ON SQUIRT SINGULARITIES

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Abstract. We consider certain singularities of hydrodynamic equations that have been proposed in the literature.

We present a kinematic argument that shows that, if a volume preserving field presents these singularities, certain integrals related to the vector field have to diverge.

We also show that, if the vector fields satisfy certain partial differential equations (Navier Stokes, Boussinesq) then the integrals have to be finite.

As a consequence, these singularities are absent in the solutions of the equations. This answers a question posed by K. Moffat in [20].

1. Introduction

One way to make progress towards settling the question of existence of singularities in incompressible fluid motion is to conjecture plausible scenarios for the formation of singularities supported by numerical evidence. Then, it becomes a natural object to develop mathematically rigorous arguments that derive quantitative consequences of the different scenarios and, possibly show that these singularities cannot occur in solutions of hydrodynamic equations.

In this note we introduce some classes of singularities which we call *"squirt"* singularities in which some portion of material is ejected from a set of positive measure. These squirt singularities include as particular cases several other singularities that had been considered in the literature (for example, the *"potato chip"* singularities, the *"saddle collapse"*, the *"tube collapse"*. See Section 2 for precise definitions.)

In Section 3 we present a very simple argument that shows that if a volume preserving vector field u presents a squirt singularity at time T, then

$$\int_0^T ||u||_{L^\infty}\, dt = \infty. \tag{1}$$

In Section 4 we show that in the vector field satisfies certain partial differential equations (e.g. Navier-Stokes in 2,3 dimensions, Boussinesq equations in 2, 3 dimensions with positive viscosity), then

$$\int_0^T ||u||_{L^\infty}\, dt < \infty. \tag{2}$$

As a consequence of the results in Sections 3 and 4, we conclude that volume preserving vector fields satisfying the partial differential equations considered in Section 4 do not experience any of the squirt singularities.

The above results include as a particular case an answer to a question proposed by K. Moffatt in [20]. We show that if a two dimensional fluid satisfies the Boussinesq equation describing a fluid moving under buoyancy forces with positive fluid viscosity $\nu > 0$, but possibly with zero molecular diffusivity $\kappa = 0$, then, it cannot have a saddle collapse.

Even if the arguments presented here exclude that the singularities happen, they give little information on how fast, the singular terms may grow. In some of the cases discussed here, these more quantitative arguments are available in the literature (see [8]). They, of course, require using more heavily the details of the equation and the singularity.

2. Squirt singularities

In this section we collect the definitions of the different types of singularities that we will be considering in this paper.

2.1. **Notation.** We denote the Lebesgue measure of a set A by $|A|$ and the ball centered at $\mathbf{x}^0$ with radius r by $B_r(\mathbf{x}^0)$.

Let $\Omega \subset \mathbb{R}^n$ be an open set. We consider a C^1 time dependent vector field $u : \Omega \times [0, T) \to \mathbb{R}^n$.

This vector field defines an evolution for trajectories $\Phi_t(x)$, where $\Phi_t(x)$ denotes the position at time t of the trajectory with initial condition x at time $t = 0$. More generally, we denote by $\Phi_{t,a}(x)$ the position at time t of the trajectory which at time $t = a$ is in x. Note that, when both sides of the formulas make sense, $\Phi_t(x) = \Phi_{t,0}(x)$, $\Phi_{t,a} = \Phi_t \circ \Phi_a^{-1}(x)$, $\Phi_{t,a} \circ \Phi_{a,b}(x) = \Phi_{t,b}(x)$.

For $\mathcal{S} \subset \Omega$, we denote by $\Phi^{\Omega}_{t,a}\mathcal{S} = \{x \in \Omega \mid x = \Phi_t(y),\ y \in \mathcal{S},\ \Phi_s(y) \in \Omega,\ 0 \le s \le t\}$. That is, $\Phi^{\Omega}_{t,a}$ is the evolution of the set $\mathcal{S}$, starting at time a, after we eliminate the trajectories which step out of Ω.

We will, henceforth assume that u is divergence free. Given the fact that u has zero divergence, we have that $|\Phi_{t,s}\mathcal{S}|$ is independent of t and $|\Phi^{\omega}_{t,a}\mathcal{S}|$ is non-increasing in t.

2.2. **Definition of singularities.**

2.2.1. *Squirt singularities.* The following definition will be the hypothesis of the main kinematic result of this paper, Theorem 3.1.

Definition 1. *Let Ω_-, Ω_+ be open and bounded sets. $\overline{\Omega_-} \subset \Omega_+$. (Therefore,* $\operatorname{dist}(\Omega_-, \mathbb{R}^d - \Omega_+ \ge r > 0$.*)*

We say that u experiences a squirt singularity in Ω_-, at time $T > 0$ when for every $0 \le s < T$. we can find a set $\mathcal{S}_s \subset \Omega_+$ such that

- *$\mathcal{S}_s \cap \Omega_-$ has positive measure, $0 \le s < T$*
- $\lim_{t \to T} |\Phi^{\Omega_+}_{t,s}\mathcal{S}_s| = 0$

The physical intuition is that there is a region of positive volume so that all the fluid occupying it gets ejected from a slightly bigger region in a finite time.

As we see in the following subsections, Definition 1 includes as particular cases, other singularities that have been considered in the literature. Of course, the conclusions of Theorem 3, which uses only Definition 1 as hypothesis, are a fortiori valid when we use as hypothesis the existence of the other singularities that we now formulate.

2.2.2. *Potato chip singularities.*

Definition 2. *We say that u experiences a potato chip singularity when we can find continuous functions*

$$\begin{aligned} f_{\pm} : \mathbb{R}^{n-1} \times [0,T) \to \mathbb{R} \\ f_{+}(x_1,\dots,x_{n-1},t) \geq f_{-}(x_1,\dots,x_{n-1},t) \quad t\in[0,T], x_1,\dots,x_{n-1}\in B_r(\Pi\mathbf{x}^0) \\ f_{+}(x_1,\dots,x_{n-1},0) > f_{-}(x_1,\dots,x_{n-1},0) \quad x_1,\dots,x_{n-1}\in B_{\frac{r}{2}}(\Pi\mathbf{x}^0) \\ \lim_{t\to T^-}\left(f_{+}(x_1,\dots,x_{n-1},t) - f_{-}(x_1,\dots,x_{n-1},t)\right) = 0 \quad \textit{for all} \quad x_1,\dots,x_{n-1}\in B_r(\Pi\mathbf{x}^0) \end{aligned}$$

The surfaces

$$\Sigma_{\pm,t} = \{x_n = f_{\pm}(x_1,\dots,x_{n-1},t)\} \subset \Omega$$

are transformed into each other by the flow

$$\Phi_t(\Sigma_{\pm,0}) \supset \Sigma_{\pm,t} \ .$$

Note that in definition 2.2.2 we are not requiring that the functions are C^1 as it was done in [12]. For us, it suffices that $f_{\pm}$ are continuous. That is, we allow the singularities to be ruffled potato chips. Since the arguments we will present in Section 3 do not depend on calculus identities, there is no need for the boundaries of the sets to be differentiable.

If we denote by $\mathcal{S}_t^{(f)} = \{x \mid f_{-}(x_1,t) \leq x_n \leq f_{+}(x_1,t)\}$, we have, by the intermediate value theorem, and the continuity of the trajectories

$$\Phi_{t,s}^{B_r(\mathbf{x}^0)}(\mathcal{S}_s^{(f)}) \subset \mathcal{S}_t. \tag{3}$$

In particular, if $f_{\pm}$ verify definition 2.2.2, then $\mathcal{S}_s^{(f)}$ verifies the assumptions of Definition 1.

Hence, if a system satisfies Definition 2.2.2, it also satisfies Definition 1.

Potato chips singularities were introduced as a conjectural mechanism (see [17] and [19]) of singularities for a 3D ideal Magnetohydrodymamic flow, in which two linked flux rings approach each other forming two-dimensional current sheets. In two dimensional cases, similar singularities were proposed by [21], [22].

The two dimensional potato chips singularities were considered in [9], where they were called *"sharp fronts"*. Using calculus identities and the fact that the fluid admits a stream function representation, it was shown that if a sharp front exits, then (1) holds. This result was generalized to three dimensions in [12]. Both of these results follow from Theorem 3.1.

2.2.3. *Tube collapse singularities.* The following definition appears for the case $n = 3$ in [11]. In the case $d = 2$ the concept was introduced in [9], [10].

Let $I_i \subset \mathbb{R}$, $i = 1, \dots n$ be bounded intervals. Let $Q = \times_i I_i \subset \mathbb{R}^n$ be a cube.

Definition 3. *A regular tube is a relatively open set $\mathcal{S} \subset Q$. characterized as*

$$\mathcal{S} = \{x \in Q \,|\, f(x) < 0\} \tag{4}$$

where $f : Q \to \mathbb{R}$ is a C^1 function that satisfies

$$f(x) = 0 \implies \nabla_{x_1,\dots,x_{n-1}} f \neq 0$$

For every $x_n \in I_n$, the set

$$\mathcal{S}(x_n) = \mathcal{S} \cap I_1 \times \cdots \times I_{n-1} \times \{x_n\}$$

is non empty and its closure is contained in the interior of $I_1 \times \cdots \times I_{n-1} \times \{x_n\}$.

We will also consider the situation when f_t is a family of functions indexed by time, $t \in [0, T)$.

Definition 4. . *We say that the vector field u experiences a tube collapse singularity at time T when the boundaries of the tube evolve with the velocity field u and $\liminf_{t\to T} |\mathcal{S}_t| = 0$.*

An example that is worth keeping in mind is when $f_t(x) = \operatorname{dist}(x, \gamma) + r(t)$, where γ is a curve, dist denotes the distance and $r(t) \to 0$ as $t \to 0$. S_t is the set of points which are at a distance less than $r(t)$ from the curve γ. (Of course, we could let the curve γ depend on time provided that it does not become too pathological.)

Again, we point that Definition 4 implies Definition 1. We can take $\Omega_- = \times_{i=1,\dots n-1} I_i \times J$, $\Omega_- = \times_{i=1,\dots n-1} I_i \times I_d$ where $J \subset I_d$ is an interval contained in the interior of I_d.

2.2.4. *Saddle collapse singularity.* This singularity is specific of two dimensional flows. We follow the definition in [8]. We refer to that paper for a comparison with alternative definitions in the literature.

Definition 5. *We consider foliations of a neighborhood of the origin (with coordinates x_1, x_2) whose leaves are given by equations of the form*

$$\rho \equiv (y_1\beta(t) + y_2) \cdot (y_1\delta(t) + y_2) = \text{cte} \tag{5}$$

and $(y_1, y_2) = F_t(x_1, x_2)$, where $\beta, \delta : [0, T) \to \mathbb{R}^+$, are C^1 functions and F is a C^2 fuction of x, t, for a fixed t, F_t is an orientation preserving diffeormorphism.

We say that the foliation experiences a saddle collapse when $\liminf_{t\to T}\beta(t)+\delta(t)=0$.

If the leaves of the foliation are transported by a vector field u, we say that the vector field u experiences a saddle collapse.

If we take as $\Omega_\pm$ balls centered at $F(0,0,T)$, and as the set $\mathcal{S}_s$ a connected component of the set $\rho<0$, we see that Definition 5 implies Definition 1.

3. Kinematic arguments

The main result of this section is:

Theorem 3.1. *If u as before has a squirt singularity, then*

$$\int_s^T \sup_x |u(x,t)|\,dt=\infty \quad \forall s\in(0,T) \tag{6}$$

Moreover, if u has a potato chip singularity, then

$$\int_s^T \sup_x |\Pi u(x,t)|\,dt=\infty \tag{7}$$

Remark. *We note that in the argument for Theorem 3.1, some of the hypothesis can be somewhat weakened*

For example, using the theory of [13]*, the hypothesis that* $u\in C^1$ *can be weakened to* $u\in H^1$.

We also note that strict volume preservation is not needed. It suffices that the volume contraction remains bounded. That is, for some constant $C\geq 1$ *and all* $M\subset\mathbb{R}^n$ *measurable,* $C^{-1}|M|\leq|\Phi_t(M)|\leq C|M|$.

Remark. *We note that if* $u(x,t)$ *experiences a squirt singularity at* $t=T$ *and* $\Gamma:[0,T)\hookleftarrow[0,T)$ *is a reparameterization, then*

$$\tilde u(x,t)=u(x,\Gamma(t))\Gamma'(t)$$

also has a potato chip singularity.

It is reassuring to note that the conclusions of Theorem 3.1 remain true for $\tilde u$. *But the observation that the existence of potato chip singularities is invariant under time reparameterizations shows that, with the present assumptions, one cannot obtain more precise rates of the blow-up of* $\sup_x|u(x,t)|$ *than* (6).

In case that we assume that singularities are somewhat more uniform, it is possible to develop more quantitative information about the rate at which happen.

Roughly speaking, we just need to assume that the exit area of the set $\mathcal{S}_s$ controls the volume of the set.

For example, in potato chip singularities (Definition 2), we say that the collapse is uniform when

$$\max_{x_1,x_2}(f_+(x_1,x_2,t)-f_-(x_1,x_2,t))\leq M\min_{x_1,x_2}(f_+(x_1,x_2,t)-f_-(x_1,x_2,t))$$

where M is a constant independent of time.

In tube collapse singularities (Definitions 4, 3 we say that the collapse is uniform when

$$\max |S(x_n)|_{n-1} \leq M \min |S(x_n)|$$

where M is a constant independent of time and $|\cdot|_{n-1}$ denotes the $n-1$ dimensional area.

Given a S_t a C^1 set, we denote by $\tilde{\partial} S_t$ the portion of the boundary which is not evolving with the fluid.

We note that by zero divergence of the fluid, the change of volume is the integral of u over $\tilde{\partial} S_t$. Hence, we always have

$$\frac{d}{dt}|\mathcal{S}_t| \geq -||u||_{L^\infty} |\tilde{\partial}\mathcal{S}_t|_{n-1}$$

In the uniform cases, we have:

$$\frac{d}{dt}|\mathcal{S}_t| \geq -M||u||_{L^\infty} |\mathcal{S}_t|$$

Integrating the above equation we have

$$|\mathcal{S}_t| \geq |\mathcal{S}_0| \exp(-M \int_0^t ||u(s)||_{L^\infty}\, ds)$$

3.1. **Proof of Theorem 3.1.** From the assumption that $|\Phi^{\Omega_+}_{T,s}\mathcal{S}_s| \to 0$, we conclude that almost all the trajectories starting in $\mathcal{S}_s$ at time s leave the set Ω_+ at a time in (s, T).

Therefore we conclude that for a trajectory $x(t)$ starting in $\Omega_- \cap \mathcal{S}_s$ at time s we have,

$$\Big| \int_s^T u(\Phi_t(x), t)\, dt \Big| \geq r > 0 \ . \tag{8}$$

Therefore,

$$\int_s^T \sup_x |u(x,t)|\, dt \geq r > 0 \ . \tag{9}$$

Since (9) holds for every $s \in (0, T)$ we conclude that (6) holds.

To establish (7) we observe that in the classical potato chip singularity, since the escape can only happen by increasing the $n-1$ first components we can sharpen (8) to

$$\Big| \int_s^T \Pi u(\Phi_t(x), t)\, dt \Big| \geq r/2 \ .$$

again for all $s \in [0, T)$.

4. A priori bounds

In this Section, we show how if the vector field u satisfies certain partial differential equations, then, (2) holds. By Theorem 3.1, we conclude immediately that these equations do not exhibit any of the singularities considered in Definition 1.

We consider, two and three dimensional Boussinesq equations and Navier-Stokes equations in three dimensions.

We note that the results on two dimensional Boussinesq equations solve the problem proposed by K. Moffatt [20]:

XXI Century Problem 3: *The problem is to examine the evolution of the θ-field for Boussinesq equations (see* [1]*) in the neighborhood of its saddle points, to determine whether singularities of $\nabla\theta$ can develop, and to examine the influence of weak molecular diffusivity k in controlling the approach to such singularities.*

The case of Navier-Stokes equations has been in the literature for a long time. See, for example [16] and the exposition in [14], where it is called the *"second F_N ladder"*.

We point out that the proofs are based on very elementary arguments. Basically, integration by parts, Sobolev and interpolation inequalities. Hence, they remain valid for all the boundary conditions that allow to carry out these operations. These includes problems defined in the whole space, in a bounded domain with periodic, Neuman and Dirichlet conditions. We will, therefore, not state this very explicitly in the calculations.

4.1. **Two dimensional Boussinesq Equations.**

The Boussinesq equations are:

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p + \nu \Delta u + (0, \theta) \tag{10}$$

$$\nabla \cdot u = 0 \tag{11}$$

$$(\partial_t + u \cdot \nabla)\, \theta = \kappa \Delta \theta \tag{12}$$

with $u = (u_1, u_2)$, $x = (x_1, x_2) \in R^2$ or R^2/Z^2 and finite energy at initial time.

The Cauchy problem for the system (10), (11) and (12) have been extensively studied in the literature, see [5], [18] and [24]. In the case $\kappa > 0$ it is known that the equation does not develop singularities in finite time. But in order to study the evolution of the level sets of θ it is reasonable to take $\kappa = 0$, where the collapse of the saddle would produce a singularity on $\nabla\theta$. This is a 2-dimensional potato chip singularity, for more details see [9].

In [6], [7] and [15] the 2-dimensional Boussinesq convection in the absence of viscous effects were studied numerically and analytically.

Theorem 4.1. *If u satisfies the two-dimensional Boussinesq equation with $\nu > 0$ and $||\theta(0)||_{L^2} \leq A < \infty$, then* (2) *holds.*

In particular, using Theorem 3.1, u does not exhibit any singularity satisfying Definition 1.

Proof We denote by C_1, C_2 constants that depend only on ν, A and the initial conditions. In particular, they can change the meaning from line to line.

From equation (12) we obtain that the L^p norms $p \geq 1$ are nonincreasing – they are conserved if $\kappa = 0$.

$$\|\theta(\cdot,t)\|_{L^p} \leq \|\theta(\cdot,0)\|_{L^p} \quad \text{for } 1 \leq p \leq \infty \quad \text{for all } t \geq 0.$$

Taking the curl of the equations (10) of the velocity field we get

$$(\partial_t + u \cdot \nabla)\, \omega = \theta_{x_1} + \nu \Delta\omega \tag{13}$$

where $\omega = \mathrm{curl}(u)$.

We multiply (13) by ω and integrate by parts to obtain:

$$\frac{1}{2}\frac{d}{dt}\int |\omega|^2\, dx + \nu \int |\nabla\omega|^2 dx = \int \omega\theta_{x_1} dx. \tag{14}$$

Integration by parts and Hölder inequality gives

$$\frac{1}{2}\frac{d}{dt}\int |\omega|^2\, dx + \nu ||\nabla\omega||_{L^2}^2 \leq ||\nabla\omega||_{L^2} ||\theta||_{L^2}.$$

This implies that

$$\frac{d}{dt}\int |\omega|^2\, dx \leq C_1$$

and, therefore $||\omega||_{L^2} \leq C_1 t + C_2$.

Substituting this into (14) gives

$$\nu ||\nabla\omega||_{L^2}^2 \leq A||\nabla\omega||_{L^2} - \frac{1}{2}\frac{d}{dt}||\omega||_{L^2}$$

An integration with respect to time and a Hölder inequality for the integration with respect to time yields

$$\begin{aligned}\nu \int_0^t ds\, ||\nabla\omega(s)||_{L^2}^2 &\leq A\int_0^t ds\, ||\nabla\omega(s)||_{L^2} - \frac{1}{2}||\omega(t)||_{L^2}^2 + \frac{1}{2}||\omega(0)||_{L^2}^2 \\ &\leq At^{1/2}\left(\int_0^t ds\, ||\nabla\omega(s)||_{L^2}^2\right)^{1/2} + \frac{1}{2}||\omega(0)||_{L^2}^2\end{aligned}$$

This yields

$$\int_0^t ds\, ||\nabla\omega(s)||_{L^2}^2 \leq C_1 t + C_2 \tag{15}$$

and using Hölder inequality again,

$$\int_0^t ds\, ||\nabla\omega(s)||_{L^2} \leq C_1 t + C_2 \tag{16}$$

The well known Biot-Savart law, recovers the velocity field from the vorticity by the integral operator

$$u(x,t) = \frac{1}{2\pi} \int K(x-y)\omega(y,t)dy$$

with $K(x) = (-\frac{x_2}{x_1^2+x_2^2}, \frac{x_1}{x_1^2+x_2^2})$ for $x \in R^2$ and a similar formula holds for R^2/Z^2.

Furthermore, ∇u is a singular integral operator of ω and Δu is a singular integral operator of $\nabla\omega$ (for details see [3]). From the classical Calderon-Zygmund theory we have

$$\|\nabla u\|_{L^2} \leq C\|\omega\|_{L^2}, \qquad \|\Delta u\|_{L^2} \leq C\|\nabla\omega\|_{L^2} \tag{17}$$

Combining estimates (15), (16), and (17) and using Sobolev inequalities we finally get

$$\begin{aligned} \int_0^t \|u\|_{L^\infty} ds &\leq C \int_0^t (\|u\|_{L^2} + \|\Delta u\|_{L^2}) ds \\ &\leq C_1 t + C_2 \end{aligned}$$

□

Remark. *The argument above works for $\nu > 0$. For $\nu = 0$ we do not have control on any norm of the derivatives of the vorticity.*

4.2. **Three dimensional Boussinesq equations.** In this section we adapt Theorem 4.1 to three dimensions.

Theorem 4.2. *If u satisfies the three-dimensional Boussinesq equation with $\nu > 0$ and $||\theta_0||_{L^2} < \infty$, then* (2) *holds.*

In particular, using Theorem 3.1, u does not exhibit any singularity satisfying Definition 1.

Compared with the proof of Theorem 4.1, the proof of Theorem 4.2 requires an extra estimate on the nonlinear term that appears on the vorticity equation. Below we give the argument which is based on the argument in [16] for Navier-Stokes.

By the usual integration by parts

$$\frac{1}{2}\frac{d}{dt}\int |u|^2\, dx + \nu \int |\nabla u|^2 dx \leq C \int |u\theta| dx.$$

therefore, proceeding as before

$$\|u\|_{L^2}^2 \leq C_1 t + C_2$$

$$\int_0^t \|\nabla u\|_{L^2}^2 ds \leq \tilde{C}_1 t + \tilde{C}_2 \tag{18}$$

The vorticity equation is

$$(\partial_t + u \cdot \nabla)\, \omega = \omega \cdot \nabla u + \theta_{x_1} - \theta_{x_2} + \nu\Delta\omega$$

Multiply the vorticity equation by ω and integrate by parts

$$\frac{1}{2}\frac{d}{dt}\int |\omega|^2\,dx + \nu\int|\nabla\omega|^2 dx < \int |(\omega\cdot\nabla u)\omega| dx + \frac{1}{2\nu}\int|\theta|^2\,dx + \frac{\nu}{2}\int|\nabla\omega|^2 dx$$

The nonlinear term can be bounded by (see [16])

$$\begin{aligned}\int |(\omega\cdot\nabla u)\omega| dx &\le C\|\omega\|_{L^2}^{\frac{3}{2}}\|\nabla\omega\|_{L^2}^{\frac{3}{2}}\\ &\le \tilde{C}\|\omega\|_{L^2}^6 + \frac{\nu}{4}\|\nabla\omega\|_{L^2}^2\end{aligned}$$

then

$$\frac{1}{2}\frac{d}{dt}\int|\omega|^2\,dx + \frac{\nu}{4}\int|\nabla\omega|^2 dx \le \tilde{C}(1+\|\omega\|_{L^2}^6)$$

and

$$\frac{\frac{1}{2}\frac{d}{dt}\|\omega\|_{L^2}^2}{(1+\|\omega\|_{L^2}^2)^2} + \nu\frac{\|\nabla\omega\|_{L^2}^2}{(1+\|\omega\|_{L^2}^2)^2} \le \tilde{C}(1+\|\omega\|_{L^2}^2)$$

we get

$$\int_0^t \frac{\|\nabla\omega\|_{L^2}^2}{(1+\|\omega\|_{L^2}^2)^2} ds \le \tilde{\tilde{C}}(1+t) \tag{19}$$

Finally we estimate $\int_0^t \|u\|_{L^\infty} ds$ applying Sobolev inequalities, Calderon-Zygmund theory, (18) and (19)

$$\begin{aligned}\int_0^t \|u\|_{L^\infty} ds &\le C\int_0^t \|\nabla u\|_{L^2}^{\frac{1}{2}}\|\Delta u\|_{L^2}^{\frac{1}{2}} ds\\ &\le C\left(\int_0^t\|\nabla u\|_{L^2}^2 ds + \int_0^t \|\Delta u\|_{L^2}^{\frac{2}{3}} ds\right)\\ &\le C\left[\int_0^t \|\omega\|_{L^2}^2 ds + \left(\int_0^t \frac{\|\nabla\omega\|_{L^2}^2}{(1+\|\omega\|_{L^2}^2)^2} ds\right)^{\frac{1}{3}}\left(\int_0^t (1+\|\omega\|_{L^2}^2) ds\right)^{\frac{2}{3}}\right]\\ &\le C(1+t)\end{aligned}$$

where C depends on the initial data and on the viscosity.

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