Controlling strong scarring for quantized ergodic toral automorphisms

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February 21, 2002

Abstract

We show that in the semi-classical limit the eigenfunctions of quantized ergodic symplectic toral automorphisms can not concentrate in measure on a finite number of closed orbits of the dynamics. More generally, we show that, if the pure point component of the limit measure has support on a finite number of such orbits, then the mass of this component must be smaller than two thirds of the total mass. The proofs use only the algebraic (*i.e.* not the number theoretic) properties of the toral automorphisms together with the exponential instability of the dynamics and therefore work in all dimensions.

1 Introduction

The Schnirelman theorem states that if a quantum system has an ergodic classical limit, then "most" of its eigenfunctions equidistribute (on the energy surface) in phase space in the semi-classical limit. This result has been proven in many different contexts: for the Laplace-Beltrami operator on compact Riemannian manifolds with an ergodic geodesic flow in [Sc] [Z1] [CdV], for ergodic billiards in [GL] [ZZ], for non-relativistic quantum mechanics in the classical limit in [HMR], for quantum maps in

[BDB] [Z2]. A precise statement in the latter context will be given below (Theorem 1.3).

The theorem raises obvious questions: do there exist exceptional sequences of eigenfunctions allowing no semi-classical limiting measure or a limit different from Liouville measure? It is well known that the limit must in that case be an invariant probability measure of the dynamics. Clearly, one would like to better characterize the class of invariant measures that are obtained as limit measures from sequences of eigenfunctions. Particularly simple candidates are delta measures concentrated on the periodic orbits of the dynamics and (finite) convex combinations thereof. Numerical and theoretical investigations for ergodic billiards and for quantum maps [Bog] [He] suggest the possibility that there exist sequences of eigenfunctions concentrating *to some degree* on (unstable) periodic orbits. This imperfectly defined enhancement phenomenon is loosely referred to as "scarring". It is not clear from the available evidence whether some sequences of eigenfunctions may concentrate *sufficiently strongly* on one or more periodic orbits to lead to a limiting Dirac measure on those orbits: no such example is known to date and many researchers in the field seem to think this should not be possible. In [CdV], for example, it is conjectured that such sequences should not exist on constant negative curvature surfaces. Partial results in this direction have been obtained using number theoretic methods for certain arithmetic hyperbolic surfaces [RS] [LS] [S3] (see [S1], [S2] for a review).

In this paper we analyze the above problem for a simpler class of models that has attracted much attention, namely the quantized ergodic automorphisms of the $2d$-torus. We prove here that for these models such sequences do not exist (Theorem 1.1). We also obtain a stronger result that controls the pure point component of the limiting measures and thereby limits the class of limit measures (Theorem 1.2). Our proofs are based on an intuitively clear argument that combines the use of the exponential instability common to all ergodic toral automorphisms (whether they are Anosov or not) with the algebraic properties of those maps and some basic semi-classical analysis. They have a distinct dynamical flavour and work in all dimensions. To put our result in perspective, we will review the previously known results for the case $d = 1$ below. In that case the ergodic automorphisms are all Anosov and are often referred to as "cat maps".

We now describe our results in detail. Unfamiliar concepts and notations are explained in Section 2. Let $\mathbb{T}^{2d} = \mathbb{R}^{2d}/\mathbb{Z}^{2d}$ be the d-dimensional torus, viewed as a symplectic manifold with the canonical two-form inherited from $\mathbb{R}^{2d}$. Let A be a symplectic and ergodic toral automorphism, *i.e.* A is a symplectic $2d \times 2d$-matrix with integer entries none of whose eigenvalues are roots of unity. It is known that in that case at least one of its eigenvalues lies outside the unit cercle so that each rational point on the torus is an unstable periodic point for A. Given such a periodic orbit $\tau = \{x_1, x_2, \dots, x_{T_\tau}\}$, we define the delta measure

$$\mu_\tau = \frac{1}{T_\tau} \sum_{i=1}^{T_\tau} \delta_{x_i},$$

which is of course an A-invariant measure on $\mathbb{T}^{2d}$. Given a finite family $\mathcal{C} = \cup_{i=1}^{K} \tau_i$ of periodic orbits, we will also consider the measures

$$\mu_{\mathcal{C},\alpha} = \sum_{j=1}^{K} \alpha_j \mu_{\tau_j}, \quad \sum_{j=1}^{K} \alpha_j = 1, \quad 0 \leq \alpha_j \leq 1, \tag{1.1}$$

which are finite convex combinations of the previous ones. These are all A-invariant pure point measures with discrete (*i.e.*) finite support. All invariant Radon measures are obtained by taking the weak closure of those [Ma]. Let $M(A)$ be the unitary quantization of A, acting on the N^d-dimensional Hilbert space $\mathcal{H}_\hbar(\kappa)$, as defined in Section 2 (We suppress the N and κ dependence of $M(A)$ in the notations). Our main results are the following.

Theorem 1.1. *Let A be an ergodic symplectic toral automorphism and let $\psi_N \in \mathcal{H}_\hbar(\kappa), N \in \mathbb{N}$ be a family of normalized eigenfunctions of $M(A)$. If the Wigner functions W_N of the ψ_N converge weakly to some measure μ on $\mathbb{T}^d$, then $\mu \neq \mu_{\mathcal{C},\alpha}$, for any choice of $\mathcal{C}$ and α. In other words, the W_N can not converge to a pure point measure with discrete support. The same is true for the Husimi functions h_N of the ψ_N.*

This result can be paraphrased by saying that the eigenfunctions can not concentrate semi-classically on a finite number of periodic orbits. It is a particular case of the following more general result.

Theorem 1.2. *Let A be an ergodic symplectic toral automorphism and let $\psi_N \in \mathcal{H}_\hbar(\kappa), N \in \mathbb{N}$ be a family of normalized eigenfunctions of $M(A)$. Suppose ν is a continuous, A-invariant probability measure on $\mathbb{T}^{2d}$ such that for some $0 \leq \beta \leq 1$ and for all $f \in C^\infty(\mathbb{T}^{2d})$*

$$\lim_{N\to\infty} \langle \psi_N | Op^W \ f \ \psi_N \rangle_{\mathcal{H}_\hbar(\kappa)} = \beta \mu_{\mathcal{C},\alpha}(f) + (1-\beta)\nu(f) \equiv \mu(f) \tag{1.2}$$

for some $\mathcal{C}$ and α. Then $0 \leq \beta \leq (1-\beta)^{1/2}$ or, equivalently, $\beta \leq (\sqrt{5}-1)/2 \sim 0,62$.

Here $Op^W f$ stands for the Weyl quantization of f. Since (1.2) is equivalent to the same statement with Weyl quantization replaced by anti-Wick quantization (see Section 2) it is easy to see that (1.2) is equivalent to saying that the absolutely continuous measures $\mu_N = h_N(x)dx$ converge weakly to μ. Here h_N is the Husimi function of ψ_N (See Section 2 for a precise definition). The result can therefore loosely be rephrased as follows.

> If the pure point component of the limiting measure μ is concentrated on a *finite* number of periodic orbits, then its mass is less than two thirds of the total mass.

In [RS], a sequence ψ_N is defined to "scar strongly on $\mathcal{C}$" if (1.2) holds with ν given by Lebesgue measure. Theorem 1.2 does not rule out the possibility of strong scarring, but limits the size of the scar. In fact, strong scarring does occur in the systems considered. Indeed, it is proven in [DBFN] that, for $d = 1$, there exists a sequence $N_k \to \infty$, so that for each choice of $\mathcal{C}$ and α as above, there exists a sequence of eigenfunctions ψ_{N_k} so that

$$\lim_{k\to\infty} \langle \psi_{N_k} | Op^W \ f \ \psi_{N_k} \rangle_{\mathcal{H}_\hbar(\kappa)} = \frac{1}{2}\mu_{\mathcal{C},\alpha}(f) + \frac{1}{2}\int_{\mathbb{T}^2} f(x)dx. \tag{1.3}$$

This also shows that, whereas the upper bound on β in the theorem is probably not optimal, one can not do better than $\beta \leq 1/2$.

Theorems 1.1–1.2 can be seen as partial results on the characterization of the measures obtained in the semi-classical limit from the eigenfunctions of quantized ergodic symplectic automorphisms of $\mathbb{T}^{2d}$. Such measures are sometimes called "quantum limits". As such these results are to be compared with previous ones for the two-torus available in the literature. Let us first recall the precise statement of the Schnirelman theorem for ergodic symplectic toral automorphisms [BDB].

Theorem 1.3. *Let A, $M(A)$ be as above. Let, for each N, $\{\psi_1, \psi_2, \ldots \psi_{N^d}\}$ be a basis of eigenfunctions of $M(A)$. Then, for each $N \in \mathbb{N}$, there exists a subset $E(N) \subset \{1, \ldots, N^d\}$ such that:*

(i) $\displaystyle\lim_{N\to\infty} \frac{\sharp E(N)}{N^d} = 1$;

(ii) For any $f \in C^\infty(\mathbb{T}^{2d})$, for any sequence $(j_N \in E(N))_{N\in\mathbb{N}}$, one has

$$\lim_{N\to\infty} \langle \psi^{(N)}_{j_N} | Op^W \ f \ \psi^{(N)}_{j_N} \rangle_{\mathcal{H}_\hbar(\kappa)} = \int_{\mathbb{T}^{2d}} f(x)dx. \tag{1.4}$$

The strongest possible statement improving on this that one may *a priori* have hoped to prove is this:

Let, for each $N \in \mathbb{N}, \psi_N \in \mathcal{H}_\hbar(\kappa)$ be a normalized eigenfunction of $M(A)$. Then, for each $f \in C^\infty(\mathbb{T}^{2d})$, one has

$$\lim_{N\to\infty} \langle \psi_N | Op^W_\kappa(f) \psi_N \rangle_{\mathcal{H}_\hbar(\kappa)} = \int_{\mathbb{T}^{2d}} f(x)dx. \tag{1.5}$$

This is sometimes referred to (in what we feel is a somewhat unfortunate terminology) as "quantum unique ergodicity" and – as already pointed out – has not been proven in any chaotic system. Of course, in view of (1.3) it is obviously not true in the present context of ergodic toral automorphisms. One nevertheless expects the sequences that satisfy (1.3) to be rather exceptional: there do indeed exist two results in the direction of (1.5), valid for a particular but rather large class of hyperbolic toral automorphisms in $d = 1$. The first one is this.

Theorem 1.4. *[KR1] If $A \in SL(2, \mathbb{Z})$ is hyperbolic and $A \equiv \mathbb{I}_2 \mod 4$, then there exists for each N a basis $\{\psi_1, \psi_2, \ldots \psi_N\}$ of eigenfunctions of $M(A)$ so that (1.4) holds with $E(N) = \{1, \ldots, N\}$.*

This obviously constitutes a strengthening of the Schnirelman theorem for the particular class of A considered. The basis for which the result holds is explicitly described in [KR1]. Note the difference between Theorem 1.4 and (1.5). Indeed, the eigenvalues of $M(A)$ may be degenerate so that it is possible that exceptional sequences of eigenfunctions not belonging to the above basis have a different semi-classical limit. This is all the more true since there exists a sequence N_k for which the eigenspaces have a $N_k/\ln N_k$ fold degeneracy [BonDB]. It is precisely this sequence of N that is used to construct the sequence of eigenfunctions in (1.3). Another result in the direction of (1.5) is the following.

Theorem 1.5. *[KR2] If $A \in SL(2,\mathbb{Z})$ is hyperbolic and $a_{11}a_{12} \equiv 0 \equiv a_{21}a_{22} \bmod 2$, then there exists a density one sequence of integers $(N_\ell)_{\ell\in\mathbb{N}}$ along which (1.5) holds.*

Theorem 1.5 states that "quantum unique ergodicity" holds along a subsequence of values of N. It is furthermore shown in [KR2] that, along this sequence, the degeneracies of the eigenspaces grow sufficiently slowly so that it is disjoint from the sequence N_k mentioned above, as it should be in order not to contradict (1.3). It is therefore seen in both theorems that the obstacle to the validity of (1.5) for all N is the existence of growing degeneracies of the eigenspaces of $M(A)$ for large N, as expected.

Our result in Theorem 1.2 is of a somewhat different nature than Theorems 1.4 and 1.5. For any A, d or N it restricts the candidate limit measures to those that have a "not too large" pure point component. In particular, it completely rules out the most "obvious" candidates, namely pure point measures with discrete support. Our result therefore shows in particular that even the very high degeneracies of the sequence N_k can not be exploited to construct eigenfunctions that concentrate *completely* on unstable periodic orbits.

Quantum mechanics on the torus is usually studied only in the case $d = 1$. Different people have different reasons for imposing this restriction. First, when doing numerics, higher dimensions quickly poses practical problems of storage size and computation speed since the dimension of the Hilbert spaces grows as $\hbar^{-d}$. Next, on the theoretical side, the Schnirelman theorem is obviously true in all dimensions d since it is proven with dimension-independent arguments exploiting only the ergodicity of the dynamics, so there is nothing to be gained from introducing the notational complications associated with the higher d problems. To prove sharper results, however, one needs to exploit finer properties of the classical maps. The proofs of Theorems 1.4 and 1.5 in [KR1] [KR2] do this by exploiting detailed number theoretic properties of a particular class of hyperbolic automorphisms of the two-torus and do therefore *not* carry over in any obvious way to higher dimensions or to general ergodic symplectic toral automorphisms. In order to stress that the sharpening of the Schnirelman theorem proven in this paper (Theorems 1.1-1.2) exploits only the exponential instability shared by *all* ergodic automorphisms of the $2d$-torus (even if they are not hyperbolic), as well as their algebraic structure, we have chosen to consider in the following the general case throughout.

The paper is organized as follows. In section 2 we describe quantum mechanics on the $2d$-torus, and the quantization of symplectic toral automorphisms, following [BDB]. We will be as brief as possible, referring to [BDB] [BonDB] [DB] and references therein for further information and motivation. In section 3 we recall some basic facts on ergodic symplectic automorphisms of the $2d$-torus. Section 4 is devoted to the proof of Theorem 1.2. In section 5, finally, we give an alternative proof of Theorem 1.1 valid only for the case $d = 1$ and for a subclass of ergodic automorphisms when $d > 1$. It is based on a result on the propagation of initially localized states (Theorem 5.1) that generalizes a result of [BonDB]. We feel this result is of interest on its own and in addition it clearly brings out the central dynamical idea underlying all the results of this paper.

2 Quantum mechanics on the $2d$-torus

In this section we will recall standard facts about quantum mechanics on the 2d–torus $\mathbb{T}^{2d} = \mathbb{R}^{2d}/\mathbb{Z}^{2d}$ as well as the quantization of the symplectic toral automorphisms which was first performed in [HB]. Further background and references, as well as proofs, which are omitted here, can be found in [BonDB, BDB].

We shall write indifferently $x = (q,p) \in \mathbb{R}^{2d}$ or $x = (q,p) \in \mathbb{T}^{2d}$, where in the latter case $q, p \in [0,1[^d$. Let $a \cdot b = \sum_i a^i b^i$, for $a, b \in R^d$ and let $\langle (q,p),(q',p')\rangle = q\cdot p' - q' \cdot p$ be the symplectic form on $\mathbb{R}^{2d}$. Let $U(a) = \exp -\frac{i}{2\hbar}\langle a, X\rangle$, for $a \in \mathbb{R}^{2d}$ and $X = (Q,P)$, be the usual representation of the Heisenberg group on $L^2(\mathbb{R}^d)$, where

$$(Q_j\psi)(y) = y_j\psi(y), \qquad (P_j\psi)(y) = \frac{\hbar}{i}\frac{\partial\psi}{\partial y_j}(y).$$

Let $n = (n_1, n_2) \in \mathbb{Z}^d \times \mathbb{Z}^d$ and $\kappa = (\kappa_2, \kappa_1) \in \mathbb{R}^{2d}/2\pi Z^{2d}$ and let us define

$$\mathcal{H}_\hbar(\kappa) = \left\{ \psi \in \mathcal{S}'(\mathbb{R}^d) \ \mid \ U(n)\psi = \exp\frac{i}{2\hbar} n_1 \cdot n_2 \exp i\langle \kappa, n\rangle \psi \right\} \ .$$

Lemma 2.1. *We have $\mathcal{H}_\hbar(\kappa) \neq \{\emptyset\}$ iff $\exists N \in \mathbb{N}^*$ such that $(2\pi\hbar)N = 1$, in which case $\dim\mathcal{H}_\hbar(\kappa) = N^d$. Moreover, in that case, $U(n/N)\mathcal{H}_\hbar(\kappa) = \mathcal{H}_\hbar(\kappa)$ for all $n \in \mathbb{Z}^{2d}$ and there is a unique Hilbert space structure such that $U(n/N)$ is unitary for each $n \in \mathbb{Z}^{2d}$.*

We shall not introduce a different notation for the restriction of $U(n/N)$ to $\mathcal{H}_\hbar(\kappa)$ and in particular not indicate its $\kappa-$dependence. If $\phi, \psi \in \mathcal{H}_\hbar(\kappa)$, we shall write $\langle\phi|\psi\rangle_{\mathcal{H}_\hbar(\kappa)}$ or simply $\langle\phi|\psi\rangle$ for their inner product.

We then define Weyl quantization of a C^∞ function $f(x) = \sum_{n\in\mathbb{Z}^{2d}} f_n e^{-i2\pi\langle n,x\rangle}$ as

$$Op^W_\kappa(f) = \sum_{n\in\mathbb{Z}^{2d}} f_n U(\frac{n}{N}) \ .$$

Recall that the map

$$S(\kappa) = (\sum_m \exp -i\kappa_2 \cdot m\; U(0,m))(\sum_n \exp i\kappa_1 \cdot n\; U(n,0))$$

defines a surjection of the space of Schwartz functions $\mathcal{S}(\mathbb{R}^d)$ onto $\mathcal{H}_\hbar(\kappa) \subset \mathcal{S}'(\mathbb{R}^d)$.

Let $\eta_x \in L^2(\mathbb{R}^d)$ denote the usual gaussian Weyl–Heisenberg coherent state centered on $x \in \mathbb{R}^{2d}$:

$$\eta_0(y) = \frac{1}{(\pi\hbar)^{\frac{d}{4}}} e^{-\frac{1}{2\hbar}y^2}, \quad \eta_x(y) = \left(U(x)\eta_0\right)(y).$$

We then define coherent states on the torus as

$$\eta_{x,\kappa} \equiv S(\kappa)\eta_x \in \mathcal{H}_\hbar(\kappa). \tag{2.6}$$

We will find it convenient to use the physicists' "bra-ket" notation and to write:

$$|x\rangle \equiv \eta_x \in L^2(\mathbb{R}^d) \quad \text{and} \quad |x,\kappa\rangle \equiv \eta_{x,\kappa} \in \mathcal{H}_\hbar(\kappa).$$

In particular, we use the notation $|x,\kappa\rangle\langle x,\kappa|$ to designate the rank one operator associated to $|x,\kappa\rangle$. Coherent states on the torus satisfy the following resolution of the identity

$$\mathrm{Id}_{\mathcal{H}_\hbar(\kappa)} = \int_{\mathbb{T}^{2d}} |x,\kappa\rangle\langle x,\kappa| \frac{dx}{(2\pi\hbar)^d} \tag{2.7}$$

and permit us to define the anti-Wick quantization $Op_\kappa^{AW}(f)$ of $f \in L^\infty(\mathbb{T}^{2d})$ as the operator on $\mathcal{H}_\hbar(\kappa)$ defined by

$$Op_\kappa^{AW}(f) = \int_{\mathbb{T}^{2d}} f(x)\; |x,\kappa\rangle\langle x,\kappa| \frac{dx}{(2\pi\hbar)^d}. \tag{2.8}$$

For each $\psi \in \mathcal{H}_\hbar(\kappa)$ we define its Wigner function as the distribution $W_\psi(x)$ such that

$$\langle\psi|Op_\kappa^W(f)\psi\rangle = \int_{\mathbb{T}^{2d}} f(x)W_\psi(x)dx \quad \forall f \in C^\infty(\mathbb{T}^{2d})$$

and its Husimi function

$$h_\psi(x) = N|\langle\psi|x,\kappa\rangle|^2 \tag{2.9}$$

which satisfies

$$\langle\psi|Op_\kappa^{AW}(f)\psi\rangle = \int_{\mathbb{T}^{2d}} f(x)h_\psi(x)dx \quad \forall f \in L^\infty(\mathbb{T}^{2d})\;.$$

Anti-Wick and Weyl quantization satisfy for each $f \in C^\infty(\mathbb{T}^{2d})$ the following estimate:

$$\left\|Op_\kappa^{AW}(f) - Op_\kappa^W(f)\right\| \leq \frac{C_f}{N}, \tag{2.10}$$

for some positive constant $C_f \geq 0$. Moreover

$$U(\frac{n}{N})|x,\kappa\rangle = e^{i\pi\langle n,x\rangle}|x+\frac{n}{N},\kappa\rangle \ . \tag{2.11}$$

Finally let us come to the quantization of an ergodic symplectic toral automorphism defined by a matrix $A \in Sp(d,\mathbb{Z})$. The metaplectic representation of $Sp(d,\mathbb{R})$ defines for each $A \in Sp(d,\mathbb{R})$ a unitary propagator $M(A)$ in $L^2(\mathbb{R}^d)$ (see [F]); up to a phase it is the unique operator which satisfies

$$M(A)^{-1}U(a)M(A) = U(A^{-1}a) \quad \forall a \in \mathbb{R}^{2d} \ . \tag{2.12}$$

The quantization of $A \in Sp(d,\mathbb{Z})$ on the torus is then straightforward:

Lemma 2.2. *For each ergodic $A \in Sp(d,\mathbb{Z})$ and each $N \in \mathbb{N}^*$ there exists at least one $\kappa \in \mathbb{R}^{2d}/\mathbb{Z}^{2d}$ such that*

$$M(A)\mathcal{H}_\hbar(\kappa) = \mathcal{H}_\hbar(\kappa) \ .$$

Proof. By applying (2.12) one can see that there exists $\kappa_A \in \mathbb{R}^{2d}/(2\pi\mathbb{Z})^{2d}$ such that, for each $n \in \mathbb{Z}^{2d}$ and $\psi \in \mathcal{H}_\hbar(\kappa)$,

$$U(n)M(A)\psi = e^{i\pi NA^{-1}n_1\cdot A^{-1}n_2}e^{i\langle A\kappa,n\rangle}M(A)\psi = e^{i\pi Nn_1\cdot n_2}e^{i\langle\kappa_A,n\rangle}M(A)\psi \ .$$

As a result $M(A)\mathcal{H}_\hbar(\kappa) = \mathcal{H}_\hbar(\kappa_A)$. Indeed let us write $A^{-1} = (\alpha_{ij})_{ij=1,2}$ with α_{ij} $d\times d$ matrices. Since A^{-1} is symplectic we have $\alpha_{11}^*\alpha_{22} - \alpha_{21}^*\alpha_{12} = 1$, $\alpha_{11}^*\alpha_{21} = \alpha_{21}^*\alpha_{11}$ and $\alpha_{12}^*\alpha_{22} = \alpha_{22}^*\alpha_{12}$. Consequently a simple computation shows that

$$A^{-1}n_1 \cdot A^{-1}n_2 - n_1 \cdot n_2 = \langle\omega_A, n\rangle \qquad \text{mod } 2$$

where

$$\omega_A = \begin{pmatrix} \mathrm{diag}(\alpha_{12}^*\alpha_{22}) \\ \mathrm{diag}(\alpha_{11}^*\alpha_{21}) \end{pmatrix}$$

and

$$\kappa_A = A\kappa + \pi N\omega_A \ \text{ mod } 2\pi \ . \tag{2.13}$$

Since A is ergodic, 1 is not an eigenvalue of A and equation (2.13) admits at least one fixed point $\kappa_A = \kappa$. □

In the following, we shall always assume that κ has been chosen as in the Lemma, but we shall not explicitly indicate the N or A dependence of κ. Similarly, we shall use the symbol $M(A)$ to indicate the restriction of $M(A)$ to $\mathcal{H}_N(\kappa)$ for a suitable κ as above, without indicating its N or κ dependence.

From this construction it follows easily that, for each $f \in C^\infty(\mathbb{T}^{2d})$,

$$M(A)^{-1}Op_\kappa^W f M(A) = Op_\kappa^W(f\circ A), \ \forall f \in C^\infty(\mathbb{T}^{2d}). \tag{2.14}$$

In other words "quantization and evolution commute".

3 Ergodic automorphisms of the torus

We collect here some rather basic facts about ergodic automorphisms of the torus. Let $A \in \mathrm{SL}(2d, \mathbb{Z})$, then A defines an ergodic toral automorphism if and only if none of the eigenvalues of A are roots of unity [M]. Ergodic toral automorphisms are automatically mixing as well [M]. In addition, their eigenvalues can not all lie on the unit circle: at least one of them has to have a modulus strictly bigger than 1. This is an immediate consequence of the Kronecker theorem ([N], Theorem 2.1), applied to the characteristic polynomial of A. As a result, in the decomposition of $\mathbb{R}^{2d}$ into A-invariant subspaces [KH] [Li] given by

$$\mathbb{R}^{2d} = E_- \oplus E_0 \oplus E_+,$$

where E_+ (respectively E_0, E_-) is the root space of A corresponding to eigenvalues of modulus strictly bigger than (respectively equal to, strictly smaller than) 1, we are sure that E_-, E_+ are non-trivial. A matrix A is said to be hyperbolic iff $E_0 = \{0\}$. The corresponding dynamical system on $\mathbb{T}^{2d}$ is then Anosov. If $E_0 \neq \{0\}$, A is called quasi-hyperbolic in [Li]. Clearly, when $d = 1$, all ergodic toral automorphisms are Anosov, but this is no longer true in higher dimension.

We will need the following result:

Lemma 3.1. *Let $n \in \mathbb{Z}^{2d}$. Then $n \notin E_0 \oplus E_-$. Moreover, there exist $\gamma > 0, C_\pm > 0$ and $0 \leq k \leq 2d-2$ so that, for all $t \in \mathbb{N}$ large enough*

$$C_- t^k e^{\gamma t} \leq \| A^t n \| \leq C_+ t^k e^{\gamma t}.$$

Proof: The first statement, namely that $E_0 \oplus E_- \cap \mathbb{Z}^{2d} = \{0\}$, can be found in [Ka][Li]. The estimate is then a simple application of the Jordan normal form. □

It should be noted that $\gamma > 0, C_\pm > 0$ and $0 \leq k \leq 2d-2$ depend on n in the above estimate. The lower bound above is an expression of the exponential instability common to all ergodic toral automorphisms and is the only information about them we shall need to prove Theorems 1.1 and 1.2.

To prove Theorem 5.1 however, we will need the following result from [Ka], which is the generalization to higher dimensions of the obvious diophantine inequality satisfied in the case $d = 1$ by the slopes of the stable and unstable directions of A.

Lemma 3.2. *Let $\mathbb{R}^{2d} = V_1 \oplus V_2$, with V_i invariant spaces for A such that*

i) $A|_{V_1}$ and $A|_{V_2}$ don't have common eigenvalues;

ii) $V_1 \cap \mathbb{Z}^{2d} = \{0\}$.

Then there exists $C > 0$ such that for each $n \in \mathbb{Z}^{2d}_*$ we have ($m = \mathrm{dim} V_1$)

$$d_{\mathbb{R}^{2d}}(n, V_1) \geq \frac{C}{\|n\|^m}.$$

4 Proof of Theorem 1.2

We will need the following simple technical lemma.

Lemma 4.1. *Let B be a Borel subset of $\mathbb{T}^{2d}$. Then, under the hypotheses of Theorem 1.2, one has*

$$\mu(\text{int } B) \leq \liminf_{N\to+\infty}\langle\psi_N|Op_\kappa^{AW}(\chi_B)\psi_N\rangle \leq \limsup_{N\to+\infty}\langle\psi_N|Op_\kappa^{AW}(\chi_B)\psi_N\rangle \leq \mu(\bar{B}). \tag{4.15}$$

where χ_B is the characteristic function of B.

Proof: This is a standard result in measure theory, we include the proof for completeness. Let us introduce, for all $\epsilon > 0$,

$$B_-^\epsilon = \{x \in B | d_{\mathbb{T}^{2d}}(x, \partial B) > \epsilon\} \subset B \subset B_+^\epsilon = \{x \in \mathbb{T}^{2d} | d_{\mathbb{T}^{2d}}(x, B) < \epsilon\},$$

where $d_{\mathbb{T}^{2d}}$ designates the Euclidean distance on the torus. Then we have $\cup_{\epsilon>0} B_-^\epsilon =$ int B and $\cap_{\epsilon>0} B_+^\epsilon = \bar{B}$, so that

$$\lim_{\epsilon\to0}\mu(B_-^\epsilon) = \mu(\text{int } B) \qquad \lim_{\epsilon\to0}\mu(B_+^\epsilon) = \mu(\bar{B}). \tag{4.16}$$

Now let $\eta \in C_0^\infty(\mathbb{R}^{2d})$ be a spherically symmetric positive function, with support in the ball of radius 1, equal to 1 on the ball of radius 1/2 and such that $\int \eta(x)dx = 1$. We set $\eta_\epsilon(x) = \frac{1}{\epsilon^{2d}}\eta(\frac{x}{\epsilon})$, and define $\chi_\pm^\epsilon(y) = \int_{B_\pm^\epsilon} \eta_\epsilon(y - x)dx$. Clearly

$$\begin{aligned} \chi_+^\epsilon(y) = 1 \text{ if } \ y \in B \qquad & \chi_+^\epsilon(y) = 0 \text{ if } \ y \in \mathbb{T}^{2d} \setminus B_+^{2\epsilon} \\ \chi_-^\epsilon(y) = 1 \text{ if } \ y \in B_-^{2\epsilon} \qquad & \chi_-^\epsilon(y) = 0 \text{ if } \ y \in \mathbb{T}^{2d} \setminus B. \end{aligned}$$

This implies in particular that $\chi_-^\epsilon \leq \chi_B \leq \chi_+^\epsilon$, so that the positivity of anti-Wick quantization implies that $Op_\kappa^{AW}(\chi_-^\epsilon) \leq Op_\kappa^{AW}(\chi_B) \leq Op_\kappa^{AW}(\chi_+^\epsilon)$. Using (1.2) and (2.10) we then find

$$\begin{aligned} \mu(B_-^{2\epsilon}) \leq \mu(\chi_-^\epsilon) \ &\leq \ \liminf_{N\to+\infty}\langle\psi_N|Op_\kappa^{AW}(\chi_B)\psi_N\rangle \\ &\leq \ \limsup_{N\to+\infty}\langle\psi_N|Op_\kappa^{AW}(\chi_B)\psi_N\rangle \leq \mu(\chi_+^\epsilon) \leq \mu(B_+^{2\epsilon}) \end{aligned}$$

so that the result follows by taking $\epsilon \to 0$ and using (4.16). □

Define, for any finite set of points $\mathcal{C}$ (not necessarily a set of periodic points of the dynamics) and for each $a > 0$

$$B_a = \{x \in \mathbb{T}^{2d} \mid d_{\mathbb{T}^{2d}}(x, \mathcal{C}) < a\}. \tag{4.17}$$

We also introduce

$$\delta_\mathcal{C} = \min\{d_{\mathbb{T}^{2d}}(x, y) | x, y \in \mathcal{C}, x \neq y\}, \tag{4.18}$$

provided $\mathcal{C}$ contains more than one point. Otherwise we define $\delta_{\mathcal{C}} = 1/\sqrt{2}$.

Proof of Theorem 1.2: Since ν is a continuous measure, the Wiener theorem says that

$$\lim_{K\to+\infty} \frac{1}{(2K+1)^d} \sum_{\|n\|\le K} |\hat{\nu}(n)|^2 = 0.$$

Here, $\hat{\nu}$ is the Fourier transform of ν. This implies immediately that there exists a density one subset G of $\mathbb{Z}^{2d}$ so that

$$\lim_{n\to\infty, n\in G} \hat{\nu}(n) = 0. \tag{4.19}$$

On the other hand, $\mathcal{C}$ is a finite collection of rational points on $\mathbb{T}^{2d}$ and we call S the least common multiple of the denominators of those points. Then, for each $n \in S\mathbb{Z}^{2d}$ and for all $x \in \mathcal{C}$, clearly $\chi_n(x) \equiv \exp 2\pi i\langle n, x\rangle = 1$ and consequently

$$\mu_{\mathcal{C},\alpha}(\chi_n) = 1.$$

Hence, for such n,

$$\lim_{N\to\infty} \langle \psi_N | U(\frac{n}{N}) \psi_N\rangle = \beta + (1-\beta)\hat{\nu}(n). \tag{4.20}$$

Since $S\mathbb{Z}^{2d}$ is a positive density subset of $\mathbb{Z}^{2d}$, it follows that $S\mathbb{Z}^{2d} \cap G$ is positive density as well. As a result, given $\epsilon > 0$, there exists $n \in S\mathbb{Z}^{2d} \cap G$, depending on ϵ, so that

$$|\hat{\nu}(n)| \le \epsilon. \tag{4.21}$$

We then have, using respectively (2.14), (2.7) and (2.11)

$$\begin{aligned}
| \langle \psi_N | U(\frac{n}{N})\psi_N\rangle | &= | \langle \psi_N | M(A)^{-t} U(\frac{n}{N}) M(A)^t \psi_N\rangle | \\
&= | \langle \psi_N \mid U(\frac{A^t n}{N})\psi_N\rangle | \\
&= | \int_{\mathbb{T}^{2d}} \langle \psi_N \mid x, \kappa\rangle \langle x, \kappa \mid U(\frac{A^t n}{N})\psi_N\rangle \frac{dx}{(2\pi\hbar)^d} | \\
&\le \int_{\mathbb{T}^{2d}} | \langle \psi_N \mid x, \kappa\rangle || \langle x - \frac{A^t n}{N}, \kappa \mid \psi_N\rangle | \frac{dx}{(2\pi\hbar)^d} \\
&\le \int_{B_a} | \langle \psi_N \mid x, \kappa\rangle || \langle x - \frac{A^t n}{N}, \kappa \mid \psi_N\rangle | \frac{dx}{(2\pi\hbar)^d} \\
&\qquad + \Big(\int_{\mathbb{T}^{2d}\setminus B_a} | \langle \psi_N \mid x, \kappa\rangle |^2 \frac{dx}{(2\pi\hbar)^d}\Big)^{1/2},
\end{aligned} \tag{4.22}$$

where B_a is defined in (4.17). Note that this inequality holds for each choice of t, N, a. Now choose $M > 3$ and such that

$$\frac{1}{\gamma}(\ln \frac{M}{C_+} - \ln \frac{3}{C_-}) > 1 + \frac{k}{\gamma}, \tag{4.23}$$

where $C_\pm, k$ and γ are defined in Lemma 3.1. We recall they depend on n. We will show below that then, for all $a > 0$, the following is true:

$$\forall N > N_a = \frac{C_- e^\gamma}{3} \frac{1}{a}, \exists t_N \in \mathbb{N} \text{ so that } \ 3a \leq \| \frac{A^{t_N} n}{N} \| \leq Ma < \delta_{\mathcal{C}}/3. \tag{4.24}$$

Introducing

$$\mathcal{A}(a, M) = \{x \in \mathbb{T}^{2d} \mid 2a \leq d_{\mathbb{T}^{2d}}(x, \mathcal{C}) \leq (M+1)a\}, \tag{4.25}$$

it is then clear that

$$x \in B_a \Rightarrow x - \frac{A^{t_N} n}{N} \in \mathcal{A}(a, M).$$

The important point here is that $\mathcal{A}(a, M)$ does not depend on N. Inequality (4.22) now yields, upon using a Schwartz inequality in the first term

$$\begin{aligned} |\langle \psi_N \mid U(\frac{n}{N})\psi_N\rangle \mid \ &\leq \ \langle \psi_N \mid Op_\kappa^{AW}(\chi_{B_a})\psi_N\rangle^{1/2}\langle \psi_N \mid Op_\kappa^{AW}(\chi_{\mathcal{A}(a,M)})\psi_N\rangle^{1/2} \\ &\qquad + \langle \psi_N \mid Op_\kappa^{AW}(\chi_{\mathbb{T}^{2d}\setminus B_a})\psi_N\rangle^{1/2} \\ &\leq \ \langle \psi_N \mid Op_\kappa^{AW}(\chi_{\mathcal{A}(a,M)})\psi_N\rangle^{1/2} \\ &\qquad + \langle \psi_N \mid Op_\kappa^{AW}(\chi_{\mathbb{T}^{2d}\setminus B_a})\psi_N\rangle^{1/2}. \end{aligned} \tag{4.26}$$

Note that, given $\epsilon > 0$, this inequality holds for n satisfying (4.21), for all a small enough (depending on n), and for all N. We now take the limsup for N to $+\infty$, and apply Lemma 4.1 in the right-hand side and (4.20)-(4.21) in the left-hand side to obtain:

$$\beta - (1-\beta)\epsilon \leq (1-\beta)^{1/2}\nu(\overline{\mathcal{A}(a, M)})^{1/2} + (1-\beta)^{1/2}\nu(\overline{\mathbb{T}^{2d} \setminus B_a})^{1/2}. \tag{4.27}$$

Finally, taking, for ϵ and M fixed, a to 0 in this inequality, the continuity of the measure ν yields

$$\beta - (1-\beta)\epsilon \leq (1-\beta)^{1/2}.$$

Since this holds for all ϵ, this is the desired result.

It remains to prove (4.24). From Lemma 3.1 we see that (4.24) will be proven provided we show there exists, for each $N \in \mathbb{N}$, $N \geq N_a$ a $t_N \in \mathbb{N}$ so that

$$3aN \leq C_- t_N^k e^{\gamma t_N}, \qquad C_+ t_N^k e^{\gamma t_N} \leq NMa,$$

or, equivalently

$$D_- \equiv \frac{1}{\gamma}[\ln N + \ln a + \ln \frac{3}{C_-}] \leq t_N + \frac{k}{\gamma}\ln t_N \leq \frac{1}{\gamma}[\ln N + \ln a + \ln \frac{M}{C_+}] \equiv D_+.$$

Introducing for $t \in \mathbb{N}_*$, $g(t) = t + \frac{k}{\gamma}\ln t$, one sees that for all $t \in \mathbb{N}_*$ $g(t+1) - g(t) \leq 1 + \frac{k}{\gamma}$. Hence, to obtain (4.24) it is sufficient that

$$D_+ - D_- = \frac{1}{\gamma}(\ln \frac{M}{C_+} - \ln \frac{3}{C_-}) > 1 + \frac{k}{\gamma},$$

and that $D_- \geq g(1)$, but this is guaranteed by condition (4.23) and the definition of N_a in (4.24).

□

5 Propagating localized states

In this section we present a generalization of the main result of [BonDB]. When $d > 1$, it only holds under some mild additional hypotheses on A specified below. Under these conditions, it provides an alternative proof of Theorem 1.1. We feel this result is of interest on its own, and in addition it clearly brings out the basic "dynamical" intuition underlying the proofs of the previous section.

We will impose, in addition to ergodicity, two more conditions on A. First, we ask that $A \in \mathrm{Sp}(d, \mathbb{Z})$ does not leave any non-trivial sublattice of $\mathbb{Z}^{2d} \subset \mathbb{R}^{2d}$ invariant. This excludes, for example, in the case $d = 2$, matrices of the form

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

where each bloc A_1, A_2 is a hyperbolic matrix in $\mathrm{SL}(2, \mathbb{Z})$. In addition, we impose the following condition. Let e^{γ_+} be the maximal modulus of the eigenvalues of A and let E_{γ_+} be the corresponding root space (*i.e.* $E_{\gamma_+} = \oplus_{|\lambda|=e^{\gamma_+}} E_\lambda$); we demand that the restriction of A to E_{γ_+} is diagonalizable. This will obviously be the case if all roots of the characteristic polynomial of A are distinct, for example. We need some more notations. Let $m_{\gamma_+} = \dim\, E_{\gamma_+}$ and $m_+ = \dim(E_+ \oplus E_0)$; of course $1 \leq m_{\gamma_+} \leq m_+ \leq d$.

We remark that in $d = 1$ any ergodic matrix $A \in SL(2, \mathbb{Z})$ satisfies these requirements.

Theorem 5.1. *Let A be as above and let μ be a pure point probability measure on $\mathbb{T}^{2d}$, with finite support $\mathcal{C}$. Let $\psi_N \in \mathcal{H}_\hbar(\kappa)$, $||\psi_N|| = 1$ be a sequence of normalized vectors in $\mathcal{H}_\hbar(\kappa)$ such that, for all $f \in C^\infty(\mathbb{T}^{2d})$*

$$\lim_{N\to\infty} \langle \psi_N \mid Op_\kappa^W(f)\psi_N\rangle = \mu(f). \tag{5.28}$$

Let a_N be a sequence of positive numbers tending to 0 with the property that

$$\int_{B_{a_N}} h_N(x)dx \to 1, \tag{5.29}$$

where B_{a_N} is defined in (4.17) and $h_N \equiv h_{\psi_N}$ in (2.9). Then, there exist $t_\pm \geq 0$ so that, for any sufficiently slowly growing sequence of integers θ_N (i.e. $\theta_N < \frac{(1-\epsilon)(1+m_+)}{m_+\gamma_+(2d+1-m_{\gamma_+})} \ln \frac{1}{a_N}$ for some $\epsilon > 0$), one has, for each $f \in C^\infty(\mathbb{T}^{2d})$,

$$\lim_{N\to\infty} \langle \psi_N \mid M(A)^{-t} Op_\kappa^W(f) M(A)^t \psi_N\rangle = \int_{\mathbb{T}^{2d}} f(x)dx \tag{5.30}$$

uniformly for all t in the region

$$\frac{1}{\gamma_+} \ln N a_N + t_- + (2d - m_{\gamma_+})\theta_N \leq t \leq \frac{1}{\gamma_+} \ln \frac{N}{a_N^{1/m_+}} - t_+ - \theta_N \tag{5.31}$$

Note that here the ψ_N are of course not assumed to be eigenvectors of the dynamics. Remark furthermore that the hypothesis (5.28) immediately implies the existence of a sequence a_N. It is finally clear from the definition of the Husimi functions in Section 2 that $a_N\sqrt{N}$ is bounded away from 0.

As an example, suppose $\mathcal{C} = \{x_1, x_2, \dots x_p\}$ is a set of p points on the torus and take

$$\psi_N = \frac{1}{\sqrt{p}} \sum_{i=1}^{p} |x_i, \kappa\rangle.$$

In that case $a_N = N^{-\frac{1}{2}+\epsilon}$ for any ϵ. It is also easy to check that the ψ_N are normalized up to an exponentially small factor. The case $p = 1$ and $d = 1$ was treated in [BonDB] but with a worse upper bound on the times in (5.31). As explained already in [BonDB], the upper bound in Theorem 5.1 is the optimal one for this case; it is obtained here using an argument borrowed from [DBFN].

Alternative proof of Theorem 1.1 for A as above: The proof goes by contradiction. Suppose a sequence of *eigenfunctions* exists, so that (5.28) holds with $\mu = \mu_{\mathcal{C},\alpha}$ (see (1.1)). Since the ψ_N are eigenfunctions, one trivially finds, for all $t \in \mathbb{N}$:

$$\langle \psi_N \mid M(A)^{-t} Op_\kappa^W(f) M(A)^t \psi_N\rangle = \langle \psi_N \mid Op_\kappa^W(f)\psi_N\rangle,$$

so that (5.30) implies that

$$\lim_{N\to\infty} \langle \psi_N \mid Op_\kappa^W(f)\psi_N\rangle = \int_{\mathbb{T}^{2d}} f(x)dx,$$

which is in obvious contradiction to the hypothesis (5.28). In other words, we have just proven that, if a sequence of eigenfunctions concentrates semi-classically on a finite family of periodic orbits, then it equidistributes. We conclude that such a sequence does not exist. $\square$

Proof of Theorem 5.1: Writing

$$f = \sum_{\|n\|\le M_N} f_n \chi_n + \sum_{\|n\|\ge M_N} f_n\chi_n,$$

the fast decrease of the f_n implies that for all $K \in \mathbb{N}$, there exists $C_f, C_{f,K} > 0$ so that

$$|\langle \psi_N \mid M(A)^{-t} Op_\kappa^W(f) M(A)^t\psi_N\rangle - \int_{\mathbb{T}^{2d}} f(x)dx| \le$$

$$C_f \sup_{0<\|n\|\le M_N} |\, \langle \psi_N \mid M(A)^{-t} U(\frac{n}{N}) M(A)^t \psi_N\rangle + C_{f,K} M_N^{-K}. \quad (5.32)$$

Hence it will be enough to show that there exist a sequence $M_N \in \mathbb{N}$ (depending on θ_N) with $M_N \to +\infty$ so that

$$\sup_{0<\|n\|\le M_N} |\, \langle \psi_N \mid M(A)^{-t} U(\frac{n}{N}) M(A)^t\psi_N\rangle \,| \to 0$$

for t in the range given in (5.31). For that purpose, first note that for each $n \in \mathbb{Z}^{2d} \setminus \{0\}$, $t \in \mathbb{Z}, a > 0$, we have (as in (4.22))

$$\begin{aligned} | \langle \psi_N \mid M(A)^{-t} U(\frac{n}{N}) M(A)^t \psi_N \rangle | &= | \int_{\mathbb{T}^{2d}} \langle \psi_N \mid x, \kappa \rangle \langle x, \kappa \mid U(\frac{A^t n}{N}) \psi_N \rangle \frac{dx}{(2\pi\hbar)^d} | \\ &\leq \int_{B_a} | \langle \psi_N \mid x, \kappa \rangle | \, | \langle x + \frac{A^t n}{N}, \kappa \mid \psi_N \rangle | \frac{dx}{(2\pi\hbar)^d} \\ &\quad + \sigma_N(a) \qquad (5.33) \end{aligned}$$

where

$$\sigma_N(a) = \Big(\int_{\mathbb{T}^{2d} \setminus B_a} h_{\psi_N}(x) dx \Big)^{1/2}.$$

The hypothesis (5.29) implies that $\sigma_N(a_N) \to 0$.

Below, we shall prove that, for each N, there exists $M_N \in \mathbb{N}$ so that

$$x \in B_{a_N}, n \in \mathbb{Z}^{2d}, 0 < \| n \| \leq M_N \Rightarrow x + \frac{A^t n}{N} \notin B_{a_N}, \qquad (5.34)$$

for all t in the region (5.31). It then follows from (5.33) and the Schwartz inequality that, for each $\| n \| \leq M_N$ and for those t

$$| \langle \psi_N \mid M(A)^{-t} U(\frac{n}{N}) M(A)^t \psi_N \rangle | \leq \sigma_N(a_N) + \sigma_N(a_N).$$

Hence

$$|\langle \psi_N \mid M(A)^{-t} Op_\kappa^W(f) M(A)^t \psi_N \rangle - \int_{\mathbb{T}^{2d}} f(x) dx| \leq 2C_f \sigma_N(a_N) + C_{f,K} M_N^{-K}. \qquad (5.35)$$

We now prove (5.34). For simplicity of notation, let us first consider the case where $\mathcal{C} = \{0\}$ so that B_{a_N} is the ball of radius a_N around $0 \in \mathbb{T}^{2d}$. We define

$$\tilde{B}_{a_N} = \{ y \in \mathbb{R}^{2d} | d_{\mathbb{R}^{2d}}(y, \mathbb{Z}^{2d}) < a_N \}.$$

Then B_{a_N} is the image of $\tilde{B}_{a_N}$ under the natural projection of $\mathbb{R}^{2d}$ to $\mathbb{T}^{2d}$ so that (5.34) is equivalent to

$$x \in \tilde{B}_{a_N}, n \in \mathbb{Z}^{2d}, 0 < \| n \| \leq M_N \Rightarrow d_{\mathbb{R}^{2d}}(x + \frac{A^t n}{N}, \mathbb{Z}^{2d}) \geq a_N.$$

But this is guaranteed if for all t in (5.31)

$$n \in \mathbb{Z}^{2d}, 0 < \| n \| \leq M_N \Rightarrow d_{\mathbb{R}^{2d}}(\frac{A^t n}{N}, \mathbb{Z}^{2d}) \geq 2a_N. \qquad (5.36)$$

This is what we now prove. First, let

$$e^{-\gamma_-} = \max\{ |\lambda| < 1 \mid \lambda \text{ is an eigenvalue of } A \}.$$

In other words, $e^{-\gamma_-}$ is the modulus of the largest eigenvalue of A strictly inside the unit disc. Since A does not leave any non-trivial sublattice invariant, it is clear that the component along E_{γ_+} of any $n \in \mathbb{Z}^{2d}_*$ is different from zero and Lemma 3.2 can therefore be used with $V_2 = E_{\gamma_+}$; since A is diagonalizable on E_{γ_+}, we conclude that there exist $C_\pm > 0$ so that for all $n \in \mathbb{Z}^{2d}_*$ and for all t one has

$$\frac{C_-}{\|n\|^{2d-m_{\gamma_+}}} e^{\gamma_+ t} \leq \| A^t n \| \leq C_+ e^{\gamma_+ t} \|n\|. \tag{5.37}$$

By using (5.37), if

$$\frac{C_-}{N M_N^{2d-m_{\gamma_+}}} e^{\gamma_+ t} \geq 2a_N,$$

it is clear that for each $n \in \mathbb{Z}^{2d}_*$ such that $\|n\| \leq M_N$ we have

$$\| \frac{A^t n}{N} \| \geq 2a_N.$$

So, if we choose $M_N = e^{\gamma_+ \theta_N}, t_- = \frac{1}{\gamma_+} \ln \frac{2}{C_-}$, this latter inequality is satisfied for all t in (5.31). We note for later reference that, since $2d - m_{\gamma_+} \geq m_+$,

$$M_N \leq \left(\frac{1}{a_N}\right)^{\frac{1+m_+}{m_+(2d+1-m_{\gamma_+})}} \leq C N^{1/2} \tag{5.38}$$

for some $C > 0$, in view of the constraint on θ_N and the fact that $a_N \sqrt{N}$ is bounded away from 0. Now, for each $n \in \mathbb{Z}^{2d}, 0 < \| n \| \leq M_N$ there exists $n_{t,N} \in \mathbb{Z}^{2d}$ so that

$$d_{\mathbb{R}^{2d}}(\frac{A^t n}{N}, \mathbb{Z}^{2d}) = \| \frac{A^t n}{N} - n_{t,N} \| .$$

Consequently, if $n_{t,N} = 0$ then $d_{\mathbb{R}^{2d}}(\frac{A^t n}{N}, \mathbb{Z}^{2d}) \geq 2a_N$ for all t in the region (5.31). Suppose therefore that $n_{t,N} \neq 0$ so that $\| \frac{A^t n}{N} \| \geq 1/2$. Let $n = n_+ + n_0 + n_- \in E_+ \oplus E_0 \oplus E_-$. Then we have

$$\begin{aligned}
d_{\mathbb{R}^{2d}}(\frac{A^t n}{N}, \mathbb{Z}^{2d}) &= \left\| \frac{A^t n}{N} - n_{t,N} \right\| \geq \left\| \frac{A^t(n_+ + n_0)}{N} - n_{t,N} \right\| - \left\| \frac{A^t n_-}{N} \right\| \\
&\geq d_{\mathbb{R}^{2d}}(n_{t,N}, E_+ \oplus E_0) - \left\| \frac{A^t n_-}{N} \right\| \geq \frac{C_o}{\|n_{t,N}\|^{m_+}} - \left\| \frac{A^t n_-}{N} \right\| \\
&\geq \frac{C_o}{(\sqrt{d/2} + \|A^t n/N\|)^{m_+}} - \left\| \frac{A^t n_-}{N} \right\| \geq \frac{C_1}{\|A^t n/N\|^{m_+}} - \left\| \frac{A^t n_-}{N} \right\| \\
&\geq C_2 \left(\frac{N}{M_N} e^{-\gamma_+ t} \right)^{m_+} - C_3 \frac{M_N}{N} t^{(d-1)} e^{-\gamma_- t} \\
&= C_2 \left(\frac{N}{M_N} e^{-\gamma_+ t} \right)^{m_+} \left[1 - \frac{C_3}{C_2} e^{(m_+ \gamma_+ - \gamma_-)t} \left(\frac{M_N}{N} \right)^{m_+ + 1} t^{(d-1)} \right] \\
&\geq 2a_N ,
\end{aligned}$$

where we used in the second line Lemma 3.2 applied to $V_1 = E_+ \oplus E_0$, in the third line the fact that $\|A^t n/N\| \geq 1/2$, in the fourth line the upper bound in (5.37) and a standard estimate on $\|A^t n_-\|$. To obtain the last line one defines t_+ via $e^{-\gamma_+ t_+ m_+} = C_2/4$, one uses $M_N = \exp \gamma_+ \theta_N$ and (5.31) to obtain

$$e^{(m_+ \gamma_+ - \gamma_-)t} \left(\frac{M_N}{N}\right)^{m_+ + 1} t^{(d-1)} \leq \frac{e^{-m_+ \gamma_+ t_+}}{a_N} \frac{M_N}{N} e^{-\gamma_- t} t^{(d-1)} \leq d_1 e^{-\gamma_- t} t^{(d-1)} < 1/2.$$

It is then clear that (5.36) holds for all t in the region (5.31).

The general case, where $\mathcal{C}$ is a finite set of points is easily treated by noting that there exists $S \in \mathbb{N}_*$ so that $S\mathcal{C} \subset \mathbb{Z}^{2d}$ so that

$$d_{\mathbb{R}^{2d}}(y, \mathcal{C} + \mathbb{Z}^{2d}) = S^{-1} d_{\mathbb{R}^{2d}}(Sy, S\mathcal{C} + S\mathbb{Z}^{2d}) \geq S^{-1} d_{\mathbb{R}^{2d}}(Sy, \mathbb{Z}^{2d}).$$

□

Acknowledgments: SDB whishes to thank S. Nonnenmacher for helpful conversations and M. Belliart for guiding him to the literature on ergodic toral automorphisms.

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