Weakly expanding skew-products of quadratic maps

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Abstract

We consider quadratic skew-products over angle doubling of the circle and prove that they admit positive Lyapunov exponents almost everywhere and an absolutely continuous invariant probability measure. This extends corresponding results of M. Viana and J.F. Alves for skew-products over linear strongly expanding map of the circle.

1 Introduction

We study the ergodic properties of the maps $F : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$

$$F(\theta, x) = (d\theta \bmod 1, a - x^2 + \alpha \sin(2\pi\theta)).$$

Here $d \geq 2$ is an integer, α is a small real number and the parameter $a \in]1, 2[$ is such that the map $f_a(x) = a - x^2$ has preperiodic (but not periodic) critical point.

It is hoped that these maps introduced by M. Viana in [5] represent a first step towards the study of perturbations of products of good unimodal maps and a conjectural 'two-dimensional Jakobson's theorem', i.e., proving that many of those perturbations admit an absolutely continuous invariant probability measure (or a.c.i.m.).

In [5], M. Viana shows that under the assumption $d \geq 16$ these maps admit two positive Lyapunov exponents at Lebesgue almost every point, provided that α is sufficiently small. J.F. Alvès [1] deduces from the underlying estimates the existence of an a.c.i.m.

One would like to weaken the uniform expansion assumptions. A first step (and the main purpose of this paper) is to replace the assumption $d \geq 16$ by the more natural one $d \geq 2$.

Theorem 1.1 *For any integer $d \geq 2$, for every sufficiently small $\alpha > 0$, F has two positive Lyapunov exponents at Lebesgue almost every point and admits an a.c.i.m. Moreover, the same is true of any sufficiently small C^∞ perturbation of F (which is not necessarily a skew product).*

Comments

1. One could consider much more general skew-products of the form $(\theta, x) \mapsto (g(\theta), h(x) + \alpha\phi(\theta))$. Especially, it wouldn't take much effort to generalize $h(x) = a - x^2$ to any unimodal map with negative Schwarzian derivative and having critical point preperiodic or non-recurrent. h could even be replaced by a "good" unimodal map of the interval, that is satisfying the so called Benedicks-Carleson conditions. Indeed, according to [3] these maps satisfy the "Building expansion" Lemmas in [5] so that an adaptation of the proof of Proposition 5.2 to this case would be sufficient.

2. Our method makes stringent demands on the coupling function $\phi(\theta) = \sin(2\pi\theta)$, demands that go much beyond the natural requirement of non-flat critical points. A likewise unnatural-looking restriction that we make is our C^∞ smoothness assumption (or C^r with r arbitrarily large as h goes closer to the full unimodal map).

3. Our proof deals only with the unperturbed case - the perturbated case is treated like in Viana's paper [5].

4. In our situation, one of the exponents is given for free by the uniform expansion of the circle map. Thus Theorem 1.1 deals with the vertical Lyapunov exponent, which is the result of 'random' compositions of perturbations of a good unimodal map. Also we are only interested in a 'dynamical strip' (the equivalent of the dynamical interval of a unimodal map), as points outside just escape to infinity and therefore have a trivially positive (in fact $+\infty$) vertical exponent.

5. We use the same strategy of proof as Viana, with some significant variations:

a) Horizontal expansion dominates vertical expansion in the dynamical strip but in our case this happens not after the first, but after a (fixed) number of iterations. This domination allows one to concentrate on nearly horizontal curves which remain so under iterations.

b) The non-flatness of the images of nearly horizontal curves is directly enforced by the non-flatness of the sin function. It ensures that crossing the critical circle $x = 0$ only removes a small proportion of any horizontal curve.

In Viana's setting, one could exploit the non-flatness just by looking at the second order derivative. Here we have to go to finite, but arbitrarily high order as a gets closer to 2.

c) This non-flatness ensured vertical dispersion of full-width curves in the strongly expanding case. In our situation the dispersion only occurs on a part of these curves.

d) Finally a large deviation estimate of the depth of returns provides all the informations needed for proving the positivity of the vertical Lyapunov exponent as well as constructing the a.c.i.m. This part requires no modification in our setting. The existence of the a.c.i.m. follows from the estimates on the vertical Lyapunov exponent just by applying verbatim [1] and [2].

2 Preliminaries

We begin with a remark on the parameter dependences. The map we consider has three parameters $1 < a < 2$, $d \geq 2$ and $\alpha > 0$. Henceforth, we fix the parameters a and d and do not specify dependence on them. On the other hand, the parameter α is not fixed, but always assumed to be sufficiently small and dependence on it is always specified.

We write, for simplicity,

$$\begin{aligned} \phi(\theta) &= \sin(2\pi\theta), \\ g(\theta) &= d\theta \mod 1, \\ f(\theta, x) &= a - x^2 + \alpha\phi(\theta), \end{aligned}$$

For $n \geq 1$, let's define $f_n(\theta, x)$ by

$$F^n(\theta, x) = (g^n(\theta), f_n(\theta, x)).$$

Notice that the functions f and f_n depend on the parameter α though the dependence is not made explicit.

The map f_a has a unique negative fixed point $p_1 = \frac{-1-\sqrt{1+4a}}{2} < 0$ which is repelling. If we take β slightly smaller than $|p_1|$, the interval $B = [-\beta, \beta]$ satisfies: $f_a(B) \subset \mathrm{int}(B)$ and $|f_a'| > 1$ on $\mathbb{R} \setminus \mathrm{int}(B)$. Then, writing $I = S^1 \times B$, for sufficiently small α, we have

- $F(I) \subset I$,
- $|\partial_x f(\theta, x)| > 1$ outside of I.

These imply that, for any point (θ, x) on $S^1 \times \mathbb{R}$, either its orbit eventually comes into the invariant strip I or the vertical Lyapunov exponent is positive. Thus it is enough to exhibit a positive vertical Lyapunov exponent at almost every point of I.

3 Upper bounds on derivatives

Note that we always have $|\partial_x f(\theta, x)| < 2|p_1| < 4$ for $(\theta, x) \in I$. We prove that the vertical expansion on the dynamic strip I is *uniformly* less than 2:

Lemma 3.1 *There exist constants $K_1 < \infty$ and $R_1 < 2$ such that, when α is sufficiently small, we have $|\partial_x f_n(\theta, x)| \leq K_1 R_1^n$ for all $(\theta, x) \in I$ and all $n \in \mathbb{N}$.*

Proof. Consider maps

$$h : [-\pi/2, \pi/2] \to [p_1, -p_1], \quad \xi \mapsto p_1 \sin \xi.$$

and

$$\psi = h^{-1} \circ f_a \circ h : [-\pi/2, \pi/2] \to [-\pi/2, \pi/2].$$

We claim that $|\psi'(x)| < \sqrt{2p_1}$ on $[-\pi/2.\pi/2]$. Take a point $\xi \in [-\pi/2, \pi/2]$ and let $\bar{\xi} = \psi(\xi)$. Using the relations $a - p_1^2 \sin^2 \xi = p_1 \sin \bar{\xi}$, $a - p_1^2 = p_1$ and $|p_1| < 2$, we obtain

$$\begin{aligned}
|\psi'(\xi)| &= \left|h'(\bar{\xi})\right|^{-1} \left|f_a'(h(\xi))\right| \left|h'(\xi)\right| = \frac{|p_1 \cos \xi| \cdot |2p_1 \sin \xi|}{|p_1 \cos \bar{\xi}|} \\
&= \frac{2|p_1^2 \cos \xi \sin \xi|}{\sqrt{p_1^2 - \left(a - p_1^2 \sin^2 \xi\right)^2}} = \frac{2|p_1^2 \cos \xi \sin \xi|}{\sqrt{p_1^2 - \left(p_1 + p_1^2 \cos^2 \xi\right)^2}} \\
&= \frac{2|p_1 \sin \xi|}{\sqrt{-2p_1 - p_1^2 \cos^2 \xi}} \leq \frac{2|p_1|}{\sqrt{2|p_1|}} \cdot \frac{|\sin \xi|}{\sqrt{1 - |p_1/2| \cos^2 \xi}} \\
&\leq \sqrt{2|p_1|} \frac{|\sin \xi|}{\sqrt{1 - \cos^2 \xi}} = \sqrt{2|p_1|} < 2.
\end{aligned}$$

Let $\sqrt{2|p_1|} < R_1 < 2$. As the map h^{-1} is a diffeomorphism on B, we have for n_0 large enough,

$$|(f_a^{n_0})'| = \frac{|h'(h^{-1}(f_a^{n_0}(x)))|}{|h'(h^{-1}(x))|} |(\psi^{n_0})'(h^{-1}(x))| < R_1^{n_0} \quad \text{on } B.$$

The integer n_0 being fixed, this implies that for α small enough, $|\partial_x f_{n_0}| < R_1^{n_0}$ on I. But it follows that $|\partial_x f_n| \leq K_1 R_1^n$ on I, $\forall n \geq 0$, for some $K_1 < \infty$. $\square$

4 Admissible curves

Following Viana's approach [5], we shall analyze the dynamics by focusing on nearly horizontal curves, the so-called *admissible curves.* Their definition is more delicate in our weakly expanding case. We fix an integer N_0 such that

$$K_1 R_1^{N_0} < d^{N_0}.$$

Put

$$A_0 = \left(1 - \frac{K_1 R_1^{N_0}}{d^{N_0}}\right)^{-1} \sum_{i=0}^{\infty} \frac{K_1 R_1^i}{d^i}.$$

Then, fix an integer $\rho \geq 1$ so large that

$$d^{\rho-1} > 100 A_0. \tag{1}$$

Consider a curve $\widehat{X} = \text{graph}(X)$, $X : S^1 \to \mathbb{R}$, which is C^ρ on $S^1 \setminus \{0\}$ and continuous to the right at 0. If X satisfies:

$$|X^{(r)}(\theta)| \leq 2A_0 (2\pi/d)^r \alpha \qquad \text{for } \theta \in S^1 \text{ and } 1 \leq r \leq \rho, \tag{2}$$

then X is called *pre-admissible.*

Also consider the sequence of Markov partitions of S^1 induced by g:

$$\begin{aligned} \mathcal{P}_1 &= \{[\theta_j, \theta_{j+1}[; 0 \leq j \leq d-1\} \qquad \text{where } \theta_j = j/d; \\ \mathcal{P}_{n+1} &= \{\text{connected components of } g^{-1}(\omega);\, \omega \in \mathcal{P}_n\}. \end{aligned}$$

Lemma 4.1 *Suppose $\alpha > 0$ is sufficiently small. If $\widehat{X} = graph(X)$ is a pre-admissible curve then, for any $\omega \in \mathcal{P}_n$ with $n \geq N_0$, the image $\widehat{Y} = F^n(\widehat{X}|_\omega)$ is again pre-admissible.*

Proof. Consider an arbitrary curve $\widehat{X} = \text{graph}(X)$. Let ω be an arbitrary element of $\mathcal{P}_{N_0}$. For $0 \leq i \leq N_0$, let $\widehat{X}_i = \text{graph}(X_i) = F^i(\widehat{X}|_\omega)$. For $i \geq 0$ and $\theta \in g^i(\omega)$, we have the obvious relation

$$X_{i+1}(g(\theta)) = f(\theta, X_i(\theta)). \tag{3}$$

Claim. Omitting obvious dependencies, we have:

$$X_{i+1}^{(r)} = \frac{1}{d^r}\partial_x f \cdot X_i^{(r)} + \frac{1}{d^r}\partial_\theta^r f + Q_r(X_i^{(1)}, \ldots, X_i^{(r-1)}) \tag{4}$$

where

$$Q_r(Y_1, \ldots, Y_{r-1}) = \sum_{j=1}^{r-1} q_{rj} Y_j Y_{r-j}$$

and q_{rj} are constants.

The claim follows from an immediate induction on r as it is obvious that the derivative of Q_{r-1} w.r.t. θ is of the form claimed.

Observe that the second term in equation (4) is $\alpha d^{-r}\phi^{(r)}(X_i(\theta))$, hence it is bounded in absolute value by $\alpha(2\pi/d)^r$.

Suppose that the curve $\widehat{X}$ satisfies the condition (2). An easy induction using (4) and Lemma 3.1 give

$$\begin{aligned} |X_n^{(r)}| &\leq \frac{1}{d^{rn}}|\partial_x f_n| \cdot |X^{(r)}| + \sum_{i=0}^{n-1} \frac{1}{d^{ri}}|\partial_x f_i| \cdot \frac{1}{d^r}|\partial_\theta^r f| + \sum_{i=0}^{n-1} \frac{1}{d^{ri}}|\partial_x f_i| \cdot |Q_r| \\ &< \frac{K_1 R_1^n}{d^{rn}} 2A_0 \left(\frac{2\pi}{d}\right)^r \alpha + \sum_{i=1}^{n-1} \frac{K_1 R_1^i}{d^{ri}} \cdot \left(\frac{2\pi}{d}\right)^r \alpha + \sum_{i=0}^{n-1} \frac{K_1 R_1^i}{d^{ri}} \cdot |Q_r|. \end{aligned} \tag{5}$$

By induction on $1 \leq r \leq \rho$, we see that there exists a constant $C > 0$, which is independent of α and $\widehat{X}$, the initial pre-admissible curve, such that

$$|Q_r(X_i^{(1)}(\theta), \ldots, X_i^{(r-1)}(\theta))| < C\alpha^2 \quad \text{and} \quad |X_i^{(r)}(\theta)| < C\alpha \tag{6}$$

for any $0 \leq i \leq N_0$ and $1 \leq r \leq \rho$. From this estimate on Q_r, the last sum of (5) is smaller than the one just before, provided that α is sufficiently small. Hence we can conclude, from (5) and the choice of A_0, that, for $n \geq N_0$ (N_0 was fixed at the beginning of this section)

$$|X_n^{(r)}| < \frac{K_1 R_1^{N_0}}{d^{rN_0}} 2A_0 \left(\frac{2\pi}{d}\right)^r \alpha + 2\sum_{i=0}^{\infty} \frac{K_1 R_1^i}{d^{ri}} \left(\frac{2\pi}{d}\right)^r \alpha = 2A_0 \left(\frac{2\pi}{d}\right)^r \alpha.$$

We proved the lemma. $\square$

This lemma motivates the following definition:

Definition 4.2 We say that a curve $\widehat{Y} = \mathrm{graph}(Y)$ is an *admissible curve* if, for each $0 < n \leq N_0$, there exist $\omega \in \mathcal{P}_n$ and a pre-admissible curve $\widehat{X} = \mathrm{graph}(X)$, $X : \omega \to \mathbb{R}$, such that $\widehat{Y} = F^n(\widehat{X}|_\omega)$.

Remarks. As we shall see

1. A flat curve $x = \text{const}$ is pre-admissible but *not* admissible (see the Lemma below).

2. The image by F^{2N_0} of a pre-admissible curve is admissible. In particular, every point in $F^{2N_0}(S^1 \times \mathbb{R})$ and so, every interesting point, lies on an admissible curve (see the corollary below).

3. In order to prove the claim of Theorem 1.1 on the vertical Lyapunov exponent, it is enough to show for an arbitrary admissible curves $\widehat{X} = \mathrm{graph}(X)$ that

$$\liminf_{n\to\infty} n^{-1} \log |\partial_x f_n(\theta, X(\theta))| > 0$$

for Lebesgue measure almost every $\theta \in S^1$.

The main property of this class of nearly horizontal curves is the

Corollary 4.3 *For any admissible curve $\widehat{X} = graph(X)$ and any $\omega \in \mathcal{P}_n$, $n \geq 0$, the image $F^n(\widehat{X}|_\omega)$ is again an admissible curve.*

The second fundamental property of admissible curves is that their image are *non-flat*:

Lemma 4.4 *If $\widehat{X} = graph(X)$ is an admissible curve, then it satisfies either*

$$|X^{(\rho-1)}(\theta)| > (\pi/d)^{\rho-1}\alpha \quad \text{or} \quad |X^{(\rho)}(\theta)| > (\pi/d)^{\rho}\alpha,$$

for each $\theta \in S^1$. More precisely S^1 can be divided into at most four (in fact three) closed intervals on each of which, one of the above two inequalities holds.

Proof. By the definition of admissible curves, there exists a pre-admissible curve $\widehat{Y} = \mathrm{graph}(Y)$ and $\omega \in \mathcal{P}_1$ such that $\widehat{X} = F(\widehat{Y}|_\omega)$. From (4) and (6), we have

$$X^{(r)}(g(\theta)) = \frac{-2Y(\theta)Y^{(r)}(\theta)}{d^r} + \alpha\frac{\phi^{(r)}(\theta)}{d^r} + \mathcal{O}(\alpha^2).$$

Thus, for sufficiently small α, we have

$$|X^{(r)}(g(\theta))| > \left|\alpha\frac{\phi^{(r)}(\theta)}{d^r}\right| - \frac{9A_0(2\pi/d)^r\alpha}{d^r}.$$

We assume that θ belongs to the subset $\omega\cap([\frac{1}{8},\frac{3}{8}]\cup[\frac{5}{8},\frac{7}{8}])$ which consists of at most two intervals, as the other case is similar. For the even number $r=\rho$ or $\rho-1$, we have $|\phi^{(r)}(\theta)|=(2\pi)^r|\sin(2\pi\theta)|\geq(2\pi)^r/\sqrt{2}$ for $\theta\in([\frac{1}{8},\frac{3}{8}]\cup[\frac{5}{8},\frac{7}{8}])$ and

$$\begin{aligned}\alpha^{-1}|X^{(r)}(g(\theta))| > \left|\frac{(2\pi)^r}{d^r\sqrt{2}}\right| - \frac{9A_0\cdot(2\pi)^r}{d^{2r}} &= \left(\frac{2\pi}{d}\right)^r\left(\frac{1}{\sqrt{2}}-\frac{9A_0}{d^r}\right)\\ &\geq \left(\frac{2\pi}{d}\right)^r\left(\frac{1}{\sqrt{2}}-\frac{9}{100}\right)\geq\left(\frac{\pi}{d}\right)^r.\end{aligned}$$

In the second inequality, we used the condition in the choice of ρ. $\square$

5 Critical return

We turn now to the central fact about the returns of admissible curves to the critical line $S^1\times\{0\}$, which corresponds to Viana's "Technical Lemma 2.6" in [5].

First we recall a fact about the map f_a from [4, Th III.6.3]:

Lemma 5.1 *There exist constants $\delta>0$, $\sigma>1$ and $\mu>0$ such that $|(f_a^n)'(x)|\geq\mu\sigma^n$ if $f_a^n(x)\in(-\delta,\delta)$.*

Define constants

$$M(\alpha)=\left[\frac{|\log\alpha|}{\log 32}\right],\qquad \eta=\frac{\log\sigma}{8\log 32}. \tag{7}$$

Fix $r\geq 0$, let $J(r)=\{x\in\mathbb{R},\ |x|\leq\sqrt{\alpha}e^{-r}\}$. Denote by $\widehat{J}(r)$ the critical strip $\widehat{J}(r)=S^1\times J(r)$.

Proposition 5.2 *There exist $C<\infty$ and $\beta>0$ such that, for all sufficiently small $\alpha>0$, for any admissible curve $\widehat{Y}$ and any $r\geq(\frac{1}{2}-2\eta)\log(\frac{1}{\alpha})$,*

$$m(\{\theta\in S^1|\ F^{M(\alpha)}(\widehat{Y}(\theta))\in\widehat{J}(r-2)\})\leq Ce^{-\beta r} \tag{8}$$

where m is the Lebesgue measure on S^1.

The main ingredients in the proof of Proposition 5.2 are the two Lemmas below. The first is a general result on the size of the pre-image of a small interval for C^s functions whose s-th derivative is bounded away from zero. The second states that the image under F of the components of an admissible curve are well-separated over a large subset of S^1.

Lemma 5.3 *Let s be a positive integer and let $I \subset \mathbb{R}$ be an interval. Suppose that $h : I \to \mathbb{R}$ is a C^s function such that $|h^{(s)}(\theta)| \geq \delta$ for all $\theta \in I$. Then,*

$$m(\{\theta \in I| \ |h(\theta)| \leq \varepsilon\}) < 2^{s+1} \left(\varepsilon/\delta\right)^{1/s}$$

for all $\varepsilon > 0$. More precisely, we have

$$C := \{\theta \in I| \ |h(\theta)| \leq \varepsilon\} \subset \bigcup_{i=1}^{2^s-1} J_i$$

where J_i are intervals with

$$m(J_i) \leq 2\Big(\frac{\varepsilon}{\delta}\Big)^{1/s}.$$

Together with Lemma 4.4, this immediately implies:

Corollary 5.4 *Let $\widehat{X}$ be an admissible curve and assume that α is sufficiently small. Then, for $0 < \epsilon < \alpha$, we have*

$$m(\{\theta \in S^1 | \widehat{X}(\theta) \in S^1 \times (-\epsilon, \epsilon)\}) \leq C_1 \cdot (\epsilon/\alpha)^{1/\rho}.$$

where $C_1 = 4 \cdot 2^{\rho+1}(d/\pi)$.

Proof of Lemma 5.3. We proceed by induction on s. The case $s = 1$ is obvious. Assume the result is true for $s - 1 \geq 1$. Let

$$A := \{\theta \in I| \ |h^{(s-1)}(\theta)| \leq \delta(\varepsilon/\delta)^{1/s}\}.$$

Then A is an interval because $h^{(s-1)}$ is monotone. Applying the case $s = 1$ to $h^{(s-1)}$ with $\varepsilon' = \delta(\varepsilon/\delta)^{1/s}$ we deduce

$$m(A) \leq 2(\varepsilon/\delta)^{1/s}.$$

The complement of A is made up of at most 2 intervals denoted by I_1 and I_2. Let $B_j = \{\theta \in I_j; \ |h(\theta)| \leq \varepsilon\}$ for $j = 1, 2$. As $|h^{(s-1)}| \geq \delta(\varepsilon/\delta)^{1/s}$ on I_j, the induction assumption at level $s - 1$ implies that B_j is contained in the union of $2^{s-1} - 1$ intervals whose length is no more than

$$2\left(\frac{\varepsilon}{\delta(\varepsilon/\delta)^{1/s}}\right)^{1/(s-1)} = 2\Big(\frac{\varepsilon}{\delta}\Big)^{1/s}.$$

Since $C \subset A \cup B_1 \cup B_2$, C is contained in at most $1 + 2(2^{s-1} - 1) = 2^s - 1$ intervals. $\square$

Let $\widehat{X} = \mathrm{graph}(X)$ be an admissible curve. Denote by $\widehat{X}_j = \mathrm{graph}(X_j) = F(\widehat{X}|_{[\theta_j, \theta_{j+1}[})$ for $0 \leq j \leq d - 1$, the d admissible curves contained in the image of $\widehat{X}$.

Lemma 5.5 *There exists a constant $\varepsilon_0 > 0$, such that, for all sufficiently small $\alpha > 0$, for any admissible curve X and any $j_1 \in \{0, \dots, d-1\}$, there exists j_2, $0 \le j_2 \le d-1$, satisfying*

$$\{\theta \in S^1 |\ |\widehat{X}_{j_1}(\theta) - \widehat{X}_{j_2}(\theta)| \le \varepsilon_0 \alpha\} \subset \bigcup_{i=1}^{2^{\rho+2}} J_i$$

where J_i are intervals with $m(J_i) \le \frac{1}{40 \times 2^\rho d}$.

Remark that if $d = 2$ there is no much choice in j_2.

Proof. It follows from (4) and (6) with $i = 0$ that for all $1 \le r \le \rho$, $0 \le j \le d-1$ and $\theta \in S^1$,

$$X_j^{(r)}(\theta) = d^{-r} \alpha \phi^{(r)}(\theta/d + \theta_j) - 2d^{-r} X(\theta/d + \theta_j) X^{(r)}(\theta/d + \theta_j) + \mathcal{O}(\alpha^2). \tag{9}$$

For $0 \le j_1 < d$, let

$$j_2 = \begin{cases} j_1 + d/2 & \mod d \quad \text{if } d \text{ is even;} \\ j_1 + (d-1)/2 & \mod d \quad \text{otherwise.} \end{cases}$$

Note that the distance between $\theta/d + \theta_{j_1}$ and $\theta/d + \theta_{j_2}$ is not smaller than $1/3$. Also note that $|\phi^{(r)}(\theta)| = (2\pi)^r |\sin(2\pi\theta)|$ if r is even and $|\phi^{(r)}(\theta)| = (2\pi)^r |\cos(2\pi\theta)|$ otherwise. Consider the intervals $I_i = [(2i-1)/8, (2i+1)/8]$, $1 \le i \le 4$, which cover S^1. Assume that $\theta/d + \theta_{j_1} \in I_i$. Let $r = \rho$ or $\rho - 1$ be the even (resp. odd) number in the case i is odd (resp. even). Then we have

$$|\phi^{(r)}(\theta/d + \theta_{j_1}) - \phi^{(r)}(\theta/d + \theta_{j_2})| \ge (2\pi)^r (\sin(\pi/4) - \sin(\pi/12)) > (2\pi)^r/10.$$

In fact, in the case $i = 1$ for example, the above inequality is clear since $\theta/d + \theta_{j_1} \in I_1 = [1/8, 3/8]$ and $\theta/d + \theta_{j_1} \notin (1/24, 11/24)$. The other cases are similar. Thus, using (9) and the condition in the choice of ρ,

$$\begin{aligned} \left| X_{j_1}^{(r)}(\theta) - X_{j_2}^{(r)}(\theta) \right| &\ge \left(\frac{2\pi}{d}\right)^r \frac{\alpha}{10} - 4d^{-r} \cdot 2A_0 \left(\frac{2\pi}{d}\right)^r \alpha - \mathcal{O}(\alpha^2) \\ &> \left(\frac{2\pi}{d}\right)^r \frac{\alpha}{20}. \end{aligned}$$

Now let $i = 0, \dots, d-1$ and $\epsilon_0 > 0$ be small (see below) and apply Lemma 5.3 to the function $X_{j_1}^{(r)}(\theta) - X_{j_2}^{(r)}(\theta)$ on each interval $g(I_i \cap [\theta_{j_1}, \theta_{j_1+1}])$ with

$\varepsilon = \varepsilon_0 \alpha$ and $\delta = (2\pi/d)^r \alpha/20$. We obtain

$$\{\theta \in g(I_i \cap [\theta_{j_1}, \theta_{j_1+1}]) \mid |X_{j_1}^{(r)}(\theta) - X_{j_2}^{(r)}(\theta)| \leq \varepsilon_0 \alpha\} \subset \bigcup_{j=1}^{2^\rho - 1} \tilde{J}_{i,j}$$

where $\tilde{J}_{i,j}$ is an interval such that

$$m(\tilde{J}_{i,j}) \leq \frac{d(20\varepsilon_0)^{1/r}}{\pi}.$$

For ε_0 sufficiently small, the right hand side is smaller than $\frac{1}{40 \times 2^\rho d}$. Finally $\bigcup_{i=0,\ldots,d-1} g(I_i \cap [\theta_{j_1}, \theta_{j_1+1}]) = S^1$, hence to get the lemma it is enough to take all the $d(2^\rho - 1)$ intervals $\tilde{J}_{i,j}$ just constructed. $\square$

Proof of Proposition 5.2 There are two cases to consider. We first assume that $r \geq (\frac{1}{2} + 2\eta)\log(\frac{1}{\alpha})$. Note that in this case,

$$1/\sqrt{\alpha} \leq \exp((1+4\eta)^{-1} r) < \exp((1-\eta) r).$$

From Corollary 5.4, we have, for any admissible curve $\widehat{X} = \mathrm{graph}(X)$,

$$m(\{\theta \in S^1 \mid \widehat{X}(\theta) \in \widehat{J}(r-2)\}) \leq C_1 \left(\frac{\sqrt{\alpha} e^{-r+2}}{\alpha} \right)^{1/\rho} < C_1 \left(e^2 e^{-\eta r} \right)^{1/\rho}.$$

Since $F^{M(\alpha)}(\widehat{Y})$ consists of $d^{M(\alpha)}$ admissible curves and since $g^{M(\alpha)}$ is linear, this implies (8) with $\beta = \eta/\rho$.

Henceforth we consider the case $(\frac{1}{2} - 2\eta)\log(\frac{1}{\alpha}) \leq r \leq (\frac{1}{2} + 2\eta)\log(\frac{1}{\alpha})$. Let $r_0(\alpha) = (\frac{1}{2} - 2\eta)\log(\frac{1}{\alpha})$. Actually, we are going to prove the claim (8) only for $r = r_0(\alpha)$, since the claim for other r's follows by replacing β by $\beta(\frac{1}{2} - 2\eta)/(\frac{1}{2} + 2\eta)$. We may and do assume that there exists a point $z_0 = (\theta_0, x_0)$ on the admissible curve $\widehat{Y}$ such that $F^{M(\alpha)}(z_0) \in \widehat{J}(0)$, since, otherwise, the claim (8) is trivial.

Let $x_i = f_a^i(x_0)$. Consider horizontal strips

$$S_i = S^1 \times [x_i - 5^i L\alpha, x_i + 5^i L\alpha], \quad i = 0, 1, 2, \cdots,$$

where $L = \max\{1, 2A_0(2\pi/d)\}$. Note that these strips are defined so that the (vertical) width $|S_i|$ satisfies

$$4|S_i| + \alpha < |S_{i+1}| \text{ and } |S_i| \leq 2 \cdot 5^{M(\alpha)} L\alpha < \sqrt{\alpha} \quad \text{for } 0 \leq i \leq M(\alpha)$$

provided that α is sufficiently small. From the first inequality, it follows that $F(S_i) \subset S_{i+1}$. Since the slope of the admissible curve $\widehat{Y}$ is bounded by $2A_0(2\pi/d)\alpha \leq L\alpha$, we have $\widehat{Y} \subset S_0$ and hence $F^i(\widehat{Y}) \subset S_i$ for $0 \leq i \leq M(\alpha)$.

The strips S_i, $0 \leq i \leq M(\alpha)-1$, do not meet the critical strip $\widehat{J}(0)$. Indeed, otherwise, $S_{M(\alpha)}$ would intersect with both of $\widehat{J}(0)$ and $F^{M(\alpha)-i}(\widehat{J}(0))$. But this is impossible. In fact, we can see inductively that for $i \geq 0$, $F^i(\widehat{J}(0))$ is contained in the strip $J_i = S^1 \times [f_a^i(0) - 5^i\alpha, f_a^i(0) + 5^i\alpha]$. The height of J_i is bounded by $\sqrt{\alpha}$ for $1 \leq i \leq M(\alpha)$. Hence the distance between S_i and $\widehat{J}(0)$ is larger than

$$|f_a^i(0)| - 2\sqrt{\alpha} \geq \text{const} - 2\sqrt{\alpha} > \sqrt{\alpha} > |S_{M(\alpha)}|.$$

From $d(S_i, S^1 \times \{0\}) > \sqrt{\alpha}$ and using $\partial_x\partial_\theta f = 0$ and $|\partial_x^2 f| \leq 2$, we obtain the following distortion estimate, for all $0 \leq i \leq M(\alpha)-1$:

$$\sum_{j=i}^{M(\alpha)-1} \sup_{w,w' \in S_j} |\log|\partial_x f(w)| - \log|\partial_x f(w')|| \leq M(\alpha)\frac{2\cdot 2\cdot 5^{M(\alpha)}L\alpha}{\sqrt{\alpha}} < \log 2 \tag{10}$$

which allows us to consider that the maps $F^{M(\alpha)-i}$ are almost linear on S_i.

Since $|f_a^{M(\alpha)}(x_0)| < |f_{M(\alpha)}(z_0)| + |S_{M(\alpha)}| < 2\sqrt{\alpha} < \delta$ if α is sufficiently small, Lemma 5.1 gives

$$|\partial_x f_{M(\alpha)-i}(\theta, x)| > \frac{1}{2}\prod_{j=0}^{M(\alpha)-i} |f_a'(x_{i+j})| > \frac{\mu\sigma^{M(\alpha)-i}}{2} \tag{11}$$

for any $(\theta, x) \in S_i$ with $0 \leq i \leq M(\alpha)$.

We introduce some more constants:

- Pick an integer m_0 such that $d^{m_0-1} \geq 80 \times 2^\rho$.
- Let $\bar{\sigma} = \sqrt{\sigma} > 1$.
- Fix a constant $\kappa > 4^{m_0}$, independent of α, so that

$$(2\kappa)(\kappa\varepsilon_0/4 - A_0(2\pi/d) - 4(1-\bar{\sigma}^{-1})^{-1}) \geq 3e^2.$$

Denote $\lambda_j = |\partial_x f_{M(\alpha)-j}(F^j(z_0))|/\bar{\sigma}^{(M(\alpha)-j)}$ for $0 \leq j \leq M(\alpha)-1$. Note that we have, from (11),

$$\lambda_j > \mu\sigma^{M(\alpha)-j}/2\bar{\sigma}^{(M(\alpha)-j)} = (\mu/2)\bar{\sigma}^{M(\alpha)-j}. \tag{12}$$

Let

$$0 = t_1 < t_2 < \ldots < t_q \leq M(\alpha)$$

be the (finite) sequence of integers defined inductively by $t_1 = 0$ and

$$t_{i+1} = \min\{s \mid \ t_i < s \leq M(\alpha), \ \lambda_{t_i} \geq 2\kappa\lambda_s\}.$$

Let $k_0(\alpha) := \max\{1 \leq i \leq q \ \mid \lambda_{t_i} \geq 2\kappa e^{-r_0(\alpha)}/\sqrt{\alpha}\}$.

We have $t_{i+1} > t_i + m_0$ for all $1 \leq i < k_0(\alpha)$, because, by the definition of λ_i's and κ,

$$4^{m_0} < \kappa \leq \frac{1}{2}\frac{\lambda_{t_i}}{\lambda_{t_{i+1}}} \leq |\partial_x f_{t_{i+1}-t_i}(F^{t_i}(z_0))|\bar{\sigma}^{t_i - t_{i+1}} < 4^{t_{i+1}-t_i}.$$

Claim 1. We have $k_0(\alpha) \geq \gamma r_0(\alpha)$ for $\gamma = \eta/\log(8\kappa)$ if α is sufficiently small.

Proof. On the one hand, $\lambda_{t_i} \leq 2\kappa\lambda_{t_{i+1}-1} \leq 8\kappa\lambda_{t_{i+1}}$, hence $\lambda_{t_1} \leq (8\kappa)^{k_0(\alpha)}\lambda_{t_{k_0(\alpha)+1}}$. From (12), it follows

$$\lambda_{t_{k_0(\alpha)+1}} \geq (8\kappa)^{-k_0(\alpha)}\lambda_{t_1} \geq (8\kappa)^{-k_0(\alpha)}(\mu/2)\cdot\bar{\sigma}^{M(\alpha)}.$$

On the other hand, by definition of $k_0(\alpha)$, $\lambda_{t_{k_0(\alpha)+1}} \leq 2\kappa e^{-r_0(\alpha)}/\sqrt{\alpha}$. Combining these two bounds on $\lambda_{t_{k_0(\alpha)+1}}$, we derive

$$-r_0(\alpha) - (\log\alpha)/2 \geq M(\alpha)\log\bar{\sigma} - k_0(\alpha)\log(8\kappa) + C$$

where $C = \log(\mu/4\kappa)$ is a constant independent of α. Recall the definitions (7) and that $r_0(\alpha) \geq (\frac{1}{2} - 2\eta)\log\frac{1}{\alpha}$, thus

$$\begin{aligned} k_0(\alpha)\log(8\kappa) &\geq r_0(\alpha) - \left(\frac{1}{2} - 4\eta\right)\log\frac{1}{\alpha} + C \geq r_0(\alpha)\left(1 - \frac{\frac{1}{2}-4\eta}{\frac{1}{2}-2\eta}\right) + C \\ &\geq r_0(\alpha)\left(\frac{2\eta}{\frac{1}{2}-2\eta}\right) + C \geq \eta r_0(\alpha), \end{aligned}$$

for sufficiently small α. This proves the claim. ☐

A sequence $\bar{l} = (l_1, \ldots, l_n)$ of n integers in $\{0, 1, \cdots, d-1\}$ will be called an *n-word* below. For an n-word, we denote by $\omega(\bar{l})$ the unique element ω

of $\mathcal{P}_n$ satisfying $g^{i-1}(\omega) \in [\theta_{l_i}, \theta_{l_i+1}[$ for $1 \leq i \leq n$. For an n-word $\bar{l}$ and $1 \leq j \leq n$ we set

$$\widehat{Y}_j(\bar{l}) = \mathrm{graph}(Y_j(\bar{l}, \cdot)) = F^j(\widehat{Y}|_{\omega(\bar{l})}).$$

Two $M(\alpha)$-words $\bar{l}$ and $\bar{m}$ are said to be *incompatible* if

$$|Y_{M(\alpha)}(\bar{l}, \theta) - Y_{M(\alpha)}(\bar{m}, \theta)| \geq 3e^{-r_0(\alpha)+2}\sqrt{\alpha} \qquad \forall \theta \in S^1. \tag{13}$$

We claim that, if two $M(\alpha)$-words $\bar{l}$ and $\bar{m}$ are incompatible, then at most one of the curves $\widehat{Y}_{M(\alpha)}(\bar{l})$ or $\widehat{Y}_{M(\alpha)}(\bar{m})$ can meet $\widehat{J}(r_0(\alpha)-2)$. Indeed, the right hand side of the inequality (13) is chosen so that it is larger than the sum of the height of $\widehat{J}(r_0(\alpha)-2)$, which is equal to $2e^{-r_0(\alpha)+2}\sqrt{\alpha}$, and the vertical width of the admissible curve $\widehat{Y}_{M(\alpha)}(\bar{l})$, which is bounded by $2A_0(2\pi/d)\alpha < e^{-r_0(\alpha)+2}\sqrt{\alpha} \sim \alpha^{1-2\eta}$. A bound on the number of mutually compatible words will give a bound of the type sought-for. To estimate this number, we are first going to establish a sufficient condition for incompatibility. Consider a (t_i-1)-word $\bar{m} = (m_1, \ldots, m_{t_i-1})$ and an integer $0 \leq m < d$. Applying Lemma 5.5 to the curve $\widehat{X} = \widehat{Y}^{t_i-1}(\bar{m})$, we find an integer $0 \leq m' < d$ such that,

$$\{\theta \in S^1 |\ |Y_{t_i}((\bar{m}, m), \theta) - Y_{t_i}((\bar{m}, m'), \theta)| \leq \varepsilon_0 \alpha\} \subset \bigcup_{\ell=1}^{2^{\rho+2}} J_\ell$$

where J_ℓ are intervals satisfying $m(J_\ell) \leq \frac{1}{40\times 2^\rho d}$. We denote by $E(\bar{m}, m)$ such an integer as m'. We write $\Omega_*(\bar{m}, m)$ for the collection of elements ω of $\mathcal{P}_{m_0}$ which meet some J_ℓ above. For a given J_ℓ,

$$\#\{\omega \in \mathcal{P}_{m_0} | \omega \cap J_\ell \neq \emptyset\} \leq 2 + \frac{m(J_\ell)}{d^{-m_0}} \leq 2 + \frac{d^{m_0}}{40 \times 2^\rho d}.$$

Hence taking the union over all J_ℓ's,

$$\#\{\omega \in \mathcal{P}_{m_0} | \omega \cap (\cup J_\ell) \neq \emptyset\} \leq 2^{\rho+3} + \frac{2^{\rho+2} d^{m_0}}{40 \times 2^\rho d} \leq \frac{1}{5d} d^{m_0}.$$

It follows that, for any t_i-word $\bar{m}$, we have

$$\#\Omega_*(\bar{m}) := \#\Omega_*((m_1, \ldots, m_{t_i-1}), m_{t_i}) \leq \frac{1}{5d} d^{m_0}. \tag{14}$$

Claim 2. A sufficient condition for incompatibility: Let $1 \leq i < k_0(\alpha)$. Any pair of $M(\alpha)$-words $\bar{l} = (l_j)_{j=1}^{M(\alpha)}$ and $\bar{l}' = (l'_j)_{j=1}^{M(\alpha)}$ such that:

(a) $l_j = l'_j$ for all $j < t_i$ and $t_i < j < t_{i+1}$,

(b) $l'_{t_i} = E((l_1, \cdots, l_{t_i - 1}), l_{t_i})$,

(c) $\omega(l_{t_i+1}, \ldots, l_{t_i+m_0}) \notin \Omega_*(l_1, \ldots, l_{t_i})$.

are incompatible.

Proof. Let $\bar{l}_j$ be $(l_1, \ldots, l_j)$. Define $\bar{l}'_j$ similarly. From assumptions (b) and (c),

$$\left| Y_{t_i}(\bar{l}_{t_i+m_0}, \theta) - Y_{t_i}(\bar{l}'_{t_i+m_0}, \theta) \right| \geq \varepsilon_0 \alpha$$

for all $\theta \in \omega(l_{t_i+1}, \ldots, l_{t_i+m_0}) = g^{t_i}(\omega(\bar{l}_{t_i+m_0}))$. We deduce from the distortion estimate (10) that

$$\left| Y_{t_{i+1}}(\bar{l}_{t_{i+1}}, \theta) - Y_{t_{i+1}}(\bar{l}'_{t_{i+1}}, \theta) \right| \geq \frac{1}{2} \frac{\lambda_{t_i}}{\lambda_{t_{i+1}}} \cdot \varepsilon_0 \alpha.$$

for the same θ's. We want now to control the vertical distance between $\widehat{Y}_{t_{i+1}}(\bar{l})$ and $\widehat{Y}_{t_{i+1}}(\bar{l}')$. To this end, note that $\widehat{Y}_{t_{i+1}}(\bar{l})$ and $\widehat{Y}_{t_{i+1}}(\bar{l}')$ are contained in the admissible curves $\widehat{Y}_{t_{i+1}}(\bar{l}_{t_{i+1}})$ and $\widehat{Y}_{t_{i+1}}(\bar{l}'_{t_{i+1}})$ respectively. Hence the vertical distance between them is bounded from below by

$$(\lambda_{t_i}/\lambda_{t_{i+1}})\varepsilon_0\alpha/2 - 2A_0(2\pi/d)\alpha \tag{15}$$

where the second term is a bound for the oscillation of the admissible curves in the vertical direction.

For $t_{i+1} \leq j \leq M(\alpha)$, let $\Delta_j = \inf_{\theta \in S^1} |Y_j(\bar{l}, \theta) - Y_j(\bar{l}', \theta)|$ and put

$$D_j = \min_{(\theta, x) \in S_j} |\partial_x f(\theta, x)| = \min_{(\theta, x) \in S_j} |f'_a(x)|.$$

Then the distances Δ_j satisfy

$$\Delta_{j+1} \geq D_j \Delta_j - 2\alpha$$

where the last term 2α is the oscillation of $\alpha\phi$. Hence

$$\Delta_{M(\alpha)} \geq \prod_{j=t_{i+1}}^{M(\alpha)-1} D_j \cdot \Delta_{t_{i+1}} - \sum_{j=t_{i+1}}^{M(\alpha)-1} \left(\prod_{\ell=j}^{M(\alpha)-1} D_\ell \right) \cdot 2\alpha.$$

From the definition of λ_j's and the distortion estimate (10), we have

$$\bar{\sigma}^{M(\alpha)-j}\lambda_j/2 \leq \prod_{\ell=j}^{M(\alpha)-1} D_\ell \leq 2\lambda_j \bar{\sigma}^{M(\alpha)-j}.$$

Since $\lambda_j \leq \lambda_{t_{i+1}}$ for $j \geq t_{i+1}$ from the definition of t_{i+1}, we obtain

$$\begin{aligned}
\Delta_{M(\alpha)} &\geq \lambda_{t_{i+1}} \bar{\sigma}^{M(\alpha)-t_{i+1}} \Delta_{t_{i+1}}/2 - \sum_{j=t_{i+1}}^{M(\alpha)-1} \lambda_j \bar{\sigma}^{M(\alpha)-j} \cdot 4\alpha \\
&\geq \lambda_{t_{i+1}} \bar{\sigma}^{M(\alpha)-t_{i+1}} \cdot \left(\Delta_{t_{i+1}}/2 - \sum_{j=t_{i+1}}^{M(\alpha)-1} 4\alpha \bar{\sigma}^{-(j-t_{i+1})} \right) \\
&\geq \lambda_{t_{i+1}} \bar{\sigma}^{M(\alpha)-t_{i+1}} \left(\Delta_{t_{i+1}}/2 - 4\alpha(1-\bar{\sigma}^{-1})^{-1} \right).
\end{aligned}$$

Recall that $\Delta_{t_{i+1}}$ is bounded from below by (15) and $\lambda_{t_{i+1}} \geq 2\kappa e^{-r_0(\alpha)}/\sqrt{\alpha}$. Using the definition of κ and $\lambda_{t_i}/\lambda_{t_{i+1}} \geq 2\kappa$, we conclude

$$\begin{aligned}
\Delta_{M(\alpha)} &\geq (2\kappa e^{-r_0(\alpha)}/\sqrt{\alpha})(\kappa \varepsilon_0 \alpha/2 - A_0(2\pi/d)\alpha - 4\alpha(1-\bar{\sigma}^{-1})^{-1}) \\
&\geq 3e^{-r_0(\alpha)+2}\sqrt{\alpha}.
\end{aligned}$$

This implies that $\bar{l}$ and $\bar{l}'$ are incompatible, as claimed. □

To conclude the proof of the proposition we have to estimate the cardinality of the set $\mathcal{C}$ of $M(\alpha)$-words $\bar{l}$ such that $F^{M(\alpha)}(\widehat{Y}|_{\omega(\bar{l})})$ meets $\widehat{J}(r_0(\alpha)-2)$. Note that no pair $\bar{l}$ and $\bar{l}'$ in $\mathcal{C}$ may be incompatible. We say that $\bar{l}, \bar{l}' \in \mathcal{C}$ are i-equivalent for some $1 \leq i \leq k_0(\alpha)$, if $l_j = l'_j$ for all $j < t_i$. Let $\mathcal{C}_i$ be the set of i-equivalence classes in $\mathcal{C}$. Obviously $\#\mathcal{C} \leq \#\mathcal{C}_{k_0(\alpha)} d^{M(\alpha)-t_{k_0(\alpha)}}$. To bound this last quantity, we make the remark that $\#\mathcal{C}_1 = 1$ and we show that

$$\#\mathcal{C}_{i+1} \leq \left(1 - \frac{4}{5}d^{-1}\right) d^{t_{i+1}-t_i} \#\mathcal{C}_i. \tag{16}$$

Indeed, consider for $1 \leq i < k_0(\alpha)$, an arbitrary i-equivalence class Q in $\mathcal{C}_i$. Split $Q = Q_* \cup Q_*^c$ with Q_* the collection of those $\bar{l} \in Q$ such that $(l_{t_i+1}, \ldots, l_{t_i+m_0}) \in \Omega_*(\bar{l}_{t_i})$ (notice that this condition depends both on $\bar{l}_{t_i}$ and $(l_{t_i+1}, \ldots, l_{t_i+m_0})$).

We first consider the number of $(i+1)$-equivalence classes into which Q_* splits. We count the number of distinct $\bar{l}_{t_{i+1}-1}$ for $\bar{l} \in Q_*$. We first choose l_{t_i} arbitrarily, then we choose $(l_{t_i+1}, \ldots, l_{t_i+m_0}) \in \Omega_*(\bar{l}_{t_i})$. Finally we choose $l_{t_i+m_0+1}, \ldots, l_{t_{i+1}-1}$ arbitrarily. This gives

$$d \times \frac{1}{5d} d^{m_0} \times d^{t_{i+1}-t_i-m_0-1} = \frac{1}{5d} d^{t_{i+1}-t_i}$$

choices according to eq. (14).

We similarly consider the number of $(i+1)$-equivalence classes into which $Q \setminus Q_*$ splits. We have at most d^{m_0} possibilities for $(l_{t_i+1}, \dots, l_{t_i+m_0})$. Then there is at most $d-1$ possibilities for l_{t_i} by claim 2. Finally we choose $l_{t_i+m_0+1}, \dots, l_{t_{i+1}-1}$ arbitrarily. This gives

$$d^{m_0} \times (d-1) \times d^{t_{i+1}-t_i-m_0-1} = (1-d^{-1})d^{t_{i+1}-t_i}.$$

Summing the two estimates for Q_* and $Q \setminus Q_*$ we obtain

$$\left(1 - \frac{4}{5}d^{-1}\right) d^{t_{i+1}-t_i}$$

This implies the inequality (16).

Iterating (16), we obtain

$$\begin{aligned}\#\mathcal{C} &\le \left(\prod_{i=1}^{k_0(\alpha)-1} (1-(4/5)d^{-1})(d^{t_{i+1}-t_i}) \right) d^{M(\alpha)-t_{k_0(\alpha)}} \\ &\le d^{M(\alpha)} \left(1-(4/5d)\right)^{k_0(\alpha)-1}.\end{aligned}$$

Therefore we obtain

$$m(\{\theta \in S^1 | \ \widehat{Y}_{M(\alpha)}(\theta) \in \widehat{J}(r_0(\alpha)-2)\}) \le d^{-M(\alpha)} \#\mathcal{C} \le \left(1-(4/5d)\right)^{k_0(\alpha)-1}.$$

As $k_0(\alpha) \ge \gamma r_0(\alpha)$ from claim 1, this concludes the proof of Proposition 5.2. □

6 Conclusion

Corollary 5.4 and Proposition 5.2 we proved above are the counterpart to Viana's Corollary 2.3 and Lemma 2.6 in [5], which were the only part in [5] where the condition $d \ge 16$ was used. In order to obtain the claim of Theorem 1.1 on the vertical Lyapunov exponent, we have only to follow the argument in section 2.4 of [5] with the following 'word for word' substitutions:

- Substitute the letter φ by F.
- Substitute Corollary 2.3 and Lemma 2.6 by our Corollary 5.4 and Proposition 5.2.
- Put $l = 2m - M(\alpha)$ instead of $l = m - M(\alpha)$ in the beginning and substitute m by $2m$ in the definition of G_q and the part below it.

- Substitute the expression $\alpha(d-\alpha)^{-l} \ll \sqrt{\alpha}e^{-m}$ on line 15 page 75 by $2A_0(2\pi/d)\alpha d^{-l} \ll \sqrt{\alpha}e^{-m}$

- The expression (16) is replaced by

$$m(B_2) \leq nC_1 \left(\frac{|J(m-1)|}{\alpha}\right)^{1/\rho} \leq \text{const } n(\sqrt{\alpha}e^{-m})^{1/\rho} \leq \text{const } e^{-\sqrt{n}/2\rho}.$$

Note that Lemmas 2.4 and 2.5 in [5] remain valid including their proof. □

The estimates in this argument are also enough to apply verbatim the proof of J.F. Alvès [1] for the existence of an a.c.i.m. This concludes the proof of our theorem.

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