HOLDER CONTINUITY OF THE INTEGRATED DENSITY OF STATES FOR SOME RANDOM OPERATORS AT ALL ENERGIES

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Abstract

We prove that the integrated density of states of random Schrödinger operators with Anderson-type potentials on $L^2(\mathbb{R}^d)$, for $d \geq 1$, is locally Hölder continuous at all energies. The single-site potential u must be nonnegative and compactly supported, and the distribution of the random variable must be absolutely continuous with a bounded, compactly supported density. We also prove this result for random Anderson-type perturbations of the Landau Hamiltonian in two-dimensions under a rational flux condition.

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1 Introduction and Main Results

There are now many results on the local continuity of the integrated density of states (IDS) for random Schrödinger operators $H_\omega(\lambda) = H_0 + \lambda V_\omega$, on $L^2(\mathbb{R}^d)$, for $d \geq 1$, and related operators describing wave propagation in random media, on certain energy intervals. The regions of the real axis where continuity has been proved lie outside the spectrum of the unperturbed, deterministic operator H_0. The first main result of this paper is the proof of the Hölder continuity of the IDS at all energies for the random operators. This applies to rather general Schrödinger operators with random Anderson-type potentials V_ω constructed with nonnegative single-site potentials, and random coupling constants having absolutely continuous probability densities of bounded support. This result requires that the background operator H_0 have an IDS that is locally Hölder continuous. Although this establishes Hölder continuity at all energies, the Hölder exponent of continuity, q, must be rather small, satisfying approximately $0 < q < 1/3$. In our second main result, we improve this estimate on the Hölder exponent of continuity. Under the hypothesis that H_0 is a periodic operator, we prove that the IDS is Hölder continuous with any exponent q of Hölder continuity satisfying $0 < q < 1$. Finally, our third result concerns the Hölder continuity of the IDS for Landau Hamiltonians in two-dimensions. We prove that if the flux through the unit square is rational, then the IDS for $H_\omega(\lambda)$, for $\lambda \neq 0$, is Hölder continuous at all energies with any exponent $0 < q < 1$. These results improve all known results about the continuity of the IDS for random Schrödinger and related, operators on $\mathbb{R}^d$. We remark that under the strong condition that supp u is sufficiently large, the IDS is known to be Lipschitz continuous at all energies [5, 7].

The family of Schrödinger operators $H_\omega(\lambda)$ on $L^2(\mathbb{R}^d)$, is constructed from a deterministic, background operator $H_0 = (-i\nabla - A_0)^2 + V_0$. We assume that this operator is self-adjoint with operator core $C_0^\infty(\mathbb{R}^d)$. We consider an Anderson-type potential V_ω constructed from the single-site potential u as

$$V_\omega(x) = \sum_{j \in \mathbb{Z}^d} \lambda_j(\omega) u(x - j). \tag{1.1}$$

The family of random Schrödinger operators is given by

$$H_\omega(\lambda) = H_0 + \lambda V_\omega. \tag{1.2}$$

As we will show below, we can normalize V_ω so that $\lambda > 0$ is a measure of the disorder. The main results are independent of the disorder (provided it is nonzero for Theorem 1.2 and in the Landau case). The results of this paper also apply to the random operators describing acoustic and electromagnetic waves in randomly perturbed media, and we refer the reader to [10, 14, 15].

We need to define local versions of the Hamiltonians and potentials associated with bounded regions in $\mathbb{R}^d$. By $\Lambda_l(x)$, we mean the open cube of side length l centered at $x \in \mathbb{R}^d$. For $\Lambda \subset \mathbb{R}^d$, we denote the lattice points in Λ by $\tilde{\Lambda} = \Lambda \cap \mathbb{Z}^d$. Let $\Lambda \subset \mathbb{R}^d$ be a cube that is a union of translated unit cubes $\Lambda_1(j) = \Lambda_1(0) + j$, $j \in \mathbb{Z}^d$, that is, $\Lambda = \cup_j \Lambda_1(j)$. For a given Λ, we take H_0^Λ and H_ω^Λ to be the restrictions of H_0 and H_ω, respectively, to the region Λ, with self-adjoint boundary conditions on the boundary of Λ, $\partial\Lambda$. We denote by $E_0^\Lambda(\cdot)$ and $E_\Lambda(\cdot)$ the spectral families for H_0^Λ and H_ω^Λ, respectively. Furthermore, for $\Lambda \subset \mathbb{R}^d$, let χ_Λ be the characteristic function for Λ. The local potential V_Λ is defined by

$$V_\Lambda(x) = \sum_{j \in \tilde{\Lambda}} \lambda_j(\omega) u(x-j) \chi_\Lambda(x). \tag{1.3}$$

We also let $\tilde{V}_\Lambda$ denote the sum of the single-site potentials in Λ, that is

$$\tilde{V}_\Lambda(x) = \sum_{j \in \tilde{\Lambda}} u(x-j) \chi_\Lambda(x). \tag{1.4}$$

The spectral family $E_\Lambda(\cdot)$, and the local potential V_Λ depend on the random variables in Λ, but we suppress this in the notation. We note that although $V_\omega | \Lambda$ differs from V_Λ in a neighborhood of the boundary of Λ, the difference does not affect the results, cf. [8].

We will always make the following two assumptions:

(H1). The single-site potential $u \neq 0$ is nonnegative with compact support, $u \in L^\infty(\mathbb{R}^d)$, and $\|u\|_\infty \leq 1$.

(H2). The random coupling constants $\{\lambda_j(\omega) \mid j \in \mathbb{Z}^d\}$, are independent and identically distributed. The distribution has a density h_0 with $\operatorname{supp} h_0 \subset [0, 1]$.

We make three important comments on these hypotheses:

1. For a given density h_0 of compact support $[m, M_1]$, we can always add the periodic potential $m \sum_j u_j$, with $u_j(x) = u(x-j)$, to the background potential V_0, so that the random coupling constants take their value in an interval $[0, M]$, with $M = M_1 - m$.

2. Because of the explicit disorder parameter $\lambda > 0$, we can rescale the coupling constants so that, without loss of generality, the support of h_0 is included in the interval $[0, 1]$.

3. Hypotheses (H1) and (H2) imply the following. There exists a finite constant $C_1(u, M, d) > 0$, depending only on the single-site potential u, and the dimension $d \geq 1$, so that for all $\Lambda \subset \mathbb{R}^d$,

$$0 \leq V_\Lambda^2 \leq C_1(u, d)\tilde{V}_\Lambda, \tag{1.5}$$

where V_Λ and $\tilde{V}_\Lambda$ are defined in (1.3) and (1.4), respectively. This simple inequality is used in the proof of Theorem 1.1.

We define the IDS $N(E)$ for H_ω using the counting function for H_ω^Λ. Let $N_\Lambda(E)$ be the number of eigenvalues of H_ω^Λ, with self-adjoint boundary conditions, less than or equal to E. This function depends on the realization ω. We define

$$N(E) = \lim_{|\Lambda| \to \infty} \frac{N_\Lambda(E)}{|\Lambda|}. \tag{1.6}$$

It is known that this limit exits and is independent of the realization ω almost surely. Furthermore, it is known that the IDS is independent of the boundary conditions taken on the finite volumes Λ, cf. [11, 18, 22].

Theorem 1.1. *In addition to hypotheses (H1) and (H2), we assume that the operator H_0 admits a density of states $N_0(E)$ that is locally Hölder continuous on $\mathbb{R}$ for some exponent $0 < q_1 \leq 1$, i. e. that satisfies*

$$|N_0(E') - N_0(E)| \leq C_0 |E' - E|^{q_1} \tag{1.7}$$

where the finite constant $C_0 > 0$ is locally uniform in energy. Then, the IDS for $H_\omega(\lambda)$ is Hölder continuous on $\mathbb{R}$ for any exponent $0 < q < q_1/(q_1 + 2)$.

For the second result, we take $H_0 = (-i\nabla - A_0)^2 + V_0$ to be a periodic Schrödinger operator with a real-valued, periodic, potential V_0, and a periodic vector potential A_0. We assume that V_0 and A_0 are sufficiently regular so that H_0 is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$. We assume that both V_0 and A_0 are periodic with respect to the group $\Gamma = \mathbb{Z}^d$ because of the form of the Anderson-type potential (1.1). We note that we could work with a nondegenerate lattice Γ, by defining a corresponding Anderson-type potential, but we will explicitly treat the case $\Gamma = \mathbb{Z}^d$. Let Γ^* denote the dual lattice, that is, $\Gamma^* = \{\gamma' \mid \gamma \cdot \gamma' \in 2\pi\mathbb{Z}, \text{ for all } \gamma \in \Gamma\}$. We let $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ be the torus, and $(\mathbb{T}^d)^* = (\mathbb{R}^d)^*/(\mathbb{Z}^d)^*$ be the dual torus. We denote by C_0 the unit cell for $\Gamma = \mathbb{Z}^d$, and by C_0^* the unit cell for $\Gamma^* = (\mathbb{Z}^d)^*$. The Floquet decomposition of H_0 yields a family of operators $H_0(\theta)$, for $\theta \in (\mathbb{R}^d)^*$. Each operator $H_0(\theta)$ is self-adjoint on $L^2(\mathbb{T}^d)$, and has a compact resolvent. We denote the eigenvalues of $H_0(\theta)$ by $E_n(\theta)$. The spectrum of H_0 is given by

$$\sigma(H_0) = \bigcup_{\theta \in (\mathbb{T}^d)^*} \sigma(H_0(\theta)) = \bigcup_{n \in \mathbb{N}} \bigcup_{\theta \in (\mathbb{T}^d)^*} E_n(\theta). \tag{1.8}$$

For a closed, bounded interval $\Delta \subset \mathbb{R}$, we let $E_0(\Delta, \theta)$ denote the spectral projector of $H_0(\theta)$ onto the eigenspace of $H_0(\theta)$ spanned by its eigenfunctions with eigenvalues $E_n(\theta) \in \Delta$. Because of the discreteness of the spectrum of $H_0(\theta)$, the dimension of $Ran E_0(\Delta, \theta)$ is finite and locally constant.

We consider Hamiltonian $H_\omega(\lambda)$ restricted to the cube Λ with periodic boundary conditions (PBC). If the cube Λ has side length n, denoted by Λ_n, we can consider it as a fundamental unit cell for the lattice $n\mathbb{Z}^d$ and take a Floquet decomposition relative to it. We let $H_0^n(\theta)$ and $E_j^{(n)}(\theta)$ denote the corresponding fibred operators and their eigenvalues. Note that the operator $H_0^{\Lambda_n}$, with PBC on $\partial\Lambda_n$, coincides with the fibred operator $H_0^n(\theta = 0)$. We define the IDS for $H_\omega(\lambda)$ as above using the counting function for H_ω^Λ with PBC.

Theorem 1.2. *Assume hypotheses (H1) and (H2), and that H_0 is $\mathbb{Z}^d$-periodic. Let $\tilde{\Delta} \subset \mathbb{R}$ be a bounded, closed interval. Suppose that for all $\theta \in (\mathbb{T}^d)^*$ there exists a finite constant $C(\tilde{\Delta}, u) > 0$, so that the single-site potential u satisfies*

$$E_0(\tilde{\Delta}, \theta) u E_0(\tilde{\Delta}, \theta) \ \geq \ C(\tilde{\Delta}, u) E_0(\tilde{\Delta}, \theta). \tag{1.9}$$

Then, the IDS for the random family $H_\omega(\lambda) = H_0 + \lambda V_\omega$, for $\lambda \neq 0$, is Hölder continuous on $\mathbb{R}$, for any Hölder exponent $0 < q < 1$.

Theorem 1.2 requires no condition on the IDS for the background operator H_0 as in (1.7). Note that, in general, for a periodic magnetic potential and no electric potential, the IDS $N_0(E)$ is not Hölder continuous at the bottom of the spectrum (see [12, 13]). This is analogous to what happens in the constant magnetic field case (see the discussion in section 6). The abstract condition (1.9) involves the single-site potential and the eigenfunctions of the background, periodic operator H_0. Concerning the abstract condition (1.9), we have the following result.

Proposition 1.3. *We assume hypotheses (H1) and (H2). In any dimension $d \geq 1$, if H_0 has the unique continuation property and $u > 0$ on a nonempty open set, then condition (1.9) holds.*

Here, we say that H_0 has the unique continuation property if for any $E \in \mathbb{R}$ and any function φ in the domain of H_0, if φ is a solution to $(H_0 - E)\varphi = 0$ and φ vanishes on some open set, then φ vanishes identically. It is well known that H_0 has the unique continuation property if A_0 and V_0 are suffciently regular; e.g. in dimension $d \geq 3$, $V_0 \in L^{d/2}_{loc}(\mathbb{R}^d)$, $A_0 \in L^{d}_{loc}(\mathbb{R}^d)$ and $\nabla A_0 \in L^{d/2}_{loc}(\mathbb{R}^d)$ are sufficient to ensure that H_0 has the unique continuation property (see e.g. [30] and references therein).

Proposition 1.3 and Theorem 1.2 establish the continuity of the IDS under rather general conditions.

Corollary 1.4. *We assume hypothesis (H2) and that H_0 is $\mathbb{Z}^d$-periodic and has the unique continuation property. If $d \geq 1$, then the IDS is Hölder continuous with any exponent $0 < q < 1$ at all energies provided (H1) holds and $u > 0$ on a nonempty open set.*

We make some comments on the distinction between Theorems 1.1 and 1.2. For any real function $f \in C_0(\mathbb{R})$, the density of states measure dN satisfies

$$\int f(s)dN(s) = \lim_{|\Lambda| \to \infty} \frac{1}{|\Lambda|} Tr(\chi_\Lambda f(H) \chi_\Lambda),$$

whenever it exists. It then follows that if H_1 and H_2 are two self-adjoint Schrödinger operators so that $V = H_2 - H_1$ satisfies the relative trace-class condition, $V(H_1 + i)^{-m} \in \mathcal{I}_1$, for some integer $m > 0$ sufficiently large, then the IDS $N_i(\lambda)$, for H_i, $i = 1, 2$ are equal: $N_1(\lambda) = N_2(\lambda)$. One consequence of this observation is the following. Let H_0 be a periodic operator and let V_ω be an Anderson-type random potential, as in (1.1), so that (H1), (H2), and (1.9) are satisfied. Then, by Theorem 1.2, the IDS for $H_\omega(\lambda)$, as in (1.2), is Hölder continuous on $\mathbb{R}$. Now let us consider a perturbation of $H_\omega(\lambda)$ by another deterministic potential V_1 satisfying the relative trace-class condition given above. We then have that the IDS for $H_\omega(\lambda) + V_1$ equals the IDS for $H_\omega(\lambda)$. This construction provides a large family of random operators with Hölder continuous IDS.

This construction raises the question of the existence of operators H_0 that are not periodic operators, nor perturbations of periodic operators by relatively trace-class perturbations, so that Theorem 1.2 cannot be applied, and for which the assumption (1.7) of Theorem 1.1 is satisfied. We offer two classes of examples. The first concern Anderson-type perturbations of one-dimensional random operators for which the Hölder continuity or better of the IDS is known by other methods. Many of these models are described in detail in the book of Pastur and Figotin [23]. Of perhaps more interest are certain background operators H_0 with magnetic fields. As an explicit example, let $H_0 = (-i\nabla - A)^2$, with vector potential $A = (B/2)(-x_2, x_1, 0)$ be the Hamiltonian describing a particle in $\mathbb{R}^3$ in the presence of a constant magnetic field in the x_3-direction with strength $B > 0$. This Hamiltonian can be as written $H_0 = H_{(1,2)} \otimes I_3 + I_{(1,2)} \otimes H_3$, where $H_{(1,2)}$ is the two-dimensional Landau Hamiltonian (see (1.10)), and $H_3 = -\partial^2/\partial_3^2$ is the free Laplacian in one-dimension. The IDS for H_0 is the convolution $N_0(\lambda) = \int N_3(\lambda - \mu) dN_{(1,2)}(\mu)$. Each IDS is known explicitly: $N_3(\lambda) = C_1(\lambda)_+^{1/2}$, and $dN_{(1,2)}(\lambda) = \sum_{j \geq 0} \delta(\lambda - (2j+1)B) d\lambda$. As simple computation of $N_0(\lambda)$ shows that it is Hölder continuous for any exponent $1/2 < q \leq 1$. Note that H_0 is not a relatively trace-class perturbation of the free Laplacian on $\mathbb{R}^3$. Consequently, Theorem 1.1 can be applied to show that $H_0 + V_\omega$ has a Hölder continuous IDS with no flux condition on B (see the discussion below and section 5). In particular, if B does not satisfy a rational flux condition, then Theorem 1.2 cannot be applied.

Our third main result concerns the IDS for random Anderson-type perturbations of Landau Hamiltonians. The unperturbed operator H_L on $L^2(\mathbb{R}^2)$ has the form

$$H_L = (-i\nabla - A)^2, \quad \text{where} \quad A(x_1, x_2) = \frac{B}{2}(-x_2, x_1), \tag{1.10}$$

where $B > 0$ is the magnetic field strength. The spectrum is pure point and consists of an increasing sequence of degenerate, isolated eigenvalues $\{E_j(B) = (2j+1)B \mid j = 0, 1, \ldots, \}$ of infinite multiplicity. The IDS for this model is a piecewise constant, monotone increasing function (cf. the example in [22]). Consequently, it does not satisfy condition (1.7). The perturbed family of operators is

$$H_\omega = H_L + \lambda V_\omega, \tag{1.11}$$

where V_ω is the Anderson-type random perturbation given in (1.1). It is known that $N(E)$ is locally Lipschitz continuous in the following sense. Given an $N > 0$, there is a $B_N > 0$ so that for $B > B_N$, the IDS $N(E)$ is Lipschitz continuous on $(0, 2(N+1)B) \backslash \{E_j(B) \mid j = 0, 1, \ldots, N\}$ [6, 27]. Under some additional conditions, Wang [28] also proved that $N(E)$ is smooth outside of a given Landau level for sufficiently large magnetic field strength. There has been some discussion as to the behavior of the IDS at the Landau energies $E_j(B)$. If the single-site potential u in (1.1) satisfies $u|\Lambda_1(0) > \epsilon\chi_{\Lambda_1(0)} > 0$, for some $\epsilon > 0$, then the IDS is locally Lipschitz continuous at all energies. We prove the following general theorem that greatly extends these known results.

Theorem 1.5. *Let H_L be the Landau Hamiltonian (1.11) with magnetic field $B \neq 0$. In addition to hypotheses (H1) and (H2), suppose that the magnetic field strength satisfies the rational flux condition for the unit square $\Lambda_1(0)$,*

$$B \in 2\pi Q. \tag{1.12}$$

Then, the IDS for the family $H_\omega = H_L + \lambda V_\omega$, for $\lambda \neq 0$, is Hölder continuous at all energies with any Hölder exponent $0 < q < 1$.

We do not believe that condition (1.12) is actually necessary; it is an artifact of our method of proof.

An immediate consequence of the proof of this theorem is the following extension.

Corollary 1.6. *Let $V_0 \in L^2_{loc}(\mathbb{R}^2)$ be a real-valued, $\mathbb{Z}^2$-periodic function. Then, under the hypotheses of Theorem 1.5, the IDS for $H_\omega = H_L + V_0 + \lambda V_\omega$ is Hölder continuous at all energies with any Hölder exponent $0 < q < 1$.*

There are very few general results on the IDS for continuous random Schrödinger operators in higher-dimensions for all energies. One of the interesting result concerns the case when V_ω is a Gaussian random processes. Fischer, Hupfer, Leschke, Müller [16] proved that the IDS is Lipschitz continuous at all energies for potentials given by certain Gaussian processes. We remark that this is very similar in spirit to the case $u|\Lambda_1(0) > \epsilon\chi_{\Lambda_1(0)} > 0$.

Extensions of this work to magnetic Schrödinger operators with certain unbounded random potentials is given in [17]. Tip [26] proved the absolute continuity of the IDS for the quantum Lorentz gas for certain repulsive single-site potentials.

The global regularity properties of the IDS for lattice models are much better understood, and we refer the reader to the monographs by Carmona and Lacroix [4], and by Pastur and Figotin [23].

The contents of this paper are as follows. We prove Theorem 1.1 in section 2. In section 3, we prove Theorem 1.2 under the assumption (1.9). We study assumption (1.9) for periodic operators H_0 in section 4. In section 5, we give the proof of Theorem 1.5 for random Landau Hamiltonians. Finally, in section 6, we give a proof of a technical estimate on the trace of the spectral projectors.

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2 Proof of Theorem 1.1.

We prove Theorem 1.1, the Hölder continuity of the IDS at all energies provided $q > 0$ is small enough, under the main assumption (1.7) on the existence and Hölder continuity of the IDS for the unperturbed operator H_0, and under the assumptions (H1)–(H2) on random potential.

Recall that $\Lambda_1(0)$ denotes the unit cube centered at the origin, and that $\Lambda_1(j)$ denotes its translate to the point $j \in \mathbb{R}^d$. Let $\Lambda = \cup_j \Lambda_1(j)$ be a cube in $\mathbb{R}^d$. We denote by $\tilde{\Lambda} = \Lambda \cap \mathbb{Z}^d$, the lattice points in Λ. By the operators H_0^Λ and H_ω^Λ, we mean the operators H_0 and H_ω restricted to Λ with self-adjoint boundary conditions. For a region $\Lambda \subset \mathbb{R}^d$, let $\tilde{V}_\Lambda \equiv \sum_{j \in \tilde{\Lambda}} u_j \chi_\Lambda$. As this quantity is independent of the random variables, it is bounded by a constant independent of the disorder.

Finally, we recall that hypotheses (H1)–(H2), and the normalization discussed in section 1, imply that the potential satisfies condition (1.5),

$$0 \;\leq\; V_\Lambda^2 \;\leq\; C_1(u,d)\tilde{V}_\Lambda. \tag{2.1}$$

To prove continuity of the IDS near $E_0 \in \mathbb{R}$, we use (1.7). We fix $\tilde{\Delta} = [E_-, E_+]$, a nonempty, closed interval. The estimate (1.7) and the definition of the density of states imply that for Λ sufficiently large (depending on $\tilde{\Delta}$)

$$TrE_0^\Lambda(\tilde{\Delta}) \ \leq \ C_2(E_+, d)|\tilde{\Delta}|^{q_1}|\Lambda|, \tag{2.2}$$

where the finite constant $C_2(E_+, d) > 0$ depends on the constant $C_0 > 0$ in (1.7), the dimension d, and is $\mathcal{O}(E^{d/2})$ as $E \to \infty$. To see this, let $N_{0,\Lambda}(E)$ be the number of eigenvalues of H_0^Λ less than E. Then, by the definition of the density of states, one has

$$N_0(E_+)(1 - |E_+ - E_-|^{q_1}) \leq \frac{N_{0,\Lambda}(E_+)}{|\Lambda|} \leq N_0(E_+)(1 + |E_+ - E_-|^{q_1}),$$
$$N_0(E_-)(1 - |E_+ - E_-|^{q_1}) \leq \frac{N_{0,\Lambda}(E_-)}{|\Lambda|} \leq N_0(E_-)(1 + |E_+ - E_-|^{q_1}).$$

Using (1.7), for Λ sufficiently large, we get

$$\begin{aligned} TrE_0^\Lambda(\tilde{\Delta}) &\leq (N_{0,\Lambda}(E_+) - N_{0,\Lambda}(E_-)) \\ &\leq |\Lambda|[N_0(E_+)(1 + |E_+ - E_-|^{q_1}) - N_0(E_-)(1 - |E_+ - E_-|^{q_1})] \\ &\leq \{C_0 + 2N_0(E_+)\}|\Lambda||E_+ - E_-|^{q_1} \\ &\leq C_2(E_+, d)|\Lambda||\tilde{\Delta}|^{q_1}. \end{aligned}$$

We used the monotonicity of the IDS and the bound $N_0(E) \leq C(E)$, where the finite constant $C(E) > 0$ satisfies $\lim_{E\to\infty} C(E)E^{-d/2} = C_d$, and $C_d > 0$ is a finite constant depending only on the dimension d.

Proof of Theorem 1.1.
1. Consider an interval $\Delta \equiv [\Delta_-, \Delta_+] \subset \mathbb{R}$, with $|\Delta| < 1$. Let $\alpha \in \mathbb{R}$ be an index satisfying $0 < \alpha < 1$. Let $\tilde{\Delta}$ be the interval $\tilde{\Delta} = [\Delta_- - |\Delta|^\alpha, \Delta_+ + |\Delta|^\alpha]$, so that $\Delta \subset \tilde{\Delta}$ and $d_0 = \text{dist}\,(\Delta, \tilde{\Delta}^c) = |\Delta|^\alpha$. We decompose the spectral projector $E_\Lambda(\Delta)$ relative to H_0^Λ as

$$TrE_\Lambda(\Delta) = TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta}) + TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta}^c). \tag{2.3}$$

We estimate the first term on the right in (2.3) using the assumption (2.2) on H_0^Λ and the positivity of the projectors. For Λ sufficiently large (depending on $\tilde{\Delta}$, this gives

$$TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta}) \ \leq \ C_2|\Delta|^{\alpha q_1}|\Lambda|, \tag{2.4}$$

where we write C_2 for the constant $C_2(E_+, d)$ appearing in (2.2).
2. To deal with the second term in (2.3), we pick an energy $E \in \Delta$ and write

$$\begin{aligned} TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta}^c) &= TrE_\Lambda(\Delta)(H_\omega^\Lambda - E)E_0^\Lambda(\tilde{\Delta}^c)(H_0^\Lambda - E)^{-1} \\ &\quad -\lambda TrE_\Lambda(\Delta)V_\Lambda E_0^\Lambda(\tilde{\Delta}^c)(H_0^\Lambda - E)^{-1} \\ &= (i) + (ii). \end{aligned} \tag{2.5}$$

For the first term (i), we use the estimate $\|E_0^\Lambda(\tilde{\Delta}^c)(H_0^\Lambda - E)^{-1}\| \leq |\Delta|^{-\alpha}$ and easily obtain:

$$|(i)| \ \leq \ |\Delta|^{1-\alpha}\, TrE_\Lambda(\Delta). \tag{2.6}$$

As long as $\alpha < 1$, this term can be absorbed into the left side of (2.3). For the term (ii) in (2.5), we insert another factor of $(H_0^\Lambda - E)(H_0^\Lambda - E)^{-1}$, and obtain,

$$\begin{aligned} (ii) &= -\lambda Tr(H_\omega^\Lambda - E)E_\Lambda(\Delta)V_\Lambda E_0^\Lambda(\tilde{\Delta}^c)(H_0^\Lambda - E)^{-2} \\ &\quad +\lambda^2 TrV_\Lambda E_\Lambda(\Delta)V_\Lambda E_0^\Lambda(\tilde{\Delta}^c)(H_0^\Lambda - E)^{-2} \\ &= (iii) + (iv). \end{aligned} \tag{2.7}$$

The term (iii) is estimated just as term (i) in (2.6), and we obtain

$$|(iii)| \ \leq \ \lambda|\Delta|^{1-2\alpha}\, \|\tilde{V}_\Lambda\|\, TrE_\Lambda(\Delta). \tag{2.8}$$

For the term (iv) in (2.7), we obtain

$$|(iv)| \ \leq \ \lambda^2|\Delta|^{-2\alpha}\, TrV_\Lambda E_\Lambda(\Delta)V_\Lambda. \tag{2.9}$$

This last term is estimated using the condition (1.5) satisfied by V_Λ,

$$TrV_\Lambda E_\Lambda(\Delta)V_\Lambda \ \leq \ C_1(u,d)Tr\tilde{V}_\Lambda E_\Lambda(\Delta). \tag{2.10}$$

We now take the expectation. From the appendix, we have

$$I\!E\{Tr(\lambda\tilde{V}_\Lambda E_\Lambda(\Delta))\} \ \leq \ C_3(q_2,u)|\Delta|^{q_2}|\Lambda|, \tag{2.11}$$

for any $0 < q_2 < 1$, so that

$$I\!E\{|(iv)|\} \ \leq \ \lambda|\Delta|^{q_2-2\alpha}C_1(u,d)C_3(q_2,u)|\Lambda|. \tag{2.12}$$

3. Gathering the estimates from (2.4), (2.6), (2.8), and (2.12), we get

$$\begin{aligned} & \{1-|\Delta|^{1-\alpha}-\lambda|\Delta|^{1-2\alpha}\|\tilde{V}_\Lambda\|\}\ I\!\!E\{TrE_\Lambda(\Delta)\} \\ \leq\ & (\lambda|\Delta|^{q_2-2\alpha}C_1(u,d)C_3(q_2,u)+C_2|\Delta|^{\alpha q_1})|\Lambda|. \end{aligned} \tag{2.13}$$

It is clear that if $\alpha < 1/2$ and $|\Delta|$ is taken sufficiently small, relative to the disorder $\lambda > 0$ (recall that $\|\tilde{V}_\Lambda\| = \mathcal{O}(1)$), then the prefactor on the left side of (2.13) is invertible. Finally, we require that $\alpha < q_2/(q_1+2) < 1/2$ in order that the right side of (2.13) be bounded above by $|\Delta|^q$, for $0 < q < q_1q_2/(q_1+2)$. As $q_2 < 1$ is arbitrary, the bound

$$\frac{1}{|\Lambda|} I\!\!E\{TrE_\Lambda(\Delta)\} \leq C|\Delta|^q$$

holds for any $0 < q < q_1/(q_1+2)$, where $0 < q_1 \leq 1$ is the Hölder exponent of continuity for N_0, and for Λ sufficiently large. Taking the limit $|\Lambda| \to +\infty$, we complete the proof of Theorem 1.1. □

3 Proof of Theorem 1.2.

We now turn to the proof of Theorem 1.2, improving the Hölder exponent given in Theorem 1.1. Let us recall that $\lambda > 0$ is a measure of the disorder, so that $\|V_\Lambda\| \leq \|\tilde{V}_\Lambda\|$, which is bounded by a constant depending only on u and independent of the volume $|\Lambda|$. The main assumption (1.9) implies that for PBC ($\theta = 0$), there exists a constant $C(\tilde{\Delta}, u) > 0$, independent of Λ, so that

$$E_0^\Lambda(\tilde{\Delta})\tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta})\ \geq C(\tilde{\Delta},u)E_0^\Lambda(\tilde{\Delta}). \tag{3.1}$$

Although the background operator H_0 is required to be periodic, no explicit information is needed on the IDS for the operator H_0, as was the case in the Theorem 1.1. For a fixed, but arbitrary, $E_0 \in \mathbb{R}$, let $E_0 \in \Delta \subset \tilde{\Delta}$ be two closed, bounded intervals centered on E_0, and let $d_0 \equiv$ dist $(E_0, \tilde{\Delta}^c)$.

Proof of Theorem 1.2.
1. As in the proof of Theorem 1.1, we need to estimate

$$I\!\!E\{TrE_\Lambda(\Delta)\}. \tag{3.2}$$

We begin with a decomposition relative to the spectral projectors $E_0^\Lambda(\cdot)$ for the operator H_0^Λ. We write

$$TrE_\Lambda(\Delta) = TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta}) + TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta}^c). \tag{3.3}$$

We will choose the intervals Δ and $\tilde{\Delta}$ below.
2. The term involving $\tilde{\Delta}^c$ is easily estimated as follows. We write

$$\begin{aligned} TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta}^c) &= TrE_\Lambda(\Delta)(H_\omega^\Lambda - E_0)(H_0^\Lambda - E_0)^{-1}E_0^\Lambda(\tilde{\Delta}^c) \\ &\quad -\lambda TrE_\Lambda(\Delta)V_\Lambda(H_0^\Lambda - E_0)^{-1}E_0^\Lambda(\tilde{\Delta}^c). \end{aligned} \tag{3.4}$$

¿From the positivity of the projector $E_\Lambda(\Delta)$, we obtain for the first term on the right of (3.4),

$$|TrE_\Lambda(\Delta)(H_\omega^\Lambda - E_0)(H_0^\Lambda - E_0)^{-1}E_0^\Lambda(\tilde{\Delta}^c)| \leq \frac{|\Delta|}{2d_0} TrE_\Lambda(\Delta). \tag{3.5}$$

As for the second term of (3.4), we expand the trace in terms of the finitely-many eigenfunctions ψ_j in the range of the projection $E_\Lambda(\Delta)$,

$$\begin{aligned} & TrE_\Lambda(\Delta)(\lambda V_\Lambda)(H_0^\Lambda - E_0)^{-1}E_0^\Lambda(\tilde{\Delta}^c) \\ &= \sum_j \langle \psi_j, (\lambda V_\Lambda)(H_0^\Lambda - E_0)^{-1}E_0^\Lambda(\tilde{\Delta}^c)\psi_j\rangle \\ &\leq \sum_j \frac{1}{d_0} \|E_0^\Lambda(\tilde{\Delta}^c)\psi_j\| \cdot \|(\lambda V_\Lambda)\psi_j\| \\ &\leq \frac{1}{2d_0} TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta}^c) + \frac{1}{2d_0} TrE_\Lambda(\Delta)(\lambda^2 V_\Lambda^2). \end{aligned} \tag{3.6}$$

In order to move the first term on the right in (3.6) to the left side of (3.4), we must choose $\tilde{\Delta}$ so that

$$1/2d_0 < 1/2, \quad \text{or} \quad d_0 > 1. \tag{3.7}$$

This being done, we find

$$TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta}^c) \leq \frac{|\Delta|}{d_0} TrE_\Lambda(\Delta) + \frac{1}{d_0} TrE_\Lambda(\Delta)(\lambda^2 V_\Lambda^2). \tag{3.8}$$

3. As for the first term on the right in (3.3), we use the fundamental assumption (3.1). We have

$$
\begin{aligned}
Tr E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}) &\leq \frac{1}{C(\tilde{\Delta}, u)} \left\{ Tr E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}) \tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta}) E_\Lambda(\Delta) \right\} \\
&\leq \frac{1}{C(\tilde{\Delta}, u)} \left\{ Tr E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}) \tilde{V}_\Lambda \right. \\
&\qquad \left. - Tr E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}) \tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta}^c) \right\} \\
&\leq \frac{1}{C(\tilde{\Delta}, u)} \left\{ Tr E_\Lambda(\Delta) \tilde{V}_\Lambda - Tr E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}^c) \tilde{V}_\Lambda \right. \\
&\qquad \left. - Tr E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}) \tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta}^c) \right\} \\
&\leq \frac{1}{C(\tilde{\Delta}, u)} \left\{ Tr E_\Lambda(\Delta) \tilde{V}_\Lambda - Tr E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}^c) \tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta}) \right. \\
&\qquad \left. - Tr E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}) \tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta}^c) \right\}, \qquad (3.9)
\end{aligned}
$$

where we dropped the positive term $Tr E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}^c) \tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta}^c) E_\Lambda(\Delta)$ since it occurs with a negative sign. We estimate the second and the third terms on the right in (3.9). As they are similar, we explicitly estimate the second term. Let $\|A\|_{HS}$ denote the Hilbert-Schmidt norm of an operator A. Using the Hölder inequality for trace norms, we have, for any $\mu_0 > 0$,

$$
\begin{aligned}
& |Tr E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}^c) \tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta})| \\
&\leq \|E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}^c)\|_{HS} \, \|\tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta}) E_\Lambda(\Delta)\|_{HS} \\
&\leq \frac{1}{2\mu_0} Tr E_0^\Lambda(\tilde{\Delta}^c) E_\Lambda(\Delta) + \frac{\mu_0}{2} Tr E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}) \tilde{V}_\Lambda^2 E_0^\Lambda(\tilde{\Delta}) E_\Lambda(\Delta).
\end{aligned} \qquad (3.10)
$$

We next estimate the second term on the right in (3.10). There exists a constant D_0, depending only on $u \geq 0$, so that $\tilde{V}_\Lambda^2 \leq D_0 \tilde{V}_\Lambda$. Using this, we find that for any $\mu_1 > 0$,

$$
\begin{aligned}
& Tr E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}) \tilde{V}_\Lambda^2 E_0^\Lambda(\tilde{\Delta}) E_\Lambda(\Delta) \\
&\leq D_0 \|E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}) \tilde{V}_\Lambda\|_{HS} \, \|E_0^\Lambda(\tilde{\Delta}) E_\Lambda(\Delta)\|_{HS} \\
&\leq \frac{D_0 \mu_1}{2} Tr E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}) \tilde{V}_\Lambda^2 E_0^\Lambda(\tilde{\Delta}) E_\Lambda(\Delta) + \frac{D_0}{2\mu_1} Tr E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}).
\end{aligned} \qquad (3.11)
$$

We choose $\mu_1 = 1/D_0 > 0$ so that $(1 - D_0\mu_1/2) = 1/2$. Consequently, we obtain

4. We now combine the two results (3.8) and (3.15) to obtain

$$\begin{aligned} TrE_\Lambda(\Delta) &= TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta}) + TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta}^c) \\ &\leq \left\{ \frac{|\Delta|}{d_0} + \frac{4D_0^2|\Delta|}{d_0 C(\tilde{\Delta},u)^2} \right\} TrE_\Lambda(\Delta) \\ &\quad + \left\{ \frac{2}{C(\tilde{\Delta},u)} + \frac{C_1(u,d)\lambda^2}{d_0}\left(1 + \frac{4D_0^2}{C(\tilde{\Delta},u)^2}\right)\right\} TrE_\Lambda(\Delta)\tilde{V}_\Lambda. \end{aligned} \tag{3.12}$$

We move the first term on the right in (3.16) to the left by choosing d_0 and $|\Delta|$ as follows. For any $E_0 \in \mathbb{R}$, we choose Δ to be an interval centered on E_0 with $|\Delta| < 1$. We choose $\tilde{\Delta}$ so that $d_0 = \mathrm{dist}(E_0, \tilde{\Delta}^c) >> \max(4, 16D_0^2)$, with D_0 defined in (3.11) (this implies, of course, condition (3.7)). Then, we have the upper bound on the coefficient

$$\left(\frac{|\Delta|}{d_0} + \frac{4D_0^2|\Delta|}{d_0 C(\tilde{\Delta},u)^2} \right) \leq \left(\frac{1}{4} + \frac{|\Delta|}{4C(\tilde{\Delta},u)^2} \right). \tag{3.13}$$

Keeping $\tilde{\Delta}$ fixed, we shrink $|\Delta|$, if necessary, so that $|\Delta|/C(\tilde{\Delta},u)^2 < 1$. In this way, we arrive at the bound

$$TrE_\Lambda(\Delta) \leq 2\left\{ \frac{2}{\lambda C(\tilde{\Delta},u)} + \frac{\lambda C_1(u,d)}{d_0}\left(1 + \frac{4D_0^2}{C(\tilde{\Delta},u)^2}\right)\right\} TrE_\Lambda(\Delta)(\lambda\tilde{V}_\Lambda). \tag{3.14}$$

5. It remains to estimate the term on the right in (3.18). For this, we use the result of the appendix (as in (2.11)), that for any $0 < q_2 < 1$, there exists a finite constant $C_3(E_0, u, q_2) > 0$, so that

$$I\!E\{TrE_\Lambda(\Delta)(\lambda\tilde{V}_\Lambda)\} \leq C_3(E_0,u,q_2)\, |\Lambda|\, |\Delta|^{q_2}. \tag{3.15}$$

Inserting this into the right side of (3.18), we obtain the upper bound

$$\begin{aligned} &I\!E\{TrE_\Lambda(\Delta)\} \\ &\leq 2\left\{ \frac{2}{\lambda C(\tilde{\Delta},u)} + \frac{\lambda C_1(u,d)}{d_0}\left(1 + \frac{4D_0^2}{C(\tilde{\Delta},u)^2}\right)\right\} C_3(E_0,u,q_2)\, |\Lambda|\, |\Delta|^{q_2}, \end{aligned} \tag{3.16}$$

provided $|\Delta|$ is sufficiently small as specified above. This proves the Hölder continuity of the IDS with any exponent $0 < q < 1$. $\square$

4 A Lower Bound Estimate for Periodic Operators

The goal of this section is the proof of the following theorem concerning spectral projections of periodic Schrödinger operators related to condition (1.9). Let Γ be a non-degenerate lattice in $\mathbb{R}^d$. Let $V_0 : \mathbb{R}^d \to \mathbb{R}$ be a real-valued, Γ-periodic potential. We consider the Γ-periodic Schrödinger operator $H_0 = -\Delta + V_0$. We assume that $V_0 \in L^p(\mathbb{R}^d/\Gamma)$, with $p = 2$ for $d \leq 3$, and $p > d/2$ for $d \geq 4$. It is known that this implies that H_0 is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$, and self-adjoint on the domain $H^2(\mathbb{R}^d)$.

We recall the Floquet decomposition of H_0 described in section 1. Let $H_0(\theta)$, for $\theta \in (\mathbb{R}^d)^*/\Gamma^*$ be the fibred operator acting on $L^2(\mathbb{R}^d/\Gamma)$, with θ-quasi-periodic boundary conditions. We also need to consider H_0 as an $n\Gamma$-periodic operator for any $n \in \mathbb{N}$. Let $H_0^n(\theta)$ be the operator H_0 restricted to the torus $\mathbb{R}^d/(n\Gamma)$, with θ-quasi-periodic boundary conditions. For $I \subset \mathbb{R}$, an interval, let $\mathbf{E}_0^n(I,\theta)$ denote the spectral projection onto the interval I for $H_0^n(\theta)$. We remark that in section 2 we took $I = \tilde{\Delta}$. Finally, for any Γ-periodic function g, we write $g^{(n)}$ for the same function understood as an $n\Gamma$-periodic function.

We remark that we can also treat the case of a periodic background operator $H_0 = (-i\nabla - A_0)^2 + V_0$, for V_0 as above, and a real-valued, Γ-periodic vector potential $A_0 : \mathbb{R}^d \to \mathbb{R}^d$. As presented in section 6, the Floquet analysis of H_0 used in the proof of the following theorem is also possible in the case of a constant magnetic field provided the flux associated with the magnetic field $B = dA$ through a unit cell of Γ is 2π times a rational.

Theorem 4.1. *Let $V : \mathbb{R}^d \to \mathbb{R}$ be a bounded, real-valued, Γ-periodic function. Consider a bounded interval $I \subset \mathbb{R}$. Then, if there exists a finite constant $C(I,V) > 0$ such that, for all $\theta \in \mathbb{R}^d$, one has*

$$\mathbf{E}_0(I,\theta) V \mathbf{E}_0(I,\theta) \geq C(I,V) \mathbf{E}_0(I,\theta), \tag{4.1}$$

then, for all $n \geq 1$ and all $\theta \in \mathbb{R}^d$, one has, with the same constant $C(I,V)$,

$$\mathbf{E}_0^n(I,\theta) V^{(n)} \mathbf{E}_0^n(I,\theta) \geq C(I,V) \mathbf{E}_0^n(I,\theta), \tag{4.2}$$

where V is considered as a Γ-periodic function in (4.1), and $V^{(n)}$ represents the corresponding $n\Gamma$-periodic function in (4.2).

Proof.
1. The spectrum of $H_0^n(\theta)$ is discrete for any n and θ. Moreover, the spectral decomposition of $H_0^n(\theta)$ is obtained from the one of $H_0^1(\theta)$ in the following way. Let $E_1(\theta) \leq E_2(\theta) \leq \cdots \leq E_j(\theta) \leq \dots$ be the sequence of Floquet eigenvalues of $H_0^1(\theta)$ ordered increasingly. Let $\{x \in \mathbb{R}^d/\Gamma \mapsto \varphi_j(x,\theta)\}_{j\geq 1}$ be the associated normalized eigenfunctions. One has

$$H_0(\theta)\varphi_j(\cdot,\theta) = E_j(\theta)\varphi_j(\cdot,\theta), \tag{4.3}$$

with the eigenfunctions φ_j satisfying the θ-quasi-periodic boundary conditions,

$$\varphi_j(x+\gamma,\theta) = e^{i\gamma\theta}\varphi_j(x,\theta), \quad \text{for all } \gamma \in \Gamma. \tag{4.4}$$

Then, the eigenvalues of $H_0^n(\theta)$ are given by the sequence

$$\{E_{j,\gamma^*}(\theta)\}_{j\geq 1,\ \gamma^*\in\Gamma^*/(n\Gamma^*)} = \{E_j(\theta+\gamma^*/n)\}_{j\geq 1,\ \gamma^*\in\Gamma^*/(n\Gamma^*)} \tag{4.5}$$

where $\Gamma^* = \{\gamma^* \in \mathbb{R}^d;\ \forall \gamma \in \Gamma,\ \gamma\cdot\gamma^* \in 2\pi\mathbb{Z}\}$. The corresponding eigenfunctions are given by $\{x \in \mathbb{R}^d/\Gamma \mapsto \varphi_j(x,\theta+\gamma^*/n)\}_{j\geq 1}$. To normalize these functions, we divide them by $n^{d/2}$ and call them $\{\varphi_{j,\gamma^*}\}_{j\geq 1,\ \gamma^*\in\Gamma^*/(n\Gamma^*)}$.
2. By definition, one has

$$\mathbf{E}_0^n(I,\theta) = \sum_{\substack{j\geq 1,\gamma^*\in\Gamma^*/(n\Gamma^*) \\ s.\ t.\ E_{j,\gamma^*}\in I}} |\varphi_{j,\gamma^*}\rangle\langle\varphi_{j,\gamma^*}|. \tag{4.6}$$

This gives

$$\mathbf{E}_0^n(I,\theta)V^{(n)}\mathbf{E}_0^n(I,\theta) = \sum_{\substack{j\geq 1,\ \gamma^*\in\Gamma^*/(n\Gamma^*) \\ s.\ t.\ E_{j,\gamma^*}\in I \\ \tilde{j}\geq 1,\ \tilde{\gamma}^*\in\Gamma^*/(n\Gamma^*) \\ s.\ t.\ E_{\tilde{j},\tilde{\gamma}^*}\in I}} \langle\varphi_{j,\gamma^*}, V^{(n)}\varphi_{\tilde{j},\tilde{\gamma}^*}\rangle_n\ |\varphi_{j,\gamma^*}\rangle\langle\varphi_{\tilde{j},\tilde{\gamma}^*}|. \tag{4.7}$$

Here the scalar product $\langle,\rangle_n$ is the usual scalar product in the x-variable over $\mathbb{R}^d/(n\Gamma)$.

3. Using the periodicity of V and the definition of the eigenfunctions, we compute

$$\begin{aligned}
\langle \varphi_{j,\gamma^*}, V^{(n)} \varphi_{\tilde{j},\tilde{\gamma}^*} \rangle_n &= \int_{\mathbb{R}^d/(n\Gamma)} V^{(n)}(x) \varphi_{j,\gamma^*}(x) \overline{\varphi_{\tilde{j},\tilde{\gamma}^*}(x)} dx \\
&= \sum_{\Gamma/(n\Gamma)} \int_{\mathbb{R}^d/\Gamma} V(x+\gamma) \varphi_{j,\gamma^*}(x+\gamma) \overline{\varphi_{\tilde{j},\tilde{\gamma}^*}(x+\gamma)} dx \\
&= \left(\sum_{\gamma \in \Gamma/(n\Gamma)} e^{i\gamma(\gamma^* - \tilde{\gamma}^*)} \right) \int_{\mathbb{R}^d/\Gamma} V(x) \varphi_{j,\gamma^*}(x) \overline{\varphi_{\tilde{j},\tilde{\gamma}^*}(x)} dx \\
&= \begin{cases} 0 & \text{if} \gamma^* \neq \tilde{\gamma}^*, \\ \int_{\mathbb{R}^d/\Gamma} V(x) \varphi_j(x, \theta + \gamma^*/n) \overline{\varphi_{\tilde{j}}(x, \theta + \gamma^*/n)} dx, & \text{otherwise} \end{cases} \\
&= \delta_{\gamma^*, \tilde{\gamma}^*} \langle V \varphi_j(\theta + \gamma^*/n), \varphi_{\tilde{j}}(\theta + \gamma^*/n) \rangle_1 \qquad (4.8)
\end{aligned}$$

Hence, (4.6) becomes

$$\mathbf{E}_0^n(I,\theta) V^{(n)} \mathbf{E}_0^n(I,\theta) = \sum_{\substack{j \geq 1,\ \tilde{j} \geq 1,\ \gamma^* \in \Gamma^*/(n\Gamma^*) \\ \text{s.t.} E_{j,\gamma^*} \in I \text{ and } E_{\tilde{j},\gamma^*} \in I}} \langle V \varphi_j(\theta + \gamma^*/n), \varphi_{\tilde{j}}(\theta + \gamma^*/n) \rangle_1 |\varphi_{j,\gamma^*}\rangle \langle \varphi_{\tilde{j},\gamma^*}|. \tag{4.9}$$

So we see that if, for some $C(I,V) > 0$, one has

$$\forall \gamma^* \in \Gamma^*, \quad \left((\langle V \varphi_j(\theta + \gamma^*/n), \varphi_{\tilde{j}}(\theta + \gamma^*/n) \rangle_1)_{j,\tilde{j}} \right) \geq C(I,V)\, Id, \tag{4.10}$$

where the inequality is meant in matrix sense, then (4.8) implies (4.2). On the other hand, equation (4.9) is nothing but condition (4.1) taken at the point $\theta + \gamma^*/n$ and rewritten in the basis $(\varphi_j(\theta + \gamma^*/n))_j$. This completes the proof of Theorem 4.1. □

We next show that, in fact, condition (4.1) holds for a wide family of periodic potentials V. We remind the reader that in the applications, we will take the potential V appearing in (4.1) to be the single-site potential u, restricted to the unit cell $\Lambda_1(0)$, viewed as a Γ-periodic function. The $n\Gamma$-periodic function appearing in (4.2) is $\tilde{V}(x) = \sum_j u(x-j)$, restricted to $\Lambda^{(n)}$, where $\Lambda^{(n)}$ is the basic $n\Gamma$-periodic cell.

Theorem 4.2. *Let $V : \mathbb{R}^d \to \mathbb{R}$ be a bounded, Γ-periodic, nonnegative function. Suppose that $V > 0$ on some open set and H_0 has the unique* *continuation property. Then, condition (4.1) holds for any compact interval $I \subset \mathbb{R}$ with a finite constant $C(I, V) > 0$.*

In order to prove Theorem 4.2, we need the following lemma. This lemma was proved in [20]. We note that the proof given there holds in all dimensions.

Lemma 4.3. **[20]** *Fix $\theta_0 \in \mathbb{T}^* = (\mathbb{R}^d)^*/\Gamma^*$, and let I be a compact interval in $\mathbb{R}$. Then, there exists N_{θ_0}, an open neighborhood of θ_0 in $\mathbb{T}^*$, J, a bounded, open interval $I \subset J$ such that*

1. *There exists a finite constant $p(\theta_0) \geq 0$ so that for any $\theta \in N_{\theta_0}$, the number of Floquet eigenvalues $E_n(\theta) \in J$ is a constant equal to $p(\theta_0)$;*
2. *There exists $p(\theta_0)$ real analytic functions $\theta \in N_{\theta_0} \mapsto v_j(\cdot, \theta) \in \mathcal{H}_\theta$, $j = 1, \ldots, p(\theta_0)$, forming an orthonormal system, such that, for $\theta \in N_{\theta_0}$, the spectral projector $\mathbf{E}_0(J, \theta)$ is the orthogonal projector on the subspace spanned by $\{v_j(\cdot, \theta) \mid 1 \leq j \leq p(\theta_0)\}$.*

Proof of Theorem 4.2. We fix $\theta_0 \in \mathbb{T}^*$ and let W_{θ_0} be an open neighborhood of θ_0 such that $\overline{W_{\theta_0}} \subset N_{\theta_0}$ (here, N_{θ_0} is given by Lemma 4.3.). Let J, $I \subset J$, be the open interval fixed by Lemma 4.3. Notice that, as $\mathbf{E}_0(J, \theta)\mathbf{E}_0(I, \theta) = \mathbf{E}_0(I, \theta)$, the inequality $\mathbf{E}_0(J, \theta)V\mathbf{E}_0(J, \theta) \geq C\mathbf{E}_0(J, \theta)$ implies $\mathbf{E}_0(I, \theta)V\mathbf{E}_0(I, \theta) \geq C\mathbf{E}_0(I, \theta)$, Let us show that there exists $C_0 > 0$ such that $\mathbf{E}_0(J, \theta)V\mathbf{E}_0(J, \theta) \geq C\mathbf{E}_0(J, \theta)$ holds for all $\theta \in \overline{W_{\theta_0}}$. Estimate $\mathbf{E}_0(J, \theta)V\mathbf{E}_0(J, \theta) \geq C\mathbf{E}_0(J, \theta)$ is equivalent to saying that the matrix $M(\theta) = ((m_{ij}(\theta)))_{1 \leq i,j \leq p}$ where

$$m_{ij}(\theta) = \int_{\mathbb{R}^d/\Gamma} V(x) v_i(x, \theta) \overline{v_j(x, \theta)} dx \tag{4.11}$$

is positive. The mapping $\theta \mapsto M(\theta)$ being continuous and $\overline{W_{\theta_0}}$ being compact, the point-wise positivity of M implies its uniform positivity in $\overline{W_{\theta_0}}$. Assume there exists $a = (a_j)_{1 \leq j \leq p}$, such that one has

$$\int_{\mathbb{R}^d/\Gamma} \left| \sum_{j=1}^{p} a_j v_j(x, \theta) \right|^2 V(x) dx = 0. \tag{4.12}$$

As $V > 0$ on some open set $O \subset \mathbb{R}^d/\Gamma$, one has

$$\sum_{j=1}^{p} a_j v_j(x, \theta) = 0 \tag{4.13}$$

By point (2) of Lemma 4.3, there exists an invertible $p \times p$ matrix, say P, such that $P(v_j)_j = (u_j)_j$ where $(u_j)_j$ is the sequence of eigenvectors of $H_0(\theta)$ spanning $\mathbf{E}_0(J, \theta)$. So that (4.12) implies that, on O, one has

$$\sum_{j=1}^{p} (P^{-1}a)_j u_j(x, \theta) = 0 \tag{4.14}$$

To fix ideas, assume $H_0(\theta)u_j(\theta) = E_j(\theta)u_j(\theta)$. Applying N times the differential operator $H_0(\theta)$ to (4.13), one obtains that, on O, one has

$$\forall N \in \mathbb{N}, \quad \sum_{j=1}^{p} (P^{-1}a)_j E_j(\theta)^N u_j(x, \theta) = 0. \tag{4.15}$$

This implies that, for every $E \in \{E_j(\theta);\ 1 \leq j \leq p\}$, on O, one has

$$\sum_{j; E_j(\theta)=E} (P^{-1}a)_j u_j(x, \theta) = 0 \tag{4.16}$$

But, by construction, $\sum_{j;E_j(\theta)=E} (P^{-1}a)_j u_j(x, \theta)$ is an eigenvector of $H_0(\theta)$ associated to the eigenvalue E. Hence, as H_0 has the unique continuation property (see e.g. [30]), we get that $\sum_{j;E_j(\theta)=E} (P^{-1}a)_j u_j(x, \theta)$ vanishes identically on $\mathbb{R}^d/\Gamma$. The $(u_j)_j$ being linearly independent, we obtain that for all $1 \leq j \leq p$, $(P^{-1}a)_j = 0$. As P is invertible, this implies $a = 0$. So the kernel of $M(\theta)$ is reduced to the vector 0; as $M(\theta)$ is obviously nonnegative, it is positive. This completes the proof of Lemma 4.2. □

5 The Integrated Density of States for Perturbed Landau Hamiltonians

The proof of Theorem 1.5 consists in reducing the problem via magnetic Floquet theory to the situation treated in section 4. The unperturbed Landau Hamiltonian H_L has the form $H_L = (p - A)^2$, where $p = -i\nabla$ and $A = (B/2)(-x_2, x_1)$. The magnetic translations for the two-dimensional, constant, transverse magnetic field problem are defined as follows. For any field strength $B \in \mathbb{R}$, any vector $\alpha \in \mathbb{R}^2$, and $f \in C_0^\infty(\mathbb{R}^2)$, we define the magnetic translation by α by

$$U_\alpha^B f(x) \equiv e^{\frac{iB}{2}(x_1\alpha_2 - x_2\alpha_1)} f(x + \alpha). \tag{5.1}$$

For $(\alpha, \beta) \in \mathbb{R}^2$, we have the relations:

$$U_\alpha^B U_\beta^B = e^{iB(\alpha_1\beta_2 - \alpha_2\beta_1)} U_\beta^B U_\alpha^B. \tag{5.2}$$

In a standard way, the family $\{U_\alpha^B \mid \alpha \in \mathbb{R}^2\}$, defined in (5.1)–(5.2), extends to a projective unitary representation of $\mathbb{R}^2$ on $L^2(\mathbb{R}^2)$. We note that

$$U_\alpha^B H_L U_{-\alpha}^B = H_L. \tag{5.3}$$

We define two fundamental vectors in $\mathbb{R}^2$,

$$\alpha_1 = (L_1, 0), \quad \text{and} \quad \alpha_2 = (0, L_2), \tag{5.4}$$

for any $L_1, L_2 \in \mathbb{R}$, and write U_j^B, for $j = 1, 2$, for the corresponding unitaries as in (5.1). If follows from (5.2) that we have the commutation rules:

$$U_1^B U_2^B = e^{iBL_1L_2} U_2^B U_1^B. \tag{5.5}$$

It is clear that $\{U_1^B, U_2^B\}$ generate a two-parameter Abelian group provided the following *integer flux condition for the square* $[0, L_1] \times [0, L_2]$ is satisfied:

$$BL_1L_2 \in 2\pi\mathbb{Z}. \tag{5.6}$$

More generally, suppose that only the *rational flux condition* for the square $[0, L_1] \times [0, L_2]$ is satisfied:

$$BL_1L_2 = 2\pi p/q, \quad \text{for} \quad p, q \in \mathbb{Z}, \tag{5.7}$$

with p, q having no common divisors. Then, the unitary operators $\{(U_1^B)^q, U_2^B\}$ satisfy the commutation relations

$$(U_1^B)^q U_2^B = e^{iqBL_1L_2} U_2^B (U_1^B)^q, \tag{5.8}$$

so it is clear that the pair generate an Abelian group.

Let us now suppose

(H3). *The rational flux condition for a unit square* $[0, 1] \times [0, 1]$ *holds for the magnetic field strength* B.

That is, we suppose (5.6) with $L_1 = L_2 = 1$. We have $[(U_1^B)^q, U_2^B] = 0$, and $[(U_1^B)^q, H_L] = 0 = [U_2^B, H_L]$. Let $\Gamma' \subset \Gamma = \mathbb{Z}^2$ be the sublattice corresponding to these translations, that is

$$\Gamma' = \{(n_1 q, n_2) \mid n_1, n_2 \in \mathbb{Z}\}. \tag{5.9}$$

We define the dual lattice to be $(\Gamma')^*$ given by

$$(\Gamma')^* = \{(2\pi m_1/q, 2\pi m_2) \mid m_1, m_2 \in \mathbb{Z}\}. \tag{5.10}$$

If $\gamma^* \in (\Gamma')^*$, and $\gamma \in \Gamma'$, then we have $\gamma^* \cdot \gamma \in 2\pi\mathbb{Z}$. For any $\gamma = (n_1 q, n_2) \in \Gamma'$, define a phase by

$$\Theta_q(\gamma) = e^{iBn_1 n_2 q/2} \in \{-1, +1\}. \tag{5.11}$$

This allows us to define a unitary representation of the sublattice Γ' by

$$W^B_{q,\gamma} = \Theta_q(\gamma) U^B_\gamma. \tag{5.12}$$

It is easy to check that

$$W^B_{q,\gamma} W^B_{q,\gamma'} = W^B_{q,\gamma+\gamma'}, \quad \gamma, \gamma' \in \Gamma'. \tag{5.13}$$

We define the transformation T^B on smooth functions by

$$(T^B f)(x, \theta) = \sum_{\gamma \in \Gamma'} e^{i\theta\cdot\gamma} (W^B_{q,\gamma} f)(x), \quad \theta \in (\mathbb{R}^2)^*/(\Gamma')^*. \tag{5.14}$$

Again, a simple calculation shows that

$$(W^B_{q,\gamma} T^B f)(x, \theta) = e^{-i\theta\cdot\gamma} (T^B f)(x, \theta). \tag{5.15}$$

We define a function space $\mathcal{H}_{B,\theta}$ by

$$\mathcal{H}_{B,\theta} = \{v \in L^2_{loc}(\mathbb{R}^2) \mid W^B_{q,\gamma} v = e^{-i\theta\cdot\gamma} v\}. \tag{5.16}$$

It then follows that T^B extends to a unitary map

$$T^B : L^2(\mathbb{R}^2) \to L^2((\mathbb{R}^2)^*/(\Gamma')^*), \mathcal{H}_{B,\theta}). \tag{5.17}$$

Given this structure, it is clear that the Hamiltonian H_L admits a direct integral decomposition (see e.g. [24]) over $(\mathbb{R}^2)^*/(\Gamma')^*$, so that

$$H_L = \int^{\oplus}_{(\mathbb{R}^2)^*/(\Gamma')^*} H_L(\theta)\, d\theta. \tag{5.18}$$

The operator $H_L(\theta)$ is self-adjoint on the Sobolev space $\mathcal{H}^2_{B,\theta}$, the local Sobolev space of order two of functions in $\mathcal{H}_{B,\theta}$. This operator has compact resolvent. Consequently, the spectrum is discrete and consists of finite multiplicity eigenvalues $E_j(B, \theta), j = 1, 2, \ldots,$ labeled in increasing order *not* including multiplicity. Unlike the case of a Schrödinger operator with a periodic potential, since

$$\bigcup_{\theta \in (\mathbb{R}^2)^*/(\Gamma')^*} E_j(B, \theta) = E_j(B), \tag{5.19}$$

the j^{th}-Landau level, the eigenvalues $E_j(B, \theta)$ are constants independent of θ.

To prove Theorem 1.5, we follow the proof of Theorem 1.2 given in section 3. We work with increasing sequences of regions $\Lambda^{(q)}$ that are integer multiples and $\mathbb{Z}^2$-translates of the basic cell $\Lambda_0^{(q)} = [0, q] \times [0, 1]$. For the IDS, we consider H_L restricted to $\Lambda^{(q)}$ with periodic boundary conditions (PBC). We note that functions satisfying PBC form the reduced Hilbert space $\mathcal{H}_{B,\theta=0}$. Let us consider the Hamiltonian $H_L(\theta)$ corresponding to the unit cell $\Lambda_0^{(q)}$. For any $I \subset \mathbb{R}$ and $\theta \in (\mathbb{R}^2)^*/(\Gamma')^*$, we define the spectral projector for $H_L(\theta)$ and I by $E_0(I, \theta)$. From the above discussion, it follows that

$$E_0(I, \theta) = 0, \quad \text{if} \quad I \cap \{E_j(B) \mid \text{for any } j = 0, 1, \ldots\} = \emptyset, \tag{5.20}$$

and

$$\begin{aligned} E_0(I, \theta) &= \sum_{j=1}^{M} \sum_{l=1}^{K} |\phi_{j,l}(\theta)\rangle\langle\phi_{j,l}(\theta)|, \\ & \text{if} \quad E_j(B) \in I, \text{ for } j = 1, \ldots, M. \end{aligned} \tag{5.21}$$

In keeping with the notation of section 4, we let $H_L^n(\theta)$ be the family of Hamiltonians on the region $\Lambda_n^{(q)} = [0, nq] \times [0, n]$. We denote by $E_0^n(I, \theta)$ the corresponding spectral projectors.

In the Landau context, assumption (3.1) means the following. Let $\tilde{\Lambda}_n^{(q)} = \Lambda_n^{(q)} \cap \mathbb{Z}^2$ be the lattice points in $\Lambda_n^{(q)}$. We define the local potential $\tilde{V}_{\Lambda_n^{(q)}} = \sum_{j \in \Lambda_n^{(q)}} u_j \chi_{\Lambda_n^{(q)}}$. For any $\theta \in \Gamma'$, let $E_L^{\Lambda_n^{(q)}}(I) = E_0^n(I, \theta)$. Then, there exists a finite constant $C(I, u) > 0$ so that for all $n \in \mathbb{N}$,

$$E_L^{\Lambda_n^{(q)}}(I) \tilde{V}_{\Lambda_n^{(q)}} E_L^{\Lambda_n^{(q)}}(I) \geq C(I, u) E_L^{\Lambda_n^{(q)}}(I). \tag{5.22}$$

With this formulation of (3.1), we can now apply all the results of section 4 to these operators.

Next, we examine the proof of the lower bound in the Landau context, as described in section 4. Theorem 4.1 is identical in this case since it is based on the Floquet decomposition. The proof of Theorem 4.2 requires addition information on the eigenfunctions as given in Lemma 4.3. The multiplicity of any eigenvalue $E_j(B,\theta)$ is locally constant due to the continuity of the spectral projectors. We note that in the case that I contains only one Landau level $E_j(B)$, which is the case when B is sufficiently large, the actual proof of the analog of Theorem 4.2 is simpler since the eigenvectors $\phi_{j,l}(\theta)$, for a given j, all correspond to the same eigenvalue $E_j(B)$. Hence, the unique continuation argument can be applied directly to the analog of (4.12).

6 Appendix: A Local Trace Estimate

We prove estimate (2.11) on the expectation of the trace of $\lambda \tilde{V_\Lambda} E_\Lambda(\Delta)$. This is based on the estimate on the spectral shift function proved in [9]. The L^p-theory of the spectral shift function for $p \in [1, \infty]$ can be viewed as an interpolation between the two well-known cases of $p = 1$ and $p = \infty$. The L^1 and L^∞-theory can be found in the review paper of Birman and Yafaev [3], and the book of Yafaev [31]. Suppose that H_0 and H are two self-adjoint operators on a separable Hilbert space $\mathcal{H}$ having the property that $V \equiv H - H_0$ is in the trace class. We denote by $\|V\|_1$ the trace norm of V. Under these conditions, we can define the Krein spectral shift function (SSF) $\xi(\lambda; H, H_0)$ through the perturbation determinant. Let $R_0(z) = (H_0 - z)^{-1}$, for $Im\ z \neq 0$. We then have

$$\xi(\lambda; H, H_0) \equiv \frac{1}{\pi} \lim_{\epsilon \to 0^+} \ \arg \det\, (1 + V R_0(\lambda + i\epsilon)). \tag{6.1}$$

We need to consider the case when $H - H_0 \in \mathcal{I}_{1/p}$, for the cases $1 < p < \infty$ (cf. [25]). Let A be a compact operator on $\mathcal{H}$ and let $\mu_j(A)$ denote the j^{th} singular value of A. We say that $A \in \mathcal{I}_{1/p}$, for some $p \geq 1$, if

$$\sum_j \ \mu_j(A)^{1/p} < \infty. \tag{6.2}$$

We define a nonnegative functional on the ideal $\mathcal{I}_{1/p}$ by

$$\|A\|_{1/p} \equiv \left(\sum_j \ \mu_j(A)^{1/p} \right)^p . \tag{6.3}$$

For $p > 1$, this functional is not a norm but satisfies

$$\|A + B\|_{1/p}^{1/p} \leq \|A\|_{1/p}^{1/p} + \|B\|_{1/p}^{1/p}. \tag{6.4}$$

If we define a metric $\rho_{1/p}(A, B) \equiv \|A - B\|_{1/p}^{1/p}$ on $\mathcal{I}_{1/p}$, then the linear space $\mathcal{I}_{1/p}$ is a complete, separable linear metric space. The finite rank operators are dense in $\mathcal{I}_{1/p}$ (cf. [2]).

Since $\mathcal{I}_{1/p} \subset \mathcal{I}_1$, for all $p \geq 1$, we refer to $A \in \mathcal{I}_{1/p}$ as being super-trace class. Consequently, we can define the SSF for a pair of self-adjoint operators H_0 and H for which $V = H - H_0 \in \mathcal{I}_{1/p}$. The main theorem on the SSF in [9] is the following.

Theorem 6.1. **[9]** *Suppose that H_0 and H are self-adjoint operators so that $V = H - H_0 \in \mathcal{I}_{1/p}$, for some $p \geq 1$. Then, the SSF $\xi(\lambda; H, H_0) \in L^p(\mathbb{R})$, and satisfies the bound*

$$\|\xi(\cdot\, ; H, H_0)\|_{L^p} \leq \|V\|_{1/p}^{1/p}. \tag{6.5}$$

We will use this in the proof of the estimate (2.11).

Proof of (2.11). We fix a closed, bounded interval $\Delta = [\Delta_-, \Delta_+] \subset \mathbb{R}$. Let $f_\Delta \geq 0$ be a monotone increasing C^1-function having so that $f_\Delta(t) = 1$, for $t > \Delta_+ + |\Delta|$, $f_\Delta(t) = 0$, for $t < \Delta_- - |\Delta|$, and such that $f'_\Delta = \mathcal{O}(1/|\Delta|)$. It follows that, for some $C > 0$, as operators

$$E_\Lambda(\Delta) \leq C|\Delta|\, f'_\Delta(H_\Lambda). \tag{6.6}$$

Here, H_Λ is H restricted to Λ with self-adjoint boundary conditions. Consequently, as the potential $\tilde{V}_\Lambda \geq 0$, we have

$$Tr(\lambda \tilde{V}_\Lambda E_\Lambda(\Delta)) \leq C|\Delta|\, Tr f'_\Delta(H_\Lambda)(\lambda \tilde{V}_\Lambda). \tag{6.7}$$

We now take the expectation of (6.8). We use the fact that

$$\sum_{k \in \tilde{\Lambda}} \frac{\partial}{\partial \lambda_k} \{Tr f_\Delta(H_\Lambda)\} = Tr f'_\Delta(H_\Lambda)(\lambda \tilde{V}_\Lambda), \tag{6.8}$$

so that

$$I\!E\{Tr(\lambda \tilde{V}_\Lambda) E_\Lambda(\Delta)\} \leq C \sum_{k \in \tilde{\Lambda}} |\Delta| I\!E \left\{ \frac{\partial}{\partial \lambda_k} Tr f_\Delta(H_\Lambda) \right\}. \tag{6.9}$$

We evaluate the expectation on the right side of (6.9) with respect to one random variable, say λ_k, using (H2):

$$\int_0^1 h_0(\lambda_k)\ d\lambda_k \left\{ \frac{\partial}{\partial \lambda_k}\ Tr f_\Delta(H_\Lambda) \right\}$$

$$\leq\ \|h_0\|_\infty\ Tr\{f_\Delta(H_\Lambda)(\lambda_k = 1) - f_\Delta(H_\Lambda)(\lambda_k = 0)\}. \tag{6.10}$$

The trace on the right in (6.10) can be expressed in terms of the spectral shift function for $H_1 \equiv H_\Lambda(\lambda_k = 1)$ and $H_0 \equiv H_\Lambda(\lambda_k = 0)$ by means of the Birman-Krein formula,

$$Tr\{f_\Delta(H_1) - f_\Delta(H_0)\}\ =\ -\int f'_\Delta(t)\xi(t; H_1, H_0)\ dt. \tag{6.11}$$

We recall that the SSF for the pair (H_0, H_1) is defined through bounded functions of these operators since the difference is not in a trace ideal. Both H_0 and H_1 are lower semibounded by, say $-M_0 > -\infty$. We define $g(s) = (s + M_0)^{-k}$, for $s > -M_0$. It follows as in [9] that the difference $g(H_1) - g(H_0) \in \mathcal{I}_{1/p}$, for any $p > 1$, provided $k > pd/2 + 2$. We have the identity for the SSF:

$$\xi(t; H_1, H_0) = sgn(g')\xi(g(s); g(H_1), g(H_0)). \tag{6.12}$$

We substitute this into the Birman-Krein trace formula and estimate the integral using Hölder's inequality as in [9]. We use the result in Theorem 6.1 to obtain an upper bound on the L^p-norm of the SSF in terms of $\|g(H_1) - g(H_0)\|_{1/p}^{1/p}$. This latter norm is bounded by a constant independent of $|\Lambda|$ and $|\Delta|$. We find that

$$Tr\{f_\Delta(H_1) - f_\Delta(H_0)\} \leq C_p|\Delta|^{p-1}, \tag{6.13}$$

for any $0 < p < 1$. Combining this with (6.10), giving an additional factor of $|\Delta|$, and with (6.11), giving a factor of $|\Lambda|$, we obtain the result (2.11). □

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