Ising models with four spin interaction at criticality

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Abstract 1. *We consider two bidimensional Ising models coupled by an interaction quartic in the spins. The model contains both the Eight vertex and the Ashkin Teller models for suitable values of the parameters. By Renormalization Group methods we write a convergent perturbative expansion for the specific heat and for the energy-energy correlation up to the critical temperature. A form of nonuniversality is proved, in the sense that the critical behaviour is described in terms of critical indices which are non trivial functions of the coupling. The logarithmic singularity of the specific heat of the Ising model is removed or changed in a power law (with a non universal critical index) depending on the sign of the interaction.*

1. Main results

1.1 Much of our understanding about phase transitions and critical behaviour of classical spin systems on a 2D lattice is based on some remarkable exact solutions. Onsager [O] solved the *Ising model*, in which the spins take two values and only nearest-neighbor two spin interactions are considered. Lieb [L] and Baxter [B] solved respectively the *Six vertex* and *Eight vertex* models; in their original formulation such models are vertex models (to each site of a bidimensional lattice is associated a vertex with four arrows) but via a suitable identification of the parameters they can be written as *two* Ising models coupled by a four spin interaction [W]. The *critical exponents* describing the behaviour of the system close to the critical point can be exactly computed; it is remarkable that the critical indices in the Ising or in the vertex models are *different*.

The exact solutions provide indeed a lot of detailed informations about such integrable models; however even very small and apparently harmless modifications of them destroy completely their integrability. On the other hand one can hope that many relevant properties of the integrable models are quite "robust" under perturbations. It is believed that a *universality property* holds for the Ising model, in the sense that by adding to it, for instance, a next to nearest neighbor or a four spin interaction the critical indices remain unchanged. A universality property is believed to hold also for the Eight vertex model; Kadanoff [K] by “operator algebra and scaling theory” found evidence that the Eight vertex model is in the same class of universality of the the *Ashkin-Teller* model [AT], which is not integrable. Other evidence for such a conclusion was found in [PB] (using second order renormalization group arguments) and [LP], [N] (by a heuristic mapping of both models into the massive Luttinger model describing interacting fermions in the continuum).

The natural method to relate non integrable models to integrable ones is given by Renormalization Group (RG); this was known since long time but the main open problem in this context was to implement such method in a rigorous way. While RG methods were generally applied to the spin variables, it was realized in recent times that it can be convenient to do this in the *fermionic representation* of spin models. The fermionic representation of the Ising model was done in [SML], [H], [Ka], [MW], [S], [ID] and it was shown that the correlations can be written as Grassman integrals formally describing *non interacting fermions* on a lattice in $d = 1 + 1$. In the same way Ising models with quartic interactions can be written as Grassman integrals formally describing *interacting* non relativistic fermions. The rigorous analysis of Grassman integrals for non relativistic fermions via RG methods is quite well developed, starting from [G] and [BG1] (see also [BG] or [GM] for extensive reviews) and one can apply such methods to classical 2D spin systems (such methods were already applied to a closely related problem, the XYZ Heisenberg spin chain [BM]; the relation between Eight vertex and XYZ model is well known, [Su], [Ba]). Fermionic RG methods for classical spin models has been applied first in [PS] to the Ising model with a small next to nearest neighbor or four spin interaction. A form of *universality* was established in the sense that the interaction does not change certain critical indices; the fermionic interaction is, in this case, *irrelevant* in the RG sense and the fixed point of the RG transformation is gaussian.

The aim of the present paper is to study two Ising models coupled by an interaction quartic in the spins, such that both the Eight vertex and the Ashkin-Teller models are included: hence the models will be, in general, non integrable. The specific heat and the energy-energy correlation are written as Grassman integrals and studied by RG methods. In such cases the fermionic interaction is *marginal* and the RG transformation has a line of fixed points. The critical behaviour is different with respect to the case of the Ising model, and it is described in terms of critical indices which are *analytic non trivial functions of* λ. In agreement with [K] we find that the behaviour of the system is quite independent from the details of the bilinear interaction. In our analysis no use is done of the Six or Eight vertex model exact solutions; we use instead some properties which can be deduced from the solution [ML] of the (massless) Luttinger model following a strategy first outlined (in purely fermionic models) in [BG1].

Our analysis establishes as a mathematically rigorous statement the statement in [K], [LP], [N], [PB] that the Gaussian boson model, the massive Luttinger model, the Eight vertex and the Ashkin-Teller models are in the same class of universality.

1.2 We consider two Ising models coupled by a four spin interaction bilinear in the energy densities of the two sublattices. Given $\Lambda \in Z^2$ a square lattice with side M and periodic boundary condition, we call $\mathbf{x} = (x, x_0)$ a site of Λ. If $\sigma^{(1)}_{\mathbf{x}} = \pm 1$ and $\sigma^{(2)}_{\mathbf{x}} = \pm 1$, we write the following Hamiltonian

$$H_\Lambda(\sigma^{(1)},\sigma^{(2)}) = H_I(\sigma^{(1)}) + H_I(\sigma^{(2)}) + V(\sigma^{(1)},\sigma^{(2)}) \equiv \sum_{x,x_0=1}^{M} H_{\Lambda,\mathbf{x}}(\sigma^{(1)},\sigma^{(2)}) \tag{1.1}$$

where, if $\alpha = 1, 2$

$$H_I(\sigma^{(\alpha)}) = -J \sum_{x,x_0=1}^{M} [\sigma^{(\alpha)}_{x,x_0}\sigma^{(\alpha)}_{x+1,x_0} + \sigma^{(\alpha)}_{x,x_0}\sigma^{(\alpha)}_{x,x_0+1}] \tag{1.2}$$

$$V(\sigma^{(1)},\sigma^{(2)}) = -\sum_{x,x_0=1}^{M} \{\lambda a[\sigma^{(1)}_{x,x_0}\sigma^{(1)}_{x+1,x_0}\sigma^{(2)}_{x,x_0}\sigma^{(2)}_{x+1,x_0} + \sigma^{(1)}_{x,x_0}\sigma^{(1)}_{x,x_0+1}\sigma^{(2)}_{x,x_0}\sigma^{(2)}_{x,x_0+1}]+$$

$$\lambda b[\sigma^{(1)}_{x,x_0}\sigma^{(1)}_{x+1,x_0}\sigma^{(2)}_{x,x_0}\sigma^{(2)}_{x,x_0+1} + \sigma^{(1)}_{x,x_0}\sigma^{(1)}_{x,x_0+1}\sigma^{(2)}_{x-1,x_0+1}\sigma^{(2)}_{x,x_0+1}]\}\ . \tag{1.3}$$

If $b = 0$ the model in eq. (1.1) coincides with the Hamiltonian of the spin representation [F] of the Ashkin-Teller model [AT]. If $a = 0$ it coincides with the spin representation [W] of the Eight vertex model.

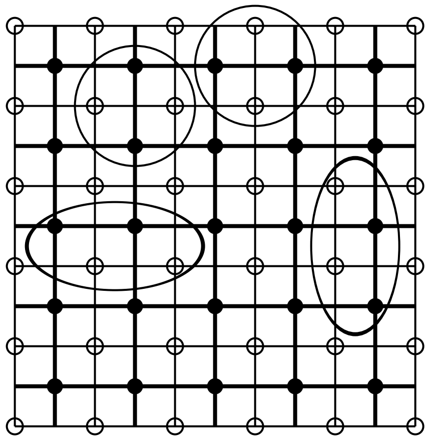

Fig.1 The spins involved in the interaction of the models in eq. (1.1). The heavy dots and lines or the light dots and lines mark the Ising lattices and the nearest neighbors Ising couplings. The ellipses symbolize the Ashkin–Teller four spins interactions (λa–couplings) and the circles the Baxter four spins interactions (λb) couplings.

For a given observable $O(\mathbf{x})$ localized near $\mathbf{x}$ we define the correlation

$$< O(\mathbf{x})O(\mathbf{y}) >_\Lambda = \frac{1}{Z_\Lambda} \sum_{\substack{\sigma^{(1)}_\mathbf{x},\sigma^{(2)}_\mathbf{x}=\pm1\\ \mathbf{x}\in\Lambda_M}} O(\mathbf{x})O(\mathbf{y}) e^{-H_\Lambda(\sigma^{(1)},\sigma^{(2)})} \tag{1.4}$$

where $Z_\Lambda = \sum_{\substack{\sigma^{(1)}_\mathbf{x},\sigma^{(2)}_\mathbf{x}=\pm1\\ \mathbf{x}\in\Lambda}} e^{-H_\Lambda(\sigma^{(1)},\sigma^{(2)})}$ is the the *partition function*. The *truncated correlation* of the observable $O(\mathbf{x})$ is

$$< O(\mathbf{x})O(\mathbf{y}) >_{\Lambda,T} = < O(\mathbf{x})O(\mathbf{y}) >_\Lambda - < O(\mathbf{x}) >_\Lambda < O(\mathbf{y}) >_\Lambda \ , \tag{1.5}$$

and the *energy-energy* truncated correlation function is given by (1.5) with $O(\mathbf{x}) = H_{\Lambda,\mathbf{x}}(\sigma^{(1)},\sigma^{(2)})$; the *specific heat* C_v^λ is

$$C_v^\lambda = \lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|} \sum_{\mathbf{x},\mathbf{y}\in\Lambda} < H_{\Lambda,\mathbf{x}}(\sigma^{(1)},\sigma^{(2)}) H_{\Lambda,\mathbf{y}}(\sigma^{(1)},\sigma^{(2)}) >_{\Lambda,T} \ . \tag{1.6}$$

If $\lambda = 0$ the model reduces to two independent *Ising models* and close to the critical temperature (equal for both) it is

$$C_v^0 \simeq -C_1 \log |\frac{J_c}{J} - 1| + C_2 \ , \tag{1.7}$$

where C_1, C_2 are positive constants and $\tanh J_c = \sqrt{2} - 1$, see [MW] eq.(3.58). The truncated correlation of the observable $O(\mathbf{x}) = H_{I,\mathbf{x}}(\sigma^{(\alpha)})$ for $\lambda = 0$ has the property $| < O(\mathbf{x})O(\mathbf{y}) >_{\Lambda,T} | \le Ce^{-A|t-t_c||\mathbf{x}-\mathbf{y}|}$ with A, C suitable constants.

We expect that the interaction changes the value of the critical temperature (*i.e.* of J_c) by quantities $O(\lambda)$. However it is convenient to keep the critical singularity at a λ-independent value J_0,with $\tanh J_0 = \sqrt{2} - 1$. We shall show that this can be easily done by choosing properly the molecular energy parameter J as a function of λ. Therefore we consider the model (1.1) with J_r replacing J, and we shall choose $J_r = J + O(\lambda)$ so that the critical coupling is J_0, *i.e.* it is precisely in correspondence of $t_c = \sqrt{2} - 1$.

Denoting by N an *arbitrary* positive integer, fixing $a+b \neq 0$ and calling $\tanh J = t$ and $t_c = \sqrt{2}-1$ we shall prove the following theorem.

Theorem 1. *There are C, C_N, C_1, C_2, τ positive λ–independent constants, such that for λ small enough one can uniquely define $\nu(\lambda)$, analytic in λ, so that the model in eq. (1.1),(1.3) with coupling $J_r = \tanh^{-1}(t+\nu(\lambda))$ is critical at $t = t_c$. This means that, for $|t-t_c|$ strictly positive and small enough*

$$\lim_{|\Lambda|\to\infty} < H_{\Lambda,\mathbf{x}}(\sigma^{(1)},\sigma^{(2)}) H_{\Lambda,\mathbf{y}}(\sigma^{(1)},\sigma^{(2)}) >_{\Lambda,T} = \Omega^a(\mathbf{x},\mathbf{y}) + \Omega^b(\mathbf{x},\mathbf{y}) \tag{1.8}$$

and the bounds

$$|\Omega^a(\mathbf{x},\mathbf{y})| \le \frac{1}{|\mathbf{x}-\mathbf{y}|^{2+2\eta_1}} \frac{C_N}{1+(\Delta|\mathbf{x}-\mathbf{y}|)^N},\ |\Omega^b(\mathbf{x},\mathbf{y})| \le \frac{1}{|\mathbf{x}-\mathbf{y}|^{2+\tau}} \frac{C_N}{1+(\Delta|\mathbf{x}-\mathbf{y}|)^N} \tag{1.9}$$

hold, with "correlation length" Δ and "critical indices" η_1, η_2 given by

$$\Delta = |t-t_c|^{1+\eta_2},\ \eta_1(\lambda) = -a_1(a+b)\lambda + O(\lambda^2) \quad \eta_2(\lambda) = -a_2(a+b)\lambda + O(\lambda^2) \tag{1.10}$$

with $a_1 > 0, a_2 > 0$ constants and η_1, η_2 analytic in λ. Furthermore if $1 \le |\mathbf{x}| \le \Delta^{-1}$ the correlation is asymptotic to Ω_a in the sense that Ω_b is neglegible because

$$\Omega^a(\mathbf{x},\mathbf{y}) = \frac{1}{\tilde{Z}_1^2} \frac{1+A(\mathbf{x},\mathbf{y})}{(\mathbf{x}-\mathbf{y})^{2+2\eta_1}}, \qquad |A(\mathbf{x})| \le C[|\lambda| + (\Delta|\mathbf{x}|)^{\frac{1}{2}} + |\mathbf{x}|^{-\tau}] \tag{1.11}$$

where $\tilde{Z}_1 > 0$ is analytic in λ and not zero at $\lambda = 0$. Finally the specific heat C_v (1.6) verifies

$$C_1 \frac{1}{2\eta_1}[1 - |\Delta|^{2\eta_1}] \le C_v^\lambda \le C_2 \frac{1}{2\eta_1}[1 - |\Delta|^{2\eta_1}] \tag{1.12}$$

where C_1, C_2 are positive constants.

1.3 The above theorem says that the interaction changes the value of the critical temperature and it qualitatively modifies the critical behaviour of the specific heat and of the energy-energy correlations. As t gets closer and closer to the critical temperature the logarithmic singularity of the specific heat in the Ising model is changed by the four spin interaction into a power law singularity with non universal critical indices if $\lambda(a+b) > 0$; if $\lambda(a+b) < 0$ the specific heat is instead continuous, but higher derivatives of the free energy are singular, as one can easily check from the proof of the Theorem.

Moreover one can distinguish two different regimes in the asymptotic behaviour of the energy-energy correlation function, discriminated by an intrinsic correlation length ξ of order $|t-t_c|^{-1-\eta_2}$ with $\eta_2 = O(\lambda)$. If $1 << |\mathbf{x}-\mathbf{y}| << \xi$, the bound for the correlation function is power-like while if $\xi << |\mathbf{x}-\mathbf{y}|$, there is a faster than any power decay with rate of order ξ^{-1}. The splitting (1.8) suggests that the fast decay is modulated by a power $|\mathbf{x}-\mathbf{y}|^{-2-2\eta_1}$ but it does not prove that because the first of (1.9) is an inequality rather than an asymptotic expression. For the same reason we can only suggest that at $t = t_c$ there is only a power law behaviour with critical index η_1.

In the case $a = 0$ our model reduces to the Eight vertex model, and our results are in agreement with the exact solution in [B] (see also [W]). The assumption $a + b \neq 0$ could be possibly removed, and we could consider easily interactions much more general than (1.3).

The paper is organized in the following way. In §2 we write, using the results of [H], [Ka],[MW], the partition function of the model (1.1) as a Grassman integral with a non quadratic formal action; to each site of the lattice are associated eight Grassman variables. We perform a linear transformation on such variables, following [ID], so that the Grassman integral formally describes four kind of interacting *Majorana fermions* on a lattice. By another linear transformation we express the Grassman integral in terms of two kinds of *Dirac fermions* on a lattice; the $\chi^\pm$ fields, with an $O(1)$ mass, and the $\psi^\pm$ fields with a mass vanishing at criticality. In §3 we integrate out the massive $\chi^{\pm}$-fields, so obtaining an effective theory in terms of the $\psi^{\pm}$-fields only. The exploitation of parity and symmetry cancellations is crucial here to show that many effective interactions, generated by the integration of the χ fields and which are apparently relevant or marginal, are indeed irrelevant; such symmetries, which are very easy to express in the original spin variables, become quite involved in the fermionic representation. The analysis of such cancellations is essential to avoid the proliferation of running coupling constants and to control the RG flow, see below.

In §4 we integrate the $\psi^{\pm}$-fields by decomposing them as a sum of independent Grassmann fields $\psi^{(h)}$, with covariance non vanishing only for momenta with modulus between γ^{h-1} and γ^{h+1}, with $\gamma > 1$ and $h = 1, 0, -1, -2, -3, ...$ We integrate each $\psi^{(h)}$ iteratively obtaining a sequence of effective potentials $\mathcal{V}^{(h)}$ describing the theory at momentum scale γ^h; at each step new contributions to the mass and the wave function renormalization are obtained which are included in the fermionic free integration; hence $\psi^{(h)}$ has a covariance with mass m_h and wave function renormalization Z_h with a non trivial dependence on h, *i.e.* $m_h \simeq C_1|t - t_c|\gamma^{\eta_2 h}$ and $Z_h \simeq C_2\gamma^{\eta h}$, with $\eta_2 = O(\lambda), \eta = O(\lambda^2)$ and C_1, C_2 constants. The iteration stops as soon as the mass m_h becomes of order γ^h; we will call h^* the last scale to be integrated (of course $h^* \to -\infty$ at the critical temperature $t = t_c$). The iterative procedure allows to write the effective potential $\mathcal{V}^{(h)}$ as sum of monomials in the Grassmann variables, with coefficients which are (convergent) perturbative expansions in terms of a few *running coupling constant*, λ_h, the effective coupling of the interaction between fermions, ν_h, which takes into account the renormalization of the value of the critical temperature and δ_h, related to the renormalization of the fermionic velocity. At the end the result of this iterative integration is an expansion for the partition function in terms of the running coupling constants which is convergent *provided that* the running coupling constants are small for any h. In §5 we write the beta function governing the flow of λ_h, δ_h as a sum of several terms, and we show that only one term is really crucial, while the other ones have a little effect on the flow in absence of the first one, if the counterterm ν is chosen in a proper way. On the other hand, one recognizes that such crucial contribution to the Beta function model is coinciding with the Beta function of the Luttinger model, and it was proved, as a consequence of its exact solution [ML], in [GS], [BGPS], [BM1] (see [BeM1] for a simplified proof) that it is vanishing; this means that the crucial contribution to the beta function is vanishing, hence λ_h, δ_h are small for any h if ν is chosen properly. Finally in §5 we define an expansion for the correlation functions and the specific heat; it is similar to the one for the partition function, with the main difference that one has to introduce new fields associated to the external fields. There is an additional marginal term in the RG expansion to which another *renormalization constant*, with a non trivial behaviour in h, is associated, *i.e.* $Z_h^{(1)} \simeq \gamma^{\eta_1 h}$. By such expansion the statements in the theorem are derived. The proof makes clear the relationship of spin models with quartic interactions like the model (1.1) and the massive Luttinger model, in agreement with what was conjectured in [LP], [N], [PB]. Our results extend a previous paper [M1] (in which (1.12) must replace (1.16) of [M1] which is incorrect) in which the analysis was restricted to the case $|t - t_c| \geq e^{-\frac{a}{\lambda^2}}$, where a is a suitable constant; the new technical ingredients (which allow us to reach the critical temperature) are the exploitation of a number of cancellations showing that a number of apparently *marginal* terms in the effective interaction were indeed *irrelevant*, and the decomposition of the Beta function in a Luttinger model part (which is vanishing) plus a rest which does not drive λ_h, δ_h too far away their initial value. The paper is self contained but it uses some technical lemmata proved in full detail in [BM].

A very important open problem is to obtain by such fermionic RG methods the asymptotic behaviour of the spin-spin correlation function; its fermionic representation is much more involved than the one for the specific heat or the energy-energy correlations. One can study also the case in which the parameter J of the two Ising model hamiltonian are different; new fermionic effective marginal interactions appear in such case. Another possibility is the analysis of *four* coupled Ising models; in this last case interacting *spinning* $d = 1$ fermions appear in the fermionic description, which are known to have a behaviour quite different from the spinless one (like in the $d = 1$ *Hubbard* model).

2. Fermionic representation

2.1 The partition function of the Ising model can be written as a Grassman integral. It is a classical result, due mainly to [Ka],[H],[MW],[S] and rederived recently in §3 of [PS] to which we refer for a detailed proof, that

$$Z_I^{(\alpha)} = (\cosh J_r)^B 2^S \frac{1}{2} \int \prod_{\mathbf{x}\in\Lambda_M} dH_{\mathbf{x}}^{(\alpha)} d\bar H_{\mathbf{x}}^{(\alpha)} dV_{\mathbf{x}}^{(\alpha)} d\bar V_{\mathbf{x}}^{(\alpha)} [-e^{S_{+,+}} + e^{S_{+,-}} + e^{S_{-,+}} + e^{S_{-,-}}] \tag{2.1}$$

where $\Lambda_M=\lambda$, B is the total number of bonds and S is the total number of sites,

$$S_{\varepsilon,\varepsilon'}^{(\alpha)} = \sum_{\mathbf{x}\in\Lambda_M} \tanh J_r [\bar H_{x,x_0}^{(\alpha)} H_{x+1,x_0}^{(\alpha)} + \bar V_{x,x_0}^{(\alpha)} V_{x,x_0+1}^{(\alpha)}] +$$

$$\sum_{\mathbf{x}\in\Lambda_M} [\bar H_{x,x_0}^{(\alpha)} H_{x,x_0}^{(\alpha)} + \bar V_{x,x_0}^{(\alpha)} V_{x,x_0}^{(\alpha)} + \bar V_{x,x_0}^{(\alpha)} \bar H_{x,x_0}^{(\alpha)} + V_{x,x_0}^{(\alpha)} \bar H_{x,x_0}^{(\alpha)} + H_{x,x_0}^{(\alpha)} \bar V_{x,x_0}^{(\alpha)} + V_{x,x_0}^{(\alpha)} H_{x,x_0}^{(\alpha)}] \tag{2.2}$$

and $H_{\mathbf{x}}^{(\alpha)}, \bar H_{\mathbf{x}}^{(\alpha)}, V_{\mathbf{x}}^{(\alpha)}, \bar V_{\mathbf{x}}^{(\alpha)}$ are *Grassman variables* such that

$$\bar H_{x,x_0+M}^{(\alpha)} = \varepsilon \bar H_{x,x_0}^{(\alpha)} \qquad \bar H_{x+M,x_0}^{(\alpha)} = \varepsilon' \bar H_{x,x_0}^{(\alpha)}$$

$$H_{x,x_0+M}^{(\alpha)} = \varepsilon H_{x,x_0}^{(\alpha)} \quad H_{x+M,x_0}^{(\alpha)} = \varepsilon' H_{x,x_0}^{(\alpha)} \tag{2.3}$$

and identical relations hold for the variables $V^{(\alpha)}, \bar V^{(\alpha)}$. The Grassman integration $\int \prod_x dH_x^{(\alpha)} d\bar H_x^{(\alpha)}$ is defined as the linear functional on the Grassmanian algebra, such that, given a monomial $Q(H^{(\alpha)}, \bar H^{(\alpha)})$ in the variables $H_{\mathbf{x}}^{(\alpha)}, \bar H_{\mathbf{x}}^{(\alpha)}$, $\mathbf{x}\in\Lambda_M$, its value is 0, except in the case $Q(H^{(\alpha)}, \bar H^{(\alpha)}) = \prod_{\mathbf{x}} H_{\mathbf{x}}^{(\alpha)} \bar H_{\mathbf{x}}^{(\alpha)}$, up to a permutation of the variables. In this case the value of the functional is determined, by using the anticommuting properties of the variables, by the condition

$$\int \left[\prod_{\mathbf{x}\in\Lambda_M} d\bar H_{\mathbf{x}}^{(\alpha)} dH_{\mathbf{x}}^{(\alpha)}\right] \prod_{\mathbf{x}\in\Lambda_M} H_{\mathbf{x}}^{(\alpha)} \bar H_{\mathbf{x}}^{(\alpha)} = 1\ . \tag{2.4}$$

In a similar way is defined the Grassman integration for $V^{(\alpha)}, \bar V^{(\alpha)}$, just exchanging $H, \bar H$ with $V, \bar V$. We write $e^{S_{\varepsilon,\varepsilon'}^{(\alpha)}} = e^{S_{\varepsilon,\varepsilon'}^{(\alpha),0}} e^{S_{\varepsilon,\varepsilon'}^{(\alpha),\nu}}$ where $S_{\varepsilon,\varepsilon'}^{(\alpha),0}$ is given by (2.2) with J replacing J_r and

$$S_{\varepsilon,\varepsilon'}^{(\alpha),\nu} = \nu \sum_{\mathbf{x}\in\Lambda_M} [\bar H_{x,x_0}^{(\alpha)} H_{x+1,x_0}^{(\alpha)} + \bar V_{x,x_0}^{(\alpha)} V_{x,x_0+1}^{(\alpha)}] \tag{2.5}$$

If J is not constant but it depends on the bonds one obtains a similar formula in which $S_{\varepsilon,\varepsilon'}^{(\alpha),0}$ is given by

$$S_{\varepsilon,\varepsilon'}^{(\alpha),0} = \sum_{\mathbf{x}} \tanh J_{x,x_0;x+1,x_0}^{(\alpha)} \bar H_{x,x_0}^{(\alpha)} H_{x+1,x_0}^{(\alpha)} + \tanh J_{x,x_0;x,x_0+1}^{(\alpha)} \bar V_{x,x_0}^{(\alpha)} V_{x,x_0+1}^{(\alpha)}] + \tag{2.6}$$

$$\sum_{\mathbf{x}} [\bar H_{x,x_0}^{(\alpha)} H_{x,x_0}^{(\alpha)} + \bar V_{x,x_0}^{(\alpha)} V_{x,x_0}^{(\alpha)} + \bar V_{x,x_0}^{(\alpha)} \bar H_{x,x_0}^{(\alpha)} + V_{x,x_0}^{(\alpha)} \bar H_{x,x_0}^{(\alpha)} + H_{x,x_0}^{(\alpha)} \bar V_{x,x_0}^{(\alpha)} + V_{x,x_0}^{(\alpha)} H_{x,x_0}^{(\alpha)}]$$

and the factor $(\cosh J)^B$ is replaced by $\prod_b \cosh J_b$, where the product is over all the possible nearest neighbor bonds.

The partition function of (1.1) is

$$Z_{2I} = \sum_{\substack{\sigma^{(1)}_{\mathbf{x}}=\pm 1\\ \mathbf{x}\in\Lambda_M}} \sum_{\substack{\sigma^{(2)}_{\mathbf{x}}=\pm 1\\ \mathbf{x}=\Lambda_M}} e^{-H_I(\sigma^{(1)})} e^{-H_I(\sigma^{(2)})} e^{-V(\sigma^{(1)},\sigma^{(2)})} \tag{2.7}$$

Let us consider first the essentially equivalent expression

$$\hat{Z}_{2I} = \sum_{\substack{\sigma^{(1)}=\pm 1\\ \mathbf{x}\in\Lambda_M}} \sum_{\substack{\sigma^{(2)}=\pm 1\\ \mathbf{x}\in\Lambda_M}} e^{-H_I(\sigma^{(1)})} e^{-H_I(\sigma^{(2)})} \tag{2.8}$$

$$\prod_{\mathbf{x}}[1+\lambda a \sigma^{(1)}_{x,x_0}\sigma^{(1)}_{x+1,x_0}\sigma^{(2)}_{x,x_0}\sigma^{(2)}_{x+1,x_0}] \prod_{\mathbf{x}}[1+\lambda a \sigma^{(1)}_{x,x_0}\sigma^{(1)}_{x,x_0+1}\sigma^{(2)}_{x,x_0}\sigma^{(2)}_{x,x_0+1}]$$

$$\prod_{\mathbf{x}}[1+\lambda b \sigma^{(1)}_{x,x_0}\sigma^{(1)}_{x+1,x_0}\sigma^{(2)}_{x,x_0}\sigma^{(2)}_{x,x_0+1}] \prod_{\mathbf{x}}[1+\lambda b \sigma^{(1)}_{x,x_0}\sigma^{(1)}_{x,x_0+1}\sigma^{(2)}_{x-1,x_0+1}\sigma^{(2)}_{x,x_0+1}] \,.$$

Noting that

$$\sigma^{(\alpha)}_{\mathbf{x}}\sigma^{(\alpha)}_{\mathbf{x}'} e^{-H_I(\sigma^{(\alpha)})} = \frac{\partial}{\partial J^{(\alpha)}_{\mathbf{x},\mathbf{x}'}} Z^{(\alpha)}_I(J^{(\alpha)}_{\mathbf{x},\mathbf{x}'})|_{\{J^{(\alpha)}_{\mathbf{x},\mathbf{x}'}\}=\{J^{(\alpha)}\}} \tag{2.9}$$

where $\mathbf{x},\mathbf{x}'$ are nearest neighbor, and from (2.6) this derivative gives an extra factor $\tanh J^{(\alpha)} + \mathrm{sech}^2 J^{(\alpha)} \bar{H}^{(\alpha)}_{x,x_0} H^{(\alpha)}_{x+1,x_0}$ in (2.1). We can write than, if $\delta_{+,+}=1$ and $\delta_{+,-}=\delta_{-,+}=\delta_{-,-}=2$

$$\hat{Z}_{2I} = \sum_{\varepsilon^{(1)},\varepsilon'^{(1)}} (-1)^{\delta_{\varepsilon^{(1)},\varepsilon'^{(1)}}} \sum_{\varepsilon^{(2)},\varepsilon'^{(2)}} (-1)^{\delta_{\varepsilon^{(2)},\varepsilon'^{(2)}}} \hat{Z}^{\varepsilon^{(1)},\varepsilon'^{(1)},\varepsilon^{(2)},\varepsilon'^{(2)}}_{2I} \tag{2.10}$$

where

$$\hat{Z}^{\varepsilon^{(1)},\varepsilon'^{(1)},\varepsilon^{(2)},\varepsilon'^{(2)}}_{2I} = (\cosh J)^{2B} 2^{2S} \frac{1}{4} \tag{2.11}$$

$$\int \prod_{\alpha=1}^{2} [\prod_{\mathbf{x}} dH^{(\alpha)}_{\mathbf{x}} d\bar{H}^{(\alpha)}_{\mathbf{x}} dV^{(\alpha)}_{\mathbf{x}} d\bar{V}^{(\alpha)}_{\mathbf{x}}] e^{S^{(1)}_{\varepsilon^{(1)},\varepsilon'^{(1)}}} e^{S^{(2)}_{\varepsilon^{(2)},\varepsilon'^{(2)}}}$$

$$\prod_{\mathbf{x}}[1+\lambda a(\tanh J + \mathrm{sech}^2 J \bar{H}^{(1)}_{x,x_0} H^{(1)}_{x+1,x_0})(\tanh J + \mathrm{sech}^2 J \bar{H}^{(2)}_{x,x_0} H^{(2)}_{x+1,x_0})]$$

$$\prod_{\mathbf{x}}[1+\lambda a(\tanh J + \mathrm{sech}^2 J \bar{V}^{(1)}_{x,x_0} V^{(1)}_{x,x_0+1})(\tanh J + \mathrm{sech}^2 J \bar{V}^{(2)}_{x,x_0} V^{(2)}_{x,x_0+1})]$$

$$\prod_{\mathbf{x}}[1+\lambda b(\tanh J + \mathrm{sech}^2 J \bar{H}^{(1)}_{x,x_0} H^{(1)}_{x+1,x_0})(\tanh J + \mathrm{sech}^2 J \bar{V}^{(2)}_{x,x_0} V^{(2)}_{x,x_0+1})]$$

$$\prod_{\mathbf{x}}[1+\lambda b(\tanh J + \mathrm{sech}^2 J \bar{V}^{(1)}_{x,x_0} V^{(1)}_{x,x_0+1})(\tanh J + \mathrm{sech}^2 J \bar{H}^{(2)}_{x-1,x_0+1} H^{(2)}_{x,x_0+1})]$$

It is easy to check that the above expression can be rewritten as

$$\hat{Z}^{\varepsilon^{(1)},\varepsilon'^{(1)},\varepsilon^{(2)},\varepsilon'^{(2)}}_{2I} = (\cosh J)^{2B} 2^{2S} \frac{1}{4}$$

$$\int [\prod_{\alpha=1}^{2} \prod_{\mathbf{x}} dH^{(\alpha)}_{\mathbf{x}} d\bar{H}^{(\alpha)}_{\mathbf{x}} dV^{(\alpha)}_{\mathbf{x}} d\bar{V}^{(\alpha)}_{\mathbf{x}} e^{S^{(\alpha),0}_{\varepsilon^{(\alpha)},\varepsilon'^{(\alpha)}}}] e^{-\mathcal{V}} \tag{2.12}$$

with

$$\mathcal{V} = \mathcal{V}_a + \mathcal{V}_b - \sum_{\alpha} S^{(\alpha),\nu}_{\varepsilon^{(\alpha)},\varepsilon'^{(\alpha)}} \tag{2.13}$$

and, if $f_i = log(1+\lambda[i]\tanh^2 J)$ and $[i]=a,b$

$$-\mathcal{V}_a=\sum_{\mathbf{x}}[f_a+\tilde\lambda_a[\bar H^{(1)}_{x,x_0}H^{(1)}_{x+1,x_0}+\bar H^{(2)}_{x,x_0}H^{(2)}_{x+1,x_0}]+\lambda_a\bar H^{(1)}_{x,x_0}H^{(1)}_{x+1,x_0}\bar H^{(2)}_{\mathbf{x}}H^{(2)}_{x+1,x_0}]$$

$$+\sum_{\mathbf{x}}[f_a+\tilde\lambda_a[\bar V^{(1)}_{x,x_0}V^{(1)}_{x,x_0+1}+\bar V^{(2)}_{x,x_0}V^{(2)}_{x,x_0+1}]+\lambda_a\bar V^{(1)}_{\mathbf{x}}V^{(1)}_{x,x_0+1}\bar V^{(2)}_{x,x_0}V^{(2)}_{x,x_0+1}] \tag{2.14}$$

$$-\mathcal{V}_b=\sum_{\mathbf{x}}[f_b+\tilde\lambda_b[\bar H^{(1)}_{x,x_0}H^{(1)}_{x+1,x_0}+\bar V^{(2)}_{x,x_0}V^{(2)}_{x,x_0+1}]+\lambda_b\bar H^{(1)}_{x,x_0}H^{(1)}_{x+1,x_0}\bar V^{(2)}_{\mathbf{x}}V^{(2)}_{x,x_0+1}]$$

$$+\sum_{\mathbf{x}}[f_b+\tilde\lambda_b[\bar V^{(1)}_{x,x_0}V^{(1)}_{x,x_0+1}+\bar H^{(2)}_{x-1,x_0+1}H^{(2)}_{x,x_0+1}]+\lambda_b\bar V^{(1)}_{x,x_0}V^{(1)}_{x,x_0+1}\bar H^{(2)}_{x-1,x_0+1}H^{(2)}_{x,x_0+1}]$$

where

$$\tilde\lambda_i(1+\lambda[i])\tanh^2 J)=\lambda[i]\mathrm{sech}^2 J\tanh J$$

$$(1+\lambda[i]\tanh^2 J)(\lambda_i+(\tilde\lambda_i)^2)=\lambda[i]\mathrm{sech}^4 J \tag{2.15}$$

An identical expression holds for Z_{2I}, the only difference being that the relation with respect $\tilde\lambda,\lambda$ is slightly more complicated than (2.15), but for small λ again $\tilde\lambda_i=\lambda[i](\tanh J\mathrm{sech}^2 J+O(\lambda))$, $\lambda_i=\lambda[i](\mathrm{sech}^4 J+O(\lambda))$.

2.2 It is convenient to perform the following change of variables [ID]

$$\bar H^{(\alpha)}_{\mathbf{x}}+iH^{(\alpha)}_{\mathbf{x}}=e^{i\frac{\pi}{4}}\psi^{(\alpha)}_{\mathbf{x}}-e^{i\frac{\pi}{4}}\chi^{(\alpha)}_{\mathbf{x}}\qquad \bar H^{(\alpha)}_{\mathbf{x}}-iH^{(\alpha)}_{\mathbf{x}}=e^{-i\frac{\pi}{4}}\bar\psi^{(\alpha)}_{\mathbf{x}}-e^{-i\frac{\pi}{4}}\bar\chi^{(\alpha)}_{\mathbf{x}}$$

$$\bar V^{(\alpha)}_{\mathbf{x}}+iV^{(\alpha)}_{\mathbf{x}}=\psi^{(\alpha)}_{\mathbf{x}}+\chi^{(\alpha)}_{\mathbf{x}}\qquad \bar V^{(\alpha)}_{\mathbf{x}}-iV^{(\alpha)}_{\mathbf{x}}=\bar\psi^{(\alpha)}_{\mathbf{x}}+\bar\chi^{(\alpha)}_{\mathbf{x}} \tag{2.16}$$

If $S^{(\alpha)}_{\varepsilon^{(\alpha)},\varepsilon'^{(\alpha)}}=\sum_{\mathbf{x}}S^{(\alpha)}_{\mathbf{x},\varepsilon^{(\alpha)},\varepsilon'^{(\alpha)}}$ we get

$$S^{(\alpha)}_{\mathbf{x},\varepsilon^{(\alpha)},\varepsilon'^{(\alpha)}}=S^{(\alpha,\psi)}_{\mathbf{x},\varepsilon^{(\alpha)},\varepsilon'^{(\alpha)}}+S^{(\alpha,\chi)}_{\mathbf{x},\varepsilon^{(\alpha)},\varepsilon'^{(\alpha)}}+Q^{(\alpha)}_{\mathbf{x},\varepsilon^{(\alpha)},\varepsilon'^{(\alpha)}}$$

where

$$S^{(\alpha,\psi)}_{\mathbf{x},\varepsilon^{(\alpha)},\varepsilon'^{(\alpha)}}=\frac{t}{4}[\psi^{(\alpha)}_{\mathbf{x}}(\partial_1-i\partial_0)\psi^{(\alpha)}_{\mathbf{x}}+\bar\psi^{(\alpha)}_{\mathbf{x}}(\partial_1+i\partial_0)\bar\psi^{(\alpha)}_{\mathbf{x}}]$$

$$+\frac{t}{4}[-i\bar\psi^{(\alpha)}_{\mathbf{x}}(\partial_1\psi^{(\alpha)}_{x}+\partial_0\psi^{(\alpha)}_{\mathbf{x}})+i\psi^{(\alpha)}_{\mathbf{x}}(\partial_1\bar\psi^{(\alpha)}_{\mathbf{x}}+\partial_0\bar\psi^{(\alpha)}_{\mathbf{x}})]$$

$$+i(\sqrt2-1-t)\bar\psi^{(\alpha)}_{\mathbf{x}}\psi^{(\alpha)}_{\mathbf{x}} \tag{2.17}$$

where

$$\partial_1\psi^{(\alpha)}_{\mathbf{x}}=\psi^{(\alpha)}_{x+1,x_0}-\psi^{(\alpha)}_{\mathbf{x}}\qquad \partial_0\psi^{(\alpha)}_{x}=\psi^{(\alpha)}_{x,x_0+1}-\psi^{(\alpha)}_{\mathbf{x}} \tag{2.18}$$

Moreover

$$S^{(\alpha,\chi)}_{\mathbf{x},\varepsilon^{(\alpha)},\varepsilon'^{(\alpha)}}=\frac{t}{4}[\chi^{(\alpha)}_{\mathbf{x}}(\partial_1-i\partial_0)\chi^{(\alpha)}_{\mathbf{x}}+\bar\chi^{(\alpha)}_{\mathbf{x}}(\partial_1+i\partial_0)\bar\chi^{(\alpha)}_{\mathbf{x}}]+$$

$$\frac{t}{4}[-i\bar\chi^{(\alpha)}_{\mathbf{x}}(\partial_1\chi^{(\alpha)}_{x}+\partial_0\chi^{(\alpha)}_{\mathbf{x}})+i\chi^{(\alpha)}_{\mathbf{x}}(\partial_1\bar\chi^{(\alpha)}_{\mathbf{x}}+\partial_0\bar\chi^{(\alpha)}_{\mathbf{x}})]-i(\sqrt2+1+t)\bar\chi^{(\alpha)}_{\mathbf{x}}\chi^{(\alpha)}_{\mathbf{x}} \tag{2.19}$$

and finally

$$Q^{(\alpha)}_{\mathbf{x},\varepsilon^{(\alpha)},\varepsilon'^{(\alpha)}}=\frac{t}{4}\{-\psi^{(\alpha)}_{\mathbf{x}}(\partial_1\chi^{(\alpha)}_{\mathbf{x}}+i\partial_0\chi^{(\alpha)}_{\mathbf{x}})-\bar\psi^{(\alpha)}_{\mathbf{x}}(\partial_1\bar\chi^{(\alpha)}_{\mathbf{x}}-i\partial_0\bar\chi^{(\alpha)}_{\mathbf{x}})$$

$$-\chi^{(\alpha)}_{\mathbf{x}}(\partial_1\psi^{(\alpha)}_{\mathbf{x}}+i\partial_0\psi^{(\alpha)}_{\mathbf{x}})-\bar\chi^{(\alpha)}_{\mathbf{x}}(\partial_1\bar\psi^{(\alpha)}_{\mathbf{x}}-i\partial_0\bar\psi^{(\alpha)}_{\mathbf{x}})+i\bar\psi^{(\alpha)}_{\mathbf{x}}(\partial_1\chi^{(\alpha)}_{\mathbf{x}}-\partial_0\chi^{(\alpha)}_{\mathbf{x}}) \tag{2.20}$$

$$+i\psi^{(\alpha)}_{\mathbf{x}}(-\partial_1\bar\chi^{(\alpha)}_{\mathbf{x}}+\partial_0\bar\chi^{(\alpha)}_{\mathbf{x}})+i\bar\chi^{(\alpha)}_{\mathbf{x}}(\partial_1\psi^{(\alpha)}_{\mathbf{x}}-\partial_0\psi^{(\alpha)}_{\mathbf{x}})+i\chi^{(\alpha)}_{\mathbf{x}}(-\partial_1\bar\psi^{(\alpha)}_{\mathbf{x}}+\partial_0\bar\psi^{(\alpha)}_{\mathbf{x}})\}\ .$$

Moreover

$$\bar H^{(\alpha)}_{x,x_0}H^{(\alpha)}_{x+1,x_0}+\bar V^{(\alpha)}_{x,x_0}V^{(\alpha)}_{x,x_0+1}=\partial_t S^{\alpha}_{\mathbf{x},\varepsilon^{(\alpha)},\varepsilon'^{(\alpha)}} \tag{2.21}$$

so that $S^{(\alpha),\nu}_{\varepsilon^\alpha,\varepsilon'^\alpha}=\nu\sum_{\mathbf{x}\in\Lambda}\partial_t S^\alpha_{\mathbf{x},\varepsilon^{(\alpha)},\varepsilon'^{(\alpha)}}$.

2.3 We find convenient to rewrite the Grassman variables in momentum space. We call $D_{\varepsilon,\varepsilon'}$ the set of $\mathbf{k}$ such that

$$k=\frac{2\pi n_1}{M}+\frac{(\varepsilon'-1)\pi}{2M}\quad k_0=\frac{2\pi n_0}{M}+\frac{(\varepsilon-1)\pi}{2M}\tag{2.22}$$

and $-[M/2]\le n_0\le[(M-1)/2]$, $-[M/2]\le n_1\le[(M-1)/2]$, $n_0,n_1\in Z$. We can write if $\mathbf{k}=(k_0,k)$

$$\psi^{(\alpha)}_{\mathbf{x}}=\frac{1}{M^2}\sum_{\mathbf{k}\in D_{\varepsilon,\varepsilon'}}\psi^{(\alpha)}_{\mathbf{k}}e^{-i\mathbf{k}\mathbf{x}}\quad\bar\psi^{(\alpha)}_{\mathbf{x}}=\frac{1}{M^2}\sum_{\mathbf{k}\in D_{\varepsilon,\varepsilon'}}\bar\psi^{(\alpha)}_{\mathbf{k}}e^{-i\mathbf{k}\mathbf{x}}\tag{2.23}$$

Let us define

$$P^{(\alpha)}_{\varepsilon,\varepsilon'}(d\psi)=\mathcal{N}_\psi^{-1}[\prod_{\mathbf{k}\in D_{\varepsilon,\varepsilon'}}d\bar\psi^{(\alpha)}_{\mathbf{k}}d\psi^{(\alpha)}_{\mathbf{k}}]\exp[\frac{t}{4M^2}\sum_{\mathbf{k}\in D_{\varepsilon,\varepsilon'}}[\psi^{(\alpha)}_{\mathbf{k}}\psi^{(\alpha)}_{-\mathbf{k}}(i\sin k+\sin k_0)$$

$$+\bar\psi^{(\alpha)}_{\mathbf{k}}\bar\psi^{(\alpha)}_{-\mathbf{k}}(i\sin k-\sin k_0)+i2m_\psi(\mathbf{k})\bar\psi^{(\alpha)}_{\mathbf{k}}\psi^{(\alpha)}_{-\mathbf{k}}]]\tag{2.24}$$

where $\mathcal{N}_\psi$ is a normalization constant and $\frac{t}{2}m_\psi(\mathbf{k})=(\sqrt2-1-t)+\frac{t}{2}(\cos k_0+\cos k-2)$.

We consider for semplicity $Z_{2I}^{-,-,-,-}$ and we set $P^{(\alpha)}_{-,-}(d\psi)=P^{(\alpha)}(d\psi)$; we will show in §5.8 that $Z_{2I}^{\varepsilon^{(1)},\varepsilon'^{(1)},\varepsilon^{(2)},\varepsilon'^{(2)}}(Z_{0,2I}^{\varepsilon^{(1)},\varepsilon'^{(1)},\varepsilon^{(2)},\varepsilon'^{(2)}})^{-1}$ is exponentially insensitive to boundary conditions, if $Z_{0,2I}^{\varepsilon^{(1)},\varepsilon'^{(1)},\varepsilon^{(2)},\varepsilon'^{(2)}}$ is (2.10) with $\lambda=0$. We can perform the following change of variables

$$\psi^-_{1,\mathbf{k}}=\frac{1}{\sqrt2}(\psi^{(1)}_{\mathbf{k}}+i\psi^{(2)}_{\mathbf{k}})\quad\psi^+_{1,-\mathbf{k}}=\frac{1}{\sqrt2}(\psi^{(1)}_{\mathbf{k}}-i\psi^{(2)}_{\mathbf{k}})\tag{2.25}$$

$$\psi^-_{-1,\mathbf{k}}=\frac{1}{\sqrt2}(\bar\psi^{(1)}_{\mathbf{k}}+i\bar\psi^{(2)}_{\mathbf{k}})\quad\psi^+_{-1,-\mathbf{k}}=\frac{1}{\sqrt2}(\bar\psi^{(1)}_{\mathbf{k}}-i\bar\psi^{(2)}_{\mathbf{k}})$$

which in coordinate space is

$$\psi^\mp_{1,\mathbf{x}}=\frac{1}{\sqrt2}(\psi^{(1)}_{\mathbf{x}}\pm i\psi^{(2)}_{\mathbf{x}})\tag{2.26}$$

$$\psi^\mp_{-1,\mathbf{x}}=\frac{1}{\sqrt2}(\bar\psi^{(1)}_{\mathbf{x}}\pm i\bar\psi^{(2)}_{\mathbf{x}})\ .$$

By this change of variables

$$P(d\psi^{(1)})P(d\psi^{(2)})\equiv P(d\psi)=\tag{2.27}$$

$$\mathcal{N}_\psi^{-1}\prod_{\mathbf{k}\in D_{-,-}}\prod_{\omega=\pm1}d\psi^+_{\mathbf{k},\omega}d\psi^-_{\mathbf{k},\omega}\exp[\frac{t}{2M^2}\sum_{\mathbf{k}\in D_{-,-}}\xi^{\mathbf{T}}_{\mathbf{k}}\mathbf{A}(\mathbf{k})\xi^{(+)}{}_{\mathbf{k}}]$$

$$A(\mathbf{k})=\begin{pmatrix}i\sin k+\sin k_0 & -im_\psi(\mathbf{k})\\ im_\psi(\mathbf{k}) & i\sin k-\sin k_0\end{pmatrix}$$

$$\xi^{\mathbf{T}}{}_{\mathbf{k}}=(\psi^-_{\mathbf{k},1},\psi^-_{\mathbf{k},-1})\quad\xi^{+,\mathbf{T}}{}_{\mathbf{k}}=(\psi^+_{\mathbf{k},1},\psi^+_{\mathbf{k},-1})\tag{2.28}$$

Remark. In the physical language, the change of variables (2.26) means that one is describing the system in terms of *Dirac fermions* instead in terms of *Majorana fermions*.

A similar computation can be done for $P(d\chi)$; proceeding exactly as above we find

$$P^{(1)}(d\chi^{(1)})P^{(2)}(d\chi^{(2)})=P(d\chi)\tag{2.29}$$

where

$$P(d\chi)=\mathcal{N}_\chi^{-1}\prod_{\mathbf{k}\in D_{-,-}}\prod_{\omega=\pm1}d\chi^+_{\mathbf{k},\omega}d\chi^-_{\mathbf{k},\omega}\exp[\frac{t}{2M^2}\sum_{\mathbf{k}\in D_{-,-}}\eta^{\mathbf{T}}_{\mathbf{k}}\mathbf{B}(\mathbf{k})\eta^{(+)}{}_{\mathbf{k}}] \tag{2.30}$$

$$B(\mathbf{k})=\begin{pmatrix} i\sin k+\sin k_0 & -im_\chi(k)\\ im_\chi(k) & i\sin k-\sin k_0\end{pmatrix}$$

$$\eta^{\mathbf{T}}{}_{\mathbf{k}}=(\chi^-_{\mathbf{k},1},\chi^-_{\mathbf{k},-1})\quad \eta^{+,\mathbf{T}}{}_{\mathbf{k}}=(\chi^+_{\mathbf{k},1},\chi^+_{\mathbf{k},-1})\ . \tag{2.31}$$

Note that $\frac{t}{2}m_\psi(\mathbf{k})=(\sqrt{2}+1+t)+\frac{t}{2}(\cos k_0+\cos k-2)$, and the mass of the χ fields never vanishes. It holds that

$$\int P(d\chi)\,\chi^-_{\mathbf{x},\omega}\chi^+_{\mathbf{y},\omega'}=g^{(\chi)}_{\omega,\omega'}(\mathbf{x}-\mathbf{y})\ , \tag{2.32}$$

where

$$g^{(\chi)}_{\omega,\omega'}(\mathbf{x}-\mathbf{y})=\frac{2}{tM^2}\sum_{\mathbf{k}}e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}[B^{-1}(\mathbf{k})]_{\omega,\omega'}\ , \tag{2.33}$$

and $B^{-1}(\mathbf{k})$ is the inverse of the $B(\mathbf{k})$ defined in (2.30).

If we set

$$A(\mathbf{k})=\det B(\mathbf{k})=-\sin^2 k_0-\sin^2 k-[m^\chi(\mathbf{k})]^2\ , \tag{2.34}$$

then

$$B^{-1}(\mathbf{k})=\frac{1}{A(\mathbf{k})}\begin{pmatrix} -\sin k_0+i\sin k & im^\chi(\mathbf{k})\\ -im^\chi(\mathbf{k}) & \sin k_0+i\sin k\end{pmatrix}\ . \tag{2.35}$$

Similar formulas hold for $g^{(\psi)}(\mathbf{x}-\mathbf{y})$ and, if $i=\psi,\chi$, the following bounds holds, for any $N>1$ and $m_i=m_i(\mathbf{0})$

$$|g^{(i)}_{\omega,\omega}(\mathbf{x}-\mathbf{y})|\le\frac{1}{1+|\mathbf{d}_M(\mathbf{x}-\mathbf{y})|}\frac{C_N}{1+|m_i\mathbf{d}_M(\mathbf{x}-\mathbf{y})|^N} \tag{2.36}$$

$$|g^{(i)}_{\omega,-\omega}(\mathbf{x}-\mathbf{y})|\le\frac{(|m_i|+|\mathbf{d}_M(\mathbf{x}-\mathbf{y})|^{-1})C_N}{1+|m_i\mathbf{d}_M(\mathbf{x}-\mathbf{y})|^N} \tag{2.37}$$

where

$$\mathbf{d}_M(\mathbf{x}-\mathbf{y})=(\frac{M}{\pi}\sin(\frac{\pi(x-y)}{M}),\frac{M}{\pi}\sin(\frac{\pi(x_0-y_0)}{M})\ . \tag{2.38}$$

Note that the following properties hold, for $i=\psi,\chi$

$$g^{(i)}_{\omega,\omega}(\mathbf{x})=-g^{(i)}_{\omega,\omega}(-\mathbf{x})\qquad g^{(i)}_{\omega,-\omega}(\mathbf{x})=g^{(i)}_{\omega,-\omega}(-\mathbf{x})\ . \tag{2.39}$$

The same change of variables are done for the term $e^{S^\nu_{\varepsilon,\varepsilon'}}$ which can be written as the product of three terms similar to (2.17), (2.19) and (2.20) with t replaced by ν and $\sqrt{2}-1$ in (2.17) and $\sqrt{2}+1$ in (2.19) replaced by 0; then (2.17),(2.19) can be written in terms of Dirac fermions as (2.28) and (2.31) with t replaced by ν and $\sqrt{2}-1$ or $\sqrt{2}+1$ replaced by 0.

At the end we have obtained the following expression

$$Z^{-,-,-,-}_{2I}=\mathcal{N}\int P(d\psi)P(d\chi)e^{Q-\mathcal{V}} \tag{2.40}.$$

2.4 Note that $\mathcal{V}$ (2.13) is an expression linear or bilinear in $\bar{H}^{(\alpha)}_{\mathbf{x}}H^{(\alpha)}_{x+1,x_0}$ or $\bar{V}^{(\alpha)}_{\mathbf{x}}V^{(\alpha)}_{x,x_0+1}$. ¿From (2.16) it holds

$$\bar{V}^{(\alpha)}_{x,x_0}V^{(\alpha)}_{x,x_0+1}=Q^{1(\alpha)}_{\mathbf{x}}+Q^{2(\alpha)}_{\mathbf{x}}+Q^{3(\alpha)}_{\mathbf{x}} \tag{2.41}$$

where

$$Q^{1(\alpha)}_{\mathbf{x}}=\frac{1}{4i}[\psi^{(\alpha)}_{x,x_0}\psi^{(\alpha)}_{x,x_0+1}-\bar{\psi}^{(\alpha)}_{x,x_0}\bar{\psi}^{(\alpha)}_{x,x_0+1}+\bar{\psi}^{(\alpha)}_{x,x_0}\psi^{(\alpha)}_{x,x_0+1}-\psi^{(\alpha)}_{x,x_0}\bar{\psi}^{(\alpha)}_{x,x_0+1}]$$

$$Q^{2(\alpha)}_{\mathbf{x}} = \frac{1}{4i}[\chi^{(\alpha)}_{x,x_0}\chi^{(\alpha)}_{x,x_0+1} - \bar\chi^{(\alpha)}_{x,x_0}\bar\chi^{(\alpha)}_{x,x_0+1} + \bar\chi^{(\alpha)}_{x,x_0}\chi^{(\alpha)}_{x,x_0+1} - \chi^{(\alpha)}_{x,x_0}\bar\chi^{(\alpha)}_{x,x_0+1}] \tag{2.42}$$

$$Q^{3(\alpha)}_{\mathbf{x}} = \frac{1}{4i}[\psi^{(\alpha)}_{x,x_0}\chi^{(\alpha)}_{x,x_0+1} - \bar\psi^{(\alpha)}_{x,x_0}\bar\chi^{(\alpha)}_{x,x_0+1} - \psi^{(\alpha)}_{x,x_0}\bar\chi^{(\alpha)}_{x,x_0+1} + \bar\psi^{(\alpha)}_{x,x_0}\chi^{(\alpha)}_{x,x_0+1} +$$
$$\chi^{(\alpha)}_{x,x_0}\psi^{(\alpha)}_{x,x_0+1} - \bar\chi^{(\alpha)}_{x,x_0}\bar\psi^{(\alpha)}_{x,x_0+1} - \chi^{(\alpha)}_{x,x_0}\bar\psi^{(\alpha)}_{x,x_0+1} + \bar\chi^{(\alpha)}_{x,x_0}\psi^{(\alpha)}_{x,x_0+1}]$$

A similar expression hold for $\bar H^{(\alpha)}_{x,x_0}H^{(\alpha)}_{x+1,x_0}$. Looking for instance to the first of (2.42) one finds

$$\psi^{(\alpha)}_{x,x_0}\psi^{(\alpha)}_{x,x_0+1} - \bar\psi^{(\alpha)}_{x,x_0}\bar\psi^{(\alpha)}_{x,x_0+1} = \psi^{(\alpha)}_{x,x_0}\partial_{x_0}\psi^{(\alpha)}_{x,x_0} - \bar\psi^{(\alpha)}_{x,x_0}\partial_{x_0}\bar\psi^{(\alpha)}_{x,x_0} \tag{2.43}$$

and

$$\bar\psi^{(\alpha)}_{x,x_0}\psi^{(\alpha)}_{x,x_0+1} - \psi^{(\alpha)}_{x,x_0}\bar\psi^{(\alpha)}_{x,x_0+1} = -\partial_{x_0}\bar\psi^{(\alpha)}_{x,x_0}\partial_{x_0}\psi^{(\alpha)}_{x,x_0} + \bar\psi^{(\alpha)}_{x,x_0}\psi^{(\alpha)}_{x,x_0} + \bar\psi^{(\alpha)}_{x,x_0+1}\psi^{(\alpha)}_{x,x_0+1}\,. \tag{2.44}$$

From (2.41) and (2.14) it is easy to verify that $\mathcal{V}$ is sum of terms of the form $\sum_{\mathbf{x}} A^{\varepsilon_1,\varepsilon_2}_{\mathbf{x};\phi,\omega_1;\phi',\omega_2}$ or $\sum_{\mathbf{x}} A^{\varepsilon_1,\varepsilon_2}_{\mathbf{x};\phi,\omega_1;\phi',\omega_2} A^{\varepsilon_1',\varepsilon_2'}_{\mathbf{x}';\phi'',\omega_1';\phi''',\omega_2'}$ where $\mathbf{x}'=\mathbf{x}$ or $\mathbf{x}'=(x-1,x_0+1)$ with $\phi\in\{\psi,\chi\}$, $\varepsilon=\pm$ and:
1)if $\omega_1=\omega_2$ then $A^{\varepsilon_1,\varepsilon_2}_{\mathbf{x};\phi,\omega;\phi',\omega}$ is equal to $a_{\varepsilon_1,\varepsilon_2,\omega,\alpha}\phi^{\varepsilon_1}_{\omega,\mathbf{x}}\partial_{x_\alpha}\phi'^{\varepsilon_2}_{\omega,\mathbf{x}}$ with $\alpha=1,2$ and:
1a)if $\alpha=1$ $\partial_{x_\alpha}=\partial_{x_0}$ and $a_{\varepsilon_1,\varepsilon_2,\omega,1}$ is imaginary;
1a)if $\alpha=2$ $\partial_{x_\alpha}=\partial_x$ and $a_{\varepsilon_1,\varepsilon_2,\omega,2}$ is real.

2)if $\omega_1=-\omega_2$ then $A^{\varepsilon_1,\varepsilon_2}_{\mathbf{x};\phi,\omega;\phi',-\omega}$ is equal to:
2a)$ib_{\varepsilon_1,\varepsilon_2,\omega,\beta}\partial_{x_\beta}\phi^{\varepsilon_1}_{\omega,\mathbf{x}}\partial_{x_\beta}\phi'^{\varepsilon_2}_{-\omega,\mathbf{x}}$, with $\beta=1,2$ and $\partial_{x_\beta}=\partial_{x_0}$ if $\beta=1$, while $\partial_{x_\beta}=\partial_x$ if $\beta=2$;
2b)or it is equal to $ic_{\varepsilon_1,\varepsilon_2,\omega,\gamma}\phi^{\varepsilon_1}_{\omega,\mathbf{x}_\gamma}\phi'^{\varepsilon_2}_{-\omega,\mathbf{x}_\gamma}$ with $\gamma=1,2,3$ and $\mathbf{x}_\gamma=\mathbf{x}$ if $\gamma=1$; $\mathbf{x}_\gamma=(x+1,x_0)$ if $\gamma=2$; $\mathbf{x}_\gamma=(x,x_0+1)$ if $\gamma=3$. The coefficients $b_{\varepsilon_1,\varepsilon_2,\omega,\beta}$, $c_{\varepsilon_1,\varepsilon_2,\omega,\gamma}$ are real.

Remark. The χ-fields will be called *heavy* fields and the ψ-fields will be called *light* fields. In the next section we will integrate out the χ fields (no multiscale analysis will be necessary as they have an $O(1)$ mass, hence no infrared problem is present) so obtaining a Grassman integral expressed only in the ψ fields which in a sense are the critical modes.

3. Integration of heavy fermions

3.1 *Local interactions.* By the change of variables in the preceding section we can write (2.12) as

$$Z^{-,-,-,-}_{2I} = \mathcal{N}\int P(d\psi)\int P(d\chi)e^{Q(\chi,\psi)}e^{-\mathcal{V}(\psi,\chi)} \tag{3.1}$$

where $\mathcal{N}$ is a normalization constant, $P(\psi)$ and $P(d\chi)$ are given by (2.27), (2.30), $\mathcal{V}$ by (2.13) and Q, $\tilde Q_{\underline{\varepsilon}}$ are obtained respectively from (2.20) and by the change of variables (2.26). We now integrate the heavy χ fields

$$\int P(d\psi)\int P(d\chi)e^{Q(\chi,\psi)}e^{-\mathcal{V}(\psi,\chi)} = \mathcal{N}\int \bar P(d\psi)e^{M^2\mathcal{N}^{(1)}-\mathcal{V}^{(1)}(\psi)} \tag{3.2}$$

where $\mathcal{N}^{(1)}$ is a constant. We prove the following result.

3.2 Theorem 2. *There exists an ε such that, for $|\lambda|,|\nu|\le\varepsilon$*

$$\mathcal{V}^{(1)} = \sum_{n\ge1}\sum_{\underline{\alpha},\underline{\omega}}\sum_{\mathbf{x}_1,..,\mathbf{x}_{2n}} W_{\underline{\alpha},\underline{\omega},n}(\mathbf{x}_1,..,\mathbf{x}_{2n})\partial^{\alpha_1}\psi^{\varepsilon_1}_{\mathbf{x}_1,\omega_1}...\partial^{\alpha_{2n}}\psi^{\varepsilon_{2n}}_{\mathbf{x}_{2n},\omega_{2n}} \tag{3.3}$$

and, for $n \geq 2$

$$|\hat{W}_{\underline{\alpha},\underline{\omega},n}(\mathbf{k}_1,...\mathbf{k}_{n-1})| \leq M^2 C^n \varepsilon^{n/2} \tag{3.4}$$

The addenda in (3.3) with $n=2$ can be written as

$$l_1 \sum_{\mathbf{x}} c_{\mathbf{x},M} \psi^+_{1,\mathbf{x}} \psi^+_{-1,\mathbf{x}} \psi^-_{-1,\mathbf{x}} \psi^-_{1,\mathbf{x}} + \tag{3.5}$$

$$\sum_{\mathbf{x}_1,..,\mathbf{x}_4} \sum_{\alpha_1+..\alpha_4\geq 1} W_{\underline{\alpha},\underline{\omega},2}(\mathbf{x}_1,..,\mathbf{x}_4) \partial^{\alpha_1}\psi^{\varepsilon_1}_{\mathbf{x}_1,\omega_1} \partial^{\alpha_2}\psi^{\varepsilon_2}_{\mathbf{x}_2,\omega_2} \partial^{\alpha_3}\psi^{\varepsilon_3}_{\mathbf{x}_3,\omega_3} \partial^{\alpha_4}\psi^{\varepsilon_4}_{\mathbf{x}_4,\omega_4}$$

with $\lim_{M\to\infty} c_{\mathbf{x},M} = 1$, $l_1 = \lambda(a+b)\mathrm{sech}^4 J + O(\varepsilon^2)$ and $l_1 \in \mathbb{R}$. The addend with $n=1$ in (3.3) can be written as

$$\sum_\omega \sum_{\mathbf{x}} \tilde{c}_{\mathbf{x},M} [i\omega\nu_1 \psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},-\omega} + \psi^+_{\mathbf{x},\omega}(i\omega a_1\partial_0 + a_2\partial_1)\psi^-_{\mathbf{x},\omega}] \tag{3.6}$$

$$+ \sum_{\mathbf{x}_1,\mathbf{x}_2} \sum_{\{\omega\}} \sum_{\alpha_1+\alpha_2\geq 2} W_{\underline{\alpha},\underline{\omega},1,a}(\mathbf{x}_1,\mathbf{x}_2) \partial^{\alpha_1}\psi^{\varepsilon_1}_{\mathbf{x}_1,\omega_1} \partial^{\alpha_2}\psi^{\varepsilon_2}_{\mathbf{x}_2,\omega_2}$$

with $\lim_{M\to\infty} \tilde{c}_{\mathbf{x},M} = 1$, $\nu_1 = \nu + O(\varepsilon)$, $a_1, a_2 = O(\varepsilon)$, $\nu_1, a_1, a_2 \in \mathbb{R}$ and $|\hat{W}_{\underline{\alpha},\underline{\omega},1,a}(\mathbf{k}_1)| \leq M^2 C|\varepsilon|$. Finally

$$\bar{P}(d\psi) = \mathcal{N}^{-1} \prod_{\mathbf{k}\in D_{-,-}} \prod_{\omega=\pm 1} d\psi^+_{\mathbf{k},\omega} d\psi^-_{\mathbf{k},\omega} \exp[-\frac{t}{M^2} \sum_{\mathbf{k}\in D_{-,-}} \psi^+_{\mathbf{k},\omega} A_{\omega,\omega'}(\mathbf{k}) \psi^-_{\mathbf{k},\omega'}] \tag{3.7}$$

with

$$A(\mathbf{k}) = \frac{1}{C_0 + f_{0,0}(\mathbf{k})} \begin{pmatrix} \tilde{Z}_1(i\sin k + \sin k_0) + f_{1,1}(\mathbf{k}) & -i(t_c - t)C_0 - i f_{1,2}(\mathbf{k}) \\ i(t_c - t)C_0 + i f_{1,2}(\mathbf{k}) & \tilde{Z}_1(i\sin k - \sin k_0) + f_{2,2}(\mathbf{k}) \end{pmatrix}$$

with $C_0 = (t+1+\sqrt{2})^2$ and $\tilde{Z}_1 = \frac{t}{2}[(2t+2\sqrt{2}t)+(2\sqrt{2}+3+t^2)]$ and $f_{i,j}(\mathbf{k})$ analytic $O(\mathbf{k}^2)$ functions, $f_{i,i}(\mathbf{k})$, $i=0,1,2$ even and $f_{1,2}(\mathbf{k})$ odd and real; moreover $C_{0,0} + f_{0,0} \geq 1$ and $\det\mathrm{A}(\mathbf{k}) \geq \det\mathrm{A}(\mathbf{0})$.

Proof. We start from the definition of truncated expectation:

$$\mathcal{E}^T_\chi(X;n) = \frac{\partial^n}{\partial\lambda^n} \log \int P(d\chi) e^{\lambda X(\chi)}|_{\lambda=0} \tag{3.8}$$

so that, calling

$$\bar{\mathcal{V}}(\chi,\psi) = -Q(\chi,\psi) + \mathcal{V}(\chi,\psi) \tag{3.9}$$

we obtain

$$M^2\mathcal{N}^{(1)} - \mathcal{V}^{(1)}(\psi) = \log \int P(d\chi) e^{-\bar{\mathcal{V}}(\chi,\psi)} = \sum_{n=0}^\infty \frac{(-1)^n}{n!} \mathcal{E}^T_\chi(V(.,\psi;n)) \tag{3.10}$$

We label each one of the monomials (whose number will be called $\tilde{C}_0$) in $\bar{\mathcal{V}}$ by an index v_i, so that each monomial can be written as

$$\sum_{\mathbf{x}_{v_i}} v(\mathbf{x}_{v_i}) \prod_{f\in\tilde{P}_{v_i}} \partial^{\alpha(f)}\psi^{\varepsilon(f)}_{\omega(f),\mathbf{x}(f)} \prod_{f\in P_{v_i}} \partial^{\alpha(f)}\chi^{\varepsilon(f)}_{\omega(f),\mathbf{x}(f)} \tag{3.11}$$

where $\mathbf{x}_{v_i}$ is the total set of coordinates associated to v_i and P_{v_i} and $\tilde{P}_{v_i}$ are set of indices labeling the χ or ψ-fields. We can write

$$\mathcal{V}^{(1)}(\psi) = \sum_{\tilde{P}_{v_0}\neq 0} \mathcal{V}^{(1)}(\tilde{P}_{v_0}) \;, \tag{3.12}$$

$$\mathcal{V}^{(1)}(\tilde P_{v_0})=\sum_{\mathbf{x}_{v_0}}[\prod_{f\in\tilde P_{v_0}}\partial^{\alpha(f)}\psi^{\varepsilon(f)}_{\omega(f),\mathbf{x}(f)}]K_{\tilde P_{v_0}}(\mathbf{x}_{v_0})\;, \tag{3.13}$$

$$K_{\tilde P_{v_0}}(\mathbf{x}_{v_0})=\sum_{n=1}^\infty\frac{1}{n!}\sum_{v_1,..,v_n}\mathcal{E}^T_\chi[\tilde\chi(P_{v_1}),\ldots,\tilde\chi(P_{v_n})]\prod_{i=1}^n v_i(\mathbf{x}_{v_i})\;, \tag{3.14}$$

where $\sum_{v_1,\ldots v_n}\le\tilde C_0^n$, $\tilde P_{v_0}=\bigcup_i\tilde P_{v_i}$ and $\mathbf{x}_{v_0}=\bigcup_i\mathbf{x}_{v_i}$. We use now the well known expression for $\mathcal{E}^T_\chi$ (see for instance [Le])

$$\mathcal{E}^T_\chi(\tilde\chi(P_1),...,\tilde\chi(P_s))=\sum_T\prod_{l\in T}g^{(\chi)}_{\omega^-,\omega^+}(\mathbf{x}_l-\mathbf{y}_l)\int dP_T(\mathbf{t})\det G^T(\mathbf{t}) \tag{3.15}$$

where:

a)P is a set of indices, and $\tilde\chi(P)=\prod_{f\in P}\partial^{\alpha(f)}\chi^{\varepsilon(f)}_{\mathbf{x}(f),\omega(f)}$.

b)T is a set of lines forming an *anchored tree* between the cluster of poins $P_1,..,P_s$ *i.e.* T is a set of lines which becomes a tree if one identifies all the points in the same clusters.

c)$\mathbf{t}=\{t_{i,i'}\in[0,1],1\le i,i'\le s\}$, $dP_T(\mathbf{t})$ is a probability measure with support on a set of $\mathbf{t}$ such that $t_{i,i'}=\mathbf{u}_i\cdot\mathbf{u}_{i'}$ for some family of vectors $\mathbf{u}_i\in\mathbb{R}^s$ of unit norm.

d)$G^T(\mathbf{t})$ is a $(n-s+1)\times(n-s+1)$ matrix, whose elements are given by $G^T_{ij,i'j'}=t_{i,i'}g_{\omega^-,\omega^+}(\mathbf{x}_{ij}-\mathbf{y}_{i'j'})$ with $(f^-_{ij},f^+_{i'j'})$ not belonging to T.

If $s=1$ the sum over T is empty, but we can still use the above equation by interpreting the r.h.s. as 1 if P_1 is empty, and $\det G(P_1)$ otherwise.
We bound the determinant using the well known *Gram-Hadamard inequality* finding

$$|G^T_{ij,i'j'}|\le C_1^n \tag{3.16}$$

where C_1 is an $O(1)$ constant. Finally we get

$$\sum_{\mathbf{x}_{v_0}}|K_{\tilde P_{v_0}}(\mathbf{x}_{v_0})|\le\sum_{n=1}^\infty\frac{1}{n!}\sum_{v_1,..,v_n}\sum_{\mathbf{x}_{v_1},\ldots,\mathbf{x}_{v_n}}C_1^n\sum_T[\prod_{l\in T}|g^{(\chi)}_{\omega^-,\omega^+}(\mathbf{x}_l-\mathbf{y}_l)|]\prod_{i=1}^n|v_i(\mathbf{x}_{v_i})| \tag{3.17}$$

where we have used that $\int dP_T(\mathbf{t})=1$. The number of addenda in $\sum_T$ is bounded by $n!C_2^n$. Finally T and the $\bigcup_i\mathbf{x}_{v_i}$ form a tree connecting all points, so that using that the propagator is massive and that the interactions are short ranged $\sum_{\mathbf{x}_{v_1},\ldots\mathbf{x}_{v_n}}\sum_T[\prod_{l\in T}|g^{(\chi)}_{\omega^-,\omega^+}(\mathbf{x}_l-\mathbf{y}_l)|]\prod_{i=1}^n|v_i(\mathbf{x}_{v_i})|\le C_3^n|\varepsilon|^{\tilde n}M^2$, where $\tilde n$ is the number of couplings $O(\varepsilon)$.
Let us consider the case $|\tilde P_{v_0}|\ge 4$. Note that if to v_i are associated only terms from $\mathcal{V}(\psi,\chi)$, then $\tilde n=n$. Let us consider now the case in which there are end-points associated to $Q(\psi,\chi)$, which have $O(1)$ coupling; there are at most $|\tilde P_{v_0}|$ end-points associated with $Q(\psi,\chi)$. In fact in $Q(\psi,\chi)$ there are only terms of the form $\psi\chi$, so at most the number of them is equal to the number of ψ fields. If we call $n_\lambda\le\tilde n$ the number of vertices quartic in the fields it is clear that $n_\lambda\ge\max\{1,|\tilde P_{v_0}|/2-1\}$; hence

$$\sum_{\mathbf{x}_{v_0}}|K_{\tilde P_{v_0}}(\mathbf{x}_{v_0})|\le M^2\sum_{\tilde n=1}^\infty C^{\tilde n+|\tilde P_{v_0}|}|\varepsilon|^{\frac{\tilde n}{2}}|\lambda|^{\max\{\frac12,|\tilde P_{v_0}|/4-1/2\}} \tag{3.18}$$

and (3.4) holds for $|\tilde P_{v_0}|\ge 4$.

Consider now the case $|\tilde P_{v_0}|=2$; in this case there are terms $O(1)$, obtained when to all the v_i are associated with elements of $Q(\psi,\chi)$. It is convenient to include all such terms in the gaussian integration, as they cannot be considered as perturbations (they are not $O(\varepsilon)$). Hence we define

$$\mathcal{N}\int \bar P(d\psi)=\int P(d\psi)\int P(d\chi)e^{Q(\psi,\chi)} \tag{3.19}$$

and, if $<X>_0=\int \bar P(d\psi)X$, it holds $<\psi^-_{\mathbf{x},1}\psi^+_{\mathbf{y},1}>_0=<\psi^{(1)}_{\mathbf{x}}\psi^{(1)}_{\mathbf{y}}>_0$, $<\psi^-_{\mathbf{x},-1}\psi^+_{\mathbf{y},-1}>_0$ $=<\bar\psi^{(1)}_{\mathbf{x}}\bar\psi^{(1)}_{\mathbf{y}}>_0$ and $<\psi^-_{\mathbf{x},1}\psi^+_{\mathbf{y},-1}>_0=<\psi^{(1)}_{\mathbf{x}}\bar\psi^{(1)}_{\mathbf{y}}>_0$. By using the explicit expressions for $<\psi^{(1)}_{\mathbf{x}}\psi^{(1)}_{\mathbf{y}}>_0$, $<\bar\psi^{(1)}_{\mathbf{x}}\bar\psi^{(1)}_{\mathbf{y}}>_0$, $<\psi^{(1)}_{\mathbf{x}}\bar\psi^{(1)}_{\mathbf{y}}>_0$ in [MPW], (3.7) follows.

The fact that $l_1=\lambda(a+b)\mathrm{sech}^4 J+O(\varepsilon^2)$ can be checked by an explicit computations of all the contributions with coupling $O(\lambda)$ to W_2, noting that they can be only obtained contracting a terms quartic in the χ fields with one of the addenda of (2.20); each of such terms carries a derivative in the coordinate space, hence the Fourier transform of such terms is vanishing at zero momentum.

3.3 To complete the proof of the theorem we need a more detailed analysis of the bilinear and quartic terms in the r.h.s of (3.3). In fact many terms bilinear or quartic in the fields are forbidden by the invariance under some symmetry transformations (for simplicity, the analysis is done in the $M\to\infty$ limit).

- There are no local terms in the r.h.s. of (3.3) of the form $\psi^+_{\mathbf{x},1}\psi^-_{\mathbf{x},1}$; such local terms can be written as $\psi^{(1)}_{\mathbf{x}}\psi^{(2)}_{\mathbf{x}}$, but the model is invariant under the transformation

$$\begin{aligned}\psi^{(1)},\bar\psi^{(1)},\chi^{(1)},\bar\chi^{(1)}&\to-\psi^{(1)},-\bar\psi^{(1)},-\chi^{(1)},-\bar\chi^{(1)}\\ \psi^{(2)},\bar\psi^{(2)},\chi^{(2)},\bar\chi^{(2)}&\to\psi^{(2)},\bar\psi^{(2)},\chi^{(2)},\bar\chi^{(2)},\end{aligned} \tag{3.20}$$

hence such terms cannot be present as they violate such symmetry.

- There are no terms in the r.h.s. of (3.3) of the form $\psi_{1,\mathbf{x}}\psi_{-1,\mathbf{x}'}$ or $\psi^+_{1,\mathbf{x}}\psi^+_{-1,\mathbf{x}'}$; in fact,

$$\psi_{1,\mathbf{x}}\psi_{-1,\mathbf{x}'}=\frac{1}{2}[\psi^{(1)}_{\mathbf{x}}\bar\psi^{(1)}_{\mathbf{x}'}-\psi^{(2)}_{\mathbf{x}}\bar\psi^{(2)}_{\mathbf{x}'}+i\psi^{(1)}_{\mathbf{x}}\bar\psi^{(2)}_{\mathbf{x}'}+i\psi^{(2)}_{\mathbf{x}}\bar\psi^{(1)}_{\mathbf{x}'}] \tag{3.21}$$

and the last two terms violates the symmetry (3.20); moreover the first two terms are *odd* in the exchange $(1),(2)\to(2),(1)$ and the model is invariant in the exchange $(1),(2)\to(2),(1)$.

- the model is invariant under complex conjugation and the exchange

$$\psi^{(\alpha)}_{\mathbf{x}},\bar\psi^{(\alpha)}_{\mathbf{x}}\to\bar\psi^{(\alpha)}_{\mathbf{x}},\psi^{(\alpha)}_{\mathbf{x}}\quad \chi^{(\alpha)}_{\mathbf{x}},\bar\chi^{(\alpha)}_{\mathbf{x}}\to\bar\chi^{(\alpha)}_{\mathbf{x}},\chi^{(\alpha)}_{\mathbf{x}}; \tag{3.22}$$

this follows from the fact that, from (2.16), $\bar H^{(\alpha)},H^{(\alpha)},\bar V^{(\alpha)},V^{(\alpha)}$, written in terms of $\bar\psi^{(\alpha)},\psi^{(\alpha)}$, $\bar\chi^{(\alpha)},\chi^{(\alpha)}$, are invariant under such transformation. Hence the coefficient of the local part of the quartic (non vanishing) terms is real; in fact $\hat w(0,0,0)\psi^+_{1,\mathbf{x}}\psi_{1,\mathbf{x}}\psi^+_{-1,\mathbf{x}}\psi_{-1,\mathbf{x}}=\hat w(0,0,0)\psi^{(1)}_{\mathbf{x}}\bar\psi^{(1)}_{\mathbf{x}}\psi^{(2)}_{\mathbf{x}}\bar\psi^{(1)}_{\mathbf{x}}$ must be equal, by the above invariance, to $\hat w^*(0,0,0)\bar\psi^{(1)}_{\mathbf{x}}\psi^{(1)}_{\mathbf{x}}\bar\psi^{(2)}_{\mathbf{x}}\psi^{(1)}_{\mathbf{x}}$, hence $\hat w(0,0,0)=\hat w^*(0,0,0)$. Finally the combination of local terms $\psi^+_{\mathbf{x},1}\psi^-_{\mathbf{x},-1}+\psi^+_{\mathbf{x},-1}\psi^-_{\mathbf{x},1}$ is equal to $\frac{1}{2}[\psi^{(1)}_{\mathbf{x}}\bar\psi^{(2)}_{\mathbf{x}}-\psi^{(2)}_{\mathbf{x}}\bar\psi^{(1)}_{\mathbf{x}}]$ so it cannot be present as it violates the symmetry (3.20). On the other hand $\psi^+_{\mathbf{x},1}\psi^-_{\mathbf{x},-1}-\psi^+_{\mathbf{x},-1}\psi^-_{\mathbf{x},1}$ is equal to $\frac{1}{2}[\psi^{(1)}_{\mathbf{x}}\bar\psi^{(1)}_{\mathbf{x}}+\psi^{(2)}_{\mathbf{x}}\bar\psi^{(2)}_{\mathbf{x}}]$; hence the coefficient of the local part is imaginary; in fact $\hat w(0)[\psi^{(1)}_{\mathbf{x}}\bar\psi^{(1)}_{\mathbf{x}}+\psi^{(2)}_{\mathbf{x}}\bar\psi^{(2)}_{\mathbf{x}}]$ must be equal to $\hat w^*(0)[\bar\psi^{(1)}_{\mathbf{x}}\psi^{(1)}_{\mathbf{x}}+\bar\psi^{(2)}_{\mathbf{x}}\bar\psi^{(2)}_{\mathbf{x}}]$, by the invariance under complex conjugation and (3.22), hence $\hat w(0)=-\hat w^*(0)$.

3.4 We consider now the addenda with $n=1$ in the rh.s. of (3.3)

$$\frac{1}{M^2}\sum_{\mathbf{k}}\hat W_{\omega_1,\omega_2}(\mathbf{k})\psi^+_{\mathbf{k},\omega_1}\psi^-_{\mathbf{k},\omega_2} \tag{3.23}$$

We can represent $W_{\omega_1,\omega_2}(\mathbf{x})$ as sum over *Feynman graphs* g in the usual way (see for instance [GM]); the external lines are associated to the ψ fields, and to the internal lines are associated the propagators $g^{\chi}_{\omega,\omega'}(\mathbf{x}-\mathbf{y})$; moreover the vertices associated to the interaction are linear or bilinear in $A^{\varepsilon_1,\varepsilon_2}_{\mathbf{x};\phi,\omega_1;\phi',\omega_2}$.

- We show that

$$\partial_k \hat{W}_{\omega,-\omega}(\mathbf{k})|_{\mathbf{k}=0} = \partial_{k_0} \hat{W}_{\omega,-\omega}(\mathbf{k})|_{\mathbf{k}=0} = 0\ . \tag{3.24}$$

Note that $\partial_k \hat{W}_{\omega,-\omega}(\mathbf{k})|_{\mathbf{k}=0} = \sum_{\mathbf{x}} W_{\omega,-\omega}(\mathbf{x})x$ and we can consider a single Feynman diagram value $\hat{W}^g_{\omega,-\omega}(\mathbf{k})$ and we call
1) n^ω_a is the number in g of terms $A^{\varepsilon_1,\varepsilon_2}_{\mathbf{x};\phi,\omega_1;\phi',\omega_2}$ with $\omega_1=\omega_2$; $n^1_a+n^{-1}_a=n_a$.
2) n_b is the number of $A^{\varepsilon_1,\varepsilon_2}_{\mathbf{x};\phi,\omega_1;\phi',\omega_2}$ with $\omega_1=-\omega_2$.
3)n^ω_+ is the number of diagonal propagators $g^{(\chi)}_{\omega,\omega}$.
4)n_- is then number of non diagonal propagators $g^{(\chi)}_{\omega,-\omega}$.
If we make the transformation $\mathbf{x}_i\to-\mathbf{x}_i$ in all the sums in $\sum_{\mathbf{x}} W^g_{\omega,-\omega}(\mathbf{x})x$ and we use (2.39) then in each Feynman graphs each propagator $g^{(\chi)}_{\omega,\omega'}(\mathbf{x})$ is replaced by $(-1)^{\delta_{\omega,\omega'}} g^{(\chi)}_{\omega,\omega'}(\mathbf{x})$. Moreover the propagators $\partial g^{(\chi)}_{\omega,\omega'}(\mathbf{x})$ are replaced by $(-1)^{\delta_{\omega,\omega'}+1}\tilde{\partial} g^{(\chi)}_{\omega,\omega'}(\mathbf{x})$, where $\tilde{\partial}_{x_0} f_{x,x_0} = f_{x,x_0}-f_{x,x_0-1}$ and $\tilde{\partial}_x f_{x,x_0} = f_{x,x_0}-f_{x-1,x_0}$; finally $g^{(\chi)}_{\omega,\omega'}(\mathbf{x}+\mathbf{a})$ is replaced by $(-1)^{\delta_{\omega,\omega'}} g^{(\chi)}_{\omega,\omega'}(\mathbf{x}-\mathbf{a})$, if $\mathbf{a}$ is a constant vector.
On the other hand we could equivalently write the interaction (1.3) as

$$V(\sigma^{(1)},\sigma^{(2)}) = -\sum_{x,x_0=1}^{M} \lambda a[\sigma^{(1)}_{x-1,x_0}\sigma^{(1)}_{x,x_0}\sigma^{(2)}_{x-1,x_0}\sigma^{(2)}_{x,x_0} + \sigma^{(1)}_{x,x_0-1}\sigma^{(1)}_{x,x_0}\sigma^{(2)}_{x,x_0-1}\sigma^{(2)}_{x,x_0}] \tag{3.25}$$

$$+\lambda b[\sigma^{(1)}_{x-1,x_0}\sigma^{(1)}_{x,x_0}\sigma^{(2)}_{x,x_0-1}\sigma^{(2)}_{x,x_0} + \sigma^{(1)}_{x,x_0-1}\sigma^{(1)}_{x,x_0}\sigma^{(2)}_{x,x_0-1}\sigma^{(2)}_{x+1,x_0-1}]\}\ ;$$

(3.25) can be found from (1.3) making the change of variables $\mathbf{x}\to-\mathbf{x}$, and then making the transformation $\sigma^{(\alpha)}_{\mathbf{x}}\to\sigma^{(\alpha)}_{-\mathbf{x}}$. Starting from this expression and repeating the computations in §2, §3 we get an expression similar to (2.12), where $\mathcal{V}$ is now an expression linear or bilinear in $\bar{H}^{(\alpha)}_{x-1,x_0}H^{(\alpha)}_{x,x_0}$ or $\bar{V}^{(\alpha)}_{x,x_0-1}V^{(\alpha)}_{x,x_0}$. From (2.16) it holds

$$\bar{V}^{(\alpha)}_{x,x_0-1}V^{(\alpha)}_{x,x_0} = \tilde{Q}^{1(\alpha)}_{\mathbf{x}} + \tilde{Q}^{2(\alpha)}_{\mathbf{x}} + \tilde{Q}^{3(\alpha)}_{\mathbf{x}} \tag{3.26}$$

where

$$\begin{aligned}
\tilde{Q}^{1(\alpha)}_{\mathbf{x}} &= \frac{1}{4i}[\psi^{(\alpha)}_{x,x_0-1}\psi^{(\alpha)}_{x,x_0} - \bar{\psi}^{(\alpha)}_{x,x_0-1}\bar{\psi}^{(\alpha)}_{x,x_0} + \bar{\psi}^{(\alpha)}_{x,x_0-1}\psi^{(\alpha)}_{x,x_0} - \psi^{(\alpha)}_{x,x_0-1}\bar{\psi}^{(\alpha)}_{x,x_0}]\\
\tilde{Q}^{2(\alpha)}_{\mathbf{x}} &= \frac{1}{4i}[\chi^{(\alpha)}_{x,x_0-1}\chi^{(\alpha)}_{x,x_0} - \bar{\chi}^{(\alpha)}_{x,x_0-1}\bar{\chi}^{(\alpha)}_{x,x_0} + \bar{\chi}^{(\alpha)}_{x,x_0-1}\chi^{(\alpha)}_{x,x_0} - \chi^{(\alpha)}_{x,x_0-1}\bar{\chi}^{(\alpha)}_{x,x_0}]\\
\tilde{Q}^{3(\alpha)}_{\mathbf{x}} &= \frac{1}{4i}[\psi^{(\alpha)}_{x,x_0-1}\chi^{(\alpha)}_{x,x_0} - \bar{\psi}^{(\alpha)}_{x,x_0-1}\bar{\chi}^{(\alpha)}_{x,x_0} - \psi^{(\alpha)}_{x,x_0-1}\bar{\chi}^{(\alpha)}_{x,x_0} + \bar{\psi}^{(\alpha)}_{x,x_0-1}\chi^{(\alpha)}_{x,x_0}+\\
&\quad \chi^{(\alpha)}_{x,x_0-1}\psi^{(\alpha)}_{x,x_0} - \bar{\chi}^{(\alpha)}_{x,x_0-1}\bar{\psi}^{(\alpha)}_{x,x_0} - \chi^{(\alpha)}_{x,x_0-1}\bar{\psi}^{(\alpha)}_{x,x_0} + \bar{\chi}^{(\alpha)}_{x,x_0-1}\psi^{(\alpha)}_{x,x_0}]
\end{aligned} \tag{3.27}$$

A similar expression hold for $\bar{H}^{(\alpha)}_{x-1,x_0}H^{(\alpha)}_{x,x_0}$. Note that, looking for instance to the first of (3.27)

$$\psi^{(\alpha)}_{x,x_0-1}\psi^{(\alpha)}_{x,x_0} - \bar{\psi}^{(\alpha)}_{x,x_0-1}\bar{\psi}^{(\alpha)}_{x,x_0} = \psi^{(\alpha)}_{x,x_0}\tilde{\partial}_{x_0}\psi^{(\alpha)}_{x,x_0} - \bar{\psi}^{(\alpha)}_{x,x_0}\tilde{\partial}_{x_0}\bar{\psi}^{(\alpha)}_{x,x_0} \tag{3.28}$$

and

$$\bar{\psi}^{(\alpha)}_{x,x_0-1}\psi^{(\alpha)}_{x,x_0} - \psi^{(\alpha)}_{x,x_0-1}\bar{\psi}^{(\alpha)}_{x,x_0} = -\tilde{\partial}_{x_0}\bar{\psi}^{(\alpha)}_{x,x_0}\tilde{\partial}_{x_0}\psi^{(\alpha)}_{x,x_0} + \bar{\psi}^{(\alpha)}_{x,x_0}\psi^{(\alpha)}_{x,x_0} + \bar{\psi}^{(\alpha)}_{x,x_0-1}\psi^{(\alpha)}_{x,x_0-1}\ . \tag{3.29}$$

It is easy to verify that $\mathcal{V}$ is sum of terms of the form $\sum_{\mathbf{x}} \tilde{A}^{\varepsilon_1,\varepsilon_2}_{\mathbf{x};\phi,\omega_1;\phi',\omega_2}$ or $\sum_{\mathbf{x}} \tilde{A}^{\varepsilon_1,\varepsilon_2}_{\mathbf{x};\phi,\omega_1;\phi',\omega_2} \tilde{A}^{\varepsilon'_1,\varepsilon'_2}_{\mathbf{x}';\phi'',\omega'_1;\phi''',\omega'_2}$ where $\mathbf{x}' = \mathbf{x}$ or $\mathbf{x}' = (x+1, x_0-1)$ and $\tilde{A}^{\varepsilon_1,\varepsilon_2}_{\mathbf{x};\phi,\omega_1;\phi',\omega_2}$ is identical to $A^{\varepsilon_1,\varepsilon_2}_{\mathbf{x};\phi,\omega_1;\phi',\omega_2}$ up to the substitutions $\partial \to \tilde{\partial}$, $x+1 \to x-1$ and $x_0+1 \to x_0-1$. Hence it holds

$$\partial_k \hat{W}_{1,-1}(\mathbf{k})|_{\mathbf{k}=0} = (-1)^{n_a+n_++1} \partial_k \hat{W}_{1,-1}(\mathbf{k})|_{\mathbf{k}=0} \ . \tag{3.30}$$

It holds $2n_a + 2n_b = 2(n_+ + n_-) + 2$ so that

$$\partial_k \hat{W}_{1,-1}(\mathbf{k})|_{\mathbf{k}=0} = (-1)^{2n_a+n_b-n_-} \partial_k \hat{W}_{1,-1}(\mathbf{k})|_{\mathbf{k}=0} = (-1)^{n_b-n_-} \partial_k \hat{W}_{1,-1}(\mathbf{k})|_{\mathbf{k}=0} \tag{3.31}$$

The number of fields with $\omega = 1$ is $2n^1_a + n_b$ and the number of external fields with $\omega = 1$ is then $2n^1_a + n_b - 2n^1_+ - n_- = 2(n^1_a - n^1_+) + n_b - n_-$ which implies that $n_b - n_-$ must be an odd number if the number of external fields with $\omega = 1$ is 1. Hence $\partial_k \hat{W}_{1,-1}(\mathbf{k})|_{\mathbf{k}=0} = (-1)\partial_k \hat{W}_{1,-1}(\mathbf{k})|_{\mathbf{k}=0}$ so that $\partial_k \hat{W}_{1,-1}(\mathbf{k})|_{\mathbf{k}=0} = 0$.

- We consider now $\hat{W}_{\omega,\omega}(\mathbf{k})$; we have already proved that $\hat{W}_{\omega,\omega}(0) = 0$. We want to show that

  $$\partial_{k_0} \hat{W}_{\omega,\omega}(\mathbf{k})_{\mathbf{k}=0} = \omega\alpha \quad \partial_k \hat{W}_{\omega,\omega}(\mathbf{k})_{\mathbf{k}=0} = i\beta$$

  with α, β real. From (2.35) we see that $g_{\omega,-\omega}(\mathbf{x})$ is even in the exchange $\mathbf{x} \to -\mathbf{x}$ and imaginary. Moreover we can write

  $$\hat{g}_{\omega,\omega}(\mathbf{k}) = \frac{-i \sin k}{\sin^2 k + \sin^2 k_0 + m^2_\chi} + \frac{\omega \sin k_0}{\sin^2 k + \sin^2 k_0 + m^2_\chi} = \hat{g}^1_{\omega,\omega}(\mathbf{k}) + \hat{g}^2_{\omega,\omega}(\mathbf{k}) \tag{3.32}$$

  with $g^1_{\omega,\omega}(\mathbf{x})$ real, odd in the exchange $x \to -x$ and even in $x_0 \to -x_0$; $g^2_{\omega,\omega}(\mathbf{x})$ is imaginary, even in the exchange $x \to -x$ and odd in $x_0 \to -x_0$. Remember that (see §2.4) the coefficient of $A^{\varepsilon_1,\varepsilon_2}_{\mathbf{x};\phi,\omega_1;\phi',\omega_2}$ is a)imaginary if $\omega_1 = \omega_2$ and $\alpha = 1$, $\partial_{x_\alpha} = \partial_{x_0}$; b) real if $\omega_1 = \omega_2$ and $\alpha = 2$, $\partial_{x_\alpha} = \partial_x$; c)imaginary if $\omega_1 = -\omega_2$. Given a Feynman diagram g contributing to $i \sum_{\mathbf{x}} x W_{\omega,\omega}(\mathbf{x})$ by parity it must by present a total odd number of $g^1_{\omega,\omega}(\mathbf{x})$ propagators and ∂_x derivatives from the interaction. Moreover by parity the number of $g^2_{\omega,\omega}(\mathbf{x})$ and ∂_{x_0} from the interaction must be even. Finally as the external lines have the same ω index, the sum of the number of non diagonal propagators $g_{\omega,-\omega}(\mathbf{x})$ plus the number of $A^{\varepsilon_1,\varepsilon_2}_{\mathbf{x};\phi,\omega_1;\phi',\omega_2}$ with $\omega_1 = -\omega_2$ must be even. Hence $\partial_k \hat{W}^g(\mathbf{k})|_{\mathbf{k}=0}$ is imaginary and ω-independent. In the same way one sees that $\partial_{k_0} W|_{\mathbf{k}=0} = \omega\alpha$

4. Renormalization Group for light fermions

4.1 *Multiscale analysis.* We continue the analysis of $Z^{-,-,-,-}_{2I}$ (2.12); after the integration of the χ fields we have to compute the Grassman integral given by the r.h.s. of (3.2), and this will be done by a multiscale analysis. We introduce a *scaling parameter* $\gamma > 1$ and a positive function $\chi(\mathbf{k}) \in C^\infty$ such that

$$\chi(\mathbf{k}) = \chi(-\mathbf{k}) = \begin{cases} 1 & \text{if } |\mathbf{k}| < 1/\gamma \ , \\ 0 & \text{if } |\mathbf{k}| > 1 \ , \end{cases} \tag{4.1}$$

where

$$|\mathbf{k}| = \sqrt{\sin k_0^2 + \sin k^2} \ . \tag{4.2}$$

We define also, for any integer $h \le 0$,

$$f_h(\mathbf{k}) = \chi(\gamma^{-h}\mathbf{k}) - \chi(\gamma^{-h+1}\mathbf{k}) \ ; \tag{4.3}$$

we have, for any $h_M < 0$,

$$\chi(\mathbf{k}) = \sum_{h=h_M+1}^{0} f_h(\mathbf{k}) + \chi(\gamma^{-h_M}\mathbf{k}) \ . \tag{4.4}$$

Note that, if $h \leq 0$, $f_h(\mathbf{k}) = 0$ for $|\mathbf{k}| < \gamma^{h-2}$ or $|\mathbf{k}| > \gamma^h$, and $f_h(\mathbf{k}) = 1$, if $|\mathbf{k}| = \gamma^{h-1}$. Therefore

$$f_h(\mathbf{k}) = 0 \quad \forall h < h_M = \min\{h : \gamma^h > \sqrt{2}(\pi M^{-1})^2\} \ , \tag{4.5}$$

and

$$1 = \sum_{h=h_M}^{1} f_h(\mathbf{k}) \qquad f_1 = 1 - \chi(\mathbf{k}) \ . \tag{4.6}$$

We define a sequence of *effective potentials* $\mathcal{V}^{(h)}(\psi)$ in the following way; assuming that we have integrated the scales $h = 1, 0, -1, -2, ..., h+1$

$$e^{-M^2 E_M} = \int P_{Z_h, m_h, C_h}(d\psi^{(\leq h)})\, e^{-\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)}) - M^2 E_h} \ , \quad \mathcal{V}^{(h)}(0) = 0 \ , \tag{4.7}$$

where

$$\begin{aligned} P_{Z_h, m_h, C_h}(d\psi^{(\leq h)}) = & \prod_{\mathbf{k}: C_h^{-1}(\mathbf{k}) > 0} \prod_{\omega = \pm 1} \frac{d\hat{\psi}^{(\leq h)+}_{\mathbf{k},\omega} d\hat{\psi}^{(\leq h)-}_{\mathbf{k}',\omega}}{\mathcal{N}(\mathbf{k})} \cdot \\ & \exp\left\{ -\frac{1}{M^2} \sum_{\mathbf{k}: C_h^{-1}(\mathbf{k}) > 0} C_h(\mathbf{k}) Z_h \sum_{\omega, \omega' = \pm 1} \hat{\psi}^{(\leq h)+}_{\mathbf{k},\omega} T^{(h+1)}_{\omega,\omega'} \hat{\psi}^{(\leq h)-}_{\mathbf{k},\omega'} \right\} \ , \end{aligned} \tag{4.8}$$

$$C_h(\mathbf{k})^{-1} = \sum_{j=h_M}^{h} f_j(\mathbf{k}') \ , \tag{4.9}$$

and the 2×2 matrix $T_h(\mathbf{k}')$ is given by

$$T_h(\mathbf{k}) = \frac{1}{C_0 + f_{0,0}(\mathbf{k})} \begin{pmatrix} \tilde{Z}_1(i \sin k + \sin k_0) + f_{1,1}(\mathbf{k}) Z_{h-1}^{-1} & -i m_{h-1}(\mathbf{k}) + f_{1,2}(\mathbf{k}) Z_{h-1}^{-1} \\ i m_{h-1}(\mathbf{k}) + f_{2,1}(\mathbf{k}) Z_{h-1}^{-1} & \tilde{Z}_1(i \sin k - \sin k_0) + f_{2,2}(\mathbf{k}) Z_{h-1}^{-1} \end{pmatrix} \tag{4.10}$$

with $m_1 = C_0(t_c - t)$ and $C_0, \tilde{Z}_1$ defined in Theorem 2; moreover $Z_1 = 1$.
Finally $\mathcal{V}^{(h)}$ is given by

$$\mathcal{V}^{(h)}(\psi^{(\leq h)}) = \sum_{n=1}^{\infty} \sum_{\substack{\mathbf{x}_1, \ldots, \mathbf{x}_{2n}, \\ \underline{\sigma}, \underline{\omega}, \underline{\alpha}}} \prod_{i=1}^{2n} \partial^{\alpha_i} \psi^{(\leq h)\sigma_i}_{\mathbf{x}_i, \omega_i} W^{(h)}_{2n, \underline{\sigma}, \underline{\alpha}, \underline{\omega}, \underline{\alpha}}(\mathbf{x}_1, ..., \mathbf{x}_{2n}) \tag{4.11}$$

4.2 *The localization operator.* We define an $\mathcal{L}$ operation, for $h \leq 0$, in the following way:

1) If $2n = 4$, then

$$\mathcal{L}\hat{W}^{(h)}_{4,\underline{\sigma},\underline{\alpha},\underline{\omega}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = \hat{W}^{(h)}_{4,\underline{\sigma},\underline{\alpha},\underline{\omega}}(\bar{\mathbf{k}}_{++}, \bar{\mathbf{k}}_{++}, \bar{\mathbf{k}}_{++}) \ , \tag{4.12}$$

where $\bar{\mathbf{k}}_{\eta\eta'} = \left(\eta \frac{\pi}{M}, \eta' \frac{\pi}{M}\right)$.

2) If $2n = 2$ then

$$\mathcal{L}\hat{W}^{(h)}_{2,\underline{\sigma},\underline{\alpha},\underline{\omega}}(\mathbf{k}) = \frac{1}{4} \sum_{\eta, \eta' = \pm 1} [\hat{W}^{(h)}_{2,\underline{\sigma},\underline{\alpha},\underline{\omega}}(\bar{\mathbf{k}}_{\eta\eta'}) + \hat{W}^{(h)}_{2,\underline{\sigma},\underline{\alpha},\underline{\omega}}(\bar{\mathbf{k}}_{\eta\eta'})(\eta \frac{M}{\pi} \sin k + \eta' \frac{M}{\pi} \sin k_0)] \ . \tag{4.13}$$

3) In all the other cases

$$\mathcal{L}\hat{W}^{h}_{2n,\underline{\sigma},\underline{\alpha},\underline{\omega}}(\mathbf{k}_1, \ldots, \mathbf{k}_{2n-1}) = 0 \ . \tag{4.14}$$

By (4.12) the operator $\mathcal{L}$ satisfies the relation $\mathcal{R}\mathcal{L} = 0$, if $\mathcal{R} \equiv 1 - \mathcal{L}$.

Remark. In the limit $M \to \infty$ (4.13) becomes simply

$$\mathcal{L}\hat{W}^{(h)}_{2,\underline{\sigma},\underline{\alpha},\underline{\omega}}(\mathbf{k}) = [\hat{W}^{(h)}_{2,\underline{\sigma},\underline{\alpha},\underline{\omega}}(\mathbf{0}) + \sin k_0 \partial_{k_0} \hat{W}^{(h)}_{2,\underline{\sigma},\underline{\alpha},\underline{\omega}}(\mathbf{0}) + \sin k \partial_k \hat{W}^{a(h)}_{2,\underline{\sigma},\underline{\alpha},\underline{\omega}}(\mathbf{0})], \tag{4.15}$$

hence $\mathcal{L}\hat W^{(h)}_{2,\underline\sigma,\underline\omega}(\mathbf{k})$ has to be understood as a discrete version of the Taylor expansion up to order 1.

4.3 By (4.12),(4.13),(4.14) and the symmetry relations in §3.3, §3.4, we can write $\mathcal{L}\mathcal{V}^{(h)}$ in the following way:

$$\mathcal{L}\mathcal{V}^{(h)}(\psi^{(\le h)}) = (s_h+\gamma^h n_h)F_\sigma^{(\le h)} + l_h F_\lambda^{(\le h)} + z_h F_\zeta^{(\le h)} + a_h F_\alpha^{(\le h)}\ , \tag{4.16}$$

where, if $|\lambda|,|\nu|\le\varepsilon$, $s_1,z_1,a_1=O(\varepsilon)$, $l_1=\lambda(a+b)\mathrm{sech}^4 J+O(\varepsilon^2)$, $\nu_1=\nu+O(\varepsilon)$ and

$$\begin{aligned}
F_m^{(\le h)} &= \frac{1}{M^2}\sum_{\mathbf{k}\in\mathcal{D}_M}\sum_{\omega=\pm1} i\omega\hat\psi^{(\le h)+}_{\mathbf{k},\omega}\hat\psi^{(\le h)-}_{\mathbf{k},-\omega}\ ,\\
F_\lambda^{(\le h)} &= \frac{1}{M^8}\sum_{\mathbf{k}_1,\dots,\mathbf{k}_4\in\mathcal{D}_M}\hat\psi^{(\le h)+}_{\mathbf{k}_1,+1}\hat\psi^{(\le h)+}_{\mathbf{k}_2,-1}\hat\psi^{(\le h)-}_{\mathbf{k}_3,-1}\hat\psi^{(\le h)-}_{\mathbf{k}_4,+1}\delta(\mathbf{k}_1-\mathbf{k}_2+\mathbf{k}_3-\mathbf{k}_4)\\
F_\alpha^{(\le h)} &= \frac{1}{M^2}\sum_{\mathbf{k}\in\mathcal{D}_M}\sum_{\omega=\pm1} i\sin k\hat\psi^{(\le h)+}_{\mathbf{k},\omega}\hat\psi^{(\le h)-}_{\mathbf{k},\omega}\\
F_\zeta^{(\le h)} &= \frac{1}{M^2}\sum_{\mathbf{k}\in\mathcal{D}_M}\sum_{\omega=\pm1} \omega\sin k_0\hat\psi^{(\le h)+}_{\mathbf{k},\omega}\hat\psi^{(\le h)-}_{\mathbf{k},\omega}\ .
\end{aligned}\tag{4.17}$$

Remark. Note that one can repeat the analysis in §3.3 and §3.4 to conclude that many terms, which could be *a priori* present in (4.16) are indeed absent. In fact the analysis of §3.3 and §3.4 apply unchanged to W^h_ω just replacing χ with $\chi+\psi^{(\ge h)}$ and ψ with $\psi^{(<h)}$. Hence the constants n_h,s_h,l_h,z_h,a_h are real and many possible marginal interactions (like $\sum_{\mathbf{k}}\sin k\hat\psi^{(\le h)+}_{\mathbf{k},\omega}\hat\psi^{(\le h)-}_{\mathbf{k},-\omega}$ or $\sum_{\mathbf{k}}\hat\psi^{(\le h)+}_{\mathbf{k},\omega}\hat\psi^{(\le h)-}_{\mathbf{k},\omega}$) are excluded. This remark is crucial in order to analyze the flow of the running coupling constant, see the next section.

At the end of our iterative construction it will appear that $\hat W^{(h)}_{2n,\underline\sigma,\underline\omega}$ can be written as

$$\hat W^{(h)}_{2,\underline\sigma,\underline\omega} = \hat W^{a(h)}_{2,\underline\sigma,\underline\omega} + \hat W^{b(h)}_{2,\underline\sigma,\underline\omega} \tag{4.18}$$

with $\hat W^{a(h)}_{2,\underline\sigma,\underline\omega}$ is vanishing if at least one $m_k=0$, for $1\ge k\ge h+1$, and $\hat W^{b(h)}_{2,\underline\sigma,\underline\omega}$ is the rest; we define

$$s_h=\delta_{\omega,-\omega}[\frac14\sum_{\eta,\eta'=\pm1}\hat W^{a(h)}_{2,\underline\sigma,\underline\omega}(\bar{\mathbf{k}}_{\eta\eta'})]\qquad \gamma^h n_h=\delta_{\omega,-\omega}[\frac14\sum_{\eta,\eta'=\pm1}\hat W^{b(h)}_{2,\underline\sigma,\underline\omega}(\bar{\mathbf{k}}_{\eta\eta'})]. \tag{4.19}$$

We *renormalize* the free integration $P_{Z_h,m_h,C_h}(d\psi^{(\le h)})$ by adding to it part of the r.h.s. of (4.16). We get

$$\begin{aligned}
&\int P_{Z_h,m_h,C_h}(d\psi^{(\le h)})\,e^{-\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\le h)})} =\\
&\qquad = e^{-M^2 t_h}\int P_{\tilde Z_{h-1},m_{h-1},C_h}(d\psi^{(\le h)})\,e^{-\tilde{\mathcal{V}}^{(h)}(\sqrt{Z_h}\psi^{(\le h)})}\ ,
\end{aligned}\tag{4.20}$$

where $P_{\tilde Z_{h-1},m_{h-1},C_h}(d\psi^{(\le h)})$ is obtained from $P_{Z_h,m_h,C_h}(d\psi^{(\le h)})$ by substituting Z_h with

$$\tilde Z_{h-1}(\mathbf{k}) = Z_h[1+C_h^{-1}\tilde Z_1^{-1}(C_0+f_{0,0}(\mathbf{k}))z_h] \tag{4.21}$$

and $m_h(\mathbf{k})$ with

$$m_{h-1}(\mathbf{k}) = \frac{Z_h}{\tilde Z_{h-1}(\mathbf{k})}[m_h(\mathbf{k})+C_h^{-1}(\mathbf{k})(C_0+f_{0,0}(\mathbf{k}))s_h]\ ; \tag{4.22}$$

moreover $\tilde{\mathcal{V}}^{(h)}=\mathcal{V}^{(h)}-Z_h s_h F_\sigma^{(\le h)}-Z_h z_h(F_\zeta^{(\le h)}+F_\alpha^{(\le h)})$. We will call $m_h(\mathbf{0})\equiv m_h$ and $Z_h(\mathbf{0})\equiv Z_h$. The r.h.s of (4.20) can be written as

$$e^{-M^2 t_h}\int P_{Z_{h-1},m_{h-1},C_{h-1}}(d\psi^{(\le h-1)})\int P_{Z_{h-1},m_{h-1},f_h^{-1}}(d\psi^{(h)})\,e^{-\tilde{\mathcal{V}}^{(h)}(\sqrt{Z_h}\psi^{(\le h)})}\;, \tag{4.23}$$

where the factor $\exp(-M^2 t_h)$ in (4.20) takes into account the different normalization of the two integrations and

$$Z_{h-1}=Z_h(1+z_h\tilde{Z}_1^{-1}C_0)\;,\qquad \tilde{f}_h(\mathbf{k})=Z_{h-1}[\frac{C_h^{-1}(\mathbf{k})}{\tilde{Z}_{h-1}(\mathbf{k})}-\frac{C_{h-1}^{-1}(\mathbf{k})}{Z_{h-1}}]\;. \tag{4.24}$$

Note that $\tilde{f}_h(\mathbf{k})$ has the same support of $f_h(\mathbf{k})$. The *single scale* propagator is

$$\int P_{Z_{h-1},m_{h-1},\tilde{f}_h^{-1}}(d\psi^{(h)})\,\psi^{(h)-}_{\mathbf{x},\omega}\psi^{(h)+}_{\mathbf{y},\omega'}=\frac{g^{(h)}_{\omega,\omega'}(\mathbf{x}-\mathbf{y})}{Z_{h-1}}\;, \tag{4.25}$$

where

$$g^{(h)}_{\omega,\omega'}(\mathbf{x}-\mathbf{y})=\frac{1}{M^2}\sum_{\mathbf{k}}e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}\tilde{f}_h(\mathbf{k})[T_h^{-1}(\mathbf{k})]_{\omega,\omega'}\;, \tag{4.26}$$

and $T_h^{-1}(\mathbf{k})$ is the inverse of the $T_h(\mathbf{k})$ defined in (4.10). The large distance behaviour of $g^{(h)}_{\omega,\omega'}(\mathbf{x}-\mathbf{y})$ is given by, if $|\tilde{Z}_1^{-1}C_0 z_h|\le\frac{1}{2}$, $|C_0 s_h|\le|m_h/2|$ and $\sup_{k\ge h}|\frac{Z_k}{Z_{k-1}}|\le e^{|\varepsilon|}$, given the positive integers N,n_0,n_1 and if $n=n_0+n_1$

$$|\partial_{x_0}^{n_0}\partial_x^{n_1}g^{(h)}_{\omega,\omega}(\mathbf{x}-\mathbf{y})|\le C_{N,n}\frac{\gamma^{h+n}}{1+(\gamma^h|\mathbf{d}(\mathbf{x}-\mathbf{y})|)^N} \tag{4.27}$$

$$|\partial_{x_0}^{n_0}\partial_x^{n_1}g^{(h)}_{\omega,-\omega}(\mathbf{x}-\mathbf{y})|\le C_{N,n}|\frac{m_h}{\gamma^h}|\frac{\gamma^{h+n}}{1+(\gamma^h|\mathbf{d}(\mathbf{x}-\mathbf{y})|)^N}\;. \tag{4.28}$$

where ∂_x denotes the discrete derivative. It will be useful to write

$$g^{(h)}_{\omega,\omega}(\mathbf{x}-\mathbf{y})=g^{(h)}_{L;\omega,\omega}(\mathbf{x}-\mathbf{y})+\tilde{g}^{(h)}_{\omega,\omega}(\mathbf{x}-\mathbf{y})+\hat{g}^{(h)}_{\omega,\omega}(\mathbf{x}-\mathbf{y}) \tag{4.29}$$

with

$$g^{(h)}_{L;\omega,\omega}(\mathbf{x}-\mathbf{y})=\frac{1}{M^2}\sum_{\mathbf{k}}e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}\frac{\tilde{f}_h(\mathbf{k})}{-\tilde{Z}_1\omega k_0+i\tilde{Z}_1 k}\;, \tag{4.30}$$

which is of course obeying to the bound (4.27). Note that $g^{(h)}_{\omega,\omega}(\mathbf{x}-\mathbf{y})-g^{(h)}_{L;\omega,\omega}(\mathbf{x}-\mathbf{y})$ is given by a part which is vanishing if $m_h=0$ (which we call $\tilde{g}^{(h)}_{\omega,\omega}(\mathbf{x}-\mathbf{y})$) plus a rest named $\hat{g}^{(h)}_{\omega,\omega}(\mathbf{x}-\mathbf{y})$ and it is easy to show that, for any N

$$|\partial_{x_0}^{n_0}\partial_x^{n_1}\tilde{g}^{(h)}_{\omega,\omega}(\mathbf{x}-\mathbf{y})|\le C_{N,n}\frac{\gamma^{(2+n)h}}{1+(\gamma^h|\mathbf{d}(\mathbf{x}-\mathbf{y}))|^N} \tag{4.31}$$

$$|\partial_{x_0}^{n_0}\partial_x^{n_1}\hat{g}^{(h)}_{\omega,\omega}(\mathbf{x}-\mathbf{y})|\le C_{N,n}|\frac{m_h}{\gamma^h}|^2\frac{\gamma^{h(1+n)}}{1+(\gamma^h|\mathbf{d}(\mathbf{x}-\mathbf{y})|)^N}\;. \tag{4.32}$$

Moreover

$$g^{(h)}_{\omega,-\omega}(\mathbf{x}-\mathbf{y})=\hat{g}^{(h)}_{\omega,-\omega}(\mathbf{x}-\mathbf{y})+\tilde{g}^{(h)}_{\omega,-\omega}(\mathbf{x}-\mathbf{y}) \tag{4.33}$$

with

$$\hat{g}^{(h)}_{\omega,-\omega}(\mathbf{x}-\mathbf{y})=\frac{1}{M^2}\sum_{\mathbf{k}}e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}\tilde{f}_h(\mathbf{k})\frac{-im_h(\mathbf{k})}{\tilde{Z}_1^2\sin^2 k_0+\tilde{Z}_1^2\sin k^2+m_h^2(\mathbf{k})} \tag{4.34}$$

verifying (4.28) and $\tilde{g}^{(h)}_{\omega,-\omega}(\mathbf{x}-\mathbf{y})$ verifying (4.31).

We now *rescale* the field so that

$$\tilde{\mathcal{V}}^{(h)}(\sqrt{Z_h}\psi^{(\le h)}) = \hat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)}) \; ; \tag{4.35}$$

it follows that

$$\mathcal{L}\hat{\mathcal{V}}^{(h)}(\psi) = \gamma^h \nu_h F_\sigma^{(\le h)} + \delta_h F_\alpha^{(\le h)} + \lambda_h F_\lambda^{(\le h)} \; , \tag{4.36}$$

where

$$\nu_h = \frac{Z_h}{Z_{h-1}} n_h \; , \quad \delta_h = \frac{Z_h}{Z_{h-1}}(a_h - z_h) \; , \quad \lambda_h = (\frac{Z_h}{Z_{h-1}})^2 l_h \; . \tag{4.37}$$

If we now define

$$e^{-\mathcal{V}^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\le h-1)}) - L\beta\tilde{E}_h} = \int P_{Z_{h-1},m_{h-1},\tilde{f}_h^{-1}}(d\psi^{(h)})\, e^{-\hat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)})} \; , \tag{4.38}$$

it is easy to see that $\mathcal{V}^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\le h-1)})$ is of the form (4.11) and that $E_{h-1} = E_h + t_h + \tilde{E}_h$. It is sufficient to use the well known identity

$$M^2\tilde{E}_h + \mathcal{V}^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\le h-1)}) = \sum_{n=1}^{\infty} \frac{1}{n!}(-1)^{n+1}\mathcal{E}_h^{T,n}(\hat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)})) \; , \tag{4.39}$$

where $\mathcal{E}_h^{T,n}$ denotes the *truncated expectation of order* n with propagator $Z_{h-1}^{-1} g^{(h)}_{\omega,\omega'}$, see (4.25), and observe that $\psi^{(\le h)} = \psi^{(\le h-1)} + \psi^{(h)}$.

Note that the above procedure allows us to write the running coupling constants $v_h = (\lambda_h, \delta_h, \nu_h)$, $h \le 0$, in terms of $v_{h'}$, $0 \ge h' \ge h+1$:

$$v_h = \beta(v_{h+1}, ..., v_0) \; . \tag{4.40}$$

The function $\beta(v_{h+1}, ..., v_0)$ is called the *Beta function.*

Let us define

$$h^* = \inf\{h : 1 \ge h \ge h_M, \gamma^{\bar{h}-1} \ge 4|m_{\bar{h}}|, \forall \bar{h} : 1 \ge \bar{h} \ge h\} \; . \tag{4.41}$$

Of course this definition is meaningful only for m_0 such that $|m_0| \le \frac{1}{4\gamma}$.

4.4 The integration of the scales from h^* to h_M will be performed "in a single step". This follows from the fact that, assuming that h^* is finite uniformly in M, so that $|m_{h^*-1}\gamma^{-h^*}| \ge \bar{\kappa}$, for a suitable constant $\bar{\kappa}$ and defining

$$\frac{\bar{g}^{(\le h^*)}_{\omega,\omega'}(\mathbf{x}-\mathbf{y})}{Z_{h^*-1}} \equiv \int P_{Z_{h^*-1},m_{h^*-1},C_{h^*}}(d\psi^{(\le h^*)})\psi^{(\le h^*)-}_{\mathbf{x},\omega}\psi^{(\le h^*)+}_{\mathbf{y},\omega'} \; . \tag{4.42}$$

it holds, given the positive integers N, n_0, n_1 and putting $n = n_0 + n_1$, that there exist a constant $C_{N,n}$ such that

$$|\partial_{x_0}^{n_0}\partial_x^{n_1} g^{(\le h^*)}_{\omega,\omega'}(\mathbf{x};\mathbf{y})| \le C_{N,n}\frac{\gamma^{h^*+n}}{1 + (\gamma^{h^*}|\mathbf{d}(\mathbf{x}-\mathbf{y})|)^N} \; . \tag{4.43}$$

Hence, we redefine $\tilde{E}_{h^*}$, so that

$$e^{-M^2\tilde{E}_{h^*}} = \int P_{Z_{h^*-1},m_{h^*-1},C_{h^*}}(d\psi^{(\le h^*)})\, e^{-\hat{\mathcal{V}}^{(h^*)}(\sqrt{Z_{h^*-1}}\psi^{(\le h^*)})} \; , \tag{4.44}$$

implying that $E_M = \sum_{h=h^*}^{1}[\tilde{E}_h + t_h]$.

4.5 We have still to give a prescription for the decomposition in (4.18). We define $\hat{W}^{a(h)}_{2,\underline{\sigma},\underline{\omega}}$ to be a sum over Feynman graphs such that there is at least a propagator $\hat{g}^{(k)}_{\omega,-\omega}$, $k \ge h$. By (4.33) and (4.22) it follows that $\hat{W}^{a(h)}_{2,\underline{\sigma},\underline{\omega}}$ is vanishing when $t = t_c$. The decomposition (4.18) respect the determinant structure of the truncated expectations.

4.6 *Remark.* Let us now explain the main motivations of the integration procedure discussed above. In a Renormalization Group framework one has to identify the relevant, marginal and irrelevant effective interactions. By a power counting argument one sees that the terms bilinear in the fields are *relevant*, and the terms quartic in the fields, or bilinear with a derivative, are marginal. The symmetry and parity considerations in §3.3, §3.4 say that the only possible non irrelevant terms are the ones in (4.16). The r.h.s. of (3.2) strongly resembles the action of interacting Dirac fermions on a lattice. There is however an important difference: in a theory of fermions if there is no mass term in the action then no mass terms are generated by the Renormalization Group iterations; the reason is that such models are invariant under the gauge transformation $\psi^{\varepsilon}_{\mathbf{x},\omega} \to e^{i\varepsilon\alpha_\omega}\psi^{\varepsilon}_{\mathbf{x},\omega}$ if there is no mass term in the action. In our spin model this is not true, as the interaction is not invariant under this symmetry; hence even if $t = t_c$ (or $m_1 = 0$) a mass term in the Renormalization Group iterations can be generated. Hence we collect all the relevant terms which are vanishing if $m_k = 0$, $1 \geq k \geq h+1$, in s_h, which we include in the fermionic iterations; the "mass" has a non trivial flow producing at the end the critical index of the correlation length. The remaining terms are left in the effective interaction; they are constituting the running coupling constant ν_h whose flow is controlled by the counterterm ν. Due to the mass gap, the propagator of the integration of all the scales between h^* and h_M has the same bound as the propagator of the integration of a single scale greater than h^*; this property is used to perform the integration of all the scales $\leq h^*$ in a single step.

4.7 **Theorem 3.** *Let $h > h^* \geq 0$ and, for some constants c_1, if*

$$\max_{h'>h} |v_{h'}| \leq \varepsilon_h, \quad \sup_{h'>h} \left|\frac{m_{h'}}{m_{h'-1}}\right| \leq e^{c_1\varepsilon_h}, \sup_{h'>h} \left|\frac{Z_{h'}}{Z_{h'-1}}\right| \leq e^{c_1\varepsilon_h^2} \tag{4.45}$$

there exists a constant $\bar{\varepsilon}$ (depending on c_1) such that, if $\varepsilon_h \leq \bar{\varepsilon}$, then, for a suitable constant c_0, independent of c_1, as well as of M, the kernels in (4.11) verify

$$|\hat{W}^{(h)}_{2n,\underline{\sigma},\underline{\omega},\underline{\alpha}}(\mathbf{k}_1,...,\mathbf{k}_{2n-1})| \leq \gamma^{-hD_k(P_{v_0})}(c_0\varepsilon_h)^{max(1,n-1)} \tag{4.46}$$

where

$$D_k(P_{v_0}) = -2 + n + k\ . \tag{4.47}$$

and $k = \sum_{i=1}^{2n}\alpha_i$. Moreover

$$[|n_h| + |z_h| + |a_h| + |l_h|] \leq (c_0\varepsilon_h)\ , \tag{4.48}$$

and $|s_h| \leq |m_h|(c_0\varepsilon_h)$, $|\tilde{E}_{h+1}| \leq \gamma^{2h}(c_0\varepsilon_h)$.

4.8 We write $\mathcal{V}^{(h)}$ in terms of a *tree expansion.*

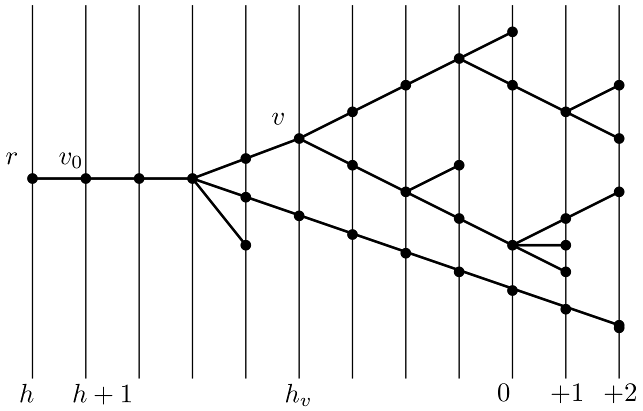

Fig. 1

Fig.2 A tree with its scale labels.

We need some definitions and notations.

1) Let us consider the family of all trees which can be constructed by joining a point r, the *root*, with an ordered set of $n \ge 1$ points, the *endpoints* of the *unlabeled tree*, so that r is not a branching point. n will be called the *order* of the unlabeled tree and the branching points will be called the *non trivial vertices*. The unlabeled trees are partially ordered from the root to the endpoints in the natural way; we shall use the symbol $<$ to denote the partial order. Two unlabeled trees are identified if they can be superposed by a suitable continuous deformation, so that the endpoints with the same index coincide. It is then easy to see that the number of unlabeled trees with n end-points is bounded by 4^n. We shall consider also the *labeled trees* (to be called simply trees in the following); they are defined by associating some labels with the unlabeled trees, as explained in the following items.

2) We associate a label $h \le 0$ with the root and we denote $\mathcal{T}_{h,n}$ the corresponding set of labeled trees with n endpoints. Moreover, we introduce a family of vertical lines, labeled by an an integer taking values in $[h, 2]$, and we represent any tree $\tau \in \mathcal{T}_{h,n}$ so that, if v is an endpoint or a non trivial vertex, it is contained in a vertical line with index $h_v > h$, to be called the *scale* of v, while the root is on the line with index h. There is the constraint that, if v is an endpoint, $h_v > h + 1$; if there is only one end-point its scale must be equal to $h + 2$, for $h \le 0$.

The tree will intersect in general the vertical lines in set of points different from the root, the endpoints and the non trivial vertices; these points will be called *trivial vertices*. The set of the *vertices* of τ will be the union of the endpoints, the trivial vertices and the non trivial vertices. Note that, if v_1 and v_2 are two vertices and $v_1 < v_2$, then $h_{v_1} < h_{v_2}$.

Moreover, there is only one vertex immediately following the root, which will be denoted v_0 and can not be an endpoint; its scale is $h + 1$.

3) With each endpoint v of scale $h_v = +2$ we associate one of the contributions to $\mathcal{V}^{(1)}$ given by (3.3); with each endpoint v of scale $h_v \le 1$ one of the terms in $\mathcal{L}V^{(h_v-1)}$ defined in (4.36). Moreover, we impose the constraint that, if v is an endpoint and $h_v \le 1$, $h_v = h_{v'} + 1$, if v' is the non trivial vertex immediately preceding v.

4) If v is not an endpoint, the *cluster* L_v with frequency h_v is the set of endpoints following the vertex v; if v is an endpoint, it is itself a (*trivial*) cluster. The tree provides an organization of endpoints into a hierarchy of clusters.

5) We introduce a *field label* f to distinguish the field variables appearing in the terms associated with the endpoints as in item 3); the set of field labels associated with the endpoint v will be called I_v. Analogously, if v is not an endpoint, we shall call I_v the set of field labels associated with the endpoints following the vertex v; $\mathbf{x}(f)$, $\varepsilon(f)$ and $\omega(f)$ will denote the space-time point, the ε index and the ω index, respectively, of the field variable with label f.

6) We associate with any vertex v of the tree a subset P_v of I_v, the *external fields* of v. These subsets must satisfy various constraints. First of all, if v is not an endpoint and $v_1, \ldots, v_{s_v}$ are the s_v vertices immediately following it, then $P_v \subset \cup_i P_{v_i}$; if v is an endpoint, $P_v = I_v$. We shall denote Q_{v_i} the intersection of P_v and P_{v_i}; this definition implies that $P_v = \cup_i Q_{v_i}$. The subsets $P_{v_i} \backslash Q_{v_i}$, whose union will be made, by definition, of the *internal fields* of v, have to be non empty, if $s_v > 1$, that is if v is a non trivial vertex.

Given $\tau \in \mathcal{T}_{j,n}$, there are many possible choices of the subsets P_v, $v \in \tau$, compatible with the previous constraints; let us call $\mathbf{P}$ one of this choices. Given $\mathbf{P}$, we consider the family $\mathcal{G}_{\mathbf{P}}$ of all connected Feynman graphs, such that, for any $v \in \tau$, the internal fields of v are paired by propagators of scale h_v, so that the following condition is satisfied: for any $v \in \tau$, the subgraph built by the propagators associated with all vertices $v' \ge v$ is connected. The sets P_v have, in this picture, the role of the external legs of the subgraph associated with v. The graphs belonging to $\mathcal{G}_{\mathbf{P}}$ will be called *compatible with* $\mathbf{P}$ and we shall denote $\mathcal{P}_\tau$ the family of all choices of $\mathbf{P}$ such that $\mathcal{G}_{\mathbf{P}}$ is not empty.

As explained for instance in §3.2 of [BM] we can write, if $h \le 0$

$$\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\le h)}) + M^2 \tilde{E}_{h+1} = \sum_{n=1}^{\infty} \sum_{\tau \in \mathcal{T}_{h,n}} \sum_{\mathbf{P} \in \mathcal{P}_\tau} \sqrt{Z_h}^{|P_{v_0}|} \sum_{\mathbf{x}_{v_0}} \tilde{\psi}^{(\le h)}(P_{v_0}) K_{\tau,\mathbf{P}}^{(h+1)}(\mathbf{x}_{v_0}) \,, \tag{4.49}$$

where

$$\tilde{\psi}^{[h,j]}(P_v) = \prod_{f \in P_v} \psi^{[h,j]\sigma(f)}_{\mathbf{x}(f),\omega(f)} \tag{4.50}$$

and $K_{\tau,\mathbf{P}}^{(j+1)}(\mathbf{x}_{v_0})$ is a suitable function, which is obtained by summing the values of all the Feynman graphs compatible with $\mathbf{P}$, see item 6) above, and applying iteratively in the vertices of the tree, different from the endpoints and v_0, the $\mathcal{R}$-operation, starting from the vertices with higher scale.

In order to control, uniformly in M, the various terms in (4.49) one has to exploit in a careful way the $\mathcal{R}$ operation acting on the vertices of the tree, as explained in full detail in [BM], §3. The result of this analysis, which apply essentially unchanged in the present case, is the following bound (see (3.105) of [BM]), if $k = \sum_i \alpha_i$

$$\begin{aligned} &\sum_{\mathbf{x}_{v_0}} |W_{\tau,\mathbf{P},\mathbf{T}}(\mathbf{x}_{v_0})| \le C^n M^2 \varepsilon_h^n \gamma^{-hD_k(P_{v_0})} \,\cdot \\ &\cdot \prod_{v \text{ not e.p.}} \left\{ \frac{1}{s_v!} C^{\sum_{i=1}^{s_v} |P_{v_i}| - |P_v|} \left(\frac{Z_{h_v}}{Z_{h_v-1}}\right)^{\frac{|P_v|}{2}} \gamma^{-[-2+\frac{|P_v|}{2}+z(P_v)]} \right\}, \end{aligned} \tag{4.51}$$

with $-2 + \frac{|P_v|}{2} + z(P_v) > 0$ and

$$z(P_v) = \begin{cases} 1 & \text{if } |P_v| = 4, \\ 2 & \text{if } |P_v| = 2, \\ 0 & \text{otherwise.} \end{cases} \tag{4.52}$$

The above bound admit a simple dimensional interpretation. If we erase the $\mathcal{R}$ operation from all the vertices of the tree, then $z_v = 0$ and (4.52) allow us to associate a factor $\gamma^{2-\frac{|P_v|}{2}}$ with any trivial or non trivial vertex of the tree. This would allow us to control the sums over the scale labels and $\mathcal{P}_\tau$, provided that $|P_v|$ were larger than 4 in all vertices, which is however not true. The effect of the $\mathcal{R}$ operation is to improve the bound with the factor $\gamma^{-z(P_v)}$, so that there there is a factor $\gamma^{-[-2+\frac{|P_v|}{2}+z(P_v)]}$ smaller than 1 associated with all the vertices.

In order to perform the sums note that the number of unlabeled trees is $\le 4^n$; fixed an unlabeled tree, the number of terms in the sum over the various labels of the tree is bounded by C^n, except the sums over the scale labels. In order to bound the sums over the scale labels and $\mathbf{P}$ we first use the inequality

$$\prod_{v \text{ not e.p.}} \gamma^{-[-2+\frac{|P_v|}{2}+z(P_v)]} \le [\prod_{\tilde{v}} \gamma^{-2\alpha(h_{\tilde{v}} - h_{\tilde{v}'})}][\prod_{v \text{ not e.p.}} \gamma^{-2\alpha|P_v|}] \tag{4.53}$$

where $\tilde{v}$ are the non trivial vertices, and $\tilde{v}'$ is the non trivial vertex immediately preceding $\tilde{v}$ or the root. The factors $\gamma^{-2\alpha(h_{\tilde{v}} - h_{\tilde{v}'})}$ in the r.h.s. of (4.53) allow to bound the sums over the scale labels by C^n; α is a suitable constant (one finds $\alpha = \frac{1}{40}$).

Finally the sum over $\mathbf{P}$ can be bounded by using the following combinatorial inequality, trivial for γ large enough. Let $\{p_v, v \in \tau\}$ a set of integers such that $p_v \le \sum_{i=1}^{s_v} p_{v_i}$ for all $v \in \tau$ which are not endpoints; then

$$\prod_{v \text{ not e.p.}} \sum_{p_v} \gamma^{-\frac{p_v}{40}} \le C^n \,. \tag{4.54}$$

It follows that

$$\sum_{\substack{\mathbf{P} \\ |P_{v_0}|=2m}} \prod_{v \text{ not e.p.}} \gamma^{-\frac{|P_v|}{40}} \le \prod_{v \text{ not e.p.}} \sum_{p_v} \gamma^{-\frac{p_v}{40}} \le C^n \,. \tag{4.55}$$

5. The flow of the running coupling constants

5.1 By the analysis of the preceding section it follows that the running coupling constants $v_h = (\lambda_h, \delta_h, \nu_h)$ and Z_k, m_k, $1 \ge k \ge h^*$, verify a set of recursive equations called *Beta function equations*, of the form

$$\begin{aligned} \nu_{h-1} &= \gamma\nu_h + \beta_\nu^h(a_h, \nu_h; ...; a_1, \nu_1) \\ a_{h-1} &= a_h + \beta_a^h(a_h, \nu_h; ...; a_1, \nu_1) \\ \frac{m_{h-1}}{m_h} &= 1 + \beta_m^h(a_h, \nu_h; ...; a_1, \nu_1) \\ \frac{Z_{h-1}}{Z_h} &= 1 + \beta_z^h(a_h, \nu_h; ...; a_1, \nu_1) \end{aligned} \tag{5.1}$$

where $a_h = (\lambda_h, \delta_h)$. By Theorem 3 such functions are bounded if (4.45) is fulfilled. Note that these functions depend on $a_h, \nu_h, \ldots, a_1, \nu_1$, directly trough the endpoints of the trees, indirectly trough z_h and the quantities $Z_{h'}/Z_{h'-1}$ and $m_{h'-1}(\mathbf{k})$, $h < h' \le 0$, appearing in the tree expansions. By explicit calculation of the lower order non zero terms one finds

$$\begin{aligned} \beta_z^h(a_h, \nu_h; ...; a_1, \nu_1) &= b_1\lambda_h^2 + O(\varepsilon_h^3) \;, \quad b_1 > 0 \;, \\ \beta_m^h(a_h, \nu_h; ...; a_1, \nu_1) &= a_2\lambda_h + O(\varepsilon_h^2) \;, \quad a_2 > 0 \;. \end{aligned} \tag{5.2}$$

Note that $\beta_m^h(a_h, \nu_h; ...; a_1, \nu_1)$ is given by a sum over terms with at least a non diagonal propagator $\hat{g}^{(k)}_{\omega,-\omega}(\mathbf{x}-\mathbf{y})$. Let us define

$$\mu_h = \sup_{k\ge h} \max\{|\lambda_k|, |\delta_k|\} \;, \quad \bar{\lambda}_h = \sup_{k\ge h} |\lambda_k| \;. \tag{5.3}$$

We want to prove the following Lemma.

5.2 LEMMA. *Let us consider the first of equations (5.1) for fixed values of a_h, Z_{h-1} and $m_{h-1}(\mathbf{k})$, $\tilde{h} \le h \le 1$, satisfying the conditions*

$$\mu_h \le \bar{\varepsilon}_1 \le \bar{\varepsilon}_0 \;, \tag{5.4}$$

$$a_0\gamma^{h-1} \ge 4|m_h| \;, \tag{5.5}$$

$$\gamma^{-c_0\mu_h} \le \frac{m_{h-1}}{m_h} \le \gamma^{+c_0\mu_h} \;, \tag{5.6}$$

$$\gamma^{-c_0\mu_h^2} \le \frac{Z_{h-1}}{Z_h} \le \gamma^{+c_0\mu_h^2} \;, \tag{5.7}$$

for some constant c_0.

Then, if $\bar{\varepsilon}_0$ is small enough, there exist some constants $\bar{\varepsilon}_1$, κ, γ', c_1, B, and a family of intervals $I^{(\bar{h})}$, $\tilde{h} \le \bar{h} \le 0$, such that $\bar{\varepsilon}_1 \le \bar{\varepsilon}_0$, $0 < \kappa < 1$, $1 < \gamma' < \gamma$, $I^{(\bar{h})} \subset I^{(\bar{h}+1)}$, $|I^{(\bar{h})}| \le c_1\bar{\varepsilon}_1(\gamma')^{\bar{h}}$ and, if $\nu = \nu_1 \in I^{(\bar{h})}$,

$$|\nu_h| \le B\bar{\varepsilon}_1[\gamma^{-\frac{1}{2}(h-\bar{h})} + \gamma^{\kappa h}] \le \bar{\varepsilon}_0 \;, \quad \bar{h} \le h \le 1 \;. \tag{5.8}$$

5.3 *Proof.* Note that, if $|\nu_h| \le \bar{\varepsilon}_0$ for $\bar{h} \le h \le 1$ and $\bar{\varepsilon}_0$ is small enough, the r.h.s. of (5.1) is well defined for $h = \bar{h}$ and we can write

$$\nu_{\bar{h}-1} = \gamma\nu_{\bar{h}} + b_{\bar{h}} + r_{\bar{h}} \;, \tag{5.9}$$

where $b_{\bar h} = c^\nu_{\bar h-1}\gamma^{\bar h-1}\lambda_{\bar h}$ and $r_{\bar h}$ collects all terms of second or higher order in $\bar\varepsilon_0$. In the tree expansion of β_ν, there is no contribution from the trees with $n\ge 2$ endpoints, which are only of type ν or δ, because of the support properties of the single scale propagators; hence by (5.4)-(5.6) $|r_{\bar h}|\le c_2\mu_{\bar h}\bar\varepsilon_0$. Let us now fix a positive constant c, consider the intervals

$$J^{(h)} = [-\frac{b_h}{\gamma-1} - c\bar\varepsilon_1, -\frac{b_h}{\gamma-1} + c\bar\varepsilon_1] \; . \tag{5.10}$$

By using (5.9) one can show by an inductive argument (see for instance §4.3 of [BM]) that there exists a decreasing family of intervals $I^{(\bar h)}$ (with size as in Lemma 5.2), $\tilde h\le\bar h\le 0$, such that, if $\nu=\nu_1\in I^{(\bar h)}$, then the sequence ν_h is well defined for $h\ge\bar h$ and satisfies the bound $|\nu_h|\le\bar\varepsilon_0$.

In order to prove the bound (5.8) we note that, if we iterate the first of (5.1), we can write, if $\bar h\le h\le 0$ and $\nu_1\in I^{(\bar h)}$,

$$\nu_h = \gamma^{-h+1}\left[\nu_1 + \sum_{k=h+1}^{1}\gamma^{k-2}\beta^\nu_k(\nu_k,\ldots,\nu_1)\right] \; , \tag{5.11}$$

where now the functions β^k_ν are thought as functions of $\nu_k,\ldots,\nu_1$ only.

If we put $h=\bar h$ in (5.11), we get the following identity:

$$\nu_1 = -\sum_{k=\bar h+1}^{1}\gamma^{k-2}\beta^\nu_k(\nu_k,\ldots,\nu_1) + \gamma^{\bar h-1}\nu_{\bar h} \; . \tag{5.12}$$

(5.11) and (5.12) are equivalent to

$$\nu_h = -\gamma^{-h}\sum_{k=\bar h+1}^{h}\gamma^{k-1}\beta^\nu_k(\nu_k,\ldots,\nu_1) + \gamma^{-(h-\bar h)}\nu_{\bar h} \; , \qquad \bar h<h\le 1 \; . \tag{5.13}$$

By construction, see §4.5, β^ν_k is given by the sum over trees with at least an end point ν_k, $k\ge h$, or at least a propagator $\tilde g_{\omega,-\omega}$, see (4.33). Hence, we can write

$$\beta^\nu_h = \mu_h\sum_{k=h}^{1}\nu_k\tilde\beta^\nu_{h,k}\gamma^{-2\kappa(k-h)} + \gamma^{\kappa h}\mu_h R^\nu_h \; , \tag{5.14}$$

where $|R^\nu_h|, |\tilde\beta^\nu_{h,k}|\le C$ and κ is a constant. The second term in (5.14) comes from the trees with at least a propagator $\tilde g_{\omega,-\omega}$ and the first term from the trees with at least a ν_k end-point. The factor $\gamma^{-2\kappa(k-h)}$ in the r.h.s. of (5.14) follows from the simple remark that the bound over all the trees contributing to ν_h, which have at least one endpoint of fixed scale $k>h$, can be improved by a factor $\gamma^{-\eta'(k-h)}$, with η' positive but small enough. It is sufficient to use (4.53), which allows to extract such factor from the r.h.s. before performing the sum over the scale indices, and to choose $\eta'=2\kappa$, which is possible if κ is small enough.

Let us now observe that the sequence ν_h, $\bar h<h\le 1$, satisfying (5.13) can be obtained as the limit as $n\to\infty$ of the sequence $\{\nu^{(n)}_h\}$, $\bar h<h\le 1$, $n\ge 0$, parameterized by $\nu_{\bar h}\in J^{(\bar h+1)}$ and defined recursively in the following way:

$$\begin{aligned}
&\nu^{(0)}_h = 0 \; , \\
&\nu^{(n)}_h = -\gamma^{-h}\sum_{k=\bar h+1}^{h}\gamma^{k-1}\beta^\nu_k(\nu^{(n-1)}_k,\ldots,\nu^{(n-1)}_1) + \gamma^{-(h-\bar h)}\nu_{\bar h} \; , \qquad n\ge 1 \; .
\end{aligned} \tag{5.15}$$

In fact, it is easy to show inductively that, if $\bar\varepsilon_1$ is small enough, $|\nu^{(n)}_h|\le C\bar\varepsilon_1\le\bar\varepsilon_0$, so that (5.15) is meaningful, and $\max_{h^*<h\le 1}|\nu^{(n)}_h - \nu^{(n-1)}_h|\le (C\bar\varepsilon_1)^n$. In fact for $n=1$ is trivial and for $n>1$

it follows by the fact that $\beta^\nu_k(\nu_k^{(n-1)},\ldots,\nu_1^{(n-1)})-\beta^\nu_k(\nu_k^{(n-2)},\ldots,\nu_1^{(n-2)})$ can be written as a sum of terms in which there are at least one endpoint of type ν, with a difference $\nu_{h'}^{n-1}-\nu_{h'}^{n-2}$, $h'\ge k$, in place of the corresponding running coupling, and one endpoint of type λ. Then $\nu_h^{(n)}$ converges as $n\to\infty$, for $\bar h<h\le 1$, to a limit ν_h, satisfying (5.13) and the bound $|\nu_h|\le\bar\varepsilon_0$, if $\bar\varepsilon_1$ is small enough. Hence, if $\bar\varepsilon_1$ is small enough, by (5.14),

$$|\beta^\nu_k|\le C\bar\varepsilon_1\left[\sum_{m=k}^{1}|\nu_m|\gamma^{-2\kappa(m-k)}+\gamma^{\kappa k}\right] . \tag{5.16}$$

Hence

$$|\nu_h^{(n)}|\le\bar c\bar\varepsilon_1\{\gamma^{-h}\sum_{k=\bar h+1}^{h}\gamma^k\left[\sum_{m=k}^{1}|\nu_m^{n-1}|\gamma^{-2\kappa(m-k)}+\gamma^{\kappa k}\right]+\gamma^{-(h-\bar h)}\} . \tag{5.17}$$

Let us now suppose that, for some constant c_{n-1},

$$|\nu_m^{(n-1)}|\le c_{n-1}\bar\varepsilon_1(\gamma^{\kappa m}+\gamma^{-\frac12(m-\bar h)})\le\bar\varepsilon_0 , \tag{5.18}$$

which is true for $n=1$, since $\nu_m^{(0)}=0$, if $\bar\varepsilon_1$ is small enough. Then, it is easy to verify that the same bound is verified by $\nu_m^{(n)}$, if c_{n-1} is substituted with $c_n=\bar c(1+c_4c_{n-1}\bar\varepsilon_1)$, where c_4 is a suitable constant. Hence, we can easily prove the bound (5.8) for $\nu_h=\lim_{n\to\infty}\nu_h^{(n)}$, for $\bar\varepsilon_1$ small enough.

5.4 Let us now consider the second of (5.1), for a fixed, arbitrary, sequence ν_h, $\bar h\le h\le 1$, satisfying the bound (5.8). Let us consider

$$\begin{aligned}\lambda_{h-1}^{(L)}&=\lambda_h^{(L)}+\beta_h^{\lambda,L}(a_h^{(L)},\ldots,a_1^{(L)}) ,\\ \delta_{h-1}^{(L)}&=\delta_h^{(L)}+\beta_h^{\delta,L}(a_h^{(L)},\ldots,a_1^{(L)}) ,\end{aligned} \tag{5.19}$$

obtained by the second of (5.1) putting $\nu_k=m_k=0$ for any $k\ge h$, by replacing $g^{(k)}_{\omega,\omega'}$ with $g^{(k)}_{L;\omega,\omega'}$ (4.30) and restricting the sum only to the trees with end-points v to which $\mathcal{L}\mathcal{V}^{(h_v-1)}$ is associated.

Let us define, for $\alpha=\lambda,\delta$,

$$r_h^\alpha(a_h,\nu_h;\ldots;a_1,\nu_1)=\beta_h^\alpha(a_h,\nu_h;\ldots;a_1,\nu_1)-\beta_h^{\alpha,L}(a_h,\ldots,a_1) . \tag{5.20}$$

Note that, in the r.h.s. of (5.20), the function $\beta_h^{\alpha,L}$ is calculated at the values of $\vec a_{h'}$, $h\le h'\le 1$, which are the solutions of the equations (5.1); these values are of course different from those satisfying the equations (5.19). It is possible to show that, if the sequence ν_h, $\bar h\le h\le 1$, satisfies the bound (5.8) and if κ is defined as in Lemma 5.2 and $\mu_h\le\bar\varepsilon_0$ and $\bar\varepsilon_0$ is small enough,

$$|r_h^\lambda|+|r_h^\delta|\le C\bar\lambda_h^2[\gamma^{-\frac12(h-\bar h)}+\gamma^{\kappa h}] ,\quad \bar h\le h\le 0 . \tag{5.21}$$

The proof of (5.21) is an inductive argument based on (5.8), (4.31),(4.32),(5.22) ; it can be found in §4.6 of [BM]. This allows to reduce the study of running couplings flow to the same problem for the flow (5.19). On the other hand one can perform a Renormalization Group analysis similar to the one of the preceding sections for the *Luttinger model*, see [BGM]. After the integration of the *ultraviolet part*, discussed in [GS], the multiscale integration of the *infrared part* leads to a set of Beta function equations essentially identical to (5.19). The Luttinger model is exactly solvable, as shown in [ML], and by using informations from the exact solution one can show, see [BGPS], [BM1] (see also [BeM1] for a new simplified proof), the following very important lemma.

5.5 Lemma. *There are $\bar\varepsilon_0$ and $\eta'>0$, such that, if $|\mu_h|\le\bar\varepsilon_0$, $\alpha=\lambda,\delta$ and $h\le 0$,*

$$|\beta_h^{\alpha,L}(a_h,\ldots,a_h)|\le C\bar\lambda_h^2\gamma^{\eta' h} . \tag{5.23}$$

We are now ready to prove the following main Theorem on the running couplings flow.

5.6 LEMMA. *There exist $\bar\varepsilon_3$ and a finite integer $h^* \le 0$, such that, if $|\lambda_1| \le \bar\varepsilon_3$ and ν belongs to a suitable interval $I^{(h^*)}$, of size smaller than $c|\lambda_1|\gamma'^{h^*}$ for some constants c and γ', $1<\gamma'<\gamma$, then the running coupling constants are well defined for $h^*-1 \le h \le 0$ and h^* satisfies the definition (4.41). Moreover, there exist positive constants c_i, $i=1,\ldots,5$, such that*

$$|\lambda_h - \lambda_1| \le c_1 |\lambda_1|^{3/2} , \qquad |\delta_h| \le c_1 |\lambda_1| , \tag{5.24}$$

$$\gamma^{-\lambda_1 c_2 h} < \frac{m_h}{m_1} < \gamma^{-\lambda_1 c_3 h} , \tag{5.25}$$

$$\gamma^{-c_4 \lambda_1^2 h} < Z_h < \gamma^{-c_5 \lambda_1^2 h} . \tag{5.26}$$

5.7 *Proof.* We shall proceed by induction. The second of (5.1) and Lemma 5.2 imply that, if λ_1 is small enough, there exists an interval $I^{(0)}$, whose size is of order λ_1, such that, if $\nu \in I^{(0)}$, then the bound (5.8) is satisfied, together with

$$|\lambda_0 - \lambda_1| \le C|\lambda_1|^2 , \quad |\delta_0 - \delta_1| = |\delta_0| \le C|\lambda_1| . \tag{5.27}$$

Let us now suppose that the solution of the first and second of (5.1) is well defined for $\bar h \le h \le 0$ and satisfies the conditions (5.5)-(5.8), for any ν belonging to an interval $I^{(\bar h)}$, defined as in Lemma 5.2. This implies, in particular, that $h^* \le \bar h$, see (5.5) and (4.41). Suppose also that there exists a constant c_0, such that

$$\bar\lambda_{\bar h} \le c_0 |\lambda_1| \qquad (5.28) .$$

We want to prove that all these conditions are verified also if $\bar h$ is substituted with $\bar h - 1$, if λ_1 is small enough. The induction will be stopped as soon as the condition (5.5) is violated for some $\nu \in I^{(\bar h)}$. We shall put ν equal to one of these values, so defining h^* as equal to $\bar h + 1$.

The fact that the condition on ν_1 and the bound (5.8) are verified also if $\bar h - 1$ takes the place of $\bar h$, follows from Lemma 5.2, since the condition (5.4) follows from (5.28), if λ_1 is small enough.

The conditions (5.6) and (5.7) follow immediately from (5.28) and the last two of (5.1). Hence, we still have to show only that (5.28) is verified also if $\bar h$ is substituted with $\bar h - 1$, if λ_1 is small enough.

By using (5.19) and (5.20), we have

$$a_{\bar h - 1} = a_{\bar h} + \beta^{\alpha,L}_{\bar h}(a_{\bar h}, \ldots, a_{\bar h}) + \sum_{k=\bar h + 1}^{1} D^\alpha_{\bar h, k} + r^\alpha_{\bar h}(a_{\bar h}, \nu_{\bar h}; \ldots; a_1, \nu_1; u) , \tag{5.29}$$

where

$$D^\alpha_{h,k} = \beta^{\alpha,L}_h(a_h, \ldots, a_h, a_k, a_{k+1}, \ldots, a_1) - \beta^{\alpha,L}_h(a_h, \ldots, a_h, a_h, a_{k+1}, \ldots, a_1) . \tag{5.30}$$

On the other hand, it is easy to see that $D^\alpha_{h,k}$ admits a tree expansion similar to that of $\beta^{\alpha,L}_h(a_h, \ldots, a_1)$, with the property that all trees giving a non zero contribution must have an endpoint of scale h, associated with a difference $\lambda_k - \lambda_h$ or $\delta_k - \delta_h$. Hence, if κ is the same constant of Lemma 5.2 and $h \le 0$,

$$|D^\alpha_{h,k}| \le C \bar\lambda_h \gamma^{-\kappa(k-h)} |a_k - a_h| . \tag{5.31}$$

Let us now suppose that $\bar h \le h \le 0$ and that there exists a constant c_0, such that

$$|a_{k-1} - a_k| \le c_0 |\lambda_1|^{3/2} [\gamma^{-\frac12 (k - \bar h)} + \gamma^{\vartheta k}] , \quad h < k \le 0 . \tag{5.32}$$

where $\vartheta = \min\{\kappa/2, \eta'\}$, η' being defined as in Lemma 5.5. (5.32) is certainly verified for $k = 0$, thanks to second of (5.1); we want to show that it is verified also if h is substituted with $h-1$, if λ_1 is small enough.

By using (5.29), (5.21),(5.23) and (5.32), we get

$$|a_{h-1} - a_h| \le C\bar\lambda_h^2 \gamma^{\eta' h} + C|\bar\lambda_h|^2[\gamma^{-\frac12(h-\bar h)} + \gamma^{\vartheta h}] + \\ + Cc_0|\bar\lambda_h|^{5/2} \sum_{k=h+1}^{1} \gamma^{-\kappa(k-h)} \sum_{h'=h+1}^{k} [\gamma^{-\frac12(h'-h^*)} + \gamma^{\vartheta h'}] \; , \tag{5.33}$$

which immediately implies (5.32) with $h \to h-1$ and (5.28) with $\bar h \to \bar h - 1$. The bound (5.33) implies also (5.24). Finally (5.25),(5.26) follows from (5.2).■

Remark. The case treated in [PS], [Spe] corresponds to V in (1.1) given by a sum of terms of the form $\sum_{\mathbf{x},\mathbf{y}} v(\mathbf{x}-\mathbf{y})\sigma^{(\alpha)}_{\mathbf{x}}\sigma^{(\alpha)}_{\mathbf{x}'}\sigma^{(\alpha)}_{\mathbf{y}}\sigma^{(\alpha)}_{\mathbf{y}'}$ with $v(\mathbf{x}-\mathbf{y})$ short ranged, so that the (1) and (2) spin systems decouple. The above analysis of course apply but $\lambda_h = 0$ for any $h \le 0$ as $\bar\psi_{\mathbf{x}}\psi_{\mathbf{x}}\bar\psi_{\mathbf{x}}\psi_{\mathbf{x}} = 0$; this means that the quartic terms are indeed irrelevant. One finds easily that $\delta_h = 1 + O(\varepsilon)$ and $Z_h = 1 + O(\varepsilon)$ and Theorem 1 holds with $\eta_1 = \eta_2 = 0$.

5.8 The above results were proved for $Z^{-,-,-,-}_{2I}$ and we have to prove that

$$\tilde Z^{\varepsilon(1),\varepsilon'(1),\varepsilon(2),\varepsilon'(2)}_{2I} = Z^{\varepsilon(1),\varepsilon'(1),\varepsilon(2),\varepsilon'(2)}_{2I} (Z^{\varepsilon(1),\varepsilon'(1),\varepsilon(2),\varepsilon'(2)}_{0,2I})^{-1} \tag{5.34}$$

where $Z^{\varepsilon(1),\varepsilon'(1),\varepsilon(2),\varepsilon'(2)}_{0,2I}$ is given by (2.11) with $\lambda = 0$, is exponentially insensitive to boundary conditions. A careful computation of $Z^{\varepsilon(1),\varepsilon'(1),\varepsilon(2),\varepsilon'(2)}_{0,2I}$ is in §4 of [MW]. We show that for $t - t_c$ fixed, λ small enough and ν chosen as in §5.2,

$$|\log \frac{\tilde Z^{\varepsilon(1),\varepsilon'(1),\varepsilon(2),\varepsilon'(2)}_{2I}}{\tilde Z^{-,-,-,-}_{2I}}| \le |\lambda| e^{-c_1|t-t_c|M} \; , \tag{5.35}$$

where $c_1 > 0$ is a suitable constant. We can write, see (2.24) and (3.9)

$$\log Z^{\varepsilon(1),\varepsilon'(1),\varepsilon(2),\varepsilon'(2)}_{2I} = \int [\prod_{\alpha=1}^{2} P^{(\alpha)}_{\varepsilon^{(\alpha)}_\alpha, \varepsilon'^{(\alpha)}_\alpha}(d\psi^{(\alpha)}) P^{(\alpha)}_{\varepsilon^{(\alpha)}_\alpha, \varepsilon'^{(\alpha)}_\alpha}(d\chi^{(\alpha)})] e^{-\bar V} \tag{5.36}$$

and proceeding as in §3 we see that $\log Z^{\varepsilon(1),\varepsilon'(1),\varepsilon(2),\varepsilon'(2)}_{2I}$ can be written as sum of terms of the form $\sum_{\mathbf{x}_1,..,\mathbf{x}_n} W_{\underline\varepsilon}(\mathbf{x}_1,..,\mathbf{x}_n)$, with $\mathbf{x}_i$ varying in $[-\frac M2, \frac M2] \times [-\frac M2, \frac M2]$ and the W are truncated expectations for which a formula like (3.15) holds. Note that $W(\mathbf{x}_1,..,\mathbf{x}_n)$ is periodic with period M in any of its coordinates, for any $\underline\varepsilon$; this follows from the fact that there is an even number of ψ, χ fields associated to any $\mathbf{x}_i$. Moreover $W(\mathbf{x}_1,..,\mathbf{x}_n)$ is translation invariant, so that we can fix one variable to $(0,0)$, for instance $\mathbf{x}_1$; hence it holds

$$\sum_{\mathbf{x}_1,..,\mathbf{x}_n} W_{\underline\varepsilon}(\mathbf{x}_1,..,\mathbf{x}_n) = \sum_{\mathbf{x}_1,..,\mathbf{x}_n} W_{\underline\varepsilon}(\mathbf{0},\mathbf{x}_2..,\mathbf{x}_n) \; . \tag{5.37}$$

We can write $\sum_{\mathbf{x}_1,..,\mathbf{x}_n} W$ as $\sum^*_{\mathbf{x}_1,..,\mathbf{x}_n} W + \sum^{**}_{\mathbf{x}_1,..,\mathbf{x}_n} W$, where $\sum^*_{\mathbf{x}_1,..,\mathbf{x}_n}$ is over $\mathbf{x}_i$ varying in $[-\frac M4, \frac M4] \times [-\frac M4, \frac M4]$. Then $\sum^{**}_{\mathbf{x}_1,..,\mathbf{x}_n} W$ is $O(e^{-c_1|t-t_c|M})$), as in W there is surely a chain of propagators exponentially decaying connecting the point $(0,0)$ with a point outside $[-\frac M4, \frac M4] \times [-\frac M4, \frac M4]$. On the other hand in $\sum^*_{\mathbf{x}_1,..,\mathbf{x}_n} W$ we can use the Poisson summation formula, stating that

$$\frac1M \sum_{n=0}^{M-1} f(\frac{n2\pi}{M} + \frac{\alpha\pi}{M}) = \sum_{n\in Z} \hat f(nM)(-1)^{\alpha n} \tag{5.38}$$,

where f is a 2π-periodic function and $\alpha = (0,1)$. From (5.38) we find, if $g^{(i)}_{\Lambda,\varepsilon,\varepsilon'}(x,x_0)$, $i=\psi,\chi$ is the propagator corresponding to $P_{\varepsilon,\varepsilon'}(d\psi)$ or $P_{\varepsilon,\varepsilon'}(d\chi)$ (2.24)

$$g^{(i)}_{\Lambda,\varepsilon,\varepsilon'}(x-y,x_0-y_0) = \sum_{n,n_0\in Z}(-1)^{n\delta_\varepsilon}(-1)^{n\delta^0_{\varepsilon'}}g^{(i)}(x-y+nM,x_0-y_0+n_0M)\equiv$$

$$g^{(i)}(x-y,x_0-y_0)+\delta g'^{(i)}_{\varepsilon,\varepsilon'}(x-y,x_0-y_0) \tag{5.39}$$

where $g^{(i)}(x,x_0)=\lim_{M\to\infty}g^{(i)}_{\Lambda,\varepsilon,\varepsilon'}(x,x_0)$ and $\delta_\varepsilon = 1$ if $\varepsilon = -$ and $\delta_\varepsilon = 0$ if $\varepsilon = +$. Note that the only dependence on boundary conditions in the r.h.s. of (5.39) is in $\delta g^{(i)}_{\varepsilon,\varepsilon'}(x-y,x_0-y_0)$ and it holds, if $|x-y|\le\frac{M}{2}$, $|x_0-y_0|\le\frac{M}{2}$

$$|\delta g^{(i)}(x-y,x_0-y_0)|\le e^{-c_2|m_i|M}\ , \tag{5.40}$$

with a proper constant c_2. Hence all the terms in $\sum^*_{\mathbf{x}_1,..,\mathbf{x}_n}W$ with at least a $\delta g^{(i)}(x-y,x_0-y_0)$ are exponentially bounded, and the part with only $g^{(i)}(x-y,x_0-y_0)$ is independent from boundary conditions.

5.9 In lemma 5.6 we have shown that by choosing $\nu\in I_{h^*}$ then (5.24),(5.25),(5.26) hold; such ν are parametrized by $\nu_{h^*}\in J^{(h^*+1)}$. Assuming (5.4),(5.5),(5.6),(5.7) and $\bar h=-\infty$, one can proceed as in Lemma 5.2 to show that there exist a sequence ν'_h, $h\le 1$ such that

$$\nu'_h = -\gamma^{-h}\sum_{k=-\infty}^{1}\gamma^{k-1}\beta^\nu_k(\nu'_k,\dots,\nu'_1)\ . \tag{5.41}$$

If ν_h, $h^*<h\le 1$ verify (5.13) with $\nu_{h^*}=0$ it holds that

$$|\nu_h-\nu'_h|\le C\bar\varepsilon_1\gamma^{\kappa h^*}\quad h^*\le h\le 1\ ; \tag{5.42}$$

this implies that one can choose $\nu=\nu'_1$ for any h^*. (5.42) is proved by induction assuming that it holds for any $k\ge h+1$ and subtracting (5.13) with $\bar h = h^*$ and $\nu_{h^*+1}=0$ from (5.41) finding

$$\nu'_h-\nu_h = -\gamma^{-h}\sum_{k=h^*+1}^{1}\gamma^{k-1}[\beta^\nu_k(\nu'_k,\dots,\nu'_1)-\beta^\nu_k(\nu_k,\dots,\nu_1)]-\gamma^{-h}\sum_{k=-\infty}^{h^*}\gamma^{k-1}\beta^\nu_k(\nu'_k,\dots,\nu'_1)\ . \tag{5.43}$$

By using (5.14) and the inductive hypothesis (5.42) follows.

6. Correlation functions and the specific heat

6.1 In the preceding sections we have found a convergent expansion for the free energy; in order to prove Theorem 1 we have do the same for the energy-energy correlation function (1.8) and the specific heat. We start by considering the following expression

$$\frac{1}{|\Lambda|}\sum_{\mathbf{x},\mathbf{y}\in\Lambda_M} < [H_{I,\mathbf{x}}(\sigma^{(1)})+H_{I,\mathbf{x}}(\sigma^{(2)})][H_{I,\mathbf{y}}(\sigma^{(1)})+H_{I,\mathbf{y}}(\sigma^{(2)})] >_{\Lambda,T} \tag{6.1}$$

where $H_I(\sigma^{(\alpha)})=\sum_{\mathbf{x}}H_{I,\mathbf{x}}(\sigma^{(1)})$, and $H_I(\sigma^{(\alpha)})$ is the Ising model hamiltonian (1.2). With the notations of (2.10), (2.21) we can write

$$< [H_{I,\mathbf{x}}(\sigma^{(1)})+H_{I,\mathbf{x}}(\sigma^{(2)})][H_{I,\mathbf{y}}(\sigma^{(1)})+H_{I,\mathbf{y}}(\sigma^{(2)})] >_{\Lambda,T}=\sum_{\varepsilon^{(1)},\varepsilon'^{(1)}}(-1)^{\delta_{\varepsilon^{(1)},\varepsilon'^{(1)}}}\sum_{\varepsilon^{(2)},\varepsilon'^{(2)}}(-1)^{\delta_{\varepsilon^{(2)},\varepsilon'^{(2)}}}$$

$$\frac{Z_{2I}^{\varepsilon^{(1)},\varepsilon'^{(1)},\varepsilon^{(2)},\varepsilon'^{(2)}} < [\partial_t S^1_{\mathbf{x},\varepsilon^{(1)},\varepsilon'^{(1)}} + \partial_t S^2_{\mathbf{x},\varepsilon^{(2)},\varepsilon'^{(2)}}][\partial_t S^1_{\mathbf{y},\varepsilon^{(1)},\varepsilon'^{(1)}} + \partial_t S^2_{\mathbf{y},\varepsilon^{(2)},\varepsilon'^{(2)}}] >_{\Lambda,T;\varepsilon^{(1)},\varepsilon'^{(1)},\varepsilon^{(2)},\varepsilon'^{(2)}}}{\sum_{\varepsilon^{(1)},\varepsilon'^{(1)}}(-1)^{\delta_{\varepsilon^{(1)},\varepsilon'^{(1)}}}\sum_{\varepsilon^{(2)},\varepsilon'^{(2)}}(-1)^{\delta_{\varepsilon^{(2)},\varepsilon'^{(2)}}} Z_{2I}^{\varepsilon^{(1)},\varepsilon'^{(1)},\varepsilon^{(2)},\varepsilon'^{(2)}}} \tag{6.2}$$

where

$$< [\partial_t S^1_{\mathbf{x},\varepsilon^{(1)},\varepsilon'^{(1)}} + \partial_t S^2_{\mathbf{x},\varepsilon^{(2)},\varepsilon'^{(2)}}][\partial_t S^1_{\mathbf{y},\varepsilon^{(1)},\varepsilon'^{(1)}} + \partial_t S^2_{\mathbf{y},\varepsilon^{(2)},\varepsilon'^{(2)}}] >_{\Lambda;\varepsilon^{(1)},\varepsilon'^{(1)},\varepsilon^{(2)},\varepsilon'^{(2)}} = \Omega_{\underline{\varepsilon},\Lambda}(\mathbf{x}-\mathbf{y}) =$$

$$\frac{\int [\prod_{\alpha=1}^2 P^{(\alpha)}_{\varepsilon^{(\alpha)},\varepsilon'^{(\alpha)}}(dH^{(\alpha)},dV^{(\alpha)})]e^{-\mathcal{V}}[\partial_t S^1_{\mathbf{x},\varepsilon^{(1)},\varepsilon'^{(1)}} + \partial_t S^2_{\mathbf{x},\varepsilon^{(2)},\varepsilon'^{(2)}}][\partial_t S^1_{\mathbf{y},\varepsilon^{(1)},\varepsilon'^{(1)}} + \partial_t S^2_{\mathbf{y},\varepsilon^{(2)},\varepsilon'^{(2)}}]}{\int [\prod_{\alpha=1}^2 P^{(\alpha)}_{\varepsilon^{(\alpha)},\varepsilon'^{(\alpha)}}(dH^{(\alpha)},dV^{(\alpha)})]e^{-\mathcal{V}}} \tag{6.3}$$

Performing the change of variable (2.16), (2.26) we see that the r.h.s. of (6.3) can be written as $\Omega_{\underline{\varepsilon},\Lambda}(\mathbf{x}-\mathbf{y}) = \frac{\partial}{\partial\phi(\mathbf{x})}\frac{\partial}{\partial\phi(\mathbf{y})}\mathcal{S}_{\underline{\varepsilon}}(\phi)|_{\phi=0}$ where, with the notation of (3.1)

$$e^{\mathcal{S}_{\underline{\varepsilon}}(\phi)} = \int P(d\psi)\int P(d\chi)e^{Q(\chi,\psi)-\mathcal{V}(\psi,\chi)}e^{\sum_{\mathbf{x}}\phi(\mathbf{x})[\partial_t S^1_{\mathbf{x},\varepsilon^1,\varepsilon'^1}+\partial_t S^2_{\mathbf{x},\varepsilon^2,\varepsilon'^2}]}. \tag{6.4}$$

We have then to slightly adapt the analysis of (3.1) for the integration of $S_{\underline{\varepsilon}}(\phi)$.
Let us consider $S_{-,-,-,-}(\phi)$. One can proceed as in §3 in order to integrate the heavy χ fields and one finds, for $|\lambda| \le \varepsilon$ and with the notations of Theorem 2,

$$e^{\mathcal{S}(\phi)} = e^{M^2\mathcal{N}}\int \bar{P}(d\psi)e^{-\mathcal{V}^{(1)}(\psi)+\mathcal{B}(\phi,\psi)} \tag{6.5}$$

where $\mathcal{N}$ is a normalization constant and

$$\begin{aligned}\mathcal{B}(\psi,\phi) = &\sum_{m=1}^\infty\sum_{n=1}^\infty\sum_{\underline{\varepsilon},\underline{\alpha},\underline{\omega}}\sum_{\mathbf{x}_1}\cdots\sum_{\mathbf{x}_m}\sum_{\mathbf{y}_1}\cdots\sum_{\mathbf{y}_{2n}}\cdot\\ &\cdot\ B_{m,2n,\underline{\varepsilon},\underline{\alpha},,\underline{\omega}}(\mathbf{x}_1,\ldots,\mathbf{x}_m;\mathbf{y}_1,\ldots,\mathbf{y}_{2n})\Big[\prod_{i=1}^m\phi(\mathbf{x}_i)\Big]\Big[\prod_{i=1}^{2n}\partial^{\alpha_i}\psi^{\varepsilon_i}_{\mathbf{y}_i,\omega_i}\Big]\,.\end{aligned} \tag{6.6}$$

where for $n \ge 2$

$$\sum_{\mathbf{y}_1,\ldots,\mathbf{y}_{2n}}|B_{m,2n,\underline{\alpha},,\underline{\omega}}(\mathbf{x}_1,\ldots,\mathbf{x}_m;\mathbf{y}_1,\ldots,\mathbf{y}_{2n})| \le C^n\varepsilon^{\frac{n}{2}} \tag{6.7}$$

and for $n = 1$

$$\sum_{\mathbf{x}} i\omega Z^{(1)}_1\phi(\mathbf{x})\psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},-\omega} + \sum_{\mathbf{y}_1,\mathbf{y}_2}\sum_{\mathbf{x}}\sum_{\{\varepsilon,\omega\}}\sum_{\alpha_1+\alpha_2\ge 1} B_{1,\underline{\alpha},\underline{\omega}}(\mathbf{x};\mathbf{y}_1,\mathbf{y}_2)\phi(\mathbf{x})\partial^{\alpha_1}\psi^{\varepsilon_1}_{\mathbf{y}_1,\omega_1}\partial^{\alpha_2}\psi^{\varepsilon_2}_{\mathbf{y}_2,\omega_2} + \tilde{B}(\phi,\psi) \tag{6.8}$$

where $Z^{(1)}_1$ is an $O(1)$ constant, $\sum_{\mathbf{y}_1,\mathbf{y}_2}|B_{1,2,\underline{\varepsilon},\underline{\alpha},\underline{\omega}}(\mathbf{x};\mathbf{y}_1,\mathbf{y}_2)| \le C$ and $\tilde{B}(\phi,\psi)$ contains the terms with $m \ge 2$. The symmetry considerations in §3.3 imply that the only possible local terms with $n = m = 1$ are of the form $\phi(\mathbf{x})\psi^+_{\mathbf{x},1}\psi^-_{\mathbf{x},-1}$.

6.2 We shall evaluate the integral in the r.h.s. of (6.4) in a way which is very close to that used for the integration in the r.h.s. of (3.2). We introduce the scale decomposition described in §4 and we perform iteratively the integration of the single scale fields, starting from the field of scale 1. After integrating the fields $\psi^{(1)},\ldots\psi^{(h+1)}$, $0 \ge h \ge h^*$, we find

$$e^{\mathcal{S}(\phi)} = e^{-M^2E_h+S^{(h+1)}(\phi)}\int P_{Z_h,m_h,C_h}(d\psi^{\le h})e^{-\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\le h)})+\mathcal{B}^{(h)}(\sqrt{Z_h}\psi^{(\le h)},\phi)}\,, \tag{6.9}$$

where $P_{Z_h,m_h,C_h}(d\psi^{(\le h)})$ and $\mathcal{V}^h$ are given by (4.8) and (4.11), respectively, while $S^{(h+1)}$ (ϕ), which denotes the sum over all the terms dependent on ϕ but independent of the ψ field, and

$\mathcal{B}^{(h)}(\psi^{(\le h)},\phi)$, which denotes the sum over all the terms containing at least one ϕ field and two ψ fields, can be represented in the form

$$S^{(h+1)}(\phi)=\sum_{m=1}^{\infty}\sum_{\mathbf{x}_1}\cdots\sum_{\mathbf{x}_m}S_m^{(h+1)}(\mathbf{x}_1,\ldots,\mathbf{x}_m)\Big[\prod_{i=1}^m\phi(\mathbf{x}_i)\Big] \tag{6.10}$$

$$\begin{aligned}\mathcal{B}^{(h)}(\psi^{(\le h)},\phi)=&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\underline{\alpha},\underline{\omega}}\sum_{\mathbf{x}_1}\cdots\sum_{\mathbf{x}_m}\sum_{\mathbf{y}_1}\cdots\sum_{\mathbf{y}_{2n}}\cdot\\ &\cdot\ B^{(h)}_{m,2n,\underline{\alpha},\underline{\omega}}(\mathbf{x}_1,\ldots,\mathbf{x}_m;\mathbf{y}_1,\ldots,\mathbf{y}_{2n})\Big[\prod_{i=1}^m\phi(\mathbf{x}_i)\Big]\Big[\prod_{i=1}^{2n}\partial^{\alpha_i}\psi^{(\le h)\varepsilon_i}_{\mathbf{y}_i,\omega_i}\Big]\ .\end{aligned} \tag{6.11}$$

Since the field ϕ is equivalent, from the point of view of dimensional considerations, to two ψ fields, the only terms in the r.h.s. of (6.11) which are not irrelevant are those with $m=1$ and $n=1$, which are marginal. Repeating the considerations in §3.3 we can conclude that the only local terms with $n=m=1$ and $\alpha_1=\alpha_2=0$ have the form $\phi(\mathbf{x})\psi^{(\le h)+}_{\mathbf{x},\omega}\psi^{(\le h)-}_{\mathbf{x},-\omega}$. Hence we extend the definition of the localization operator $\mathcal{L}$, so that its action on $\mathcal{B}^{(h)}(\psi^{(\le h)},\phi)$ is defined by its action on the kernels $B^{(h)}_{m,2n,\underline{\omega}}(\mathbf{x}_1,\ldots,\mathbf{x}_m;\mathbf{y}_1,\ldots,\mathbf{y}_{2n})$:

1) if $m=1$, $n=1$, $\alpha_1=\alpha_2=0$ then

$$\mathcal{L}B^{(h)}_{1,2,\underline{\omega}}(\mathbf{x}_1;\mathbf{y}_1,\mathbf{y}_2)=\delta(\mathbf{y}_1-\mathbf{x}_1)\delta(\mathbf{y}_2-\mathbf{x}_1)\int d\mathbf{z}_1d\mathbf{z}_2B^{(h)}_{1,2,\underline{\sigma},\underline{\omega}}(\mathbf{x}_1;\mathbf{z}_1,\mathbf{z}_2) \tag{6.12}$$

2) in all the other cases

$$\mathcal{L}B^{(h)}_{m,2n,\underline{\varepsilon},\underline{\omega}}(\mathbf{x}_1,...\mathbf{x}_m;\mathbf{y}_1,...,\mathbf{y}_{2n})=0\ . \tag{6.13}$$

Hence for the symmetry reasons discussed in §3.3

$$\mathcal{L}\mathcal{B}^{(h)}(\psi^{(\le h)},\phi)=\frac{Z_h^{(1)}}{Z_h}F_1^{(\le h)}\ , \tag{6.14}$$

where $Z_h^{(1)}$ is a real number and

$$F_1^{(\le h)}=\sum_{\mathbf{x}}\phi(\mathbf{x})i[\psi^{(\le h)+}_{\mathbf{x},1}\psi^{(\le h)-}_{\mathbf{x},-1}-\psi^{(\le h)+}_{\mathbf{x},-1}\psi^{(\le h)-}_{\mathbf{x},1}]\ . \tag{6.15}$$

By using the notation of the preceding section we can write the integral in the r.h.s. of (6.9)

$$\begin{aligned}&e^{-M^2t_h}\int P_{\tilde Z_{h-1},m_{h-1},C_h}(d\psi^{(\le h)})e^{-\tilde{\mathcal{V}}^{(h)}(\sqrt{Z_h}\psi^{(\le h)})+\mathcal{B}^{(h)}(\sqrt{Z_h}\psi^{(\le h)},\phi)}=\\ &=e^{-M^2t_h}\int P_{Z_{h-1},m_{h-1},C_{h-1}}(d\psi^{(\le h-1)})\cdot\\ &\cdot\int P_{Z_{h-1},m_{h-1},\tilde f_h^{-1}}(d\psi^{(h)})e^{-\hat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)})+\hat{\mathcal{B}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)},\phi)}\ ,\end{aligned} \tag{6.16}$$

where $\hat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)})$ is defined as in (4.35) and $\hat{\mathcal{B}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)},\phi)=\mathcal{B}^{(h)}(\sqrt{Z_h}\psi^{(\le h)},\phi)$; moreover $\mathcal{B}^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\le h-1)},\phi)$ and $S^{(h)}(\phi)$ are then defined through the analogous of (4.38), that is

$$\begin{aligned}&e^{-\mathcal{V}^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\le h-1)})+\mathcal{B}^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\le h-1)},\phi)-M^2\tilde E_h+\tilde S^{(h)}(\phi)}=\\ &=\int P_{Z_{h-1},m_{h-1},\tilde f_h^{-1}}(d\psi^{(h)})e^{-\hat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)})+\hat{\mathcal{B}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)},\phi)}\ .\end{aligned} \tag{6.17}$$

The definitions (6.16) and (6.14) easily imply that

$$\frac{Z^{(1)}_{h-1}}{Z^{(1)}_h} = 1 + z^{(1)}_h \ , \tag{6.18}$$

where $z^{(1)}_h$ is of order ε_h and it can be written in terms of a tree expansion similar to that described in §5, as we shall explain below.

As in §4.4, the fields of scale between h^* and h_M are integrated in a single step, so we define, in analogy to (4.44),

$$\begin{aligned} & e^{\tilde{S}^{(h^*)}(\phi) - M^2 \tilde{E}_{h^*}} = \\ & \int P_{Z_{h^*-1}, m_{h^*-1}, C_{h^*}}(d\psi^{(\leq h^*)}) e^{-\hat{\mathcal{V}}^{(h^*)}(\sqrt{Z_{h^*-1}}\psi^{(\leq h^*)}) + \hat{\mathcal{B}}^{(h^*)}(\sqrt{Z_{h^*-1}}\psi^{(\leq h^*)}, \phi)} \ . \end{aligned} \tag{6.19}$$

It follows that

$$S(\phi) = -M^2 E_{L,\beta} + S^{(h^*)}(\phi) = -M^2 E_M + \sum_{h=h^*}^{1} \tilde{S}^{(h)}(\phi) \ ; \tag{6.20}$$

hence, if $\tilde{S}^{(h)}_2(\mathbf{x},\mathbf{y}) = \frac{\partial}{\partial\phi(\mathbf{x})}\frac{\partial}{\partial\phi(\mathbf{y})}\tilde{S}^{(h)}(\phi)|_{\phi=0}$

$$\Omega_\Lambda(\mathbf{x},\mathbf{y}) = S^{(h^*)}_2(\mathbf{x},\mathbf{y}) = \sum_{h=h^*}^{1} \tilde{S}^{(h)}_2(\mathbf{x},\mathbf{y}) \ . \tag{6.21}$$

The functionals $\mathcal{B}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)}, \phi)$ and $S^{(h)}(\phi)$ can be written in terms of a tree expansion similar to the one described in §4. We introduce, for each $n \geq 0$ and each $m \geq 1$, a family $\mathcal{T}^m_{h,n}$ of trees, which are defined as in §4.7, with some differences.

1) First of all, if $\tau \in \mathcal{T}^m_{h,n}$, the tree has $n+m$ (instead of n) endpoints. Moreover, among the $n+m$ endpoints, there are n endpoints, which we call *normal endpoints*, which are associated with a contribution to the effective potential on scale $h_v - 1$. The m remaining endpoints, which we call *special endpoints*, are associated with a local term of the form (6.14); we shall say that they are of type $Z^{(1)}$.

2) We associate with each vertex v a new integer $l_v \in [0,m]$, which denotes the number of special endpoints following v, *i.e.* contained in L_v.

6.3 We write $\Omega_\Lambda(\mathbf{x},\mathbf{y}) = \Omega^a_\Lambda(\mathbf{x},\mathbf{y}) + \Omega^b_\Lambda(\mathbf{x},\mathbf{y})$, where the $\Omega^a_\Lambda(\mathbf{x},\mathbf{y})$ is given by the sum over trees belonging to $\mathcal{T}^2_{h,n}$ with endpoints v to which are associated one of the terms in $\mathcal{L}\mathcal{V}^{(h_v-1)}$ or $\mathcal{L}\mathcal{B}^{(h_v-1)}$), and $\Omega^b_\Lambda(\mathbf{x},\mathbf{y})$ is the sum over the remaining trees. We can single out from $\Omega^a_\Lambda(\mathbf{x},\mathbf{y})$ the contribution from the trees with $n=0$ so that

$$\begin{aligned} \Omega^a_\Lambda(\mathbf{x},\mathbf{y}) = & \sum_{h,h'=h^*}^{1} \sum_{\omega=\pm1} \Big\{ \frac{(Z^{(1)}_{h\vee h'})^2}{Z_{h-1}Z_{h'-1}} [g^{(h)}_{\omega,\omega}(\mathbf{x}-\mathbf{y}) g^{(h')}_{-\omega,-\omega}(\mathbf{y}-\mathbf{x}) - \\ & g^{(h)}_{+1,-1}(\mathbf{x}-\mathbf{y}) g^{(h')}_{-1,+1}(\mathbf{y}-\mathbf{x})] \Big\} + \sum_{h=h^*}^{1} \left(\frac{Z^{(1)}_h}{Z_h}\right)^2 G^{(h),a}_\Lambda(\mathbf{x},\mathbf{y}) \end{aligned} \tag{6.22}$$

where $h \vee h' = \max\{h, h'\}$ and $g^{(h^*)}_{\omega_1,\omega_2}(\mathbf{x})$ has to be understood as $g^{(\leq h^*)}_{\omega_1,\omega_2}(\mathbf{x})$; moreover $(\frac{Z^{(1)}_h}{Z_h})^2 G^{(h)}_\Lambda(\mathbf{x})$ is given by the sum of trees with $n \geq 1$. It holds that (see §5.9 of [BM]) for λ small enough, for any N there exist a constant N such that

$$|\partial^{m_1}_x \partial^{m_0}_{x_0} G^{(h),a}_\Lambda(\mathbf{x},\mathbf{y})| \leq \gamma^{(2+m_0+m_1)h} |\lambda_1| \frac{C_N}{1 + (\gamma^h |\mathbf{d}(\mathbf{x}-\mathbf{y})|)^N} \ . \tag{6.23}$$

For $\Omega^b_\Lambda(\mathbf{x},\mathbf{y})$ the following bound holds

$$|\partial_x^{m_1}\partial_{x_0}^{m_0}\Omega^b_\Lambda(\mathbf{x},\mathbf{y})| \le \sum_{h=h^*}^{1}\left(\frac{Z_h^{(1)}}{Z_h}\right)^2 \gamma^{(2+m_0+m_1+\tau)h}\frac{C_N}{1+(\gamma^h|\mathbf{d}(\mathbf{x}-\mathbf{y})|)^N}\ , \tag{6.24}$$

where $0<\tau<1$ is a constant; the extra factor $\gamma^{h\tau}$ in (6.24) (with respect to (6.23)) is due to the fact that the bound over all the trees which have at least one endpoint v of fixed scale $h_v=2$ can be improved by a factor $\gamma^{\tau h}$. It is sufficient to use (4.53), which allows to extract such factor from the r.h.s. before performing the sum over the scale indices.

Note also that, by reasoning as in §5.8, for $\mathbf{x},\mathbf{y}$ and $t-t_c$ fixed

$$\lim_{M\to\infty}[\Omega_{\underline{\varepsilon},\Lambda}(\mathbf{x},\mathbf{y})-\Omega_{-,-,-,-,\Lambda}(\mathbf{x},\mathbf{y})]=0. \tag{6.25}$$

Then (6.2) is equal to the limiting value of $\Omega_{-,-,-,-,\Lambda}(\mathbf{x},\mathbf{y})$

6.4 In order to complete the proof of Theorem 1, first note that the energy-energy correlation (1.8) differs from (6.2) for terms which are faster decaying, as it is easy to check by a trivial dimensional argument. Moreover $z_h^{(1)}$ in (6.18) is given by a sum over trees $\tau\in\mathcal{T}^1_{h,n}$; an explicit computation shows that

$$z_h^{(1)}=a_1\lambda_h+O(\mu_h^2) \tag{6.26}$$

so that there exist two constant c_1,c_2 such that, under the same hypothesis of Lemma 5.6, $\gamma^{-\lambda_1 c_1 h}<\frac{Z_h^{(1)}}{Z_h}<\gamma^{-\lambda_1 c_2 h}$. If we define

$$\eta=\log_\gamma(1+z_{[h^*/2]})\ , \tag{6.27}$$

with $C_0\tilde Z_1^{-1}z_h=\frac{Z_{h-1}}{Z_h}-1$, it is easy to check that $|\frac{\gamma^{-\eta h}}{Z_h}-1|\le C\lambda_1^2$ and, from (5.2), $\eta=b_1\lambda_1^2+O(\lambda^3)$. In a similar way, if we define $\tilde\eta_1=\log_\gamma(1+z^{(1)}_{[h^*/2]})$, it holds $|\frac{Z_1^{(1)}\gamma^{-\tilde\eta_1 h}}{Z_h^{(1)}}-1|\le C|\lambda_1|$ and $\tilde\eta_1=a_1\lambda_1+O(\lambda^2)$. In order to prove the first inequality in (1.9), we write, if $m_0+m_1=n$ and $\eta_1=\eta-\tilde\eta_1$

$$\begin{aligned}&|\sum_{h=h^*}^{1}\left(\frac{Z_h^{(1)}}{Z_h}\right)^2\partial_x^{m_1}\partial_{x_0}^{m_0}G^{(h),a}_\Lambda(\mathbf{x},0)|\\ &\le C_{N,n}\sum_{h=h^*}^{0}\frac{\gamma^{(2+2\eta_1+n)h}}{[1+(\gamma^h|\mathbf{d}(\mathbf{x})|)^N]}\le\frac{C_{N,n}}{|\mathbf{d}(\mathbf{x})|^{2+2\eta_1+n}}H_{N,2+2\eta_1+n}(|\mathbf{d}(\mathbf{x})|)\end{aligned}\ , \tag{6.28}$$

where

$$\eta_1=\eta-\tilde\eta_1\ , \tag{6.29}$$

$$H_{N,\alpha}(r)=\sum_{h=h^*}^{0}\frac{(\gamma^h r)^\alpha}{1+(\gamma^h r)^N}\ . \tag{6.30}$$

On the other hand, it is easy to see that, if $\alpha\ge 1/2$ and $N-\alpha\ge 1$, there exists a constant $C_{N,\alpha}$ such that

$$H_{N,\alpha}(r)\le\frac{C_{N,\alpha}}{1+(\Delta r)^{N-\alpha}}\ ,\quad \Delta=\gamma^{h^*}\ . \tag{6.31}$$

and this easily implies the first inequality in (1.9). Proceeding in the same way by using (6.24) one can prove the second inequality in (1.9). Moreover by writing the propagators in the first two sums in the r.h.s. of (6.22) as in (4.29), (4.33) and using (4.30),(4.31),(4.32),(4.34) it follows (1.11).

Finally first note that the specific heat C_v^λ differs, by trivial dimensional arguments, from $\sum_{\mathbf{x}} \Omega_{\underline{\varepsilon},\Lambda}(\mathbf{x},\mathbf{0})$ by terms which are $O(\lambda)$. By (6.22),(6.23),(6.24) it holds

$$\sum_{\mathbf{x}} |\Omega^a_\Lambda(\mathbf{x},\mathbf{0})| \leq C \sum_{h=h^*}^{1} \left(\frac{Z^{(1)}_h}{Z_h}\right)^2 \leq C_2 \sum_{h=h^*}^{1} \gamma^{2\eta_1 h} \leq C_2 \frac{1-\gamma^{2\eta_1 h^*}}{2\eta_1} \; . \tag{6.32}$$

On the other hand, by (6.22),(6.23),(6.24)

$$|\sum_{\mathbf{x}} \Omega_\Lambda(\mathbf{x},\mathbf{0}) - \sum_{h,h'=h^*}^{1} \sum_{\omega=\pm 1} \sum_{\mathbf{x}} \frac{(Z^{(1)}_{h\vee h'})^2}{Z_{h-1}Z_{h'-1}} g^{(h)}_{L,\omega,\omega}(\mathbf{x}) g^{(h')}_{L,-\omega,-\omega}(-\mathbf{x})| \leq$$

$$C \sum_{h=h^*}^{1} \left(\frac{Z^{(1)}_h}{Z_h}\right)^2 [|\lambda| + \gamma^{\tau h} + \frac{|m_h|}{\gamma^h}] \tag{6.33}$$

so the the first of the two inequalities in (1.12) follows.
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