Mel'nikov method revisited

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Abstract

We illustrate a completely analytic approach to Mel'nikov theory, which is based on a suitable extension of a classical method, and which is parallel and – at least in part – complementary to the standard procedure. This approach can be also applied to some "degenerate" situations, as to the case of nonhyperbolic unstable points, or of critical points located at the infinity (thus giving rise to unbounded orbits, e.g. the Keplerian parabolic orbits), and it is naturally "compatible" with the presence of general symmetry properties of the problem. These peculiarities may clearly make this approach of great interest in celestial mechanics, as shown by some classical examples.

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1 Introduction

It is certainly impossible to give a fairly complete list of the papers devoted to the applications of the Mel'nikov method [1, 2, 3] for evaluating the onset of chaos arising in perturbed homoclinic (or heteroclinic) orbits. We will quote only some of the papers which are more directly connected with the present approach.

The main purpose of this paper is to illustrate a completely analytic procedure, based on a suitable extension of a classical method [4, 5], which is parallel and – at least in part – complementary to the standard approach (see e.g. [2, 3, 6]). This procedure can be also applied to some "degenerate" situations, as to the case of unstable nonhyperbolic points, or of critical points located at the infinity (thus giving rise to unbounded orbits, e.g. the Keplerian parabolic orbits), and it is naturally "compatible" with the presence of general symmetry properties of the problem. For these reasons, apart from a clear "unifying" aspect, this method could be of great interest in celestial mechanics, and it could be a contribution to the study of some of the questions about the onset of chaos in unbounded phase space systems and in the presence of unbounded orbits. It has been remarked indeed that in this situation chaos manifests itself in a particularly dramatic way [7].

Let us remark immediately that, in the above mentioned degenerate cases, i.e. in the lack of the hypothesis of hyperbolicity, standard results of perturbation theory cannot be directly applied; for instance, to preserve the criticality, we will have to impose a sufficiently rapid vanishing of the perturbation at the critical point. We can then extend the introduction of Mel'nikov functions, and show not only the existence of smooth solutions of the perturbed problem, approaching the critical points and playing in this context the role of stable and unstable manifolds, but also the possible presence of infinitely many intersections of these asymptotic sets on the Poincaré sections, thus leading to a complicate dynamics typical of the homoclinic chaotic behaviour [2, 3, 6, 8, 9].

Being mainly interested in the methodological aspects, we will not devote special emphasis on new applications, but rather we will show, in the two last sections of this paper, how the method can be concretely applied in some typical situations, arising especially in celestial mechanics and general relativity.

2 Statement of the Problem

Although the procedure is completely general (indeed, we will state some results in a quite general setting in sections 3 and 4), we have actually in mind applications to celestial mechanics or to gravitational problems in general relativity; therefore we will restrict our attention mainly to problems described by Hamiltonians of the classical form

$$H = \frac{1}{2}\Big(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2 \sin^2\theta}\Big) + V(r) \tag{1}$$

where $V(r)$ is some "long range" potential. Concretely, we have in mind potentials of the form

$$V(r) = -\frac{1-\beta_1}{r} - \frac{\beta_2}{r^2} - \frac{\beta_3}{r^3} \tag{2}$$

where β_i are given parameters, which model several interesting situations, including standard Kepler problem, Kepler problems plus quadrupolar effects [7], Manev problem [10], or the motion of a charged particle in the field of a Schwarzschild black hole in general relativity [11, 12], and many other situations.

Let us write the equations of the motion, assuming for a moment that it lies in the plane $\theta = \pi/2$,

$$\dot r = p_r \quad , \quad \dot p_r = -\frac{dV}{dr} + \frac{L^2}{r^3} \quad , \quad \dot\varphi = \frac{L}{r^2} \quad , \quad \dot p_\varphi = 0 \tag{3}$$

where we have put $L = p_\varphi = \text{const} \neq 0$. We are interested in this paper in the appearance of chaotic behaviour related to the presence of homoclinic (or heteroclinic) orbits subjected to perturbations; therefore, the relevant situations which may occur, depending on the values of the parameters β_i in the potential (2), and which we are going to deal with in this paper, are the following:

i) the presence of an unstable equilibrium point $r_u \neq 0$ for the first two equations in the system (3), which involve the variables r, p_r. This point corresponds in the plane $\theta = \pi/2$ to a unstable circular orbit γ of radius r_u. In this case, we have also the presence of a $1-$parameter family of homoclinic bounded orbits biasymptotic to γ (but see also case *iii)* below).
ii) the degenerate situation where the unstable equilibrium point is located at the infinity, i.e. $r = \infty$, $\dot{r} = 0$; the homoclinic orbits are in this case a family of parabolas.
iii) a "critical" case, with an unstable equilibrium point $r_u \neq 0$ and another unstable equilibrium point located at $r = \infty$, and a family of heteroclinic orbits connecting these points.

It is clear that – due to the spherical symmetry of the problem – all conclusions and properties stated for the plane $\theta = \pi/2$ are equally true for any plane for the origin in $\mathbf{R}^3$.

Let us now choose and single out the following homoclinic (heteroclinic in case *iii)*) orbit in the plane $\theta = \pi/2$, denoted by $\widehat{\chi}(t)$, written in the spherical variables

$$u := (r, p_r, \varphi, p_\varphi, \theta, p_\theta)$$

as follows

$$\widehat{\chi}(t) := (R(t), \dot{R}(t), \Phi(t), L, \pi/2, 0) \tag{4}$$

where $R(t)$ and $\Phi(t)$ solve (3) with the conditions $R(\pm\infty) = r_u$ in case *i)* or respectively $R(\pm\infty) = \infty$ in case *ii)*, and $R(-\infty) = r_u$, $R(+\infty) = +\infty$ in case *iii)*, and with $\Phi(0) = \pi$. It is not necessary, for our purposes, to know explicitly the expression of the functions $R(t)$ and $\Phi(t)$; it will be useful only to know that choosing $R(0) = r_0$ (the turning point) in cases *i)* and *ii)*, $R(t)$ is an even function and $\Phi(t)$ an odd function of the time t. Let us remark that any other homoclinic orbit can be transformed by means of a rotation into the $\widehat{\chi}(t)$ given by (4).

We now introduce a smooth (analytic) perturbation depending in general on all the variables u and time-periodic; the equations of the motion we are considering are then, in general,

$$\begin{aligned}
\dot{r} &= p_r + \epsilon g_r(u,t) \\
\dot{p}_r &= \frac{p_\theta^2}{r^3} + \frac{p_\varphi^2}{r^3 \sin^2\theta} - \frac{dV}{dr} + \epsilon g_{p_r}(u,t) \\
\dot{\varphi} &= \frac{p_\varphi}{r^2 \sin^2\theta} + \epsilon g_\varphi(u,t) \\
\dot{p}_\varphi &= \epsilon g_{p_\varphi}(u,t) \\
\dot{\theta} &= \frac{p_\theta}{r^2} + \epsilon g_\theta(u,t) \\
\dot{p}_\theta &= \frac{p_\varphi^2 \cos\theta}{r^2 \sin^3\theta} + \epsilon g_{p_\theta}(u,t)
\end{aligned} \tag{5}$$

where $\epsilon \ll 1$. Let us also write (5) in a more convenient compact form

$$\dot{u} = f(u) + \epsilon g(u,t) \tag{6}$$

where

$$f = J\nabla_u H \tag{7}$$

and J is the standard symplectic matrix. Given any homoclinic orbit $\chi(t)$ of the unperturbed problem, one has that $\chi_{t_0} := \chi(t-t_0)$ satisfies, for all $t_0 \in \mathbf{R}$,

$$\frac{d\chi_{t_0}}{dt} = f(\chi_{t_0}) \tag{8}$$

In order to find conditions ensuring the occurrence of intersections of stable and unstable manifolds of the critical point for the perturbed problem, and hence the appearance of chaotic behaviour, we follow a (suitably extended) procedure which has been first used in this context (to the best of our knowledge) by Chow, Hale and Mallet-Paret [4] in a problem with 1 degree of freedom. We have first to look for smooth solutions near the homoclinic orbit; we then put

$$u(t) = \chi(t-t_0) + v(t-t_0) \tag{9}$$

Substituting in (6), we find for $v(t)$ the equation (with the time shift $t \to t + t_0$)

$$\begin{aligned}
\dot{v} &= A(t)v + \epsilon g(\chi(t), t+t_0) + \text{higher order terms in } v \text{ and } \epsilon \\
&:= A(t)v + G(t, t_0, \epsilon)
\end{aligned} \tag{10}$$

having used the shorthand notation $G(t, t_0, \epsilon)$, and where

$$A(t) := (\nabla_u f)(\chi(t)) \tag{11}$$

This equation, or more often its first-order approximation

$$\dot{v} = A(t)v + \epsilon g(\chi(t), t + t_0) \tag{12}$$

is called the variational equation associated to (6) and to the orbit $\chi(t)$.

All the solutions $v(t)$ of (10) can be written, in implicit form (see [5, 13])

$$v(t) = v_h(t) + \Psi(t) \int_{t_1}^{t} \Psi^{-1}(s)\, G(s, t_0, \epsilon)\, ds \tag{13}$$

where t_1 is arbitrary, $v_h(t)$ is any solution of the homogeneous linear problem

$$\dot{v}_h = A(t) v_h \tag{14}$$

and Ψ is a fundamental matrix of solutions of (14). We now have to look for the solutions of the homogeneous equation (14) and to control their behaviour for $t \to \pm\infty$. As well known [5, 9, 13], this equation admits some solutions which remain bounded for all $t \in \mathbf{R}$ and other solutions which diverge for $t \to \pm\infty$. The asymptotic behaviour is exponential in the case of hyperbolic unstable points, is like some power $|t|^\mu$ in the case of unstable point at $r = \infty$, $\dot{r} = 0$, and more in general of nonhyperbolic critical points.

An obvious bounded solution is given by $d\chi/dt$. To find other solutions, the following results, which we will state for convenience in a quite general form, will be useful.

3 General Results. The Role of Symmetry

Let us start with the following definition (cf. [14, 15])

Definition 1 *A vector field*

$$S := \sigma(u) \cdot \nabla = \sum_{i=1}^{n} \sigma_i(u) \frac{\partial}{\partial u_i} \tag{15}$$

is said to be a Lie-point symmetry (or – more exactly – the Lie generator of a Lie-point symmetry) for a general dynamical system (not necessarily Hamiltonian)

$$\dot{u} = f(u) \tag{16}$$

if

$$[\sigma \cdot \nabla, f \cdot \nabla] = 0 \tag{17}$$

where $[\cdot,\cdot]$ *denotes the usual Lie commutator.*

We then can state:

Lemma 1 *Let* $\chi(t)$ *be any given homoclinic orbit, solution of the problem (16), and assume that this problem admits a Lie-point symmetry* $S = \sigma(u)\cdot\nabla$*. Then*

$$\zeta(t) := \sigma(\chi(t))$$

is a solution of the homogeneous part (14) of the variational equation associated to this problem.

Proof. From direct calculation, using (17) and (11),

$$\dot{\zeta} = \frac{d}{dt}\sigma(\chi(t)) = \nabla_u\sigma \cdot \dot{\chi} = (f \cdot \nabla_u \sigma)(\chi(t)) = (\nabla_u f \cdot \sigma)(\chi(t)) = A(t)\zeta.$$

Remark. The obvious solution $d\chi/dt$ of (14) may be included into the solutions given by the above Lemma, indeed $f \cdot \nabla = d/dt$ is a (trivial) Lie-point symmetry of any autonomous dynamical system (16), expressing simply its time-flow invariance: if $u_0(t)$ is a solution, then $u_0(t+t_0)$ is also a solution.

Lemma 2 *Given an Hamiltonian* H*, let* K *be a constant of motion for the Hamiltonian, i.e.* $\{K, H\} = 0$*. Then*

$$S := J\nabla K \cdot \nabla \tag{18}$$

is a Lie-point symmetry for the dynamical system $\dot{u} = J\nabla_u H$*.*

The proof is a straightforward verification.

4 General Results. Mel'nikov-type Conditions

Still considering, from a general point of view, eq. (6) as arising from an Hamiltonian with n degrees of freedom, assume that there are n bounded solutions $\zeta^{[\alpha]}(t)$ and n divergent solutions $\eta^{[\alpha]}(t)$ of the homogeneous part (14) of the variational equation. Let us construct the fundamental matrix Ψ in (13) putting the solutions $\zeta^{[\alpha]}$ in the first n columns and the $\eta^{[\alpha]}$ in the remaining columns of Ψ; recalling that $\det \Psi = 1$ and observing that Ψ^{-1} is a matrix having the $\eta^{[\alpha]}$ in the first rows and $-\zeta^{[\alpha]}$ in the remaining rows, one easily sees (cf. [5]) that (13) can be written (still in implicit form, see (10))

$$\begin{aligned} v(t) = v_h(t) + \quad \textstyle\sum_\alpha \quad & \zeta^{[\alpha]}(t) \int_{t_1}^{t} \Big(\eta^{[\alpha]}(s) \cdot J\, G(s, t_0, \epsilon) \Big)\, ds \\ + \quad & \eta^{[\alpha]}(t) \int_{t_1}^{t} \Big(\zeta^{[\alpha]}(s) \cdot J\, G(s, t_0, \epsilon) \Big)\, ds \end{aligned} \tag{19}$$

where $v_h(t)$ is a linear combination of the $\zeta^{[\alpha]}(t), \eta^{[\alpha]}(t)$.

We now have to look for the existence of bounded solutions of (10); more precisely we have to look for solutions $v^{(-)}(t)$ (and resp. $v^{(+)}(t)$) with the property of being bounded for $t \to -\infty$ (resp. $t \to +\infty$); these will provide precisely those solutions

$$u^{(\pm)}(t) = \chi(t - t_0) + v^{(\pm)}(t - t_0) \tag{20}$$

of (6) which belong, by definition, to the unstable (resp. stable) manifold of the critical point.

If we linearize (19) around the solution $v(t) \equiv 0$ (which amounts to deleting the higher-order terms in (10), or to considering the variational equation in the form (12)), and take into account the different asymptotic behaviour of the solutions $\zeta^{[\alpha]}(t)$ and $\eta^{[\alpha]}(t)$, it can be seen, as a consequence of the implicit-function theorem [4, 5, 9], that the existence of bounded solutions *both* for $t \to -\infty$ and $t \to +\infty$ is ensured if the following Mel'nikov-type conditions are verified

$$M^{[\alpha]}(t_0) = \int_{-\infty}^{+\infty} \Big(\zeta^{[\alpha]}(t) \cdot J\, g(\chi(t), t + t_0) \Big)\, dt = 0 \tag{21}$$

Let us remark that if the perturbation is Hamiltonian, $g = J\nabla_u W$, as often happens in celestial mechanics, choosing $\zeta^{[1]}(t) = d\chi/dt$, then the first of these conditions becomes

$$M^{[1]}(t_0) = \int_{-\infty}^{+\infty} \{H, W\}(\chi(t), t + t_0)\, dt \;=\; 0 \tag{22}$$

Similarly, if, e.g., $\zeta^{[2]}(t)$ comes from a constant of motion K, according to Sect. 3, then

$$M^{[2]}(t_0) = \int_{-\infty}^{+\infty} \{K, W\}(\chi(t), t + t_0)\, dt \;=\; 0 \tag{23}$$

These conditions are identical to the conditions given e.g. in [3, 6, 16], where they are obtained by means of different procedures and hypotheses. It can be significant to remark that the present method then provides an extension of these formulas also to "degenerate" cases (nonhyperbolic points and possibly unbounded orbits), and to symmetries of more general nature, as in (15).

We have only to notice that, whereas the convergence of Mel'nikov integrals (21) is granted in the case of standard hyperbolic and isolated unstable points, the convergence is only "conditional", i.e. along a suitable sequence of intervals (see [3] for any details), in cases *i)* and *iii)* of our classification in Sect. 2, and finally it must be controlled "by hand" in the non-hyperbolic case or in the case of critical point at the infinity. In the last cases one has to impose a sufficiently rapid vanishing of the contribution of the perturbation $g(\chi(t), t)$ when $t \to \pm\infty$, i.e., as expected, when approaching the critical point. The precise rate of this vanishing will depend on the specific problem in consideration (see Examples in the next sections, and also [17] for examples in 1 degree of freedom). For what concerns the regularity of the solutions and of the asymptotic manifolds, see e.g. [8, 18, 19].

Changing now the point of view, and considering the Poincaré sections of the $u^{(-)}$ and $u^{(+)}$ solutions, the above arguments show that, once conditions (21) are satisfied, there occurs a crossing of the negatively and positively asymptotic sets on the Poincaré section [2, 3, 6]. One usually imposes that the intersection is transversal; actually this condition is not strictly necessary, indeed it can be shown that it is sufficient that the crossing is "topological" [20], i.e., roughly, that there is really a "crossing", from one side to the other, but we do not insist on this point, which goes beyond the scope of the present paper.

Thanks to the periodicity of the perturbation, one immediately deduces [2, 3, 4, 20] that there is an infinite sequence of intersections of the positively and negatively asymptotic sets of the critical point in the Poincaré section, leading to a situation typical of the homoclinic chaos. The presence of such infinitely many intersections is clearly reminiscent of the chaotic behaviour expressed by the Birkhoff-Smale theorem in terms of the equivalence to the symbolic dynamics of Smale horseshoes. Actually, this theorem cannot be directly used in the present context because its standard proof is intrinsically based on hyperbolicity properties [2, 3]. However, several arguments can be invoked even in the "degenerate" cases, which allow us to conclude that, if the conditions (21) are satisfied, the perturbed problem exhibits a chaotic behaviour. We can refer e.g. to the classical arguments used in [21], and reconsidered by many others (see e.g. [7, 22, 23, 24]), possibly resorting to singular coordinate transformations, such as the McGehee transformation in the case of critical point at the infinity, or the "blowing-up" method [25, 26]. More specifically, an equivalence to a "nonhyperbolic horseshoe" has been proved in [8], in which the contracting and expanding actions are not exponential but "polynomial" in time. The presence of Smale horseshoes and of a positive topological entropy has been also proved by means of a quite general geometrical or "topological" procedure [20] which holds, in the presence of area-preserving perturbations, even in the nonhyperbolic case.

5 Applications to Celestial Mechanics. Two Degrees of Freedom

Coming back to the initial problems as stated in Sect. 2, the first step, according to the above discussion, is to write down the appropriate Mel'nikov conditions. It is clear that, apart from a rotation (this will change the expression of the perturbation g in equations (5,6) into some new $\widetilde{g}$ which will also depend in general on the parameters of the rotation: this will be explained in detail later), we can always assume that the homoclinic orbit lies in the plane $z = 0$ and in particular is given precisely by the orbit $\widehat{\chi}(t)$ defined in

(4), Sect. 2. This greatly simplifies the variational equation, and in particular its homogeneous part (14); indeed it is immediate to verify that, with this choice, the two equations for the variations v_5, v_6 of θ and p_θ are separated from the first four equations and admit regular bounded solutions; therefore the problem turns out to be 4-dimensional. An obvious symmetry for it is given by the rotations around the $z-$axis, generated by

$$S = \frac{\partial}{\partial \varphi} \tag{24}$$

Then, as discussed in Sect. 3, two bounded solutions of the homogeneous part of the variational equation are

$$\begin{aligned} \zeta^{[1]} &= \frac{d\widehat{\chi}}{dt} = (\dot{R}(t), \ddot{R}(t), \dot{\Phi}(t), 0, 0, 0) \\ \zeta^{[2]} &= (0, 0, 1, 0, 0, 0) \end{aligned} \tag{25}$$

where $R(t)$, $\Phi(t)$ have been defined in Sect. 2 (see (3,4)). We then get from (21) the two Mel'nikov conditions

$$\begin{aligned} M^{[1]} &= \int_{-\infty}^{+\infty} \Big[\dot{R}(t)\widetilde{g}_{p_r}(\widehat{\chi}(t), t+t_0) - \ddot{R}(t)\widetilde{g}_r(\ldots) \\ &\qquad + \dot{\Phi}(t)\widetilde{g}_{p_\varphi}(\ldots)\Big] dt \ = \ 0 \end{aligned} \tag{26}$$

$$M^{[2]} = \int_{-\infty}^{+\infty} \widetilde{g}_{p_\varphi}(\widehat{\chi}(t), t+t_0) \ dt \ = \ 0 \tag{27}$$

Notice that usually one has $\widetilde{g}_r = 0$. In the case where the problem, included the perturbation, is completely planar, and that the perturbation is generated by an Hamiltonian W of the form

$$W = W(r, \varphi, t) \tag{28}$$

then, under rotation, W is changed simply into $W(r, \varphi + \varphi_0, t)$ and the above conditions become

$$M^{[1]}(t_0, \varphi_0) = \int_{-\infty}^{+\infty} \Big[\dot{R}(t)\frac{\partial W(R(t), \Phi(t) + \varphi_0, t + t_0)}{\partial r} + \dot{\Phi}(t)\frac{\partial W(\ldots)}{\partial \varphi}\Big] dt \ = \ 0 \tag{29}$$

and

$$M^{[2]}(t_0, \varphi_0) = \int_{-\infty}^{+\infty} \frac{\partial W(R(t), \Phi(t) + \varphi_0, t + t_0)}{\partial \varphi} \, dt \ = \ 0 \tag{30}$$

and it is easily seen that these can be also written according to the general form as given in (22,23).

As a first simple application of this situation, consider the classical Kepler problem with unstable equilibrium point at $r = \infty$ and $\dot{r} = 0$ and a perturbation not depending on φ, e.g.

$$W = \frac{\sin 2\pi\nu t}{r^\delta} \qquad\qquad (\delta > 1/2) \tag{31}$$

as in the classical Gyldén problem [17, 23]. Then, condition (30) is automatically satisfied, and the integral in (29) becomes (thanks to the parity of the function $R(t)$)

$$\cos 2\pi\nu t_0 \int_{-\infty}^{+\infty} \frac{\dot{R}(t)}{R(t)^{\delta+1}} \sin 2\pi\nu t \; dt$$

which clearly admits simple zeroes (the integral is easily seen to be $\neq 0$). In this example, it is also simple to evaluate explicitly the asymptotic behaviour of the solutions of the variational equation: one has indeed [9]

$$\zeta = \dot{R}(t) \sim |t|^{-1/3} \quad , \quad \eta(t) \sim |t|^{4/3} \qquad \text{for} \quad |t| \to +\infty \tag{32}$$

and all conditions for the procedure given in Sect. 4 are satisfied, with – in particular – the condition $\delta > 1/2$ which ensures in this case the correct rate of vanishing of the perturbation at $r = \infty$. Notice that this same condition would also guarantee that under the McGehee transformation [21, 22, 23], the perturbation is not singular. Then chaotic behaviour is expected for this problem.

A similar result holds for the more general (time-periodic) perturbations of the form (28) occurring e.g. in the restricted 3-body problems [16, 24]. These cases can be conveniently dealt with in this way. Assuming that $W(R(t), \Phi(t), t) \to 0$ for $t \to \pm\infty$, then (29) (or (22)) becomes

$$M^{[1]}(t_0, \varphi_0) = \int_{-\infty}^{+\infty} \frac{\partial W}{\partial t}(R(t), \Phi(t) + \varphi_0, t + t_0) \, dt \;=\; 0 \tag{33}$$

(where clearly the derivative $\partial/\partial t$ must be performed only with respect to the explicit time-dependence of W); introducing then the "Mel'nikov potential" $\mathcal{W} = \mathcal{W}(\varphi_0, t_0)$ (cf. [27]), corresponding to the perturbation $W(r, \varphi, t)$:

$$\mathcal{W}(t_0, \varphi_0) := \int_{-\infty}^{+\infty} W(R(t), \Phi(t) + \varphi_0, t + t_0) \, dt \tag{34}$$

then one gets from this definition and from (33,30)

$$M^{[1]}(t_0,\varphi_0)=\frac{\partial\mathcal{W}}{\partial t_0}=0 \qquad , \qquad M^{[2]}(t_0,\varphi_0)=\frac{\partial\mathcal{W}}{\partial\varphi_0}=0 \tag{35}$$

In other words, the two Mel'nikov conditions are equivalent to the existence of stationary points for the Mel'nikov potential $\mathcal{W}(\varphi_0,t_0)$. On the other hand, $\mathcal{W}$ is a smooth doubly-periodic function, and such a function certainly possesses points $\overline{t_0},\overline{\varphi_0}$ where the two partial derivatives in (35) vanish, and this implies that the two conditions (33,30) are certainly satisfied (one has only to check that these stationary points of $\mathcal{W}$ are isolated).

6 Applications to Celestial Mechanics. Three Degrees of Freedom

Let us consider finally a perturbation depending on both angles φ,θ. As already stated, to obtain Mel'nikov condition in the form (26,27), one has to transform the generic homoclinic orbit into the orbit $\widehat{\chi}(t)$ given by (4). This is obtained by means of a rotation defined by the following Euler angles (with the conventions and notations as in [28]):

$$-\Omega,\ -i,\ -\omega \tag{36}$$

where, with the language of celestial mechanics, i is the inclination of the plane of the orbit, ω the angle of the perihelion with the line of nodes in the orbital plane, and Ω is the longitude of the ascending node.

Considering for simplicity a perturbation generated by an Hamiltonian not depending on the variables p, which we denote here, with a little abuse of notation, either by $W(r,\varphi,\theta,t)$ or by $W(\mathbf{x},t)$, $\mathbf{x}=(x,y,z)$, and denoting by B the matrix of this rotation, the new expression $\widetilde{W}$ of the perturbation is obtained by replacing $\mathbf{x}$ with $B\mathbf{x}$. It is then easy to verify that Mel'nikov conditions (26,27) become

$$M^{[1]}(t_0,\omega,\Omega) \;=\; \int_{-\infty}^{+\infty}\Big[\dot{R}\frac{\partial W}{\partial r}+\dot{\Phi}\Big(\frac{\partial\widetilde{W}}{\partial\varphi}\Big)_0\Big]\,dt=0 \tag{37}$$

$$M^{[2]}(t_0,\omega,\Omega) \;=\; \int_{-\infty}^{+\infty}\Big(\frac{\partial\widetilde{W}}{\partial\varphi}\Big)_0\,dt=0 \tag{38}$$

where

$$\Big(\frac{\partial\widetilde{W}}{\partial\varphi}\Big)_0 = R(t)\Big(- C_1(R(t),\Phi(t),\omega,\Omega)\ \sin\Phi(t) + C_2(\ldots)\cos\Phi(t)\Big) \qquad (39)$$

with

$$\begin{aligned}
C_1 &= \Big(\frac{\partial\widetilde{W}}{\partial x_1}\Big)_0(\cos\omega\cos\Omega - \cos i\sin\omega\sin\Omega) + \\
&+ \Big(\frac{\partial\widetilde{W}}{\partial x_2}\Big)_0(-\cos\omega\sin\Omega - \cos i\sin\omega\cos\Omega) + \Big(\frac{\partial\widetilde{W}}{\partial x_3}\Big)_0\sin i\sin\omega \\
C_2 &= \Big(\frac{\partial\widetilde{W}}{\partial x_1}\Big)_0(\sin\omega\cos\Omega + \cos i\cos\omega\sin\Omega) + \\
&+ \Big(\frac{\partial\widetilde{W}}{\partial x_2}\Big)_0(-\sin\omega\sin\Omega + \cos i\cos\omega\cos\Omega) - \Big(\frac{\partial\widetilde{W}}{\partial x_3}\Big)_0\sin i\cos\omega
\end{aligned}$$

and where $(\partial\widetilde{W}/\partial x_i)_0$ means that in the derivative of the given W with respect to x_i one has to replace $\mathbf{x}$ with $B\mathbf{x}$ and finally put $z=0$ (or $\theta=\pi/2$).

If $i=0$, i.e. if the problem is completely planar, including the perturbation, or if the perturbation is "generic", i.e. has no "preferred" direction in the space (as often happens, see below for an example), and therefore it is not restrictive to choose $i=0$, then one gets

$$C_1 = \Big(\frac{\partial\widetilde{W}}{\partial x_1}\Big)_0\cos\varphi_0 + \Big(\frac{\partial\widetilde{W}}{\partial x_2}\Big)_0\sin\varphi_0\ , \quad C_2 = \Big(\frac{\partial\widetilde{W}}{\partial x_1}\Big)_0\sin\varphi_0 - \Big(\frac{\partial\widetilde{W}}{\partial x_2}\Big)_0\cos\varphi_0$$

where $\varphi_0 = -(\omega+\Omega)$, and the above expressions (29,30) are recovered.

An example is provided by a problem in general relativity [11, 12]. Consider indeed the motion of a relativistic charged particle in a gravitational field produced by a Schwarzschild black hole, perturbed by a homogeneous constant electric field. It can be shown [12] that the perturbation is given by

$$W = \mathcal{F}(r)(l_1\cos\varphi\sin\theta + l_2\sin\varphi\sin\theta + l_3\cos\theta) \qquad (40)$$

where $\mathcal{F}(r)$ is suitable function, and $\mathbf{l} = (l_1,l_2,l_3)$ is the direction of the perturbing field. This direction is generic and therefore no rotation is required (it would simply change the values of l_i, which are not fixed).

In this case, the first Mel'nikov condition in (37) is identically satisfied, because the perturbation is independent of the time (cf. (33)), and the other one becomes

$$M^{[2]} = M^{[2]}(\varphi_0) = (l_1 \sin\varphi_0 - l_2 \cos\varphi_0) \int_{-\infty}^{+\infty} \mathcal{F}(R(t)) \cos\Phi(t)\ dt \qquad (41)$$

which clearly admits simple zeroes, if only one can show that the integral is different from zero (see [11]).

Notice that in this example one has that the effective unperturbed potential admits bounded homoclinic orbits biasymptotic to an unstable point $r_u \neq 0$ (which corresponds to unstable circles, as in case *i)* of our initial classification), therefore the above integral is expected to converge only conditionally [3, 16].

Similar results hold if the perturbation is produced by a constant homogeneous magnetic field [12]. It can be observed that, whereas the component of the electric and of the magnetic field on the plane of motion leads to a chaotic dynamics, the component normal to the plane does not. Since the problem is spherically symmetric, this argument can be applied to every plane for the origin. Thus, given an electric or magnetic field, on each plane for the origin (except at most the one normal to the field), chaos appears for a suitable choice of initial conditions.

References

[1] V.K. Mel'nikov, On the stability of the center for time periodic perturbations, Trans. Moscow Math. Soc. **12** (1963) 1-56

[2] J. Guckenheimer and P.J. Holmes, *Nonlinear oscillations, dynamical systems and bifurcations of vector fields* (Springer, Berlin 1983)

[3] S. Wiggins, *Global Bifurcations and Chaos* (Springer, New York, 1988).

[4] S.-N. Chow, J.K. Hale and J. Mallet-Paret, An example of bifurcations to homoclinic orbits, J. Diff. Eq. **37** (1980) 351-373

[5] J. Gründler, The existence of homoclinic orbits and the method of Mel'nikov for systems in $\mathbf{R}^n$, SIAM J. Math. Anal. **16** (1985) 907-931

[6] P.J. Holmes and J.E. Marsden, Horseshoes in perturbations of Hamiltonian systems with 2 degrees of freedom, Comm. Math. Phys., **82** (1982) 523-544

[7] P.S. Letelier and A.E. Motter, Homoclinic crossing in open systems: Chaos in periodically perturbed monopole plus quadrupolelike potentials, Phys. Rev. E, **60** (1999) 3920-3927

[8] H. Dankowicz and P. Holmes, The existence of transverse homoclinic points in the Sitnikov problem, J. Diff. Eq. **116** (1995) 468-483

[9] G. Cicogna and M. Santoprete, An approach to Mel'nikov theory in celestial mechanics, J. Math. Phys., **41** (2000) 805-815

[10] F. Diacu and M. Santoprete, Nonintegrability and chaos in the anisotropic Manev problem, *Physica D*, **156** (2001) 39-52

[11] P. S. Letelier and W. M. Vieira, Chaos in black holes surrounded by gravitational waves, Class. and Quantum Gravity **14** (1997) 1249-1257

[12] M. Santoprete and G. Cicogna, Chaos in black holes surrounded by electromagnetic fields, in preparation

[13] E.A. Coddington and N. Levinson, *Theory of ordinary differential equations* (McGraw-Hill, New York 1955)

[14] P.J. Olver, *Application of Lie groups to differential equations* (Springer, Berlin, 1986, and 1998, second Edition)

[15] G. Cicogna, G. Gaeta: *Symmetry and Perturbation Theory in Nonlinear Dynamics*, Lecture Notes in Physics, LNP monographs Vol. 57 (Springer, Berlin 1999)

[16] C. Robinson, Mel'nikov method for autonomous Hamiltonians, Contemp. Math. **198** (1996) 45-53

[17] G. Cicogna and M. Santoprete, Nonhyperbolic homoclinic chaos, Phys. Letters A **256** (1999) 25-30

[18] C. Robinson, Homoclinic orbits and oscillations for planar 3-body problrem, J. Diff. Eq. **52** (1984) 356-377

[19] J. Casasayas, E. Fontich and A. Nunes, Invariant manifolds for a class of parabolic points, Nonlinearity **5** (1992) 1193-1210

[20] K.Burns and H.Weiss, A geometric criterion for positive topological entropy, *Comm. Math. Phys.*, **172** (1995) 95-118

[21] J. Moser, *Stable and random motions in dynamical systems* (Princeton Univ. Press, Princeton 1973)

[22] R. McGehee, A stable manifold theorem for degenerate fixed points with applications, J. Diff. Eq. **14** (1973) 70-88

[23] F. Diacu and D. Selaru, Chaos in the Gyldén problem, J. Math. Phys. **39** (1998) 6537-6546

[24] Z. Xia, Mel'nikov method and transversal homoclinic points in the restricted three-body problem, J. Diff. Eq. **96** (1992) 170-184

[25] F. Dumortier, Singularities of vector fields on the plane, J. Diff. Eq. **23** (1977) 53-106

[26] M. Brunella and M. Miari, Topological equivalence of a plane vector field with its principal part defined through Newton polyedra, J. Diff. Eq. **85** (1990) 338-366

[27] A. Delshams and R. Ramírez-Ros, Mel'nikov potential for exact symplectic maps, Comm. Math. Phys. **190** (1997) 213-245

[28] H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, Ma, 1980)