Exact analytical solutions for an ideal elastic infinite line having one heterogeneity transition

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This paper is one of a large cycle of papers on obtaining the exact analytical solutions for vibrant elastic systems. This was published first in the electronic journal of laboratory SELF "SELF Transactions", vol.2, issue 1, pp.60-70. Visiting this journal for free at http://angelfire.lycos.com/la3/selftrans you will find many other important results both on this subject and on many applications of this new method, as well as on many other subjects of mathematical physics and related themes. You will also find there that we check our results experimentally and yield the full coincidence of the calculation and experimental curves. And the main, this method gives exact solutions, and this is non-matrix. Already now we have solved many practically important problems which cannot be even approached properly by conventional methods. This is the future of dynamics out of which any development in any area of basic or applied wave physics is impossible.

You are welcomed to possess this wealth and contact us for discussions and joint further development.

Abstract

In this paper we will present some results of investigation of an infinite 1D elastic lumped line having one section of heterogeneity. We have obtained these results, using the original non-matrix method to find exact analytical solutions of the infinite system of differential equations. We will present few features important for the practical use, conditioned by the transition of an elastic line section to the anti-phase damping regime. We also will consider the conditions of solutions transformation, when transiting to the models related to basic, as well as to the related elastic distributed line. The results of this investigation can be extended to the rotary vibrations of lumped and distributed elastic lines, as well as with the help of original dynamical electromechanical analogy (DEMA) they can be applied to the electrical filters calculation.

Keywords: Heterogeneous elastic lines; Finite deformation; Oscillation theory; Dynamical systems.

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1. Introduction

In [1] – [4], on the basis of ideal elastic finite and infinite, lumped and distributed lines, we showed that the vibration process is not limited by an undercritical frequency band (lower than critical frequency ω_0). At the overcritical band we can see the aperiodical regime of damping anti-phase vibrations; with them the mechanical line behaves as a natural damper.

In a heterogeneous line, the influence of aperiodical process on the vibration pattern substantially complicates, when the frequency of external excitation exceeded the local critical frequency ω_{0i} for some sections of a line, while for the others the undercritical vibration regime retains. It leads to the complex amplitude and phase transitions whose pattern depends both on the local parameters of section and on the features of particular line as the whole. In this case, to analyse completely a heterogeneous line, it is deficient to know the natural frequencies and eigenfunctions. It appears important to know an integral pattern of vibration process running with the amplitude and phase vibration characteristics at the undercritical and overcritical bands determined in the analytical form.

These features cannot be completely described by the conventional methods. "In the vibration theory, the solution of boundary vibrations reduces basically to the determination of eigenvalues connected with natural frequencies or other parameters of the studied system and to finding the eigenfunctions (of vibration forms). If the eigenvalues and eigenfunctions have been found, we can think the boundary problem solved... At present a great amount of approximate methods has been developed for finding the eigenvalues, but they all are quite laborious, give only the first eigenvalues, and the main, do not unite the studying of systems with discrete and continuous mass distribution" [5, pp.3 – 4]. Only in some particular cases we have incomplete, limitedly applicable solutions or mentionings that the overcritical vibrations are possible (see, e.g., [6, pp.272 – 275], [7, p.294], [8, p.109]). However they also are deficient to analyse the system completely, especially in heterogeneous lines.

In this paper we will partly fill this gap in. We will present some results of the analysis carried out for the particular case of an ideal elastic line with one heterogeneity transition. With it we will suppose that the main regularities of these phenomena may be extended to the more complex heterogeneous elastic lines, when using the method on whose basis we have obtained the solutions analysed in this paper.

2. General solution for a heterogeneous ideal elastic line having one heterogeneity transition

As the base model, consider an infinite lumped line having one transition of mass heterogeneity under longitudinal external harmonic force action. Suppose that the external force acts on the *k*th line element, and $k \le n$, where *n* is the number of the boundary element of the heterogeneous section. The modelling system of differential equations for this line is

$$
\begin{aligned}
& m_1 \frac{d^2\Delta_{k-1}}{dt^2} = s\left(\Delta_k + \Delta_{k-2} - 2\Delta_{k-1}\right); \\
& m_1 \frac{d^2\Delta_k}{dt^2} = F(t) + s\left(\Delta_{k+1} + \Delta_{k-1} - 2\Delta_k\right); \\
& m_1 \frac{d^2\Delta_{k+1}}{dt^2} = s\left(\Delta_{k+2} + \Delta_k - 2\Delta_{k+1}\right); \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& m_1 \frac{d^2\Delta_n}{dt^2} = s\left(\Delta_{n+1} + \Delta_{n-1} - 2\Delta_n\right); \\
& m_2 \frac{d^2\Delta_{n+1}}{dt^2} = s\left(\Delta_{n+2} + \Delta_n - 2\Delta_{n+1}\right); \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots ,
\end{aligned}
\qquad (1)
$$

where Δ_i is the longitudinal displacement of the ith mass of a line ($-\infty \le i \le \infty$); s is the stiffness coefficient of a line; m_1 and m_2 are the element masses of related sections.

We can see from (1) that the external force application point and heterogeneity transition divide the line into three sections. According to it, the solution also divides into three intervals, and each has its distinctions. This solution has the following form:
for the first section $i \le k$

$$\begin{aligned}\Delta_i &= -j\frac{F_0 e^{j\{\omega t-[2(n-i)+1]\tau_1\}}}{4s\beta_1 \sin(\tau_1+\tau_2)\cos\tau_1}\times \\ &\times\left\{\sin(\tau_1+\tau_2)e^{j[2(n-k)+1]\tau_1}+\sin(\tau_1-\tau_2)e^{-j[2(n-k)+1]\tau_1}\right\};\end{aligned} \tag{2}$$

for the second section $k \le i \le n$

$$\begin{aligned}\Delta_i &= -j\frac{F_0 e^{j\{\omega t-[2(n-k)+1]\tau_1\}}}{4s\beta_1 \sin(\tau_1+\tau_2)\cos\tau_1}\times \\ &\times\left\{\sin(\tau_1+\tau_2)e^{j[2(n-i)+1]\tau_1}+\sin(\tau_1-\tau_2)e^{-j[2(n-i)+1]\tau_1}\right\};\end{aligned} \tag{3}$$

and for the third one $i \ge n+1$

$$\Delta_i = -j\frac{F_0}{2s\sin(\tau_1+\tau_2)}e^{j\{\omega t-[2(n-k)+1]\tau_1-[2(i-n)11]\tau_2\}}, \tag{4}$$

where

$$\beta_1=\sqrt{\frac{\omega^2 m_1}{4s}};\ \beta_2=\sqrt{\frac{\omega^2 m_2}{4s}}; \tag{5}$$

$$\tau_1 = \arcsin\beta_1;\ \tau_2=\arcsin\beta_2. \tag{6}$$

As this solution shows, in the first and third sections the progressive wave propagates. It has its specific phase delays depending on the parameters of both sections. Besides, the entire vibration process shifts by ($-\pi$); it is determined by the unit imaginary number in the right-hand parts of both solutions. In the second section there forms the standing wave with common phase delay $[2(n-k)+1]\tau_1$ and additional element-to-element phase delay determined by the expression in braces in (3). In this connection the vibration pattern in the second section of a line complicates. As we will show further, this is the cause, why (3) describes not only standing but also progressive waves.

Furthermore, we see from (5) that in the line two boundary frequencies

$$\omega_{01}=\sqrt{\frac{4s}{m_1}}\ \text{и}\ \omega_{02}=\sqrt{\frac{4s}{m_2}} \tag{7}$$

have appeared. These frequencies determine two conditions transitive to the overcritical band. Due to it, the frequency range divides into three bands; in each the solutions (2) – (4) transform dependently on the conditions

$$\beta_1 \ge 1\ \text{and/or}\ \beta_2 \ge 1. \tag{8}$$

With it the section whose masses are heavier transits to the aperiodical regime first, while in the second section the periodical regime of non-damping vibrations retains until (8) is true.

To analyse the solution transformation pattern when (8) was satisfied for some section, it is sufficient to transform the parameters in (2) – (4) relevant to this section, using an evident relation following from (6):

$$e^{-j\tau_p}=\sqrt{1-\beta_p^2}-j\beta_p;\quad p=1,2. \tag{9}$$

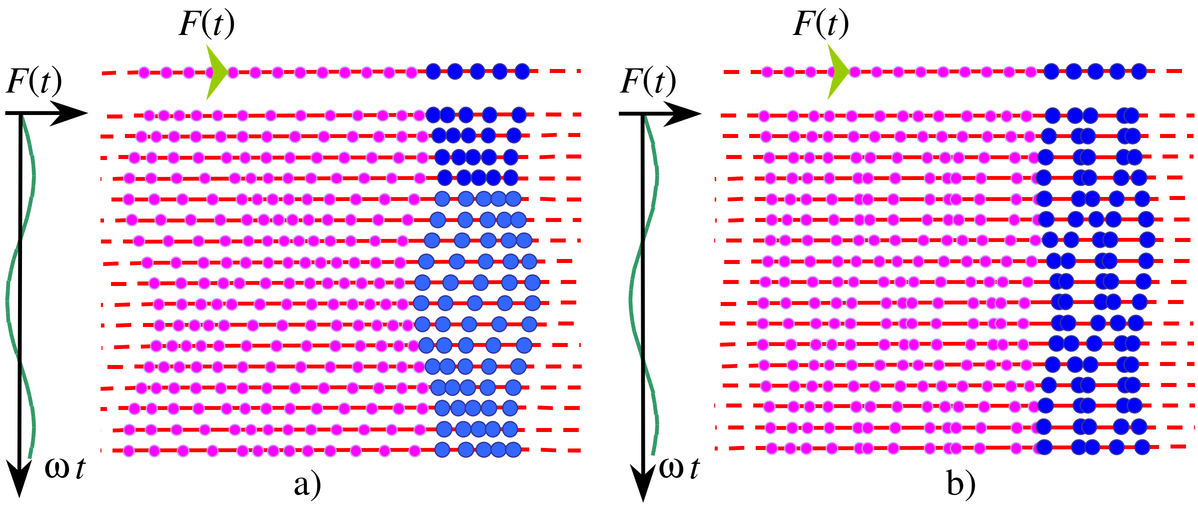

Fig. 1. The vibration diagrams at 8 Hz (a) and 22 Hz (b) in the heterogeneous elastic line. The external force frequency is lower than the boundary frequencies for both parts of the line; F_0 = 0,6 N; s = 100 N/m; m_1 = 0,01 kg; ω_{01} = 200 sec^{-1}; m_2 = 0,02 kg; ω_{02} = 141,42 sec^{-1}.

Below we will consider the possible transformations of (2) – (4) dependently on parameters m_1, m_2 and ω and analyse the most typical distinctions of vibration process accompanying them.

3. Analysis of typical pattern of vibration process in the infinite heterogeneous elastic lumped line

Begin with the most simple and evident case, when (8) is not true for all sections.

3.1 $m_1 < m_2$; $\beta_1 < 1$; $\beta_2 < 1$

At these conditions both sections vibrate periodically and (2) – (4) retain unchanged. The process in the second section is especially interesting. According to the expression in the braces in (3), the vibration pattern in this section is the superposition of two progressive waves propagating to each other and having the amplitudes proportional to $\sin(\tau_1 + \tau_2)$ and $\sin(\tau_1 - \tau_2)$ correspondingly. At the same time, the expression in braces can be presented otherwise:

$$\{\ \} = \left\{ e^{j\tau_2} \sin 2(n-i+1)\tau_1 - e^{-j\tau_2} \sin 2(n-i)\tau_1 \right\}. \tag{10}$$

In accord to (10), the process in the middle section can be presented as the superposition of two standing waves having different amplitudes and phase shifts. These two presentations are equivalent.

The typical vibration pattern is shown in Fig. 1 for two different values of the external force frequency. In Fig. 1a, in case of less frequency, we see the progressive wave in the middle section; it propagates into both directions from the external force application point. In the second case shown in Fig. 1b, the external force frequency is some less than the critical frequency ω_{02}. With it in the middle section we see practically standing waves, but in the heavy section they are almost anti-phase but not damping. This last corroborates the explanation of the vibration damping, when transiting to the aperiodical regime given by K. Magnus – true, only in case of two elastically connected masses. “The damping phenomenon can be explained in the following way. At the correct tuning, the second mass vibrates in anti-phase to the excitation and has namely such amplitude that the force of the second spring acting on the first mass counterpoises the excitation force transmitting through the first spring" [6, p.268]. It means, the vibrations damp not due to the anti-phase pattern of neighbouring masses, but due to the combination of the anti-phasing and vibration amplitude. Rather, not amplitude itself but the degree of element mass responses to the elastic link excitation. In absence of this correspondence, the damping will be absent, as we can see in Fig. 1b.

Now raise the frequency so that the condition (8) was violated for the heavy part of a line, and consider the following typical pattern.

3.2 $\mathbf{m_1 < m_2 \,;\, \beta_1 < 1 \,;\, \beta_2 > 1}$

When β_2 exceeds the unit value, in the heavy part of line the aperiodical vibration regime settles. With it all the terms in the right-hand parts of the system (2) – (4) transform as according to (8), and this system takes the following form:
for $i \le k$

$$\Delta_i = \frac{F_0 e^{j\{\omega t - [2(n-i)+1]\tau_1 + \varphi_2\}}}{4s\beta_1\sqrt{\beta_2^2-\beta_1^2}\cos\tau_1} \times \\ \times\left[\gamma_{2+}\sin 2(n-k+1)\tau_1 + \gamma_{2-}\sin 2(n-k)\tau_1\right], \tag{11}$$

for $k \le i \le n$

$$\Delta_i = \frac{F_0 e^{j\{\omega t - [2(n-k)+1]\tau_1 + \varphi_2\}}}{4s\beta_1\sqrt{\beta_2^2-\beta_1^2}\cos\tau_1} \times \\ \times\left[\gamma_{2+}\sin 2(n-i+1)\tau_1 + \gamma_{2-}\sin 2(n-i)\tau_1\right], \tag{12}$$

and for $i \ge n+1$

$$\Delta_i = (-1)^{i-n+1}\frac{F_0 e^{j\{\omega t - [2(n-k)+1]\tau_1 + \varphi_2\}}}{2s\sqrt{\beta_2^2-\beta_1^2}}\gamma_{2-}^{2(i-n)-1}\,, \tag{13}$$

where

$$\varphi_2 = \arcsin\frac{\beta_1\sqrt{\beta_2^2-1}}{\sqrt{\beta_2^2-\beta_1^2}}\,; \tag{14}$$

$$\gamma_{2+} = \beta_2 + \sqrt{\beta_2^2-1}\,;\quad \gamma_{2-} = \beta_2 - \sqrt{\beta_2^2-1}\,. \tag{15}$$

In the first section related to $i \le k$, there has retained the periodical process of progressive wave propagation with the phase delay $[2(n-i)+1]\,\tau_1$ and the amplitude depending on the parameters of both sections of the line. However, the general phase delay has changed. If in (2) the vibration process has delayed by π relatively to the external force phase, then in (11) the general delay is determined by the parameter $(-\varphi_2)$. At $\beta_2 \to +1$ we have $\varphi_2 \to \pi$, and at $\beta_2 \to \infty$ we have $\varphi_2 \to \arcsin\sqrt{\dfrac{m_1 m_2}{m_2 - m_1}}$, not zero.

The less difference between the element masses m_1 and m_2 is, the more vibration amplitude is. Basically, at $m_1 \to m_2$ the amplitude of $|\Delta_i|$ tends to infinity too, this is determined by the multipliers $\sqrt{\beta_2^2-\beta_1^2}$ and $\cos\tau_1$. But the amplitude does not reach the infinite value, since at $m_1 = m_2$ in the light section the aperiodical vibration regime will also settle, and the solution (11) will transform accordingly.

When the expression in square brackets in (11) vanished, i.e. at

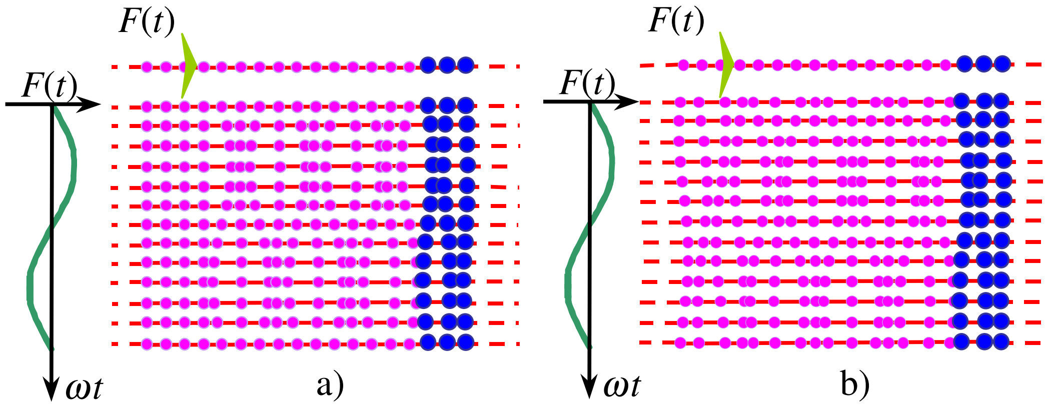

Fig. 2. The diagrams of vibration in the heterogeneous elastic line at frequencies 23,0515 Hz (a) and 23,5 Hz (b). The value of external force frequency is between the boundary frequencies of the light and heavy parts of line. $F_0 = 0{,}6$ N; $s = 100$ N/m; $m_1 = 0{,}01$ kg; $\omega_{01} = 200$ sec^{-1}; $m_2 = 0{,}02$ kg; $\omega_{02} = 141{,}42$ sec^{-1}.

$$\gamma_{2+}\sin 2(n-i+1)\tau_1 = -\gamma_{2-}\sin 2(n-i)\tau_1 , \tag{16}$$

the vibration amplitude in the first section vanishes, as it is visual in Fig. 2a. And in the above case, when all sections have vibrated periodically, such phenomenon was impossible, because of the phase shift presence in the summands in the braces of (2). In (14) the phase shift vanishes, due to the heavy section transition to the aperiodical regime.

In the second section of line ($k \le i \le n$), according to (12), there form the complex-form standing waves determined by the superposition of two standing waves. Their form also distinguishes from that considered, so that between the summands in the square brackets of (12) the phase shift is absent too. Therefore, with all the vibration pattern complexity, (12) cannot describe the progressive wave. Furthermore, we can see from (12) that the standing waves forming here are not able to exhibit the resonance, and this is one more typical feature of the studied case. The diagrams in Figures 2a and 2b corroborate the said.

As is expected, in the third section the anti-phase damping vibration forms. The regularity of damping is power-type but not exponential. The base of power diminishes with growing external force frequency, therefore the damping increases. One more distinction of the studied section is, the phase shift between the neighbouring elements vibration never exceeds π. It essentially refines the settled opinion concerning the pattern of vibration process in the overcritical band (see e.g. [8], [9]).

Solutions (11) – (13) get some other form, if the external force acts on the heavy part of elastic line. Since this case is important for the practical use, consider it separately.

3.3 $\mathbf{m_1 > m_2 \, ; \; \beta_1 > 1 \, ; \; \beta_2 < 1}$

When the indicated conditions were realised, the terms containing β_1 and τ_1 in (2) – (4) transform. Modifying them and noting (9), we obtain:

for $i \le k$:

$$\Delta_i = (-1)^{r+i-1}\frac{F_0\gamma_{1-}^{2(n-i)+1}e^{j(\omega t-\varphi_1)}}{8s\sqrt{\beta_1^2-\beta_2^2}\sqrt{\beta_1^2-1}}\times \\ \times\left[\left(\gamma_{1+}^{2(n-k+1)}-\gamma_{1-}^{2(n-k+1)}\right)e^{j\tau_2}+\left(\gamma_{1+}^{2(n-k)}-\gamma_{1-}^{2(n-k)}\right)e^{-j\tau_2}\right], \tag{17}$$

for $r \le i \le n$

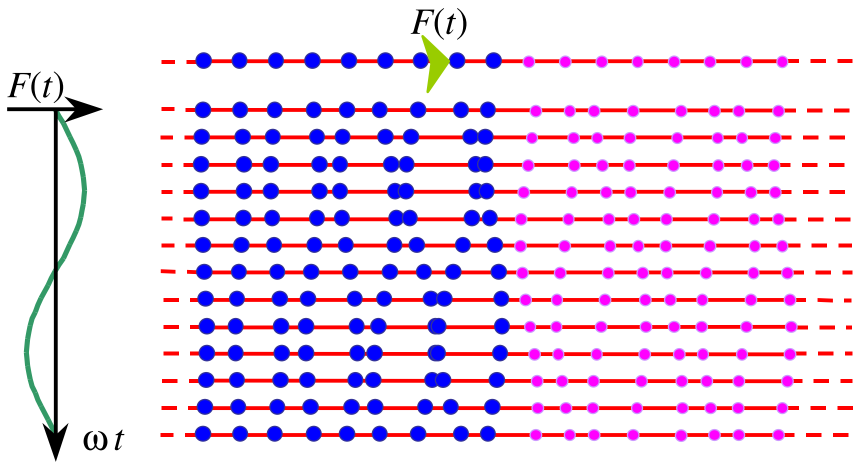

Fig. 3. The vibration diagram at 22,7 Hz in the heterogeneous elastic line. The external force acts on the heavy part of line; the frequency value is between the boundary frequencies of the light and heavy parts of line. F_0 = 0,5 N; s = 100 N/m; m_1 = 0,02 kg; ω_{01} = 141,42 sec^{-1}; m_2 = 0,01 kg; ω_{02} = 200 sec^{-1}.

$$\Delta_i = (-1)^{r+i-1} \frac{F_0 \gamma_{1-}^{2(n-k)+1} e^{j(\omega t-\varphi_1)}}{8s\sqrt{\beta_1^2-\beta_2^2}\sqrt{\beta_1^2-1}} \times \\ \times\left[\left(\gamma_{1+}^{2(n-i+1)}-\gamma_{1-}^{2(n-i+1)}\right)e^{j\tau_2}+\left(\gamma_{1+}^{2(n-i)}-\gamma_{1-}^{2(n-i)}\right)e^{-j\tau_2}\right], \tag{18}$$

and for $i \geq n+1$

$$\Delta_i = (-1)^{r+i-1} \frac{F_0 \gamma_{1-}^{2(n-k)+1}}{2s\sqrt{\beta_1^2-\beta_2^2}} e^{j\{\omega t-[2(i-n)-1]\tau_2-\varphi_1\}}, \tag{19}$$

where

$$\gamma_{1+} = \beta_1 + \sqrt{\beta_1^2-1}, \quad \gamma_{1-} = \beta_1 - \sqrt{\beta_1^2-1}. \tag{20}$$

The interchange between the light and heavy parts of a line has led to appearing the anti-phase damping along-the-line vibrations in the first section. The base of power of γ_{1-} depending on the external action frequency determines the degree of damping. With raising frequency, the damping grows. The square-brackets expression in the right-hand part of (17) has no effect on the anti-phase pattern of vibrations, as in the item 3.1, but together with the phase φ_1 it determines the general delay of vibration process.

At the same time, similarly to the item 3.2, the amplitude in the near of external force application point depends on the value of multiplier $\sqrt{\beta_1^2-\beta_2^2}$. When small difference between m_1 and m_2 , it can take large values, despite the damping. In real models this distinction can lead to the failure of the system of elastically linked masses, while out of the region of external force effect the vibrations can be practically absent.

In the second section of elastic line, the pattern of aperiodical process is much more complex. This is conditioned by the influence of the square-brackets expression in the right-hand part of solution (18). The presence of phase shifts τ_2 departs from the rigorous anti-phase pattern of neighbouring elements of section and the regularity of the frequency damping degree will complicate.

In the third section the progressive wave propagates. Its amplitude is proportional to $\gamma_{1-}^{2(n-k)+1}$, i.e. to the damping degree in the heavy part of line. As we see in Fig. 3, the vibration amplitude in this section can be well less than the vibration amplitude in the region of external force action, despite when plotting, we chose $m_1 = m_2$, so the external force acting point was closed to the utmost to the heterogeneity transition. It evidences the residual pattern of vibration process in the third section of line.

As the next typical transformation of basic solutions, consider the case when condition (8) is true for the entire line.

3.4. $\beta_1 > 1$; $\beta_2 > 1$

In this case we should apply the transformation (9) both to the light and heavy parts of line. The result will be the following:

for $i \le k$

$$\Delta_i = (-1)^{k+i}\, j \frac{F_0 \gamma_{1-}^{2(n-i)+1} e^{j\omega t}}{8s\left(\beta_1\sqrt{\beta_2^2-1}+\beta_2\sqrt{\beta_1^2-1}\right)\sqrt{\beta_1^2-1}} \times \left[\gamma_{2+}\left(\gamma_{1+}^{2(n-k)+1}-\gamma_{1-}^{2(n-k)+1}\right)-\gamma_{2-}\left(\gamma_{1+}^{2(n-k)}-\gamma_{1-}^{2(n-k)}\right)\right], \tag{21}$$

for $k \le i \le n$

$$\Delta_i = (-1)^{k+i}\, j \frac{F_0 \gamma_{1-}^{2(n-k)+1} e^{j\omega t}}{8s\left(\beta_1\sqrt{\beta_2^2-1}+\beta_2\sqrt{\beta_1^2-1}\right)\sqrt{\beta_1^2-1}} \times \left[\gamma_{2+}\left(\gamma_{1+}^{2(n-i)+1}-\gamma_{1-}^{2(n-i)+1}\right)-\gamma_{2-}\left(\gamma_{1+}^{2(n-i)}-\gamma_{1-}^{2(n-i)}\right)\right], \tag{22}$$

and for $i \ge n+1$

$$\Delta_i = (-1)^{k+i}\, j \frac{F_0\, \gamma_{2-}^{2(n-i)-1}\, \gamma_{1-}^{2(n-k)+1}}{2s\left(\beta_1\sqrt{\beta_2^2-1}+\beta_2\sqrt{\beta_1^2-1}\right)} e^{j\omega t}. \tag{23}$$

We see from (21) – (23) that in all three sections of an elastic line the aperiodical regime settles. None the less, in each section it has its distinctions. In the first section the damping degree along the line is determined by the multiplier $\gamma_{1-}^{2(n-i)+1}$, in the second section – by the square-brackets expression in the right-hand part of solution (22), and in the third section – by the multiplier $\gamma_{2-}^{2(n-i)-1}$. In each case this influence reflects not only on the base of power but also on its index and even on the function type.

The common salient feature of solutions (21) – (23) in comparison with, e.g., [1] and [2] is the vibration phase shift by ($-\pi/2$) in the first and second sections; the unit imaginary number in the right-hand parts of (21) and (22) determines it.

Thus, having studied the transformation of solutions (2) – (4) dependently on complete or partial satisfying the condition (8), we see that if taking into consideration the aperiodical regime in case of elastic infinite line, it allows to study much more completely the entire variety of vibration pattern possible in a line. It helps analyse better the possible transformations of vibration pattern dependently on the parameters of elastic system and external force, as well as to reveal more completely the most dangerous and stressed sections.

4. Solutions transformation when transforming the model

We can essentially complicate the basic model using the standard procedures. By the superposition, we can take into account the external force acting simultaneously on a few line elements. By the spectral expansion, we can model the response of an elastic line to the external unharmonic action. We can complicate the vibration, extending (2) – (4) to the inclined force action. In this case the solution will describe in the implicit form the inclined waves propagating from the external force action point. The main regularities of vibration process will correspond to those considered in the items 3.1 – 3.4. When transiting to a line with distributed parameters, the inclined pattern of wave process will remain in complete correspondence with results presented in [4].

A heterogeneous elastic infinite line can serve the basic model also for a number of other types of elastic lines having an alike structure. We can obtain for them also the exact analytic solutions by way of corresponding transformation of solutions (2) – (4). For example, using the conventional analogy between the linear and rotary motion of solid body, we can obtain solutions for a system of elastically linked disks. Extending additionally the limiting process to a distributed line, we can investigate the torsion vibrations of heterogeneous shafts, cables and so on. Using the dynamical electromechanical analogy DEMA described in [10] and [12], we can obtain a full complex of solutions for complex electric filters. By way of modelling system (1) simple modification, we can take into account the resistance of an elastic line, remaining the solutions complete, analytical and exact, etc.

As an example, consider a few simple and visual models.

4.1. $m_1 = m_2$

The transformation of a heterogeneous elastic line into that homogeneous equalises the parameters characterising both parts of a heterogeneous line. It means that in this case

$$\beta_1 = \beta_2 = \beta; \qquad \tau_1 = \tau_2 = \tau . \tag{24}$$

If noting (24), the basic system (2) – (4) takes the following form:
for $i \le k$

$$\Delta_i = -j\frac{F_0}{4s\beta\cos\tau}e^{j[\omega t - 2(k-i)\tau]}, \tag{25}$$

for $k \le i \le n$

$$\Delta_i = -j\frac{F_0}{4s\beta\cos\tau}e^{j[\omega t - 2(i-k)\tau]}, \tag{26}$$

and for $i \ge n+1$

$$\Delta_i = -j\frac{F_0}{4s\beta\cos\tau}e^{j[\omega t - 2(i-k)\tau]}. \tag{27}$$

Solutions (25) – (28) completely coincide with the results presented in [1], where the similar solutions were obtained by way of direct use of the presented method to obtain exact analytical solutions. This coincidence of results serves to a definite extent the check of correctness of solutions given here.

4.2. $m_2 = 0$

In this case we can consider the elastic line as a semi-finite line with unfixed end. With it the equalities

$$\tau_2 = 0; \quad \beta_2 = 0; \quad \tau_1 = \tau; \quad \beta_1 = \beta \tag{28}$$

are true. With regard to (28), the solutions (2) – (4) will take the form:
for $i \le k$

$$\Delta_i = -j\frac{F_0\cos\left[2(n-k)+1\right]\tau}{2s\beta\cos\tau}e^{j\{\omega t - [2(n-i)+1]\tau\}}, \tag{29}$$

for $k \le i \le n$

$$\Delta_i = -j\frac{F_0\cos\left[2(n-i)+1\right]\tau}{2s\beta\cos\tau}e^{j\{\omega t - [2(n-k)+1]\tau\}}, \tag{30}$$

and for $i \ge n+1$

$$\Delta_i = -j\frac{F_0}{2s\beta}e^{j\{\omega t-[2(n-k)+1]\tau\}}. \tag{31}$$

At $i = n$ the expression (31) will coincide with (30) and will not depend on the index i. Thus we can ignore it in the general system of solutions (29) – (30). The rest two expressions for the first and second sections fully coincide with the corresponding results obtained in [3]. If we go on with this transformation and take additionally $k = n$, we will obtain the solutions for a homogeneous semi-finite elastic line under external force acting on its free end:

$$\Delta_i = -j\frac{F_0}{2s\beta}e^{j\{\omega t-[2(n-i)+1]\tau\}}. \tag{32}$$

This result coincides with the solution presented in [1].

4.3. $m_2 = \infty$

By the same principle we can obtain the solution for a semi-finite line with a fixed end. First of all we should note that when the masses m_2 turn into infinity, vibrations in the section containing these masses turn into periodical regime. Besides, it follows from this condition that at $m_2 \to \infty$

$$\beta_2 \to \infty,\quad \gamma_{2+} \approx 2\beta_2 \to \infty,\quad \gamma_{2-} \to 0,\quad \varphi_2 \to \arcsin\beta_1 = \tau_1 . \tag{33}$$

To transform (2) – (4), conveniently use the system (11) – (13) where the transition of the third section vibrations into aperiodical regime was taken into account. Substituting (33) sequentially to the expressions of this system, we will obtain:

for $i \le k$

$$\Delta_i = -j\frac{F_0\sin 2(n-k-i)\tau_1}{4s\beta_1\cos\tau_1}e^{j[\omega t-2(n-i)\tau_1]}, \tag{34}$$

for $k \le i \le n$

$$\Delta_i = -j\frac{F_0\sin 2(n-i-i)\tau_1}{4s\beta_1\cos\tau_1}e^{j[\omega t-2(n-k)\tau_1]}, \tag{35}$$

and for $i \ge n+1$

$$\Delta_i = 0 . \tag{36}$$

In the first section of a studied line there propagates the progressive wave whose amplitude depends in a complex way on the external force frequency and the line parameters. In the second section the standing wave with some phase delay $2(n-k)\tau_1$ has formed. In that third, just as it was expected, the vibration amplitude is zero.

If we continue transforming the solutions (34) – (36), taking $k = n$, we will obtain the solution for a semi-infinite elastic line on whose first after fixation element the external force acts. With it only one solution of three will remain:

$$\Delta_i = \frac{F_0}{s}e^{j[\omega t-2(n-i)\tau_1]}. \tag{37}$$

Comparing (37) with (32), we see that at the same external force parameters the vibration amplitude in a line with fixed ends is less at low frequencies, when the condition

$$2\beta_1 < 1,\ \text{or}\ \ \omega < \sqrt{\frac{s}{m_1}} \tag{38}$$

is valid. At the frequencies higher than indicated in (38), the vibration amplitude in the line with fixed end will be higher than in a line with unfixed end.

Thus we see that the basic solutions transform easy into solutions for the models generalised by the basic model. This is a very important property of the complete analytical solutions. Should the initial base solutions be incomplete, or should it be presented only numerically, such transformation would be impossible.

4.4. The limit passing to a distributed line

To make a complete analysis, trace the transformation of solutions (2) – (4) at the limit passing to a distributed line. To transform the base solutions for this case, present the parameters of elastic line in the form, corresponding to a distributed line. To do so, introduce

$$\rho_{01} = \frac{m_1}{a}, \quad \rho_{02} = \frac{m_2}{a}, \quad T = sa, \tag{39}$$

where T is the stiffness of a line with lumped parameters, a is the distance between the unexcited elements in a lumped line, ρ_{01} and ρ_{02} are the densities of relative sections of heterogeneous unexcited line with lumped parameters.

Suppose that zero of the distributed line is the location of $n+1$-st mass of a lumped line in unexcited state. Then

$$\begin{aligned} &\lim_{a\to 0}\left[i-(n+1)\right] = x_0; \\ &\lim_{a\to 0}\Delta_i = x - x_0. \end{aligned} \tag{40}$$

Using (39) and (40), transform some multipliers of the system (2) – (4), noting that a is small:

$$\begin{aligned} &\sin(\tau_1+\tau_2) \approx \frac{a\omega}{2}\left(\frac{v_1+v_2}{v_1 v_2}\right); \\ &\sin(\tau_1-\tau_2) \approx \frac{a\omega}{2}\left(\frac{v_2-v_1}{v_1 v_2}\right); \\ &\left[2(n-k)+1\right]\tau_1 \approx (a-2x_k)\frac{\omega}{2v_1}; \\ &\left[2(n-i)+1\right]\tau_1 \approx (a-2x_0)\frac{\omega}{2v_1}; \\ &\left[2(i-n)-1\right]\tau_1 \approx (2x_0+a)\frac{\omega}{2v_1}, \end{aligned} \tag{41}$$

where x_k is the external force application point, and $v_1 = \sqrt{T/\rho_{01}}$, $v_2 = \sqrt{T/\rho_{02}}$ are the wave propagation velocities in the sections having related densities ρ_{01} and ρ_{02}.

Substituting (41) to the base system and finding the limit at $a \to 0$, we yield the sought solution:

for the section $x_0 \le x_k$

$$x = x_0 - \frac{F_0\left[v_1\sin(\omega x_k/v_1) + jv_2\cos(\omega x_k/v_1)\right]}{\omega\sqrt{\rho_{01}T}\,(v_1+v_2)} e^{j\omega[t+(x_0/v_1)]}, \tag{42}$$

for the section $x_k \le x_0 \le 0$

$$x = x_0 - \frac{F_0\left[v_1\sin(\omega x_0/v_1) + jv_2\cos(\omega x_0/v_1)\right]}{\omega\sqrt{\rho_{01}T}\,(v_1+v_2)} e^{j\omega[t+(x_k/v_1)]}, \tag{43}$$

and for the section $x_0 \ge 0$

$$x = x_0 - j\frac{F_0 v_2}{\omega\sqrt{\rho_{01}T}\,(v_1+v_2)} e^{j\omega[t-(x_0/v_2)-(x_0/v_2)]}. \tag{44}$$

We see from (42) – (44) that at the limit passing, the line with distributed parameters remains its distinctions considered above for a lumped line.

In the first section there propagates the progressive wave whose phase shift is determined by the complex multiplier in the square brackets of (42):

$$\varphi_k = \arcsin \frac{v_2 \cos\left(\omega x_k / v_1\right)}{\sqrt{v_1^2 \sin^2\left(\omega x_k / v_1\right) + v_2^2 \cos^2\left(\omega x_k / v_1\right)}}. \tag{45}$$

In the second section we see a complex vibration process determined by the similar expression in brackets in (43). On one hand, the wave process in this section has the phase

$$\varphi_x = \arcsin \frac{v_2 \cos\left(\omega x_0 / v_1\right)}{\sqrt{v_1^2 \sin^2\left(\omega x_0 / v_1\right) + v_2^2 \cos^2\left(\omega x_0 / v_1\right)}}, \tag{46}$$

depending on parameter x_0 nonlinearly. On the other hand, the wave amplitude depends on this parameter nonlinearly too. Its value is proportional to

$$\sqrt{v_1^2 \sin^2\left(\omega x_k / v_1\right) + v_2^2 \cos^2\left(\omega x_k / v_1\right)}. \tag{47}$$

The amplitude reaches its extreme values at the points

$$\begin{aligned} &\left(\omega x_0 / v_1\right) = p\pi, \\ &\left(\omega x_0 / v_1\right) = \left(\pi/2\right) + p\pi, \end{aligned} \tag{48}$$

where $p = 0, 1, 2, \ldots$. In the first case the expression (47) is equal to v_2, and in the second case – to v_1. Depending on the relationship between the wave propagation velocities in related sections, at these points we observe the maximums and minimums of the wave process.

The results obtained for the first and second sections essentially differ from the conventional concept based on a simple superposition of the direct and reverse waves. According to this concept,

"the amplitude reflection coefficient

$$\frac{B_1}{A_1} = \frac{Z_1 - Z_2}{Z_1 + Z_2}, \tag{49}$$

the amplitude transmission coefficient

$$\frac{A_2}{A_1} = \frac{2Z_1}{Z_1 + Z_2} \tag{50}$$

(where A_1, A_2, B_1 are respectively the amplitude coefficients of direct, reflected and transmitted waves; $Z_1 = \rho_{01} v_1 = T/v_1$ is the mechanical impedance of the first section of an elastic line; $Z_2 = \rho_{02} v_2 = T/v_2$ is the mechanical impedance of the second section of an elastic line,– Authors). These coefficients do not depend on ω and are the same for the waves of any frequency. They are real and do not introduce any phase shifts except the shift by π changing the sign of the term" [11, p.123]. As we see from this analysis, when reflecting from the heterogeneity transition, there forms a complex wave process whose phase and amplitude depend directly on the external force frequency, the same as on elastic line parameters. These conclusions are true also for the transmitted wave.

In the third section of elastic line we observe the progressive wave having the along-the-line phase delay $\omega x_0 / v_2$ and common phase shift determined in (44) by the unit imaginary number and $\omega x_k / v_1$.

To see the typical vibration pattern, conveniently pass from the longitudinal to transverse vibrations. It will be sufficient for it to direct the external force perpendicularly to the line axis and to substitute in the solutions (42) – (44) the longitudinal shift $x - x_0$ by the transversal shift y . The typical diagram of transversal vibrations in a heterogeneous elastic dis-

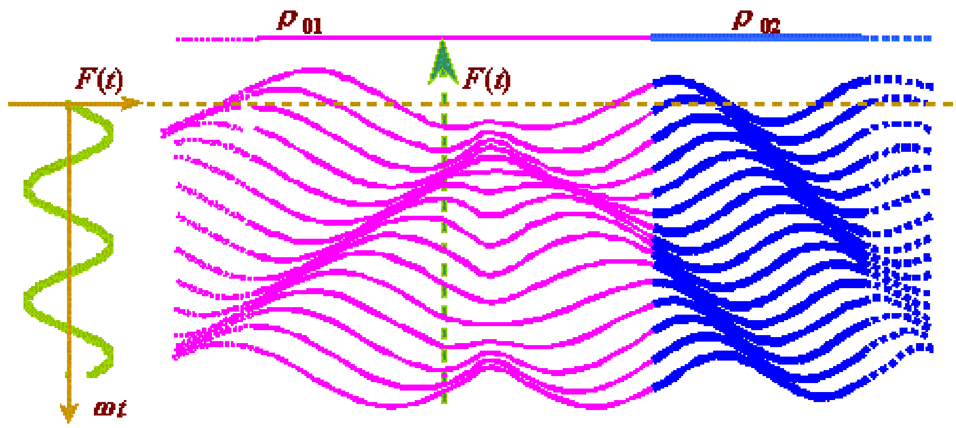

Fig. 4. The diagram of transverse vibrations in the heterogeneous elastic line with distributed parameters at the external force frequency $f = 5$ Hz and amplitude $F_0 = 1$ N. The parameters of system: $\rho_{01} = 1{,}7$ kg/m; $\rho_{02} = 4{,}5$ kg/m; $T = 1$ N.

tributed line is shown in Fig. 4. The pattern presented in it completely corroborates the above analysis of solutions (42) – (44).

Thus the solutions obtained for the heterogeneous elastic distributed line show that the full pattern of the process in a line is not limited by a simple superposition of the direct and reverse waves. The reason is, in the first section the wave reflected from the heterogeneity is shifted relatively to the wave produced directly by the external force, by the angle depending on the second section length x_k and on the complex conditions of reflection from the heterogeneity. When transiting to the section having another density, the wave process also transforms dependently on the parameters of this section. Basically, this method reveals the indicated distinctions, however it needs the additional investigation being out of frames of present paper.

The results presented here essentially extend the understanding of variety of vibration process distinctions in the heterogeneous elastic lines, if noting the overcritical regime, and establish the most important and typical transformations of the vibration pattern under external and internal factors action. This was just the target of presented investigation.

5. Conclusions

In this paper we have revealed that the aperiodical regime of anti-phase damping vibrations, having been not taken into consideration before, essentially effects on the pattern and parameters of vibration process running in a heterogeneous elastic lumped line.

We have observed that each section of a heterogeneous line has its own boundary frequency ω_0. Dependently on frequency and parameters of each section, three vibration regimes may occur in it:

- the *periodical* regime of non-damping vibrations;
- the *aperiodical* regime of vibrations, fast-damping along the line;
- the transient, *critical* regime.

In the semi-finite sections of line there arise the progressive waves having specific phase delays and vibration amplitudes. These phase delays depend nonlinearly on the frequency and parameters of corresponding sections of an elastic line. In the region between the application point of external force and the heterogeneity transition, there form the complex-form standing waves being the result of superposition of the waves having different amplitudes and delay phases. When the hard section of a line vibrated aperiodically, in the light section of line, at certain frequencies, the amplitude of progressive wave can vanish. But when the hard section vibrates in the periodical regime, the vanishing is possible at no conditions.

We showed that the presented solutions are base for a number of models analogous by structure of their lines with lumped parameters and easy transform into solutions for the lines with distributed parameters. The obtained vibration pattern essentially differs from the conventional concept. Particularly, in a distributed line, when reflecting from the heterogeneity transition, the wave amplitude and phase gain the dependence on the external force frequency. In the section between the external force application point and the heterogeneity transition, the standing complex-structure waves settle.

The results of this investigation can be extended to the torsion vibrations of elastic rods, and with the help of dynamical electromechanical analogy DEMA they can be applied to find the solutions for the electric filters.

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