := H_{u,\beta}(f). \end{aligned} \tag{2.4}$$

Let $\mathcal{B}(\mathcal{M},\mathbb{C})$ and $\mathcal{B}_w(\mathcal{M},\mathbb{C})$ be the completion of $\mathcal{C}^1(\mathcal{M},\mathbb{C})$ with respect to the norms $\|\cdot\|$ and $\|\cdot\|_w$ respectively. Note that such spaces are separable by construction and are all contained in $(\mathcal{C}^\beta)^*$, the dual of the β-Hölder functions.

It is well known that the strong stable and unstable foliations for an Anosov flow are τ-Hölder (see Appendices A, B for quantitative estimates of τ and Remark B.4 for the use of τ in this paper). Moreover the Jacobian of the holonomies associated to the stable and unstable foliations are τ-Hölder. From now on we will assume[5]

$$\beta < \tau^2. \tag{2.5}$$

The **main result** of the paper is the following.

[4]That is, they can attain the value $+\infty$.

[5]The square is needed only in Lemma 4.3. In fact, employing the strategy used in [2], section 3.6, and refining Lemma B.7, it may be possible to replace τ^2 by τ. I do not pursue this possibility since it would complicate the proofs without any substantial addition to the present results.

Theorem 2.4. *For a $\mathcal{C}^4$ Anosov Contact flow T_t satisfying Conditions 1 and 2 the operators $\mathcal{L}_t$ form a strongly continuous group on $\mathcal{B}(\mathcal{M},\mathbb{C})$.*[6] *In addition, there exists $\sigma, C_1 > 0$ such that, for each $f \in \mathcal{C}^1$, $\int f = 0$, the following holds true*

$$\|\mathcal{L}_t f\| \leq C_1 e^{-\sigma t}|f|_{\mathcal{C}^1}.$$

Clearly the above theorem implies exponential decay of correlations for $\mathcal{C}^1$ function:

$$\int f\varphi\circ T_t = \int \mathcal{L}_t\left[f - \int f\right]\varphi + \int f\int \varphi\mathcal{L}_t 1 = \int f\int\varphi + O(e^{-\sigma t}|f|_{\mathcal{C}^1}|\varphi|_{s,\beta}).$$

In fact, a standard approximation argument extends the result to all Hölder functions.

Corollary 2.5. *For each $\alpha \in (0,1)$ there exists $C_\alpha > 0$ such that, for each $f, \varphi \in \mathcal{C}^\alpha$,*

$$\left|\int f\varphi\circ T_t - \int f\int\varphi\right| \leq C_\alpha |f|_{\mathcal{C}^\alpha}|\varphi|_{\mathcal{C}^\alpha} e^{-\frac{\alpha\sigma}{2-\alpha}t}.$$

Remark 2.6. *Note that Theorem 2.4 does not imply that $\mathcal{L}_1$ is a quasicompact operator neither that it enjoys a spectral gap. This is a reflection of the impossibility, with the ideas at hand, to investigate directly the time one map and indicates that the result must be pursued in a more roundabout way.*

The proof of Theorem 2.4 is achieved via a careful study of the spectral properties of the generator of the group. The first step consists in the following result proven in section 3.

Lemma 2.7. *The operators $\mathcal{L}_t$ extend to a group of bounded operators on $\mathcal{B}(\mathcal{M},\mathbb{C})$ and $\mathcal{B}_w(\mathcal{M},\mathbb{C})$; they form a strongly continuous group. In addition, for each $\beta' < \beta$ there exists a constant $B \geq 0$ such that, for each $f \in \mathcal{B}_w(\mathcal{M},\mathbb{C})$, $t \geq 0$,*

$$\|\mathcal{L}_t f\|_w \leq \|f\|_w$$

and, for each $f \in \mathcal{B}(\mathcal{M},\mathbb{C})$, $t \geq 0$,

$$\|\mathcal{L}_t f\| \leq \|f\|; \quad \|\mathcal{L}_t f\| \leq 3e^{-\lambda\beta' t}\|f\| + B\|f\|_w.$$

From now on let β' be fixed.

Accordingly the spectral radius of $\mathcal{L}_t$, $t \geq 0$, is bounded by one. In addition, it is possible to define the generator X of the group. Clearly, the domain $D(X) \supset \mathcal{C}^2(\mathcal{M},\mathbb{C})$ and restricted to $\mathcal{C}^2(\mathcal{M},\mathbb{C})$ it is nothing else but the action of the vector field defining the flow.

The spectral properties of the generator depend on the resolvent $R(z) = (z\mathbf{Id} - X)^{-1}$. It is well known (e.g. see [5]) that for all $z \in \mathbb{C}$, $\Re(z) > 0$, the following holds

$$R(z)f = \int_0^\infty e^{-zt}\mathcal{L}_t f dt. \tag{2.6}$$

Thanks to (2.6) it is possible to obtain the analogue of Lemma 2.7 for the resolvent.

Lemma 2.8. *For each $z \in \mathbb{C}$, $\Re(z) = a > 0$, holds*

$$\|R(z)\|_w \leq a^{-1}\ ; \quad \|R(z)\| \leq a^{-1}\ ; \quad \|R(z)^n f\| \leq \frac{3}{(a+\lambda\beta')^n}\|f\| + a^{-n}B\|f\|_w.$$

Proof. The first two inequalities follow directly from formula (2.6) and the first two inequalities of Lemma 2.7:

$$\|R(z)f\| \leq \int_0^\infty e^{-at}\|\mathcal{L}_t f\|\, dt \leq a^{-1}\|f\|.$$

By induction one easily obtains the formula

$$R(z)^n f = \frac{1}{(n-1)!}\int_{\mathbb{R}^+} t^{n-1}e^{-zt}\mathcal{L}_t f dt. \tag{2.7}$$

[6] In fact the only place in which the $\mathcal{C}^4$ hypothesis is used is in the estimate (C.5). With a bit more work, adopting the alternative approach used in [2] Sub-lemma 3.1.3, it is possible to reduce the needed smoothness to $\mathcal{C}^3$, possibly $\mathcal{C}^{2+\alpha}$, but to reduce it further some new ideas seem to be needed.

Using again Lemma 2.7

$$\|R(z)^n f\| \leq \frac{1}{(n-1)!}\int_0^\infty t^{n-1}e^{-at}(3e^{-\lambda\beta' t}\|f\| + B\|f\|_w) \leq \frac{3\|f\|}{(a+\lambda\beta')^n} + a^{-n}B\|f\|_w.$$

□

The next basic result (proven in section 4) is a compactness property for the operators $R(z)$.

Lemma 2.9. *For each $a = \Re(z) > 0$ the operator $R(z)$, seen as an operator from $\mathcal{B}(\mathcal{M},\mathbb{C})$ to $\mathcal{B}_w(\mathcal{M},\mathbb{C})$, is compact.*

Proposition 2.10. *For each $a = \Re(z) > 0$ the operator $R(z)$, seen as an operator on $\mathcal{B}(\mathcal{M},\mathbb{C})$, is quasi compact, has spectral radius a^{-1} and essential spectral radius bounded by $(a+\lambda\beta')^{-1}$.*

Proof. The bound on the spectral radius of $R(z)$ follows trivially from the second inequality of Lemma 2.8. While, by the third inequality of Lemma 2.8, Lemma 2.9 and the usual Hennion's argument [12] based on Nussbaum's formula [25], it follows that the essential spectral radius is bounded by $(a+\lambda\beta')^{-1}$. Let us recall the argument. Nussbaum's formula asserts that if r_n is the inf of the r such that $\{R(z)^n f\}_{\|f\|\leq 1}$ can be covered by a finite number of balls of radius r, then the essential spectral radius of $R(z)$ is given by $\liminf_{n\to\infty}\sqrt[n]{r_n}$. Let $B_1 := \{f \in \mathcal{B} \mid \|f\| \leq 1\}$. By Lemma 2.9, $R(z)B_1$ is relatively compact in $\mathcal{B}_w$. Thus, for each $\epsilon > 0$ there are $f_1,\dots,f_{N_\epsilon} \in R(z)B_1$ such that $R(z)B_1 \subseteq \bigcup_{i=1}^{N_\epsilon} U_\epsilon(f_i)$, where $U_\epsilon(f_i) = \{f \in \mathcal{B} \mid \|f - f_i\|_w < \epsilon\}$. For $f \in R(z)B_1 \cap U_\epsilon(f_i)$, Lemma 2.8 implies that

$$\|R(z)^{n-1}(f-f_i)\| \leq \frac{3}{(a+\lambda\beta')^{n-1}}\|f-f_i\| + \frac{B}{a^{n-1}}\|f-f_i\|_w \leq a^{-n+1}\left\{\frac{3}{(1+\lambda\beta' a^{-1})^{n-1}} + B\epsilon\right\}.$$

Choosing $\epsilon = (1+\lambda\beta' a^{-1})^{-n+1}$ we can conclude that for each $n \in \mathbb{N}$ the set $R(z)^n(B_1)$ can be covered by a finite number of $\|\cdot\|$–balls of radius $(3+B)(a+\lambda\beta')^{-n+1}$. □

For each $\zeta \in \mathbb{R}^+$ let $U_\zeta := \{z \in \mathbb{C} \mid \Re(z) > -\zeta\}$. Proposition 2.10 implies the following corollary.[7]

Corollary 2.11. *The spectrum $\sigma(X)$ of the generator is contained in the left half plane. The set $\sigma(X) \cap U_{\lambda\beta'}$ consists of, at most, countably many isolated points of point spectrum with finite multiplicity. Zero is the only eigenvalue on the imaginary axis and has multiplicity one.*

Proof. If $F_z(w) := z - w^{-1}$, then $\sigma(X) = F_z(\sigma(R(z)))$. Thus the essential spectrum of X must lie outside $\bigcup_{\Re(z)>0}\{w \in \mathbb{C} \mid |z-w| \leq a + \lambda\beta'\}$. This is exactly $U_{\lambda\beta'}$.

Since $\mathcal{L}_t 1 = 1$, and the space $V_0 := \overline{\{f \in \mathcal{C}^1(\mathcal{M},\mathbb{C}); \mid \int f = 0\}}^{\mathcal{B}(\mathcal{M},\mathbb{C})}$ is invariant, it follows $\sigma(X) = \{0\} \cup \sigma(X|_{V_0})$. Next, suppose $Xf = ibf$ for some $b \in \mathbb{R}$ and $f \in V_0$, $f \neq 0$, then $R(z)f = (z+ib)^{-1}f$, thus for $z = a - ib$ holds (see equation (3.2))

$$\|f\|_u \leq \frac{|z+ib|}{a+\beta\lambda}\|f\|_u = \frac{a}{a+\beta\lambda}\|f\|_u,$$

that is $\|f\|_u = 0$. Let $\{f_n\} \subset \mathcal{C}^1$ be an approximating sequence for f, $\varphi \in \mathcal{D}_\beta$, and $t \in \mathbb{R}^+$,

$$\left|\int f\varphi\right| = \left|e^{-ibt}\int fT_t\varphi\right| \leq \left|\int f_nT_t\varphi\right| + \|f - f_n\|.$$

Contact Anosov flows are mixing (see Corollary B.6), hence $\lim_{t\to\infty}\int f_nT_t\varphi = 0$. The arbitrariness of t and n implies then $\int f\varphi = 0$, that is $\|f\|_s = 0$, which implies the contradiction $f \equiv 0$. □

The above result, although rather interesting, does not suffice to investigate the statistical properties of the system, to do so it is necessary to exclude the presence of spectrum near the imaginary axis (apart from 0). This follows form the next result proven in sections 5, 6.

[7] This is the equivalent of the statement that the Laplace transform of the correlation function can be extend to a meromorphic function in a neighborhood of the imaginary axes, see [28].

Proposition 2.12. *There exists $b_* > 0$, $\bar c > 1$ and $\nu \in (0,1)$ such that for each $z = a + ib$, $a \in [\bar c^{-1}, \bar c]$, $|b| \geq b_*$, the spectral radius of $R(z)$ is bounded by νa^{-1}. More precisely, there exists $c^* > 0$ such that, for $\bar n = \lceil c^* \ln |b|\rceil$,*

$$\|R(z)^{\bar n}\| \leq \left(\frac{\nu}{a}\right)^{\bar n}.$$

Corollary 2.13. *The exists $\zeta_1 < 0$ such that $\sigma(X) \cap U_{\zeta_1} = \{0\}$.*

Proof. By the same argument at the beginning of Corollary 2.11, setting $\zeta_0 = \min\{\lambda\beta', \nu^{-1} - 1\}$, $U_{\zeta_0} \cap \sigma(X) \subset \{z \in \mathbb{C} \mid \Re(z) \in [-\zeta_0, 0], |\Im(z)| \leq b_*\}$. By Corollary 2.11 it follows that $U_{\zeta_0} \cap \sigma(X)$ contains only finitely many points, from this the result follows. □

To conclude we need to transfer the knowledge gained on the spectrum of X into an estimate on the behavior of the semigroup. A typical way to do so would be to use the Weak Spectral Mapping Theorem ([24], page 91) stating that, for all $t \in \mathbb{R}$, $\sigma(T_t) = \overline{\exp(t\sigma(X))}$, provided the semigroup is polynomially bounded for all times. Unfortunately, our semigroup grows exponentially in the past. Thus we need to argue directly. For this purpose a silly preliminary fact is needed.

Lemma 2.14. *For each $z \in \rho(X)$ (the resolvent set) and $f \in D(X^2)$ the following holds true*

$$\|R(z)f - z^{-1}f - z^{-2}Xf\| \leq |z|^{-2}\|R(z)\|\,\|X^2 f\|.$$

Proof. This follows from the identity $R(z)f = z^{-1}f + z^{-2}Xf + z^{-2}R(z)X^2 f$, for all $f \in D(X^2)$. □

Next notice that, for each $a > 0$ and $f \in D(X^2) \cap \mathcal{C}^0(\mathcal{M}, \mathbb{C})$,[8]

$$\mathcal{L}_t f = \frac{1}{2\pi} \lim_{w\to\infty} \int_{-w}^{w} db\, e^{at+ibt} R(a+ib)f. \tag{2.8}$$

We can now conclude the section with the proof of Theorem 2.4.

Proof of Theorem 2.4. Let $\nu_1 = \max\{\nu, \frac{4c^*}{3+4c^*}\}$ and $3\omega = \min\{\zeta_1, (\nu_1^{-1} - 1)\bar c\}$.[9] First of all by equation (3.2) it follows that

$$\|\mathcal{L}_t f\|_u \leq e^{-\lambda\beta t}\|f\|_u, \tag{2.9}$$

so we need only worry about the stable part of the norm.

Since $\int f = 0$, Corollary 2.13 implies that the function $R(z)f$ is analytic in the domain $\{\Re(z) \geq -\zeta_1\}$. Then $M := \sup_{a\in[-2\omega,0];\, |b|\leq b_*} \|R(a+ib)f\| < \infty$, moreover, for $a \in [-2\omega, 0]$ and $|b| \geq b_*$, it follows

$$R(a+ib) = \left[\mathbf{Id} + (a - \bar c)R(\bar c + ib)\right]^{-1} R(\bar c + ib).$$

To see that the above formula is well defined consider that, by hypothesis and Lemma 2.8,

$$\|(a - \bar c)R(\bar c + ib)\| \leq (1 + \frac{|a|}{\bar c}) \leq 1/3 + 2/3\nu_1^{-1}.$$

In addition, for $\bar n = \lceil c^* \ln |b|\rceil$ Proposition 2.12 implies

$$\|(a - \bar c)^{\bar n} R(\bar c + ib)^{\bar n}\| \leq \left[\nu_1\left(1 + \frac{|a|}{\bar c}\right)\right]^{\bar n} \leq \left[\frac{2}{3} + \frac{\nu_1}{3}\right]^{\bar n}.$$

[8] Just notice that, for $f \in D(X^2)$, $\|R(z)f\|_\infty \leq |z|^{-1}(\|X^2 f\| + \|Xf\| + \|f\|)$ (see Lemma 2.14). Hence for each $x \in \mathcal{M}$, $a > 0$, $R(a+ib)f(x)$ is in L^2 as a function of b. This means that for $f \in D(X^2)$ and $x \in \mathcal{M}$ one can apply the inverse Laplace transform formula and obtain the formula (2.8) point wise. Note that this implies only that the limit in (2.8) takes place in the $L^2([0,\infty], e^{-at}dt)$ sense as a function of t. On the other hand $\mathcal{L}_t f$ is a continuous function of t and, again by Lemma 2.14, $R(a+ib)f - \frac{1}{a+ib}f$ is in $L^1(\mathbb{R}, \mathcal{B})$, as a function of b. From this it follows that the limit in (2.8) converges in the $\mathcal{B}$ norm for each $t \in \mathbb{R}^+$.

[9] The constants $\nu, c^*, \bar c$ are defined in Proposition 2.12, ζ_1 is defined in Corollary 2.13.

Accordingly,

$$\begin{aligned}\| [\mathbf{Id} + (a-\bar c)R(\bar c+ib)]^{-1} \| &\leq \sum_{n=0}^{\infty} \|(a-\bar c)^n R(\bar c+ib)^n\| \\ &\leq \sum_{k=0}^{\infty} \|[(a-\bar c)^{\bar n}R(\bar c+ib)^{\bar n}]^k\| \sum_{j=0}^{\bar n-1} \|[(a-\bar c)R(\bar c+ib)]^j\| \\ &\leq \frac{9}{2(1-\nu_1)^2}|b|^{c^*\ln[\frac{1}{3}+\frac{2}{3\nu_1}]} \leq \frac{9}{2(1-\nu_1)^2}|b|^{1/2}.\end{aligned}$$

Thus there exists $M_1>0$ such that, for $a\in[-2\omega,0]$ and $b\in\mathbb{R}$,

$$\|R(a+ib)\| \leq M_1\sqrt{|b|}+M. \tag{2.10}$$

To conclude we use (2.8) and shift the contour of integration. For each $f\in D(X^2)\cap\mathcal{C}^0$,

$$\mathcal{L}_t f = \frac{1}{2\pi i}\int_{-2\omega+i\mathbb{R}} dz\, e^{zt}R(z)f = \frac{1}{2\pi i}\int_{-2\omega+i\mathbb{R}} dz\, e^{zt}\left(R(z)-\frac{1}{z}\right)f.$$

By using Lemma 2.14 and (2.10) we have that for each $\varphi\in\mathcal{D}_\beta$ and $f\in D(X^2)\cap\mathcal{C}^0$ holds

$$\begin{aligned}\left|\int_{\mathcal{M}} \mathcal{L}_t f\varphi\right| &\leq \frac{1}{2\pi}\int_{\mathbb{R}} db \left\|R(-2\omega+ib)f - \frac{1}{-2\omega+ib}f\right\| e^{-2\omega t} \\ &\leq C\left\{\|X^2 f\|+\|Xf\|+\|f\|\right\}e^{-2\omega t}.\end{aligned}$$

We have thus completed the proof for all $f\in D(X^2)\cap\mathcal{C}^0$; to obtain the announced result for $f\in\mathcal{C}^1$ it suffices a standard approximation argument. Let $\phi:\mathbb{R}^+\to\mathbb{R}^+$ be a $\mathcal{C}^\infty$ function such that $\operatorname{supp}(\phi)\subset(0,1)$ and $\int\phi=1$. For each $\varepsilon>0$ define $\phi_\varepsilon(t):=\varepsilon^{-1}\phi(\varepsilon^{-1}t)$ and, for each $f\in\mathcal{B}(\mathcal{M},\mathbb{C})$,

$$f_\varepsilon := \int_0^\infty \phi_\varepsilon(t)\mathcal{L}_t f.$$

Clearly $f_\varepsilon\in D(X^n)\cap\mathcal{C}^1$ for each $n\in\mathbb{N}$. More to the point

$$\|X^2 f_\varepsilon\| \leq \int |\phi_\varepsilon''(t)|\|\mathcal{L}_t f\| \leq \varepsilon^{-2}|\phi''|_{L^1}|f|_{\mathcal{C}^1}.$$

In addition, if $f\in\mathcal{C}^1$,

$$\|f_\varepsilon - f\| \leq \int \phi_\varepsilon(t)|f\circ T_{-t}-f|_{\mathcal{C}^\beta} \leq \varepsilon^{1-\beta}|f|_{\mathcal{C}^1}\sup_{t\in[0,1]}|T_{-t}|_{\mathcal{C}^1}.$$

Accordingly, for each $f\in\mathcal{C}^1(\mathcal{M},\mathbb{C})$, $\int f=0$, we have

$$\|\mathcal{L}_t f\| \leq \|\mathcal{L}_t f_\varepsilon\|+\|f-f_\varepsilon\| \leq C_1 e^{-2\omega t}\varepsilon^{-2}|f|_{\mathcal{C}^1}+C_2\varepsilon^{1-\beta}|f|_{\mathcal{C}^1},$$

and the wanted results follows by choosing $\varepsilon=e^{-2\omega(3-\beta)^{-1}t}$, hence $\sigma=2\omega(1-\beta)(3-\beta)^{-1}$. □

3. Proofs: Lasota–Yorke inequality

Proof of Lemma 2.7. By Lemma 2.3, for each $\alpha\in(0,1]$

$$|T_t\varphi|_\infty = |\varphi|_\infty;\quad H_{s,\alpha}(T_t\varphi) \leq e^{-\lambda\alpha t}H_{s,\alpha}(\varphi). \tag{3.1}$$

The first inequalities of Lemma 2.7 are immediate since, for $f\in\mathcal{C}^1(\mathcal{M},\mathbb{C})$ and $\varphi\in\mathcal{D}_\beta$ or $\varphi\in\mathcal{D}_1$,

$$\int_{\mathcal{M}}\varphi\mathcal{L}_t\, f = \int_{\mathcal{M}} fT_t\varphi.$$

In addition, again by Lemma 2.3

$$\|\mathcal{L}_t f\|_u = H_{u,\beta}(\mathcal{L}_t f) \leq e^{-\beta\lambda t}H_{u,\beta}(f) = e^{-\lambda\beta t}\|f\|_u. \tag{3.2}$$

To conclude the argument we need the averaging operator[10]

$$\mathbb{A}_\delta^s\varphi(x) := \frac{1}{m^s(W_\delta^s(x))}\int_{W_\delta^s(x)}\varphi(z)m^s(dz). \tag{3.3}$$

The basic properties of such an operator consist in the following

Sub-lemma 3.1. *There exists $C>0$ such that for each $\varphi\in\mathcal{D}_\beta$ one has*

$$\begin{aligned}&|\mathbb{A}_\delta^s\varphi-\varphi|_\infty\leq C\delta^\beta|\varphi|_{s,\beta}\\ &H_{s,\beta}(\mathbb{A}_\delta^s\varphi-\varphi)\leq(2+C\delta)H_{s,\beta}(\varphi)+C\delta^{1-\beta}|\varphi|_\infty\\ &H_{s,1}(\mathbb{A}_\delta^s\varphi)\leq C\delta^{-1}|\varphi|_\infty\end{aligned}$$

The above Sub-Lemma is hardly surprising, yet its proof is a bit technical and it is postponed to Appendix C. By Sub-Lemma 3.1 it follows that, given $\varphi\in\mathcal{D}_\beta$ and $f\in\mathcal{C}^1$, holds

$$\begin{aligned}\int_{\mathcal{M}}f\varphi=\int_{\mathcal{M}}f\{\varphi-\mathbb{A}_\delta^s\varphi\}+\int_{\mathcal{M}}f\mathbb{A}_\delta^s\varphi\leq|\varphi-\mathbb{A}_\delta^s\varphi|_{s,\beta}\|f\|_s+|\mathbb{A}_\delta\varphi|_{s,1}\|f\|_w\\ \leq(C(\delta^\beta+\delta^{1-\beta})|\varphi|_\infty+(2+C\delta)H_{s,\beta}(\varphi))\|f\|_s+C\delta^{-1}\|f\|_w.\end{aligned}$$

Accordingly, remembering (3.1), for each $\varphi\in\mathcal{D}_\beta$,

$$\begin{aligned}\int_{\mathcal{M}}\mathcal{L}_tf\varphi=\int_{\mathcal{M}}fT_t\varphi\leq(C(\delta^\beta+\delta^{1-\beta})|\varphi|_\infty+(2+C\delta)H_{s,\beta}(\varphi\circ T_t))\|f\|_s+C\delta^{-1}\|f\|_w\\ \leq(C(\delta^\beta+\delta^{1-\beta})|\varphi|_\infty+(2+C\delta)e^{-\lambda\beta t}H_{s,\beta}(\varphi))\|f\|_s+C\delta^{-1}\|f\|_w.\end{aligned}$$

We start by requiring $2+C\delta\leq 3$, then let $T_0\in\mathbb{R}^+$ be such that $3e^{-\lambda\beta T_0}\leq e^{-\lambda\beta' T_0}$; at last we choose δ so that $C(\delta^\beta+\delta^{1-\beta})\leq e^{-\lambda\beta' T_0}$. Thus, for each $t\leq T_0$,

$$\begin{aligned}\|\mathcal{L}_tf\|_s\leq 3e^{-\lambda\beta' t}\|f\|_s+C\delta^{-1}\|f\|_w\\ \|\mathcal{L}_{T_0}f\|_s\leq e^{-\lambda\beta' T_0}\|f\|_s+C\delta^{-1}\|f\|_w.\end{aligned} \tag{3.4}$$

For each $t\in\mathbb{R}^+$ we write $t=kT_0+s$, $k\in\mathbb{N}$, $s\in(0,T_0)$, and we use (3.4) iteratively to obtain

$$\|\mathcal{L}_tf\|_s\leq 3e^{-\lambda\beta' t}\|f\|_s+B\|f\|_w \tag{3.5}$$

with $B=C\delta^{-1}(1-e^{-\lambda\beta' T_0})^{-1}$.

The strong continuity of the group follows trivially since, for each $f\in\mathcal{C}^1(\mathcal{M},\mathbb{C})$,[11]

$$\lim_{t\to 0}\|\mathcal{L}_tf-f\|=0$$

and $\mathcal{C}^1(\mathcal{M},\mathbb{C})$ is dense in $\mathcal{B}(\mathcal{M},\mathbb{C})$ and $\mathcal{B}_w(\mathcal{M},\mathbb{C})$ by construction. □

4. Proofs: Quasi-compactness of the resolvent

Proof of Lemma 2.9. The idea is to introduce approximate operators $R_\varepsilon(z)$ (close in norm to $R(z)$ as operators from $\mathcal{B}(\mathcal{M},\mathbb{C})$ to $\mathcal{B}_w(\mathcal{M},\mathbb{C})$) and then consider the following sequence of maps (for some $\tau^2\geq\beta_*>\beta>0$)

$$\mathcal{B}(\mathcal{M},\mathbb{C})\xmapsto{Id}\mathcal{C}^\beta(\mathcal{M},\mathbb{C})^*\xhookrightarrow{Id}\mathcal{C}^{\beta_*}(\mathcal{M},\mathbb{C})^*\xmapsto{R_\varepsilon(z)}\mathcal{B}_w(\mathcal{M},\mathbb{C}). \tag{4.1}$$

The first map is clearly continuous since for each $\varphi\in\mathcal{C}^\beta(\mathcal{M},\mathbb{C})$ and $f\in\mathcal{B}(\mathcal{M},\mathbb{C})$ one has

$$\int_{\mathcal{M}}f\varphi\leq\|f\|\,|\varphi|_{s,\beta}\leq\|f\|\,|\varphi|_{\mathcal{C}^\beta}$$

and thus $\|f\|_{(\mathcal{C}^\beta)^*}\leq\|f\|$. The second is well known to be compact. Hence it suffices to prove that the last map is continuous and the compactness of $R_\varepsilon(z)$ as an operator from $\mathcal{B}$ to $\mathcal{B}_w$ immediately follows. Let us postpone the proof of this fact to Lemma 4.4.

[10]By $W_\delta^s(x)$ we mean a ball of radius δ, centered at x, with respect to the metric obtained by restricting the Riemannian metric to $W^s(x)$. By m^s we designate the corresponding volume form.

[11]Indeed, $|f\circ T_{-t}-f|_\infty+H_{u,\beta}(f\circ T_{-t}-f)\to 0$ as $t\to 0$.

To define the approximate operators let us introduce the averaging operator

$$\mathbb{A}^u_\varepsilon f(x) := Z_\varepsilon(x)\int_{W^u_\varepsilon(x)} f(\xi)m^u(d\xi), \tag{4.2}$$

where $Z_\varepsilon(x)$ is determined by the equation $\mathbb{A}^u_\varepsilon 1 = 1$. We set $R_\varepsilon(z) := R(z)\mathbb{A}^u_\varepsilon$.

Sub-lemma 4.1. *The operators $R_\varepsilon(z)$ satisfy*[12]

$$|||R(z) - R_\varepsilon(z)||| \le C\varepsilon^\beta.$$

Proof. For each $f \in \mathcal{C}^1(\mathcal{M},\mathbb{C})$ and $\varphi \in \mathcal{C}^0(\mathcal{M},\mathbb{C})$, we have

$$\left|\int_{\mathcal{M}} \mathbb{A}^u_\varepsilon f\varphi - \int_{\mathcal{M}} f\varphi\right| \le |\varphi|_\infty \int_{\mathcal{M}} dx Z_\varepsilon(x)\int_{W^u_\varepsilon(x)} d\xi |f(\xi) - f(x)| \le C\varepsilon^\beta \|f\|_u|\varphi|_\infty.$$

Accordingly, $\|\mathbb{A}^u_\varepsilon f - f\|_w \le C\varepsilon^\beta\|f\|$, that is $|||\mathbb{A}^u_\varepsilon - \mathbf{Id}||| \le C\varepsilon^\beta$. From Lemma 2.8 it follows $|||R_\varepsilon(z) - R(z)||| \le Ca^{-1}\varepsilon^\beta$. □

Form Sub-Lemma 4.1 and the compactness of $R_\varepsilon(z)$ the compactness of $R(z) : \mathcal{B}(\mathcal{M},\mathbb{C}) \to \mathcal{B}_w(\mathcal{M},\mathbb{C})$ is obvious since the compact operators form a closed set. □

In the previous Lemma we have postponed the proof of Lemma 4.4. Before giving such a proof some preparatory work is needed.

Definition 2. *Given an operator $B : \mathcal{B} \to \mathcal{B}$ we define $B^* : \mathcal{B}^* \to \mathcal{B}^*$ as usual. Notice that if $\varphi \in \mathcal{D}_\beta \subset \mathcal{B}^*$ and $B^*\varphi \subset L^\infty$ then, for each $f \in \mathcal{C}^1$, $Bf \in L^1$, one has*

$$\int Bf\varphi = \int fB^*\varphi. \tag{4.3}$$

Similar definitions hold for $\mathcal{B}_w$ and $\mathcal{D}_1$.

Remark 4.2. *In the following we will never need to investigate the duals $\mathcal{B}^*$, $\mathcal{B}^*_w$; it will suffice to consider elements of $\mathcal{D}_\beta$ and $\mathcal{D}_1$. Accordingly we will always use* (4.3).

Next we isolate a result needed in the present argument but useful also in the following.

Lemma 4.3. *There exists $c > 0$ such that for each $\alpha \in (0,\tau^2)$, $\varphi \in \mathcal{D}_1$, $z \in \mathbb{C}$ with $|b| = |\Im(z)| > 1$ and $a = \Re(z) > 0$, $\mathbb{A}^{u*}_\varepsilon R(z)^*\varphi \in \mathcal{C}^\alpha$. More precisely*

$$|\mathbb{A}^{u*}_\varepsilon R(z)^*\varphi|_{\mathcal{C}^\alpha} \le c(|b| + \varepsilon^{-1})|\varphi|_{s,1}$$

Proof. Let $f \in \mathcal{C}^1(\mathcal{M},\mathbb{C})$ and $\varphi \in \mathcal{D}_1$, then

$$\int_{\mathcal{M}} R_\varepsilon(z)f\varphi = \int_{\mathcal{M}} fR_\varepsilon(z)^*\varphi$$

where $R_\varepsilon(z)^* = \mathbb{A}^{u*}_\varepsilon R(z)^*$,

$$R(z)^*\varphi(x) = \int_0^\infty e^{-zt}T_t\varphi(x)\,dt \tag{4.4}$$

by the definition of $\mathcal{L}_t$. On the other hand in Appendix C it is shown that

$$\mathbb{A}^{u*}_\varepsilon\varphi(x) = \int_{W^u_\varepsilon(x)} \tilde{Z}_\varepsilon(x,\xi)\varphi(\xi)m^u(d\xi) \tag{4.5}$$

for some appropriate τ-Hölder function $\tilde{Z}_\varepsilon$ (see Lemma C.2). Since by (4.4)

$$\frac{d}{dt}(R(z)^*\varphi)\circ T_t\bigg|_{t=0} = zR(z)^*\varphi - \varphi$$

it follows that $R(z)^*\varphi$ is Hölder along the strong stable direction and differentiable along the flow direction. Let us set $\varphi_* := R(z)^*\varphi$.

[12]By $|||\cdot|||$ we mean the norm of an operator viewed as an operator from $\mathcal{B}(\mathcal{M},\mathbb{C})$ to $\mathcal{B}_w(\mathcal{M},\mathbb{C})$.

Let x, y be two points on the same strong stable manifold, and let Ψ be the stable holonomy between $W^{uc}(x)$ and $W^{uc}(y)$. According to Lemma C.1, for each $z \in W^u_\delta(x)$ holds $d(z, \Psi(z)) \leq Cd(x,y)^\tau$. Moreover, $|1 - J\Psi(z)| \leq Cd(x,y)^\tau$.

If $\delta^{\tau^2} \geq d(x,y)^{\tau^2} \geq \varepsilon$ then, see Lemma C.2,

$$|\mathbb{A}^{u*}_\varepsilon \varphi_*(x) - \mathbb{A}^{u*}_\varepsilon \varphi_*(y)| \leq 2\bar{c}|\varphi|_\infty d(x,y)^\alpha \varepsilon^{-\tau^2}.$$

Suppose instead $d(x,y)^{\tau^2} \leq \varepsilon$. Let $\hat{\Psi} : W^u(x) \to W^u(y)$ be the weak stable holonomy ($\{\hat{\Psi}(\xi)\} = W^{sc}(\xi) \cap W^u(y)$). The distance along the flow between $\hat{\Psi}(\xi)$ and $\Psi(\xi)$ is nothing else than the temporal distance $\Delta(y,\xi)$, (see definition at the end of appendix A or Figure 2). Accordingly, Lemma B.7 yields $d(\hat{\Psi}(\xi), \Psi(\xi)) \leq Cd(x,y)^{\tau^2}$. In addition, $W^{uc}_{\varepsilon - c\varepsilon d(x,y)^{\tau^2}}(y) \subset \Psi(W^{uc}_\varepsilon(x)) \subset W^{uc}_{\varepsilon + c\varepsilon d(x,y)^{\tau^2}}(y)$.[13] This, together with the uniform transversality between the unstable manifold and the flow direction, implies that the symmetric difference between $W^u_\varepsilon(y)$ and $\hat{\Psi}(W^u_\varepsilon(x))$ has a volume bounded by a $\varepsilon^{-1}d(x,y)^\alpha$ times the volume of $W^u_\varepsilon(x)$. Finally, it is easy to verify that $J\hat{\Psi} = J\Psi$. Hence, remembering Lemma C.2,

$$\begin{aligned}|\mathbb{A}^{u*}_\varepsilon \varphi_*(x) - \mathbb{A}^{u*}_\varepsilon \varphi_*(y)| &\leq C\left\{|\varphi|_\infty d(x,y)^\alpha(\varepsilon^{-1} + |b|) + \int_{W^u_\varepsilon(x)} \tilde{Z}_\varepsilon(x,\xi)|\varphi_*(\Psi(\xi)) - \varphi_*(\xi)| m^u(d\xi)\right\} \\ &\leq C\left\{|\varphi|_\infty(\varepsilon^{-1} + |b|) + H_{s,1}(\varphi)\right\} d(x,y)^\alpha.\end{aligned}$$

To conclude note that the arguments in the proof of Sub-Lemma 3.1 hold unchanged for $\mathbb{A}^{u*}_\varepsilon$ instead of $\mathbb{A}^s_\varepsilon$. Accordingly,

$$H_{u,\alpha}(\mathbb{A}^{u*}_\varepsilon \varphi_*) \leq C\varepsilon^{-1}|\varphi_*|_\infty \leq C\varepsilon^{-1}|\varphi|_\infty.$$

While a direct computation shows

$$\left|\frac{d}{dt}(\mathbb{A}^{u*}_\varepsilon \varphi_*) \circ T_t\Big|_{t=0}\right|_\infty \leq C\left(|\varphi_*|_\infty + \left|\frac{d}{dt}(\varphi_*) \circ T_t\Big|_{t=0}\right|_\infty\right) \leq C|b|\,|\varphi|_\infty.$$

Since any point in a δ-neighborhood of x can be reached by a path along the stable, unstable and flow direction of length less than const.δ, the Lemma follows. □

We are finally able to prove the continuity of the operator $R_\varepsilon : \mathcal{C}^{\beta_*}(\mathcal{M}, \mathbb{C}) \to B_w(\mathcal{M}, \mathbb{C})$.

Lemma 4.4. *For each $\varepsilon > 0$ an $z \in \mathbb{C}$, $\Re(z) > 0$, the operators $R_\varepsilon(z)$ are bounded operators from $\mathcal{C}^{\beta_*}(\mathcal{M}, \mathbb{C})^*$ to $\mathcal{B}_w(\mathcal{M}, \mathbb{C})$.*

Proof. By Lemma 4.3 it follows that, for each $f \in \mathcal{C}^1$ and $\varphi \in \mathcal{D}_1$,

$$\int_{\mathcal{M}} R_\varepsilon(z) f\varphi \leq |f|_{(\mathcal{C}^{\beta_*})^*}\ |R_\varepsilon(z)^*\varphi|_{\mathcal{C}^{\beta_*}} \leq C(|z| + \varepsilon^{-1})|\varphi|_{s,1}|f|_{(\mathcal{C}^{\beta_*})^*}$$

which means $\|R_\varepsilon(z)f\|_w \leq C(|z| + \varepsilon^{-1})|f|_{(\mathcal{C}^{\beta_*})^*}$ and the required result follows by an obvious density argument. □

[13] By introducing a coordinate system in which $W^{uc}(x)$ and $W^s(x)$ are linear spaces one can represent $W^{uc}(y)$ as $\{(\xi, F(\xi))\}$ where, by the Hölder continuity of the unstable foliation and setting $U(\xi) := D_\xi F$, one has $\|U(\xi)\| \leq c\|F(\xi)\|^\tau$ and, by the Hölder continuity of the unstable holonomy, $\|F(\xi)\| \leq cd(x,y)^\tau$. Thus, setting $\gamma(t) = (vt, F(vt))$, with $v := z - x$, and $z' := (z, F(z))$, one can estimate

$$\begin{aligned}\operatorname{dist}(y, z') &= \int_0^1 \|\gamma'(t)\| dt = \int_0^1 \sqrt{\langle (v, U(vt)v), g((vt, F(vt)))(v, U(vt)v)\rangle} dt \\ &= \int_0^1 \sqrt{\langle (v,0), g((vt,0))(v,0)\rangle + O(d(x,z)^2 d(x,y)^{\tau^2})}\, dt = d(x,z)(1 + O(d(x,y)^{\tau^2})\end{aligned}$$

where g is the matrix defining the Riemannian metric. On the other hand one can represent $W^s(z)$ as $\{(G(\zeta), \zeta)\}$, where $V(\zeta) := D_\zeta G$ is bounded in norm by $c\varepsilon^\tau$. Setting $\Psi(z) =: (a,b) = (a, F(a)) = (G(b), b)$ it follows $\|b\| \leq cd(x,y)^\tau$, hence (provided $d(x,z) \geq d(x,y)^\tau$) $\operatorname{dist}(z', \Psi(z)) \leq c \operatorname{dist}((a,0), z) \leq \int_0^1 \|V(bt)b\| dt \leq c\, d(x,z)^\tau d(x,y)^\tau \leq c\, d(x,z) d(x,y)^{\tau^2}$.

5. Proofs: Resolvent bound for large $\Im(z)$

Proof of Proposition 2.12. Lemma 2.8 states that, for each $m, n \in \mathbb{N}$ and $f \in \mathcal{C}^1(\mathcal{M}, \mathbb{C})$,

$$\begin{aligned}\|R(z)^{n+m} f\| &\leq \frac{3}{(a+\lambda\beta')^m}\|R(z)^n f\| + a^{-m} B\|R(z)^n f\|_w \\ &\leq \frac{3}{(a+\lambda\beta')^m a^n}\|f\| + a^{-m} B\|R(z)^n f\|_w\end{aligned} \tag{5.1}$$

hence all we need is to estimate more precisely the weak norm of $R(z)^n f$.

Remembering (4.2)

$$\int f\varphi = \int \mathbb{A}^u_\delta f\varphi + O(\delta^\beta \|f\|_u |\varphi|_\infty) = \int f \mathbb{A}^{u*}_\delta \varphi + O(\|f\|_u |\varphi|_\infty). \tag{5.2}$$

Thus, for each $k, l \in \mathbb{N}$, $k + l = n$, and $\varphi \in \mathcal{D}_1$ holds, by equation (5.2),

$$\int_{\mathcal{M}} R(z)^n f\varphi = \int_{\mathcal{M}} R(z)^k f R(z)^{*l}\varphi = \int_{\mathcal{M}} R(z)^k f \mathbb{A}^{u*}_\delta R(z)^{*l}\varphi + a^{-l} O(\|R(z)^k f\|_u).$$

To continue let

$$\Phi_l(\varphi) := \mathbb{A}^{u*}_\delta R(z)^{*l}\varphi.$$

Thus, taking into account (2.7) and (2.9),

$$\int_{\mathcal{M}} R(z)^n f\varphi = \int_{\mathcal{M}} R(z)^k f\Phi_l(\varphi) + a^{-n}(1 + a^{-1}\lambda\beta)^{-k} O(\|f\|_u). \tag{5.3}$$

Lemma 5.1. *There exists $c > 0$ such that, for each $l \in \mathbb{N}$ and $\varphi \in \mathcal{D}_1$,*

$$H_{s,\beta}(\Phi_l(\varphi)) \leq c|b|a^{-l}|\varphi|_{s,1}.$$

Proof. The proof follows immediately from Lemma 4.3 and formulae (2.7), (3.1). □

The above estimate is not particularly impressive and clearly it can have some interest only if we can get good bounds on $|\Phi_l(\varphi)|_\infty$. This can be achieved by using an inequality due to Dolgopyat.[14]

Lemma 5.2 (Dolgopyat inequality)**.** *There exists $c_*, c_1, \gamma > 0$ such that, for each $\varphi \in R(z)^*(\mathcal{D}_1)$ and $l \geq \lceil c_* \ln |b| \rceil$, the following holds*

$$a^l |\Phi_l(\varphi)|_\infty \leq c_1 |b|^{-\gamma} l |\varphi|_{s,1}.$$

The proof of the above Lemma can be found in Section 6.

Since equation (3.1) implies that, for each $q \in \mathbb{N}$, $a^q R(z)^{*q}\varphi \in \mathcal{D}_{s,1}$ and $H_{s,\beta}(R(z)^{*q}\varphi) \leq (a+\beta\lambda)^{-q} H_{s,\beta}(\varphi)$ by Lemma 5.2 and Lemma 5.1 it follows that

$$|R(z)^{*k}\Phi_l(\varphi)|_{s,\beta} \leq c_4\{(1 + a^{-1}\lambda\beta)^{-k}|b| + |b|^{-\gamma} l\} a^{-n}|\varphi|_{s,1}.$$

Choose $l := \lceil c_* \ln b \rceil$, then there exists $c' > 0$ and $\nu_0 \in (0,1)$ such that setting $k = \lceil c' \ln b \rceil$ equation (5.3) yields

$$\|R(z)^n f\|_w \leq c_5 a^{-n} \nu_0^n \|f\|. \tag{5.4}$$

The Proposition follows by (5.1), (5.4), choosing $m = n = \bar{n}/2$ (hence $c^* = 2(c_* + c')$), $\bar{c} = 2$, $\nu \in (\sqrt{\nu_0}, 1)$ and b_* such that $c_5(\nu_0\nu^{-2})^n \leq 1$. □

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[14]Actually the original Dolgopyat estimate, [7], holds for the L^2 norm and it is done for a different operator in a different functional space, yet the key cancellation mechanism due to the oscillations of the exponential and the non joint integrability of the foliation remains substantially identical in the two settings.

6. Dolgopyat Inequality

This section is devoted to the proof of Lemma 5.2. The strategy is based on the representation (2.7) (actually on the obvious adjoint representation obtained by (4.4)) and a careful estimate of the corresponding integral.

The following simple preliminary Lemma shows that we need to worry about only a part of the integral defining $\Phi_l(\varphi)$.

Lemma 6.1. *There exists $\nu_* < 1$ such that*

$$\left|\frac{1}{(l-1)!}\int_0^{e^{-1}a^{-1}l} t^{l-1}e^{-zt}\mathbb{A}_\delta^{u*}(T_t\varphi)dt\right| \leq \nu_*^l a^{-l}|\varphi|_\infty.$$

The straightforward proof is left to the reader.

Thus we can limit ourselves to consider

$$\frac{1}{(l-1)!}\int_{e^{-1}a^{-1}l}^\infty t^{l-1}e^{-zt}\mathbb{A}_\delta^{u*}(T_t\varphi).$$

To continue it is useful to localize in time. To do so we introduce a $\mathcal{C}^\infty$ function $\mathbf{p}:\mathbb{R}\to\mathbb{R}$ such that $0\leq\mathbf{p}\leq 1$, $\text{supp}(\mathbf{p})\subset[-1/2,3/2]$ and with the property that $\sum_{k=-\infty}^\infty \mathbf{p}(t-k)=1$ for each $t\in\mathbb{R}$. Using such a partition of unity and setting $p_0:=\lceil a^{-1}e^{-1}l\rceil$, we can write

$$\left|\int_0^\infty t^{l-1}e^{-zt}\mathbb{A}_\delta^{u*}(T_t\varphi)\right| \leq \left|\sum_{k=p_0}^\infty\int_{\mathbb{R}} t^{l-1}e^{-zt}\mathbf{p}(t-k)\mathbb{A}_\delta^{u*}(T_t\varphi)\right| + \nu_*^l a^{-l}(l-1)!|\varphi|_\infty.$$

Let us analyze each of the above addenda separately.

For each $k\in\mathbb{N}$ holds (see (4.5))

$$\begin{aligned}&\int_{\mathbb{R}} t^{l-1}e^{-zt}\mathbf{p}(t-k)\mathbb{A}_\delta^{u*}(T_t\varphi)\\ &=\int_{\mathbb{R}}\mathbf{p}(t-k)t^{l-1}e^{-zt}\int_{T_kW_\delta^u(x)}\tilde{Z}(x,T_{-k}\xi)\varphi(T_{t-k}\xi)J_uT_{-k}(\xi),\end{aligned}$$

where by J_uT_t we designate the unstable Jacobian of the map T_t.

To compute the above quantity it is convenient to localize in space as well. To this end we fix a sequence of smooth partitions of unity. There exists $c_d>0$ such that, for each $r\in(0,1)$ one can consider a $\mathcal{C}^4$ partition of unity $\{\phi_{r,i}\}_{i=1}^{q(r)}$ enjoying the following properties[15]

- **(i)** for each $i\in\{1,\dots,q(r)\}$, there exists $x_i\in\mathcal{M}$ such that $\phi_{r,i}(\xi)=1$ for all $\xi\in B_r(x_i)$ (the ball of radius r centered at x_i) and $\phi_{r,i}(\xi)=0$ for all $\xi\notin B_{c_dr}(x_i)$;
- **(ii)** there exists a $K>0$ such that for each r,i holds $\|\phi_{r,i}'(x)\|\leq Kr^{-1}\chi_{B_{c_dr}(x_i)}(x)$;[16]
- **(iii)** there exists $C>0$ such that $q(r)\leq Cr^{-2d-1}$.

Accordingly, we can write

$$\begin{aligned}&\int_{\mathbb{R}} t^{l-1}e^{-zt}\mathbf{p}(t-k)\mathbb{A}_\delta^{u*}(T_t\varphi)\\ &=\sum_{i=1} q(r)e^{-zk}\int_{\mathbb{R}}\mathbf{p}(t)(t+k)^{l-1}e^{-zt}\int_{T_kW_\delta^u(x)}\phi_{r,i}(T_t\xi)\tilde{Z}(x,T_{-k}\xi)\varphi(T_t\xi)J_uT_{-k}(\xi).\end{aligned}$$

From now on we will assume $b>0$, the case $b<0$ being identical.

In the following we choose $\rho\in(0,\tau/8)$ and we fix

$$r:=b^{-\varrho};\quad \varrho:=\frac{1-\tau+2\rho}{2-\tau}. \tag{6.1}$$

It is useful to partition $T^kW_\delta^u(x)$ into submanifolds. For each x_i let us consider the connected pieces of $T^kW_\delta^u(x)\cap B_{\theta c_dr}(x_i)$ intersecting $B_{c_dr}(x_i)$ (θ is specified shortly). Call them $\{W_{k,i,m}^u\}$. Among such local manifolds discard the ones such that $\partial W_{k,i,m}^u\not\subset\partial B_{\theta c_dr}(x_i)$, see Figure 1.

[15]It is an easy exercise to verify that partitions with the properties below do exist.

[16]Here, and in the following, χ_A is the characteristic function of the set A.

Clearly, if W is a discarded manifold, then $T_{-k}W$ belongs to a $\theta c_d r\lambda^{-k}$-neighborhood of $\partial W^u_\delta(x)$, hence the total measure of the preimages of the discarded manifolds is bounded by const.λ^{-k}. The constant θ is chosen so that if $\xi \in W^u_{k,i,m} \cap B_{c_d r}(x_i)$, then $W^s_\delta(\xi) \cap W^u_{k,i,j} \neq \emptyset$, for all j.

Let us define $W^{uc}_{k,i,m} := \cup_{t\in[-2,2]} T_t W^u_{k,i,m}$.

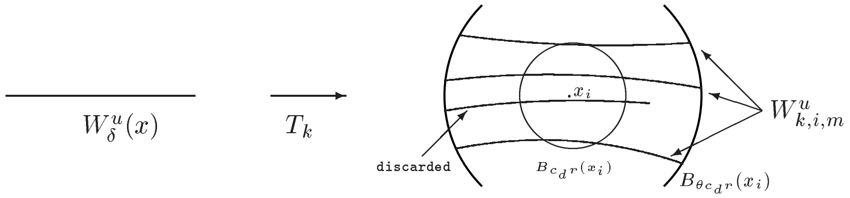

FIGURE 1. The manifolds $W^u_{k,i,m}$.

For each $\xi \in W^{uc}_{k,i,j}$ let $t(\xi)$ be such that $T_{t(\xi)}\xi \in W^u_{k,i,j}$ and let $u(\xi) := T_{t(\xi)}\xi$. Then

$$\begin{aligned}\int_{\mathbb{R}} t^{l-1}e^{-zt}\mathbf{p}(t-k)\mathbb{A}^{u*}_\delta(T_t\varphi) &= \sum_{ij} e^{-zk}k^{l-1}\int_{W^{uc}_{k,i,j}} \mathbf{p}(t(\xi)) \\ &\times \left(1+\frac{t(\xi)}{k}\right)^{l-1} e^{-zt(\xi)}\tilde{Z}(x, T_{-k}u(\xi))\varphi(\xi)\phi_{r,i}(\xi)J_uT_{-k}(u(\xi)) + k^{l-1}e^{-ak}O(\lambda^{-k}|\varphi|_\infty).\end{aligned} \tag{6.2}$$

Next, for each $W^{uc}_{k,i,j}$ let $\Psi_{k,i,j}$ be the stable holonomy between $W^{uc}_{k,i,j}$ and $W^{uc}_{k,i,0}$. By the general theory of the holonomy maps (see Appendix A) it follows that $\Psi_{k,i,j}$ is a τ-Hölder function with τ-Hölder Jacobian $J\Psi_{k,i,j}$.

Notice that $T_{-k}W^u_{k,i,j}$ has size smaller than $2c_d\lambda^{-k}r$ and thus (see Lemma C.2)

$$\tilde{Z}(x, T_{-k}u(\xi)) = Z_{k,i,j} + O(\lambda^{-k\tau}r^\tau).$$

To simplify notations let us introduce the functions

$$\begin{aligned}F_{k,i,j}(\xi) =& \mathbf{p}(t(\xi))\left(1+\frac{t(\xi)}{k}\right)^{l-1}\phi_{r,i}(\xi)J_uT_{-k}(u(\xi))e^{-at(\xi)}Z_{k,i,j} \\ \hat{F}_{k,i,j}(\xi) =& F_{k,i,j}(\Psi_{k,i,j}(\xi)))J\Psi_{k,i,j}(\xi)\end{aligned} \tag{6.3}$$

Using the above formulae we can rewrite (6.2) as

$$\begin{aligned}\int_{\mathbb{R}} t^{l-1}\mathbf{p}(t-k)e^{-zt}\mathbb{A}^{u*}_\delta(T_t\varphi) = \sum_{ij} e^{-zk}k^{l-1}\int_{W^{uc}_{k,i,0}} e^{-ibt(\Psi_{k,i,j}(\xi))}\hat{F}_{k,i,j}(\xi)\varphi(\xi) \\ + k^{l-1}e^{-ak}O(\lambda^{-k\tau}|\varphi|_\infty + rH_{s,1}(\varphi)).\end{aligned} \tag{6.4}$$

The last preparatory step is to apply Schwartz inequality. More precisely, for each k, i, we can compute

$$\begin{aligned}\left|\int_{W^{uc}_{k,i,0}} \varphi(\xi)\left[\sum_j e^{-ibt(\Psi_{k,i,j}(\xi))}\hat{F}_{k,i,j}(\xi)\right]\right| \leq C|\varphi|_\infty r^{(d+1)/2} \\ \times\left[\sum_{j,j'}\int_{W^{uc}_{k,i,0}} e^{-ibg^0_{j,j'}(\xi)}\hat{F}_{k,i,j}(\xi)\overline{\hat{F}_{k,i,j'}(\xi)}\right]^{\frac{1}{2}},\end{aligned} \tag{6.5}$$

where

$$g^0_{j,j'}(\xi) := t(\Psi_{k,i,j}(\xi)) - t(\Psi_{k,i,j'}(\xi)). \tag{6.6}$$

We are finally approaching the end of the story: to conclude we must only show that the above integral is small.

Let us perform the sum on j' for each j. Fixed j it is convenient to express the integral on the manifold $W^{uc}_{k,i,j}$:

$$\int_{W^{uc}_{k,i,0}} e^{-ibg^0_{j,j'}(\xi)}\hat F_{k,i,j}(\xi)\overline{\hat F_{k,i,j'}(\xi)} = \int_{W^{uc}_{k,i,j}} e^{-ibg_{j,j'}(\xi)}F_{k,i,j}(\xi)\overline{F_{k,i,j,j'}(\xi)}$$

where

$$\begin{aligned} g_{j,j'}(\xi) &:= t(\xi) - t(\Psi_{k,i,j,j'}(\xi)) \\ \Psi_{k,i,j,j'}(\xi) &:= \Psi_{k,i,j'}\circ\Psi^{-1}_{k,i,j}(\xi) \\ F_{k,i,j,j'}(\xi) &:= F_{k,i,j'}(\Psi_{k,i,j,j'}(\xi))J\Psi_{k,i,j'}\circ\Psi^{-1}_{k,i,j}(\xi), \end{aligned} \tag{6.7}$$

clearly $\Psi_{k,i,j,j'}$ is nothing else than the holonomy between $W^{uc}_{k,i,j}$ and $W^{uc}_{k,i,j'}$.

Finally, it is convenient to divide the sum over j' into two part: the sum over nearby manifolds and the sum over manifolds at a useful distance. Let us be more precise.

Let $y_{k,i,j} := W^s_{2c_dr}(x_i)\cap W^{uc}_{k,i,j}$. We define the sets of indexes $A_{k,i,j} := \{j' \mid d(y_{k,i,j}, y_{k,i,j'}) < b^{-\varsigma}\}$ and $B_{k,i,j} := \{j' \mid d(y_{k,i,j}, y_{k,i,j'}) \geq b^{-\varsigma}\}$. In the following we choose

$$\varsigma := \frac{1-4\rho}{2-\tau}. \tag{6.8}$$

Notice the the assumption $\rho < \tau/6$ implies that $b^{-\varsigma}$ is much smaller than r, as b increases.

The first step is to estimate the sum with indexes in $A_{k,i,j}$. To do so we need the next Lemma whose proof is postponed to the end of the section.

Lemma 6.2. *For each $\epsilon > 0$, let W^{uc}_ϵ be an unstable disk of radius ϵ. Then there exit constants $C, r_0 > 0$ such that for each $k \in \mathbb{N}$, $r_1 > 0$ and $x \in \mathcal{M}$, calling $\{W_j\}$ the connected components of $T_kW^{uc}_\epsilon \cap B_{2r_1}(x)$,*

$$\sum_{j\in\Omega}\sup_{\xi\in W_j} J^u_\xi T_{-k} \leq C\, m^s(W^s_{r_1+\lambda^{-k}r_0}(x)),$$

where $\Omega := \{j \mid W_j \cap W^s_{r_1}(x) \neq \emptyset\}$.

We then require

$$\lambda^{-l}r_0 \leq b^{-\varsigma}. \tag{6.9}$$

Using the above Lemma and standard distortion arguments we readily obtain

$$\left|\sum_{j'\in A_{k,i,j}}\int_{W^{uc}_{k,i,j}} e^{-ibg_{j,j'}}F_{k,i,j}\overline{F_{k,i,j,j'}}\right| \leq CJ^uT_{-k}(y_{k,i,j})b^{-d\varsigma}r^{d+1}. \tag{6.10}$$

We are then left with the estimate of the indexes in $B_{k,i,j}$. To this end it is useful to make a connection with the *temporal function* introduced at the end of Appendix A and shown pictorially in Figure 2. For each $\xi \in W^{uc}_{k,i,j}$ holds[17]

$$g_{j,j'}(\xi) = t(\Psi_{k,i,j,j'}(\xi)) - t(\xi) = \Delta(y_{k,i,j'}, T_{-t(y_{k,i,j})}u(\xi)) - t(y_{k,i,j}) + t(y_{k,i,j'}). \tag{6.11}$$

All the above work was just preparation to apply the following Lemma (the proof can be found at the end of the section).

Lemma 6.3. *For each function $G \in \mathcal{C}^\alpha(W^{uc}_{k,i,j})$, $0 < \alpha < 1$, $j' \in B_{k,i,j}$, and setting $\bar\phi(u) := \phi_{i,r}(u)\phi_{i,r}(\Psi_{k,i,j,j'}(u))$, the following holds*

$$\left|\int_{W^{uc}_{k,i,j}} du\, e^{-ibg_{j,j'}(u)}G(u)\bar\phi(u)\right| \leq Cb^{-\alpha\rho}r^{d+1}|G|_{\mathcal{C}^\alpha}.$$

[17]To apply Figure 2 to the present case set: $y = y_{k,i,j'}$, $x = y_{k,i,j}$ and $y' = T_{-t(y_{k,i,j})}u(\xi)$.

Remembering (6.3), (6.4), (6.5), (6.7), using (6.10) with Lemma 6.3 and taking (A.3) into account yields[18]

$$
\begin{aligned}
\left|\int_{\mathbb{R}} t^{l-1}\mathbf{p}(t-k)e^{-zt}\mathbb{A}_\delta^{u*}(T_t\varphi)(x)\right| &\leq Ck^{l-1}e^{-ak}\sum_i |\varphi|_\infty r^{\frac{d+1}{2}}\\
&\times\left[\sum_j J_uT_{-k}(y_{k,i,j})\left\{r^{d+1}b^{-d\varsigma}+r^{2d+1}b^{-\alpha\rho}l\right\}\right]^{\frac12}+Ck^{l-1}e^{-ak}b^{-\rho}|\varphi|_{s,1}\\
&\leq Ck^{l-1}e^{-ak}\left\{\sum_i|\varphi|_\infty r^{\frac{d+1}{2}}[r^{3d+1}b^{-\alpha\rho}l]^{\frac12}+b^{-\rho}|\varphi|_{s,1}\right\}\\
&\leq Ck^{l-1}e^{-ak}b^{-\frac{\alpha\rho}{2}}l^{\frac12}|\varphi|_{s,1}\,.
\end{aligned}
\tag{6.12}
$$

We can finally sum over k and the result follows.

We are left with the postponed proofs.

Proof of Lemma 6.2. Note that the Jacobian of T_k must be equal one, on the other hand it must also be equal to the product of the stable and unstable Jacobian times a function θ which express the "angle" between the stable and unstable manifold (and hence it is Hölder). Thus, setting $\{\xi_j\} = W_j \cap W^s_{2r_1}(x)$,

$$\sum_{j\in\Omega} J^u_{\xi_j}T_{-k} = \sum_{j\in\Omega}\theta(T_{-k}\xi_j)J^s_{T_{-k}\xi_j}T_k.$$

Now, consider $T_{-k}W^s_{r_1}(x)$, clearly it will intersect W^{uc} at the points $T_{-k}\xi_j$, $j\in\Omega$. Obviously, if we consider disks $D_j\subset T_{-k}W^s_{r_1}(x)$ centered at $T_{-k}\xi_j$ and with radius r_0 sufficiently small, but depending only on T, they will be all disjoint. Moreover the diameter of each T_kD_j must be smaller than $\lambda^{-k}r_0$. This means that $\cup_{j\in\Omega}T_kD_j$ is a collection of disjoint sets contained in the disk $W^s_{r_1+\lambda^{-k}r_0}(x)$. In addition, by the usual distortion arguments, there exists $c>0$ such that

$$J^s_{T_{-k}\xi_j}T_k \leq c\, m^s(T_kD_j).$$

Using again distortion it follows

$$\sum_{j\in\Omega}\sup_{\xi\in W_j} J^u_\xi T_{-k} \leq c^2|\theta|_\infty\sum_{j\in\Omega}m^s(T_kD_j)\leq Cm^s(D_{r_1+\lambda^{-k}r_0}(x)).$$

□

Proof of Lemma 6.3. The Lemma rests on smoothness estimates for $g_{j,j'}$ which, in turn, are obtained by estimates on Δ. Indeed, by looking at Figure 2 again it follows that for each $\xi,\eta\in W^{uc}_{k,i,j}$ (see also footnote 17)

$$g_{j,j'}(\xi)-g_{j,j'}(\eta)=\Delta(\Psi_{k,i,j,j'}(T_{-t(y_{k,i,j})}u(\eta)),T_{-t(y_{k,i,j})}u(\xi))=\Delta(\Psi_{k,i,j,j'}(\eta),T_{t(\eta)-t(\xi)}\xi).\tag{6.13}$$

For each $y\in W^{uc}_{k,i,j}$ define $w_{j'}(y)\in E^s(y)$ by $\exp_y(w_{j'}(y))=\Psi_{k,i,j,j'}(y)$. Then the normalized vectors $\hat w_{j'}(y):=w_{j'}(y)|w_{j'}(y)|^{-1}$ are uniformly continuous functions. It follows that there exists a uniformly smooth coordinate system $\{u_1,u_2,\dots,u_{d+1}\}:=\{u_1,\bar u)\}$ for $W^{uc}_{k,i,j}$ such that $d\alpha(\partial_{u_1},\hat w_{j'}(y))\geq c_-\|\partial_{u_1}\|$ for all $y\in W^{uc}_{k,i,j}$. Without loss of generality we can assume $t(u)=u_{d+1}$. Let $v(u):=\|\partial_{u_1}\|^{-1}\partial_{u_1}$. All is needed in the following are bounds on the dependence of $g_{jj'}$ from the coordinate u_1 keeping fixed the other coordinates.

[18]We remark that if we choose $\alpha<\tau^2$, then

$$
\begin{aligned}
\sum_{j'\in B_{k,i,j}}|\bar\phi^{-1}F_{k,i,j}F_{k,i,j,j'}|_{\mathcal{C}^\alpha}&\leq Cl\sum_{j'\in B_{k,i,j}}|J_uT_{-k}J_uT_{-k}\circ\Psi_{k,i,j,j'}|_{\mathcal{C}^\alpha}\\
&\leq Cl\sum_{j'\in B_{k,i,j}}|J_uT_{-k}J_uT_{-k}\circ\Psi_{k,i,j,j'}|_\infty\leq Clr^dJ_uT_{-k}(y_{k,i,j}).
\end{aligned}
$$

For each $\bar{u}$, let us consider a partition $\{[a_q, a_{q+1}]\}$ of $[-2c_d r, 2c_d r]$, such that

$$b(a_{q+1} - a_q)\, d\alpha(w_{j'}(a_q, \bar{u}), v((a_q, \bar{u}))) = 2\pi.$$

This implies

$$2\pi c_+ b^{-1} |w_{j'}(a_q, \bar{u})|^{-1} \geq a_{a+1} - a_q \geq 2\pi c_- b^{-1} |w_{j'}(a_q, \bar{u})|^{-1}. \tag{6.14}$$

Now, since $j' \in B_{k,i,j}$ it follows that, by the Hölder continuity of the foliation,

$$|w_{j'}(u_q)| \geq b^{-\varsigma} - Cb^{-\varsigma\tau} r \geq b^{-\varsigma} - Cb^{-\frac{1-2\rho}{2-\tau}} \geq \frac{1}{2} b^{-\varsigma} \tag{6.15}$$

provided b is large enough. Hence, our choices imply $|a_{q+1} - a_q| << r$ provided b is large.

Accordingly, if $u_q = (a_q, \bar{u})$ and $u' = (u_1, \bar{u})$, with $u_1 \in [a_q,\, a_{q+1}]$, setting $\delta_q = a_{q+1} - a_q$, by Lemma B.7 and (6.13) the following holds

$$|g_{j,j'}(u') - g_{j,j'}(u_q) - (u_1 - a_q) d\alpha(w_{j'}(u_q), v(u_q))| \leq C \left(|w_{j'}(u_q)|^2 \delta_q^{\tau} + |w_{j'}(u_q)|^{\tau} \delta_q^2 \right). \tag{6.16}$$

Indeed, $|w_{j'}(u_q)|^{\tau -} \geq \delta_q$ and $\delta_q^{\tau -} \geq |w_{j'}(u_q)|$. This follows readily from (6.14) and $\theta c_d r \geq |w_{j'}(u_q)| \geq \frac{1}{2} b^{-\varsigma}$. By (6.14) we have $\delta_q \leq Cb^{-\frac{1-\tau+4\rho}{2-\tau}}$, therefore (6.1), (6.15) and (6.16) yield

$$|g_{j,j'}(u') - g_{j,j'}(u_q) - (u_1 - a_q) d\alpha(w_{j'}(u_q), v(u_q))| \leq Cb^{-1-2\rho}. \tag{6.17}$$

Hence,[19]

$$\begin{aligned}
\int_{W^{uc}_{k,i,j}} du\; m(u) e^{-ibg_{j,j'}(u)} G(u) \bar{\phi}(u) &= \int d\bar{u} \sum_q \int_{a_q}^{a_{q+1}} du_1\; m(u_1, \bar{u}) e^{-ibg_{j,j'}(u_1, \bar{u})} G(u_1, \bar{u}) \bar{\phi}(u_1, \bar{u}) \\
= \int d\bar{u} \sum_q \Bigg\{ m(u_q) G(u_q) \bar{\phi}(u_q) e^{-ibg_{j,j'}(u_q)} &\int_{a_q}^{a_{q+1}} du_1\; e^{-ib(u_1 - a_q) d\alpha(w_{jj'}(u_q), v(u_q))} \\
&+ O\left((|G|_\infty b^{-2\rho} + |G|_{\mathcal{C}^\alpha} \delta_q^{\alpha}) \int_{a_q}^{a_{q+1}} \bar{\phi} + |G|_\infty \delta_q r^{-1} \int_{a_q}^{b_q} \chi_{B_{c_d r}(x_i)} \right) \Bigg\}
\end{aligned}$$

where we have used the fact that, for each $|h| \leq \delta_q$, $d(\Psi_{k,i,j,j'}(u_q),\, \Psi_{k,i,j,j'}(a_q + h, \bar{u})) \leq C\delta_q$ thanks to the Hölder continuity of the stable foliation, our choice of the parameters and since the maximal distance between u and $\Psi_{k,i,j,j'}(u)$ is bounded by a constant times r.[20] Continuing the above chain of inequalities yields

$$\begin{aligned}
&= \int d\bar{u} \sum_q \left\{ m(u_q) G(u_q) \bar{\phi}(u_q) e^{-ibg_{j,j'}(u_q)} \int_0^{\delta_q} du_1\; e^{-ibu_1 d\alpha(w_{jj'}(u_q), v(u_q))} \right\} \\
&\quad + |G|_{\mathcal{C}^\alpha} r^{d+1} O(b^{-\alpha\rho} + b^{-\frac{2\rho}{2-\tau}}) \\
&= \int d\bar{u} \sum_q \left\{ m(u_q) G(u_q) \bar{\phi}(u_q) e^{-ibg_{j,j'}(u_q)} \delta_q \int_0^1 ds\; e^{-2\pi i s} \right\} + |G|_{\mathcal{C}^\alpha} r^{d+1} O(b^{-\alpha\rho}).
\end{aligned}$$

Since the inner integral equals zero exactly, the Lemma is proven. □

[19] Let $m^{uc}(du) =: m(u)du$ be the measure on the manifold, clearly m is uniformly smooth.

[20] Here is a more detailed argument: consider a coordinate chart based at u_q in which $W^{uc}_{k,i,j}$ and $W^s(u_q)$ are linear spaces. Then $W^s(u')$, $u' = (a_q + h, \bar{u})$, can be represented as $\{(G(\xi), \xi\}_{\xi \in \mathbb{R}^d}$ and if $\Psi_{k,i,j,j'}(u') =: (a, b)$, then $G(b) = a$. Setting $d(t) := \|G(bt)\|$, by the Hölder continuity of the foliation it follows

$$|d'(t)| \leq C\|b\| d(t)^{\tau}.$$

The above differential inequality yields $\|a\| \leq [\delta_q^{1-\tau} + C\|b\|]^{\frac{1}{1-\tau}} \leq \delta_q [1 + C\delta_q^{\tau-1} r]^{\frac{1}{1-\tau}}$. The result follows since $r < \delta_q^{1-\tau}$, the maximal "angle" between $W^{uc}_{k,i,j}$ and $W^{uc}_{k,i,j'}$ is bounded by Cr^{τ}, and the metric in the chart is equivalent to the Riemannian metric.

Appendix A. Basic facts (Anosov flows)

In this appendix we collect, for the reader's convenience, some information on the smoothness properties of the invariant foliations in Anosov flows that are used in the paper.

First of all, as already mentioned, for $\mathcal{C}^2$ Anosov flows the invariant distributions (sometimes called *splittings*) are known to be uniformly Hölder continuous. Let us be more precise.

For each invertible linear map L let $\theta(L) := \|L^{-1}\|^{-1}$. We define $\|d^s_x T_t\| = \|d_x T_t|_{E^s_x}\|$, $\|d^u_x T_t\| = \|d_x T_t|_{E^u_x}\|$, $\theta(d^s_x T_t) = \theta(d_x T_t|_{E^s_x})$ and $\theta(d^u_x T_t) = \theta(d_u T_t|_{E^u_x})$. Then the following holds ([10, 37])

(A.1)
- If there exists τ_d such that, for each $x \in \mathcal{M}$, and some $t \in \mathbb{R}^+$, $\|d^s_x T_t\|\, \|d^u_x T_t\|^{\tau_d} < \theta(d^u_x T_t)$, then $E^{sc} \in \mathcal{C}^{\tau_d}$.
- If there exists $\tau_d > 0$ such that, for each $x \in \mathcal{M}$, and some $t \in \mathbb{R}^+$, $\theta(d^s_x T_t)^{\tau_d}\, \theta(d^u_x T_t) > \|d^s_x T_t\|^{-1}$, then $E^{uc} \in \mathcal{C}^{\tau_d}$.

Moreover the Hölder continuity is uniform (that is the τ_d-Hölder norm of the distributions is bounded). The above conditions are often called *τ_d-pinching* or *bunching* conditions.

The next relevant fact is that the above splittings are integrable. The integral manifolds are the stable and unstable manifolds, respectively. Clearly, this implies the existence of the weak stable and weak unstable manifolds as well. They form invariant continuous foliations. Each leaf of such foliations is as smooth as the map and it is tangent, at each point, to the corresponding distribution, [17]. In addition, for $\mathcal{C}^r$ maps, the $\mathcal{C}^r$ derivatives of such manifolds (viewed as graphs over the corresponding distributions) are uniformly bounded, [14]. Finally, the foliations are uniformly transversal and $\mathcal{C}^{\tau_d}$.

In the case in which both distribution are $\mathcal{C}^{1+\alpha}$, it follows by Frobenius' theorem that the Holonomy maps are $\mathcal{C}^{1+\alpha}$ (section six of [29]). If the splitting is only Hölder the situation is more subtle.

We will call *stable holonomy* any holonomy constructed via the strong stable foliations and *unstable holonomy* holonomies constructed by the strong unstable foliation. The basic result on holonomies is given by the following [29].

(A.2)
- If there exists $\tau_h > 0$ such that for some $t \in \mathbb{R}^+$ and for each $x \in \mathcal{M}$ $\|d^s_x T_t\|\, \|d^u_x T_t\|^{\tau_h} < 1$, then the stable holonomies are uniformly $\mathcal{C}^{\tau_h}$.
- If there exists $\tau_h > 0$ such that for some $t \in \mathbb{R}^+$ and for each $x \in \mathcal{M}$ $\theta(d^s_x T_t)^{\tau_h}\, \theta(d^u_x T_t) > 1$, then the unstable holonomies are uniformly $\mathcal{C}^{\tau_h}$.

The relation between smoothness of holonomies and smoothness of the foliation (in the sense that the local foliation charts are smooth) is discussed in detail in [29, section 6]. Here we restrict ourselves to what is needed in the present paper.

This is not yet enough for our purposes: we need to talk about the smoothness of the Jacobian of the holonomies between two manifolds $W^{uc}(x)$ and $W^{uc}(y)$.[21]

(A.3)
- The stable and unstable holonomies are absolutely continuous.
- There exists $\tau_j > 0$ such that $|1 - J\Psi|_\infty \le C d(x,y)^{\tau_j}$
- There exists $\tau_j > 0$ such that for each $x \in \mathcal{M}$ the Jacobian of the stable holonomies are uniformly $\mathcal{C}^{\tau_j}$.
- There exists $\tau_j > 0$ such that for each $x \in \mathcal{M}$ the Jacobian of the unstable holonomies are uniformly $\mathcal{C}^{\tau_j}$.

The last, but not least important, object on which we need smoothness informations is the so called *temporal distance*.

Fix any point $x \in \mathcal{M}$ and a small neighborhood $B_\delta(x)$. Consider a smooth $2d$ dimensional manifold $\mathcal{W}$ containing $W^u(x)$ and $W^s(x)$, clearly the flow is transversal to such a manifold. On $\mathcal{W}$ choose a smooth coordinate system (u,s) such that $\{(u,0)\} = W^u(x)$, and $\{(0,s)\} =$

[21] These are a direct consequence of the formula [22]

$$J\Psi(x) = \prod_{n=0}^{\infty} \frac{J^u T_{-1}(T_n \Psi(x))}{J^u T_{-1}(T_n x)}.$$

$W^s(x)$. Although it is not necessary, for further convenience we can assume that the coordinate system, restricted to the stable and unstable manifolds, is the one given by the exponential map (corresponding to the metric restricted to such manifolds). Define then a coordinate system (u,t,s) in $B_\delta(x)$ as follows: $T_{-t}(\xi) \in \mathcal{W}$ and (u,s) are the coordinates of $T_{-t}(\xi)$, clearly such coordinates locally trivialize the flow. Let $y \in B_\delta(x) \cap W^s(x)$ and $y' \in B_\delta(x) \cap W^u(x)$. Moreover let $z' = W^u(y) \cap W^{sc}(y')$ and $z = W^s(y') \cap W^{uc}(y)$. By construction z and z' are on the same flow orbit. Thus there exists $\Delta(y,y')$ such that $T_{\Delta(y,y')} z = z'$. The function $\Delta(y,y')$ is called *temporal distance*, see Figure 2 for a pictorial description.

In general the only thing that can be said is that the temporal distance is as smooth as the strong stable and unstable foliation (see (A.2)), but we will see in appendix B that, if some geometric structure is present, more can be said.

Appendix B. Basic facts (Contact flows)

Given an odd dimensional (say $2d+1$) connected compact manifold $\mathcal{M}$ a contact form is a $\mathcal{C}^1$ differential 1-form such that the $(2d+1)$-form $\alpha \wedge (d\alpha)^d$ is non zero at every point.

Given a flow T_t on $\mathcal{M}$ we call it *contact flow* if its associated vector field V ($V(x) := \frac{d\,T_t x}{dt}\big|_{t=0}$) is such that $d\alpha(V,v) = 0$ for all vector fields v and $\alpha(V) = 1$, for some contact form α.

Clearly the contact flow preserves the contact form and hence also the contact volume.

Let us start with some trivial facts showing that, for contact flows, a bit more can be said on the quantities introduced in the previous appendix.

Lemma B.1. *For a contact flow there exists a constant $C > 0$ such that, for each $x \in \mathcal{M}$,*

$$C_0^{-1} \le \|d_x^s T\| \theta(d_x^u T) \le C_0 \ ; \quad C_0^{-1} \le \|d_x^u T\| \theta(d_x^s T) \le C_0.$$

Proof. Let $v \in E^u(x)$, $|v| = 1$, clearly there must exist $w \in E^s(x)$, $|w| = 1$, such that $|d\alpha(v,w)| \ge c_-|v|\,|w|$. Accordingly,

$$c_-|v|\,|w| \le |d\alpha(d_x T_t v, d_x T_t w)| \le c_+ |d_x T_t v|\,|d_x T_t w| \le c_+ |d_x T_t v|\,\|d_x^s T_t\|,$$

taking the inf on v we have

$$\theta(d_x^u T_t)\,\|d_x^s T_t\| \ge c_- c_+^{-1}. \tag{B.1}$$

On the other hand, given $w \in E^s(x)$, $|w| = 1$, there must be $v \in E^u(x)$, $|v| = 1$, such that $|d\alpha(d_x T_t w, d_x T_t v)| \ge c_- |d_x T_t w|\,|d_x T_t v|$. Hence,

$$c_+ \ge c_- |d_x T_t w|\,|d_x T_t v| \ge c_- |d_x T_t w|\,\theta(d_x^u T_t)$$

taking the sup over w we have

$$\|d_x^s T_t\|\,\theta(d_x^u T_t) \le c_-^{-1} c_+. \tag{B.2}$$

The first inequality of the Lemma is then obtained putting together (B.1) and (B.2). The second inequality follows similarly. □

Another trivial, but helpful, property of contact flows is the following.

Lemma B.2. *The contact form α restricted to a stable or unstable manifold must be identically zero. In addition, the form $d\alpha$ is identically zero when restricted to a weak stable or weak unstable manifold.*

Proof. The first statement is a consequence of the invariance of α, for example if v is a stable vector then $\alpha(v) = \lim_{t\to+\infty} \alpha(dT_t v) = 0$. The second statement is proved again by invariance. Let v,w be weak stable vectors and write them as $v = v' + aV$ and $w = w' + bV$ where v' and w' are stable vectors. Then $d\alpha(v,w) = \lim_{t\to+\infty} d\alpha(dT_t v, dT_t w) = ab\, d\alpha(V,V) = 0$. □

Corollary B.3. *The distributions are smoother than indicated in Appendix A: $E^u, E^s \in \mathcal{C}^{\tau_d}$.*

Proof. Since $E^{uc} \in \mathcal{C}^{\tau_d}$ and $E^u = \{v \in E^{uc} \mid \alpha(v) = 0\}$ the result follows trivially. □

Remark B.4. *A bit more work should show that A.2 and A.3 hold with τ_d instead than τ_h and τ_j. This it is not important for the task at hand and we will ignore it. Throughout the paper τ will designate the best constant (less or equal one) for which the properties in A.1, A.2 and A.3 hold.*

The first really interesting fact concerning contact flow is given by the following result proved in [16], Theorem 3.6.

Theorem B.5 (Katok-Burns)**.** *Let $\mathcal{M}$ be a contact manifold as above. Let E be an ergodic component of the contact flow T which has positive measure and non-zero Lyapunov exponents except in the flow direction. Then the flow on E is Bernoulli.*

Accordingly, by the usual Hopf argument [15, 21], it follows immediately.

Corollary B.6. *Let $\mathcal{M}$ be a connected compact contact manifold as above and let T_t be an Anosov contact flow. Then the flow is Bernoulli (and hence mixing).*

The proof of Theorem B.5 is based, among other things, on a Lemma concerning the temporal function (see the definition at the end of the previous appendix) which, at least for us, has an interest in itself. Since we need it in a slightly different, stronger and more explicit, from we will state and prove it here again.

Lemma B.7. *Assume $\alpha \in \mathcal{C}^2$ and conditions* (A.1), (A.2) *for some $t > 0$. Let $\bar{v} \in E^u(x)$, $\bar{w} \in E^s(x)$ be such that $exp_x(\bar{v}) = y'$ and $exp_x(\bar{w}) = y$,[22] then*

$$\Delta(y, y') = d\alpha(\bar{v}, \bar{w}) + O(\|\bar{v}\|^{\tau^2}\|\bar{w}\|^2 + \|\bar{w}\|^{\tau^2}\|\bar{v}\|^2).$$

In addition,

$$\Delta(y, y') = d\alpha(\bar{v}, \bar{w}) + O(\|\bar{v}\|^{\tau}\|\bar{w}\|^2 + \|\bar{w}\|^{\tau}\|\bar{v}\|^2),$$

provided $\|\bar{v}\|^{\frac{1}{\tau_-}} \leq \|\bar{w}\| \leq \|\bar{v}\|^{\tau_-}$, $\tau_- := \min\{\tau, (1-\tau)\}$.[23]

Proof. Consider the coordinate system introduced at the end of appendix A to define the temporal distance. Notice that the Euclidean metric in such coordinates gives the right measure for the temporal distance and the distance from x of points in $W^u(x)$ or $W^s(x)$, at the same time it is uniformly equivalent to the Riemannian metric, we can then use it without any further comment.

Let $y = (0, 0, w)$ and $y' = (v, 0, 0)$. In coordinates the manifold $W^u(x)$ has the form $\{(u, 0, 0)\}$, the manifold $W^s(x)$ $\{(0, 0, s)\}$ and the manifolds $W^{uc}(y)$, $W^{sc}(y')$ have the form $\{(u, t, F(u))\}$, $\{(G(s), t, s\}$, respectively. In addition, on the one hand the smoothness of the holonomies implies $\|F\|_\infty \leq C\|w\|^\tau$ and $\|G\|_\infty \leq C\|v\|^\tau$. On the other hand the smoothness of the distributions implies $\|D_uF\| \leq C\|F(u)\|^\tau$ and $\|D_sG\| \leq C\|G(s)\|^\tau$. Finally, the uniform smoothness of the manifolds implies $\|F(u) - w - D_0Fu\| \leq C\|u\|^2$, $\|G(s) - v - D_0Gs\| \leq C\|s\|^2$.[24]

Our aim is to introduce a two dimensional manifold that captures the essential geometric features related to Δ. To do so we introduce two smooth foliations: $\mathcal{W}_u := \{W_u(b) \mid b \in [0, 1]\}$, $W_u(b) := \{(u, 0, bF(u))\}$, and $\mathcal{W}_s := \{W_s(a) \mid a \in [0, 1]\}$, $W_s(a) := \{(aG(s), 0, s)\}$.[25] Notice that the two above foliations are transversal, hence for all $(a, b) \in \Sigma_0 := [0, 1]^2$ is uniquely defined the point $\{\Xi(a, b)\} := W_u(b) \cap W_s(a)$. In fact, if we define the function $\Phi : \mathbb{R}^{2d+2} \to \mathbb{R}^{2d}$ by $\Phi(u, s, a, b) := (u - aG(s), s - bF(u))$, then $\Phi(\Xi(a, b), a, b) \equiv 0$. Since

$$\frac{\partial \Phi}{\partial(u, s)} = \begin{pmatrix} \mathbf{Id} & -aDG \\ -bDF & \mathbf{Id} \end{pmatrix} =: \mathbf{Id} - \Lambda, \tag{B.3}$$

where $\|\Lambda\| < 1$, provided the coordinate neighborhood has been chosen small enough. It follows that we can apply the implicit function theorem. Accordingly Ξ is a uniformly $\mathcal{C}^4$ chart for the

[22]The exponential function is with respect to the restriction of the metric to $W^u(x)$ and $W^s(x)$, respectively.

[23]The latter limitation–although compatible with our needs– is certainly excessive and, possibly, completely redundant. Yet, as it will be clear from the proof, to remove it effectively it would be necessary to have some informations on the Hölder continuity of the foliation in $\mathcal{C}^r$ topology, which seem not to be readily available in the literature but it does hold true–at least to some extent, see footnotes 24 and 28.

[24]Actually, here we use a very rough bound on the second derivative, one can certainly do better. For example, since $F(u) = F(0) + D_0F(0)u + \frac{1}{2}D_0^2F(u, u) + O(\|u\|^3) \leq C\|w\|^\tau$, it must be, at least, $|D_0^2F| \leq C\|w\|^{\frac{\tau}{3}}$.

[25]These are just a linear interpolation between the manifolds at x ant the manifolds at y and y', respectively.

surface $\Sigma := \Xi(\Sigma_0)$. Such a surface is bounded by the curves $\gamma_1 := \{\Xi(a,0)\} = \{(av,0,0)\}$, $\gamma_2 := \{\Xi(1,b)\}$ that belongs to $W^{sc}(y')$, $\gamma_3 := \{\Xi(a,1)\}$ that belongs to $W^{uc}(y)$ and $\gamma_4 := \{\Xi(0,b)\} = \{(0,0,bw)\}$. Moreover, let us set $\hat z := \Xi(1,1)$, clearly $\hat z$ lies on the same flow orbit of z and z'. At last, consider the curves $\gamma \subset W^u(y)$ and $\gamma' \subset W^s(y')$ obtained by transporting, along the flow direction, γ_3 and γ_2 respectively.[26] Clearly γ, γ_3 and the flow line between $\hat z$ and z' bound a two dimensional manifold (contained in $\bigcup_{t\in\mathbb{R}} T_t\gamma_3$), let us call it $\Omega' \subset W^{uc}(y)$; analogously we define Ω. See Figure 2 for a visual description.[27]

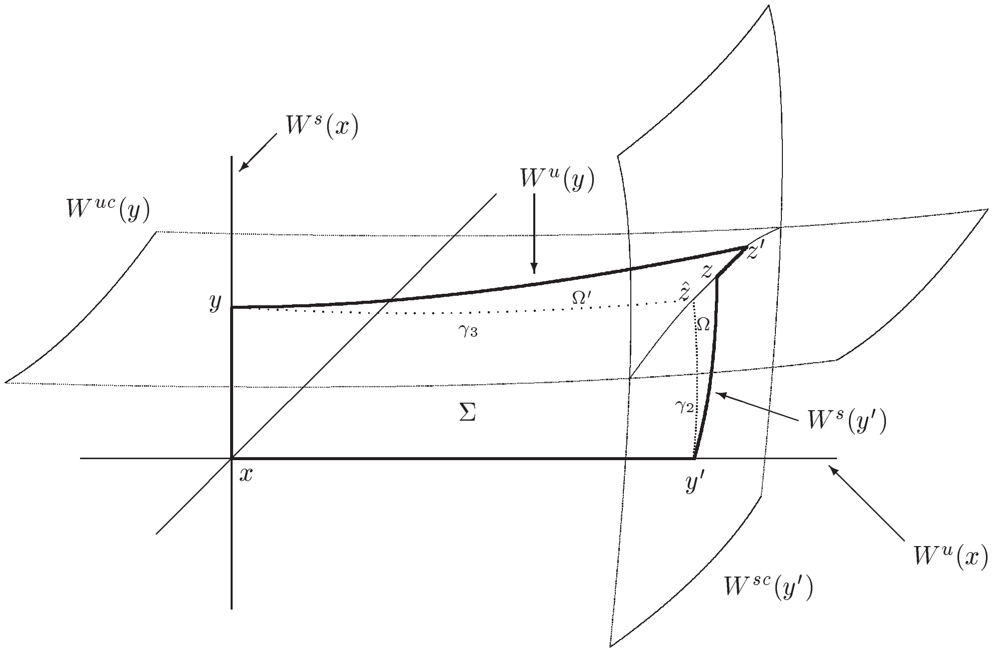

FIGURE 2. Definition of the *temporal function* $\Delta(y,y')$ and related quantities

We can now compute the required quantity. Consider the closed curve Γ following γ_1, γ' then going from z to z' along the flow direction and finally coming back to x via γ and γ_4 (the bold path in Figure 2) then

$$\int_\Gamma \alpha = \Delta(y,y'). \tag{B.4}$$

This is because α is identically zero when restricted to a stable or unstable manifold (see Lemma B.2). On the other hand

$$\int_\Gamma \alpha = \int_{\partial\Sigma} \alpha + \int_{\partial\Omega} \alpha + \int_{\partial\Omega'} \alpha = \int_\Sigma d\alpha \tag{B.5}$$

where we have used Stokes theorem and the fact that $d\alpha$ is identically zero when restricted to a weak stable or unstable manifold (see Lemma B.2). To continue it is better to change coordinates.

$$\begin{aligned}\int_\Sigma d\alpha &= \int_{\Sigma_0} \Xi^* d\alpha = \int_{\Sigma_0} d_{\Xi(a,b)}\alpha(D\Xi e_1, D\Xi e_2)dadb\\ &= \int_{\Sigma_0} d_x\alpha(D\Xi e_1, D\Xi e_2)dadb + O(\|\Xi\|_\infty \|D\Xi e_1\|_\infty \|D\Xi e_2\|_\infty),\end{aligned} \tag{B.6}$$

[26]The smoothness of $W^u(y)$ and γ_3 imply trivially the smoothness of γ. The same considerations apply to γ'.
[27]Of course the picture is a bit misleading due to a lack of dimensions. For example, the picture does not differentiate between the d-dimensional manifold $W^u(y)$ and the curve γ.

where we have used the fact that α is $\mathcal{C}^2$. By the implicit function theorem,

$$D\Xi = -(\mathbf{Id}-\Lambda)^{-1}\frac{\partial\Phi}{\partial(a,b)} = \sum_{k=0}^{\infty}\Lambda^k\begin{pmatrix}G & 0\\ 0 & F\end{pmatrix} \tag{B.7}$$

Since all the following arguments are restricted to the hypersurface $t\equiv 0$, from now on we will forget the t coordinate. Accordingly,

$$\begin{aligned}D\Xi e_1 &= (\mathbf{Id}-\Lambda)^{-1}(G,0) = (G,0)+\sum_{k=0}^{\infty}\Lambda^{2k}\{\Lambda(G,0)+\Lambda^2(G,0)\}\\ &= (G,0)+b(\mathbf{Id}-\Lambda^2)^{-1}(aDG\,DF\,G, DF\,G)\\ &= (G,0)+b(a(\mathbf{Id}-abDG\,DF)^{-1}DG\,DF\,G,(\mathbf{Id}-abDF\,DG)^{-1}DF\,G)\\ &=: (v,0)+(\Delta_u v,\Delta_s v) =: \bar v+\Delta v\\ D\Xi e_2 &= (0,F)+a((\mathbf{Id}-abDG\,DF)^{-1}DG\,F, b(\mathbf{Id}-abDF\,DG)^{-1}DF\,DG\,F)\\ &=: (0,w)+(\Delta_u w,\Delta_s w) =: \bar w+\Delta w.\end{aligned} \tag{B.8}$$

Since $d_x\alpha$ is identically zero on the weak stable and weak unstable manifold of x, we have

$$\begin{aligned}d_x\alpha(D\Xi e_1, D\Xi e_2) &= d_x\alpha(\bar v,\bar w)+d_x\alpha(\bar v,\Delta w)+d_x\alpha(\Delta v,\bar w)+d_x\alpha(\Delta v,\Delta w)\\ &= d_x\alpha(\bar v,\bar w)+O((\|v\|+\|\Delta_u v\|)\|\Delta_s w\|+\|w\|\|\Delta_u v\|+\|\Delta_s v\|\|\Delta_u w\|).\end{aligned} \tag{B.9}$$

The last needed estimate concerns the variation of the functions F, G.

$$\Delta G(a,b) := G(\Xi_s(a,b))-v = G(\Xi_s(a,b))-G(\Xi_s(a,0)) = \int_0^b DG D\Xi_s e_2 = \int_0^b DGw+DG\Delta_s w$$

$$\Delta F(a,b) := F(\Xi_u(a,b))-w = F(\Xi_u(a,b))-F(\Xi_u(0,b)) = \int_0^a DF D\Xi_u e_1 = \int_0^a DFv+DF\Delta_u v$$

Remembering (B.8) we can estimate

$$\begin{aligned}\|\Delta_u v\| &\leq \|\Delta G\|+Cab\|DG\,DF(v+\Delta G)\|_\infty \leq C\|\Delta G\|_\infty+Cab\|DG\,DFv\|_\infty\\ \|\Delta_s v\| &\leq Cb\|DF\,G\|_\infty \leq Cb\|DFv\|_\infty+Cb\|DF\Delta G\|_\infty\\ \|\Delta_u w\| &\leq Ca\|DGw\|_\infty+Ca\|DG\Delta F\|_\infty\\ \|\Delta_s w\| &\leq C\|\Delta F\|_\infty+Cab\|DF\,DGw\|_\infty\end{aligned} \tag{B.10}$$

Therefore,

$$\begin{aligned}\|\Delta G\|_\infty &\leq \|DG\|_\infty\|w\|+C\|DG\|_\infty\|\Delta F\|_\infty+C\|DG\|_\infty\|DF\,DGw\|_\infty\\ &\leq C\|DG\|_\infty\|w\|+C\|DG\|_\infty\|\Delta F\|_\infty\\ \|\Delta F\|_\infty &\leq C\|DF\|_\infty\|v\|+C\|DF\|_\infty\|\Delta G\|_\infty\end{aligned}$$

Substituting the first in the second yields

$$\|\Delta F\|_\infty \leq C\|DF\|_\infty\|v\|+C\|DF\|_\infty\|DG\|_\infty\|w\|+C\|DF\|_\infty\|DG\|_\infty\|\Delta F\|_\infty,$$

that is

$$\begin{aligned}\|\Delta F\|_\infty &\leq C\|DF\|_\infty\|v\|+C\|DF\|_\infty\|DG\|_\infty\|w\|\\ \|\Delta G\|_\infty &\leq C\|DG\|_\infty\|w\|+C\|DG\|_\infty\|DF\|_\infty\|v\|\end{aligned} \tag{B.11}$$

Using estimates (B.11) and (B.10) in (B.9) yields

$$d_x\alpha(D\Xi e_1, D\Xi e_2) = d_x\alpha(\bar v,\bar w)+O(\|DF\|_\infty\|v\|^2+\|DG\|_\infty\|w\|^2).$$

Remembering that, by definition, $\Xi_u = aG\circ\Xi_s$ and $\Xi_s = bF\circ\Xi_u$ we can use the above estimate in (B.6), (B.4) and finally obtain

$$\Delta(y,y') = d_x\alpha(\bar v,\bar w)+O(\|v\|^2\|w\|+\|w\|^2\|v\|+\|DF\|_\infty\|v\|^2+\|DG\|_\infty\|w\|^2) \tag{B.12}$$

Since $\|DF\|_\infty \leq C\|F\|_\infty^\tau \leq C\|w\|^{\tau^2}$ and $\|DG\|_\infty \leq C\|v\|^{\tau^2}$ the first inequality of the Lemma is proven. To prove the second let us assume $\|w\|\geq\|v\|$, the other situation being symmetric with respect to the exchange of the stable and unstable direction (which corresponds to a time reversal). Remember that $DF \in \mathcal{C}^1$, hence $\|DF\|_\infty \leq C\|w\|^\tau + C\|\Xi_u\|_\infty$, thus

$$\|DF\|_\infty \leq C\|w\|^\tau + C\|v\| + C\|DG\|_\infty\|w\|$$
$$\|DG\|_\infty \leq C\|v\|^\tau + C\|w\| + C\|DF\|_\infty\|v\|$$

which yields $\|DF\|_\infty \leq C\|w\|^\tau + C\|v\|$; $\|DG\|_\infty \leq C\|v\|^\tau + C\|w\|$.[28] This proves the lemma provided $\|v\|^\tau \geq \|w\|$. Clearly this condition is less stringent as τ decreases, while such a situation should be the worst case. Obviously the previous estimates must have been inefficient for "large" τ. Indeed, it is possible to do a different estimate for ΔF, ΔG. Suppose $\|v\|^\tau \leq \|w\|$.

$$\frac{d\|F \circ \Xi_u\|}{da} = \frac{\langle DF D\Xi_u e_1, F\rangle}{\|F\|} \leq \|DF\|\|D\Xi_u e_1\| \leq C\|F\|^\tau\|G\|$$

Integrating the above differential inequality (and the analogous one for G) yields

$$\left[\|w\|^{1-\tau} - C\|G\|_\infty\right]^{\frac{1}{1-\tau}} \leq \|F\|_\infty \leq \left[\|w\|^{1-\tau} + C\|G\|_\infty\right]^{\frac{1}{1-\tau}}$$
$$\left[\|v\|^{1-\tau} - C\|F\|_\infty\right]^{\frac{1}{1-\tau}} \leq \|G\|_\infty \leq \left[\|v\|^{1-\tau} + C\|F\|_\infty\right]^{\frac{1}{1-\tau}}$$

Clearly the above equations imply $\|\Delta F\| \leq \|w\|^\tau\|v\|$ and $\|\Delta G\| \leq \|v\|^\tau\|w\|$, provided $\|v\|^{1-\tau} \geq \|w\|$. This implies again $\|DF\|_\infty \leq C(\|w\|^\tau + \|v\|)$ and $\|DG\|_\infty \leq C(\|v\|^\tau + \|w\|)$. In addition, $\|F\|_\infty \leq C\|w\|$, $\|G\|_\infty \leq C\|v\|$ and $\|D\Xi e_1\|_\infty \leq C\|G\|_\infty \leq C\|v\|$, $\|D\Xi e_2\|_\infty \leq C\|w\|$. Using such estimates in (B.10), (B.9) and (B.6) yields

$$\Delta(y, y') = d_x\alpha(\bar{v}, \bar{w}) + O(\|v\|^2\|w\|^\tau + \|w\|^2\|v\|^\tau).$$

□

Remark B.8. *It may be possible to optimize Lemma B.7 by pushing forward (or backward) the picture until $d(T_k x, T_k y) = d(T_k x, T_k y')$; of course one would need to be rather careful by properly estimating distortion. At any rate, the best result one can hope for is that if $\tau > \sqrt{3} - 1$, then $\Delta(y, y') = d\alpha(v, w) + o(|v|)$. That is, Δ is differentiable with respect to y' and the derivative is $\mathcal{C}^\tau$. We do not push matters in such a direction since it is not necessary for the purpose at hand.*

Appendix C. Averages

We start with a long overdue proof.

Proof of Sub-Lemma 3.1. Clearly

(C.1) $$|\mathbb{A}^s_\delta\varphi|_\infty \leq |\varphi|_\infty; \quad |\mathbb{A}^s_\delta\varphi - \varphi|_\infty \leq \delta^\beta H_{s,\beta}(\varphi).$$

The estimate of the smoothness of $\mathbb{A}^s_\delta\varphi$ is a bit more subtle, to investigate it is convenient to introduce an appropriate coordinate system.

Since all the quantities are related to the same stable manifold, form now on we will consider the Riemannian metric restricted to the stable manifold.

Given x, y belonging to the same stable manifold, we first identify the tangent spaces at x and y by parallel transport, then we consider normal coordinates at x and at y. Clearly in such coordinates the balls $W^s_\delta(x)$ and $W^s_\delta(y)$ are actual balls of radius δ; of course this it is not the case for $W^s_\delta(y)$ in the normal coordinates at x. We call $I_{xy} : \mathcal{T}_x\mathcal{M} \to \mathcal{T}_y\mathcal{M}$ the isometry that identifies the tangent spaces and we define the map $\Upsilon_{xy} : \mathcal{M} \to \mathcal{M}$ as

$$\Upsilon_{xy}(z) = \exp_y(I_{xy}\exp_x^{-1}(z)).$$

where exp is the exponential map defined by the metric on the stable manifold.

First of all notice that, by construction

(C.2) $$\Upsilon_{xy}(W^s_\delta(x)) = W^s_\delta(y).$$

[28]Here again a better knowledge of the size of the second derivative would improve the result, see footnote 24.

Next, to study Υ_{xy} we describe it in the normal coordinates of the point x, we will then identify all the tangent spaces by the Cartesian structure of such a chart. Calling, as usual, Γ^k_{ij} the Christoffel symbols, the equation of parallel transport for a vector v along the curve γ reads

$$\frac{dv^k}{dt} = -\sum_{ij} \Gamma^k_{ij} v^i \frac{d\gamma^j}{dt}.$$

Moreover, in the normal coordinates of the point x,[29]

$$|\Gamma^k_{ij}(\xi)| \leq C|\xi|.$$

Assuming $d(x, y) \leq \delta$, we are interested only in a region contained in the ball $W^s_{2\delta}(x)$, thus $|\Gamma^k_{ij}|_\infty \leq C\delta$. Hence, by a standard use of Gronwald inequality,

$$|I_{x,y}v - v| \leq C_1 d(x, y)^2 |v|. \tag{C.3}$$

Arguing in the same manner on the equations defining the geodesics, and taking into account (C.3), it follows that

$$d(\Upsilon_{xy}(z), z) \leq (1 + C_2\delta)d(x, y). \tag{C.4}$$

This implies that the symmetric difference $W^s_\delta(x)\Delta W^s_\delta(y)$ is contained in the spherical shell $W^s_{\delta+C_2 d(x,y)}(x)\backslash W^s_{\delta-C_2 d(x,y)}(x)$ whose measure is proportional to $\delta^{d-1}d(x, y)$.

To see that the Jacobian is close to one a bit more work is needed. Namely we must linearize the geodesic equations along the geodesic. This is a standard procedure and it is best done via the Jacobi fields [6]. By using Gronwald again, and the fact that the manifolds are uniformly $\mathcal{C}^4$, yields

$$|J\Upsilon_{xy} - 1|_\infty \leq C_3 d(x, y). \tag{C.5}$$

From this it follows immediately

$$H_{s,1}(\mathbb{A}^s_\delta \varphi) \leq C\delta^{-1}|\varphi|_\infty. \tag{C.6}$$

We can then conclude by using (C.2), (C.4) and (C.5),

$$\begin{aligned} \left| \int_{W^s_\delta(x)} \varphi - \int_{W^s_\delta(y)} \varphi \right| &\leq \int_{B_\delta(0)} |\varphi(\xi)\rho(x, \xi) - \varphi \circ \Upsilon_{xy}(\xi)\rho(y, \Upsilon_{xy}(\xi))J\Upsilon_{xy}(\xi)|\, d\xi \\ &\leq \left[(1 + c_2\delta)^\beta H_{s,\beta}(\varphi)d(x, y)^\beta + C_4|\varphi|_\infty d(x, y)\right] m^s(W^s_\delta(x)). \end{aligned} \tag{C.7}$$

□

Next we need an estimate of how much two nearby manifolds can drift apart.

Lemma C.1. *There exists a constant $C > 0$ such that for each $x \in \mathcal{M}$ and $y \in W^s_\delta(x)$ holds*[30]

$$\operatorname{dist}(W^u_\delta(x), W^u_\delta(y)) \leq Cd(x, y)^\tau.$$

Proof. Clearly $d(W^u_\delta(x), W^u_\delta(y))$ is bounded by the distance computed along the stable manifold. For each $\xi \in W^u_\delta(x)$ consider the unstable holonomy between $W^{sc}(x)$ and $W^{sc}(\xi)$. Let $\{\eta\} := W^{sc}(\xi) \cap W^u_\delta(y)$. By A.3 it follows $d_s(\xi, \eta) \leq Cd_s(x, y)^\tau$. From this the Lemma follows. □

The other needed results concerning averages are all based on a sort of change of order of integration formula. Although such a result may already exist in some form in the literature (after all it is a sort of Fubini with respect to a foliation with Hölder smoothness), I find it more convenient to derive it in the following.

To proceed it is helpful to choose special coordinates in which the unstable, or the stable manifolds, are straight. Let us do the construction for the unstable manifold, the one for the stable being similar.

[29]Clearly the smoothness of the metric will depend on the smoothness of the tangent planes (that in our case are uniformly $\mathcal{C}^3$), see [17]. Accordingly Γ will be uniformly $\mathcal{C}^2$.

[30]Here by "dist" we mean the Hausdorff distance.

First notice that such a straightening can be only local, we can then choose an appropriate covering $\{U_i\}$ of $\mathcal{M}$ (appropriate means that the open sets must be sufficiently small) and perform the wanted construction in each open set U_i.

Let U be a sufficiently small open ball. Let us choose a coordinate system in U, since the Euclidean norm in the coordinate is equivalent to the Riemannian length we will use it instead and we will, from now on, confuse U with its coordinate representation.

It is particularly convenient to choose the chart in such a way that, given a preferred point $\bar{x} \in U$, $\{(u,0)\}_{u\in\mathbb{R}^{d_u}} = W^u(\bar{x})$ and $\{(0,s)\}_{s\in\mathbb{R}^{d_s+1}} = W^{sc}(\bar{x})$.

At this point we can define the function $H : \mathbb{R}^{d_u+d_s+1} \to \mathbb{R}^{d_s+1}$ by the requirement

$$\{(u,\, H(u,s))\}_{u\in\mathbb{R}^{d_u}} = W^u((0,s)).$$

Clearly this implies $H(0,s) = s$; $\quad H(u,0) = 0$. We define then the change of coordinates

$$\Psi(\bar{u},\bar{s}) = (\bar{u},\, H(\bar{u},\bar{s}))$$

in the coordinates $(\bar{u},\bar{s})$ the unstable manifolds are just all the vector spaces of the type $\{(\bar{u},a)\}$ for some $a \in \mathbb{R}^{d_s+1}$.

In addition, a trivial computation shows that, calling $J\Psi$ the Jacobian of the change of coordinates Ψ, we have that $J\Psi(\bar{u},\bar{s})$ is nothing else than the Jacobian of the unstable holonomy between $\{(0,\xi)\}_{\xi\in\mathbb{R}^{d_s+1}}$ and $\{(\bar{u},\xi)\}_{\xi\in\mathbb{R}^{d_s+1}}$.

Lemma C.2. *There exists $\bar{c} > 0$ such that the kernel $\tilde{Z}_\varepsilon$, defined in* (4.5)*, satisfies*

$$|\tilde{Z}_\varepsilon|_{\mathcal{C}^\tau} \leq \bar{c}|Z_\varepsilon|_\infty,$$

moreover $\tilde{Z}(x,\xi)$ is Lipschitz with respect to the second variable, limited to the flow direction, with Lipschitz constant $\bar{c}|Z_\varepsilon|_\infty$.

Proof. Since all the relevant quantities are local quantities, we can compute in a chart Ψ as above. Let U be an open set in the chart and consider $f : \mathcal{M}^2 \to \mathbb{C}$ supported in U^2. Then

$$\int_{\mathcal{M}} m(dx) \int_{W^u_\delta(x)} f(x,\xi) m^u(d\xi) = \int_{\{(x,\xi)\in U^2 \;|\; d^u(x,\xi)\leq\delta\}} f(x,\xi) m^u(d\xi) m(dx).$$

Now we set $\Xi_\delta := \{(x,\xi) \in U^2 \mid d^u(x,\xi) \leq \delta\}$ and we change variables: $x = \Psi(u,s)$ and $\xi = \Psi(u',s)$.

$$\int_{\mathcal{M}} m(dx) \int_{W^u_\delta(x)} f(x,\xi) m^u(d\xi) = \int_{\Xi_\delta} f(\Psi(u,s),\Psi(u',s))\rho(u_1,s) J\Psi(u,s) du'\, du\, ds,$$

where $\rho \circ \Psi^{-1}$ is a uniformly τ-Hölder function. Accordingly,

$$\tilde{Z}_\varepsilon(x,\xi) = \frac{J\Psi(x)\rho(\Psi^{-1}(\xi))}{J\Psi(\xi)\rho(\Psi^{-1}(x))} Z_\varepsilon(x).$$

The smoothness of $\tilde{Z}_\varepsilon(x,\xi)$ follows then from previous results on holonomy smoothness and the smoothness of Z_ε. In turn, the latter is proven exactly as in equation (C.7) exchanging the rôle of the stable and unstable manifolds and setting $\varphi = 1$. $\square$

References

[1] D.V.Anosov, Ya.G. Sinai, *Certain smooth ergodic systems*, Russian Math. Surveys 22 (1967), no. 5, 103–167.

[2] M.Blank, G. Keller, C. Liverani, *Ruelle-Perron-Frobenius spectrum for Anosov maps*, Nonlinearity, 15, n.6, 1905-1973 (2002).

[3] N.Chernov, *Markov approximations and decay of correlations for Anosov flows*, Annals of Mathematics, **147** (1998), 269–324.

[4] P.Collet, H. Epstein, G. Gallavotti, *Perturbations of geodesic flows on surfaces of constant negative curvature*, Communications in Mathematical Physics, **95** (1984), 61–112.

[5] E.B.Davies, *One-Parameter semigroups*, Academic Press, London, (1980).

[6] M.P. Do Carmo, *Riemannian Geometry,* Birkhäuser, Boston (1992).

[7] D.Dolgopyat, *Decay of correlations in Anosov flows*, Annals of mathematics, **147** (1998), 357-390.

[8] D.Dolgopyat, *Prevalence of rapid mixing in hyperbolic flows*, Ergodic Theory and Dynamical Systems, **18**, 1097-1114 (1998).

[9] D.Dolgopyat, *Prevalence of rapid mixing-II: topological prevalence*, Ergodic Theory and Dynamical Systems, **20** (2000), no. 4, 1045–1059.
[10] B.Hasselblatt, *Regularity of the Anosov splitting and of horospheric foliations*, Ergod. Th.& Dynam. Sys. **14** (1994), 645-666.
[11] B.Hasselblatt, *Horospheric foliation and relative pinching*, J.Differential Geometry, **39**, (1994) 57-63.
[12] H. Hennion, *Sur un théorème spectral et son application aux noyaux Lipchitziens*, Proceedings of the American Mathematical Society, **118** (1993), 627–634.
[13] M.Hirsch, C.Pugh, *Smoothness of horocycle foliations*, J.Differential Geometry, **10**, (1975) 225-238.
[14] M.Hirsch, C.Pugh, M.Shub, *Invariant Manifolds*, Lecture Notes in Math. **583** (1977).
[15] E. Hopf, *Statistik der geodtischen Linien in Mannigfaltigkeiten negativer Krmmung*, Ber. Verh. Schs. Akad. Wiss. Leipzig 91, (1939). 261–304.
[16] A.Katok, K.Burns, *Infinitesimal Lyapunov functions, invariant cone families and stochastic properties of smooth dynamical systems*, Ergodic Theory and Dynamical Systems, **14**, 757-785, (1994).
[17] A.Katok, B.Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Encyclopedia of Mathematics and its Applications, **54**, G.-C.Rota editor, Cambridge University Press, Cambridge (1995).
[18] W.P.A.Klingenberg, *Riemannian Geometry*, second edition, Walter de Gruyter, Berlin, New York (1995).
[19] C.Liverani, *Decay of Correlations*, Annals of Mathematics, **142** (1995), 239-301.
[20] C.Liverani, *Flows, Random Perturbations and Rate of Mixing*, Ergodic Theory and Dynamical Systems, **18** (1998), 1421–1446.
[21] C.Liverani, M.Wojtkowski, *Ergodicity in Hamiltonian Systems*, Dynamics Reported, **4**, C.K.R.T. Jones, U.Kirchgraber, H.O.Walther edts., Springer-Verlag, Berling, Heidelgerg, New York (1995) 130-202.
[22] R.Mañe, *Ergodic Theory and Differentiable Dynamics*, Springer-Verlag, Berlin Heidelberg (1987).
[23] C.Moore, *Exponential decay of correlation coefficients for geodesic flows*, in *Group Representation Ergodic Theory*, Operator Algebra and Mathematical Physics, Springer, Berlin, (1987).
[24] G.Greiner, R. Nagel, *Spectral theory*, in *One-Parameter semigroups of Positive Operators*, Nagel edt., Lecture Notes in Mathematics, **1184**, Springer-Verlag, Berlin, (1980).
[25] R.D. Nussbaum, *The radius of the essential spectrum*, Duke Math. J., **37** (1970), 473-478.
[26] M.Pollicott, *Exponential mixing for geodesic flow on hyperbolic three manifold*, Journal of Statistical Physics, **67** (1992), 667-673.
[27] M.Pollicott, *A complex Ruelle-Perron-Frobenius theorem and two counterexamples*, Eergodic Theory and Dynamical Systems, **4** (1984), 135-146.
[28] M.Pollicott, *On the rate of mixing of Axiom A flows*, Inventiones Mathematicae, **85** (1985), 413–426.
[29] C.Pugh, M.Shub, A.Wilkinson, *Hölder foliations*, Duke Math. J. 86 (1997), no. 3, 517–546. *Correction to: "Hölder foliations" [Duke Math. J. 86 (1997), no. 3, 517–546]*, Duke Math. J. 105 (2000), no. 1, 105–106.
[30] M. Ratner, *The rate of mixing for geodesic and horocycle flows*, Ergodic Theory and Dynamical Systems, **7** (1987), 267–288.
[31] M.Reed, B.Simon, *Methods of modern mathematical physics. Fouries-Analisys, self-Adjointness*, vol. **2**, Academic Press, New York, San Francisco, London (1972).
[32] D.Ruelle, *A measure associated with Axiom A attractors*, Amer. J. Math., **98**, 616–654 (1976).
[33] D.Ruelle, *Thermodynamics formalism*, Addison-Wesley, New York, 1978.
[34] D.Ruelle, *Flots qui ne mélange pas exponentialment*, C.R.Acad. Sc. Paris, **296** (1983), 191–193.
[35] Ya.G.Sinai, *Geodesic flows on compact surfaces of negative curvature*, Soviet Math. Dokl. 2 (1961) 106–109.
[36] Ya.G.Sinai, *Gibbs measures in ergodic theory*, Russian Math. Surveys **27**, 21-69 (1972).
[37] J.Schmeling, R.Siegmund–Schultze, *Hölder Continuity of the Holonomy Map for Hyperbolic Basic Sets I*, in *Ergodic Theory and Related Topics III*, Proceedings International Conference (Güstrow, Germany, 1990), Springer Lecture notes in Mathematics, eds. U.Krengel, K.Richter and V.Warstat, Springer, Berlin, **1514** (1992), 174–191.

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