# Absence of transport in Anderson localization

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**Abstract**

We consider the charge transport in the tight-binding Anderson model. Under a mild condition on the Fermi projection, we show that it is zero almost surely. This result has wider applicability than our previous work[12], while the definition of charge transport is slightly different. It also applies to the computation of non-diagonal component of the conductivity tensor which recovers the famous result of quantization of Hall conductivity in quantum Hall systems.

## 1 Introduction

Since the pioneering work of Anderson[5], where he discussed that a certain disorder may cause materials to have insulating property, there have been much development in the theory of Anderson localization. From mathematical point of view, this phenomenon is represented as the exponential decay of Green's functions which implies the exponential decay of eigenfunctions, and, together with Kubo formula, vanishing of the electrical conductivity (*e.g.,* [1, 3, 9, 16]). On the other hand, Bellissard *et.al.* [8, 13, 14] developed a theory to compute the electrical conductivity by using $C^*$-algebraic approach. They used the random kick-rotor term to introduce dissipation mechanism. In this paper, we study essentially the same quantity, which we call charge transport to compare it with that treated in [12]. We do not use dissipation term, which turns out not to be necessary to consider in the study of Anderson localization, and develop the theory by the use of spectral theory and functional analysis. This approach has the advantage of being mathematically simple, while it would not be straightforward to extend to more general systems, without suitable improvements in the analysis. We will show that, when the Fermi energy lies in the "localized states", then the charge transport is equal to zero almost surely. This fact has a clear physical interpretation: the current vanishes in the localized regime. However, to prove this fact seems not to be trivial as discussed in [12].

Our model is the standard, tight-binding random Hamiltonian on $l^2(\mathbb{Z}^d)$:

$$(H_\omega\varphi)(x) := \sum_{|y-x|=1} \varphi(y) + \lambda V_\omega(x)\varphi(x), \quad \varphi \in l^2(\mathbb{Z}^d). \tag{1.1}$$

$\lambda > 0$ is the coupling constant, and $\{V_\omega(x)\}_{x\in\mathbb{Z}^d}$ are independent, identically distributed random variables on a probability space $(\Omega, \mathcal{B}, \mathbf{P})$. We assume

(1) the probability distribution of $V_\omega(0)$ has the density $r(v)dv$. supp $r$ is bounded from below, and $r \in L^p(\mathrm{I\!R})$ for some $1 < p \leq \infty$.

(2) $\mathbf{E}|V_\omega(0)|^2 < \infty$, where $\mathbf{E}$ stands for taking expectations.

The assumption that supp $r$ is bounded from below is purely technical to use almost analytic continuation (section 2, Proposition 2.8). However, this assumption is satisfied in most situations, and is natural to study the conducting properties. The condition $\mathbf{E}|V_\omega(0)|^2 < \infty$ is assumed in order that the Liouville operator of $H_{\mathcal{E},\omega}$ is densely defined.

It is known that, the spectrum of $H_\omega$ is deterministic almost surely, and $\sigma(H_\omega) = [-2d, 2d] + \text{supp } r$, for a.e. $\omega$[10]. Moreover, there appears some region in the spectrum depending on $\lambda$ where we have pure point spectrum with exponentially decaying eigenfunctions which, as we already mentioned, is a mathematical presentation of Anderson localization (*e.g.*,[1, 3, 9, 16]). In [12], we considered the charge transport caused by the slowly varying potential. We used adiabatic approximation, and proved that this quantity is equal to zero almost surely when $\lambda$ is sufficiently large (*i.e.,* at high disorder). In this paper, we consider the charge transport caused by the discrete analogue of the constant electric field $\mathcal{E}(\in \mathrm{I\!R})$

$$H_{\omega,\mathcal{E}} := H_\omega + \mathcal{E}\cdot x_1, \tag{1.2}$$

where $x_1$ is the first component of $x = (x_1, x_2, \cdots, x_d) \in \mathbb{Z}^d$. We define the charge transport by carrying out the following three operations successively: (1) take the thermal average of current operator at zero temperature, (2) divide it by $\mathcal{E}$ and let $\mathcal{E} \downarrow 0$, (3) take the time average. That is,

$$\sigma_1(\omega, \epsilon_F) := \lim_{T\uparrow\infty} \int_0^T \frac{dt}{T} \lim_{\mathcal{E}\downarrow 0} \frac{1}{\mathcal{E}} \mathcal{T}(e^{itH_{\omega,\mathcal{E}}} i[H_\omega, x_1] e^{-itH_{\omega,\mathcal{E}}} P_{\epsilon_F}), \tag{1.3}$$

where $\mathcal{T}$ is the trace per volume for an operator $A$:

$$\mathcal{T}(A) := \lim_{L\uparrow\infty} \frac{1}{|\Lambda_L|} \text{trace}(\chi_{\Lambda_L} A \chi_{\Lambda_L}). \tag{1.4}$$

$\chi_C$ is the characteristic funtion of a set $C$, and $\Lambda_L$ is a finite box in $\mathbb{Z}^d$:

$$\Lambda_L := \{x \in \mathbb{Z}^d : |x_j| \le L, j = 1, \cdots, d\}.$$

$e^{-itH_{\omega,\mathcal{E}}}$ is the unitary propagator of $H_{\omega,\mathcal{E}}$, and $P_{\epsilon_F}$ is the Fermi projection operator: $\chi_{(-\infty,\epsilon_F]}(H_\omega)$, corresponding to the Fermi energy $\epsilon_F \in \mathbb{R}$ which we fix arbitrary. In the definition of $\sigma_1(\omega, \epsilon_F)$, it might be more natural if we took $\lim_{\mathcal{E}\downarrow 0} \frac{1}{\mathcal{E}}$ after taking the time average $\lim_{T\uparrow\infty} \int_0^T \frac{dt}{T}$. However, we can only deal with the Abel limit of that, which can be regarded as a sort of approximation.

$$(1.5)\ \sigma_2(\omega, \epsilon_F) := \lim_{\delta\downarrow 0} \lim_{\mathcal{E}\downarrow 0} \frac{1}{\mathcal{E}} \int_0^\infty dt\, \delta e^{-t\delta} \mathcal{T}(e^{itH_{\omega,\mathcal{E}}} i[H_\omega, x_1] e^{-itH_{\omega,\mathcal{E}}} P_{\epsilon_F}),$$

**Remark 1.1** *(1) $\sigma_1(\omega, \epsilon_F)$ was originally defined in [8]. Instead of studying $\sigma_1(\omega, \epsilon_F)$, they computed $\sigma_2(\omega, \epsilon_F)$ by using the relaxation time approximation.*

*(2) We are not able to discuss the deep question that whether the discrete analogue of the constant electric field $\mathcal{E} \cdot x_1$ would be appropriate to compute the electrical conductivity [7, 15]. In fact, the spectrum of $H_{\omega,\mathcal{E}}$ is known to be purely discrete. However, we believe that taking $\mathcal{E} \downarrow 0$ limit $\lim_{\mathcal{E}\downarrow 0} \frac{1}{\mathcal{E}}$ is not trivial in any case.*

*(3) In non-zero temperature, $P_{\epsilon_F}$ is replaced by $f_{\beta,\epsilon_F}(H_\omega)$, where $f_{\beta,\mu}(\lambda)$ is the Fermi-Dirac distribution function*

$$f_{\beta,\mu}(\lambda) := \Big(1 + \exp[\beta(\lambda - \mu)]\Big)^{-1}, \quad \lambda \in \mathbb{R},$$

*and $\beta > 0$ is the inverse temperature.*

Throughout this paper, we assume the following decaying property of the matrix element of the Fermi projection.

**Assumption**

$$(1.6) \qquad \mathbf{E} \sum_{x\in\mathbb{Z}} |x_1|^2 |P_{\epsilon_F}(\omega\,; x, 0)|^2 < \infty.$$

$P_{\epsilon_F}(\omega\,; x, y) :=< \delta_x, P_{\epsilon_F}\delta_y >_{l^2(\mathbb{Z}^d)}$, where $< \cdot, \cdot >_{l^2(\mathbb{Z}^d)}$ is the inner product on $l^2(\mathbb{Z}^d)$, and $\delta_x(z) = 1$, if $z = x$, and $\delta_x(z) = 0$, if $z \neq x$. It is shown that [2], (1.6) follows from the fractional moment bound of the Green's function

$$\mathbf{E}|G_{\epsilon_F+i\epsilon}(\omega\,;x,y)|^s \le Ce^{-\mu|x-y|}, \quad x,\ y \in \mathbb{Z}^d, \tag{1.7}$$

for some constants $0 < s < 1$, $C > 0$, $\mu > 0$, uniformly w.r.t. $\epsilon \in \mathbb{R}$. $G_E(\omega\,;x,y) :=< \delta_x, (H_\omega - E)^{-1}\delta_y >_{l^2(\mathbb{Z}^d)}$ is the matrix element of Green's function. (1.7) is one of the key observation of Anderson localization [1, 3], and known to hold if (1) $\lambda > 0$ is sufficiently large (high disorder case), or (2) $|\epsilon_F|$ is sufficiently large (extreme energy case), or (3) $\epsilon_F$ is away from the spectrum of free Laplacian for $|\lambda|$ small (weak disorder case). Under the above assumption(1.6), we have following results.

**Theorem 1** $\sigma_2(\omega, \epsilon_F) = 0, \quad a.e.\ \omega.$

**Theorem 2** $\sigma_1(\omega, \epsilon_F) = 0, \quad a.e.\ \omega.$

**Remark 1.2** *(1) We can extend Theorem 2 to some long-range hopping Hamiltonians. However, it seems not to be a trivial task to extend the resolvent equation (Proposition 3.1) to those Hamiltonians, which is important to prove Theorem 1.*

*(2) It would require further studies to see under which conditions the various limits involved in the definitions of $\sigma_1(\omega, \epsilon_F)$, $\sigma_2(\omega, \epsilon_F)$ may be interchanged. However, in Theorem 1, the exchange of two operations, $\lim_{\mathcal{E}\downarrow 0} \frac{1}{\mathcal{E}}$ and $\int_0^\infty dt\, \delta e^{-\delta t}$ is allowed (Remark 4.1) if, in addition, we assume the following dynamical localization estimate*

$$\mathbf{E}|e^{-itH_\omega}(\omega\,;x,y)| \le Ce^{-\mu|x-y|}, \quad x,\ y \in \mathbb{Z}^d, \tag{1.8}$$

*for some constants $C > 0$, $\mu > 0$, independent of $t \in \mathbb{R}$. (1.8) is known to be true when $\lambda > 0$ is sufficiently large (high disorder case[2, 4]).*

*(3) In [12], essentially the same result is proved under more restricted situations (i.e., bounded random potential, bounded electrical potential, and high disorder). However, there are some differences between the definition of $\sigma_1(\omega, \epsilon_F)$ and the charge transport defined in [12]. For instance, in the definition of $\sigma_1(\omega, \epsilon_F)$, we take $\mathcal{E} \downarrow 0$ limit first and then take the time average, while, in [12], we take these two operations at the same time in a sense. It would not be easy to take these two limits at the same time in the approach of this paper.*

*(4) In [8], they argue that the conductivity is always zero or infinity without taking dissipation mechanism into account. Moreover, they show that the conductivity is always zero if the Hamiltonian is bounded. Our results imply that this is also true in case Anderson localization happens, even when the Hamiltonian is unbounded. On the other hand, in [2], they discuss the derivation of Kubo formula, and obtained the same quantity as what we will have in the proof of above two theorems.*

Theorem 1, 2 are still true when we impose constant magnetic field to $H_\omega$. Moreover, the above results have some implications to the theory of integral Hall effect. Let $d = 2$, and we consider the following Hamiltonian which has constant magnetic flux $-B(\in \mathbb{R})$ penetrating each plaquettes on $\mathbb{Z}^2$.

$$(H_\omega^B \varphi)(x) := \sum_{|y-x|=1} h_B(x,y)\varphi(y) + \lambda V_\omega(x)\varphi(x), \quad \varphi \in l^2(\mathbb{Z}^2),$$

where,

$$h_B(x,y) := \begin{cases} e^{\frac{iB}{2}<y,x-y>_{\mathbb{R}^2}}, & \text{if } |x-y| = 1, \\ 0, & \text{otherwise,} \end{cases} \qquad x, y \in \mathbb{Z}^2.$$

$< \cdot, \cdot >_{\mathbb{R}^d}$ is the inner product on $\mathbb{R}^d$. We define the Hall conductivity $\sigma_1^H(\omega, \epsilon_F)$, $\sigma_2^H(\omega, \epsilon_F)$, by replacing $x_1$ by $x_2$ in the definition of $\sigma_1(\omega, \epsilon_F)$, $\sigma_2(\omega, \epsilon_F)$, respectively. We note that dissipation mechanism is not necessary to consider in the definition of the Hall conductivity. Under Assumption (1.6), we have the following result which can be proved by mimicking the proof of Theorem 1, 2.

**Theorem 3**
(1) $\sigma_1(\omega, \epsilon_F) = \sigma_2(\omega, \epsilon_F) = 0, \quad a.e.\ \omega.$
(2) $\sigma_1^H(\omega, \epsilon_F) = \sigma_2^H(\omega, \epsilon_F) = i\mathcal{T}(P_{\epsilon_F}[\partial_1 P_{\epsilon_F}, \partial_2 P_{\epsilon_F}]P_{\epsilon_F}), \quad a.e.\ \omega,$
where, $\partial_j P_{\epsilon_F} := [x_j, P_{\epsilon_F}]$, $j = 1, 2$.

The quantity $i\mathcal{T}(P_{\epsilon_F}[\partial_1 P_{\epsilon_F}, \partial_2 P_{\epsilon_F}]P_{\epsilon_F})$ is known to have an interpretation as a "topological invariant" ([6, 8] and references therein). Therefore, Theorem 3 implies the vanishment of the direct conductivity and the quantization of the Hall conductivity if the Fermi energy lies in the localized regime, which is well-known in the theory of integral quantum Hall effect.

The key ingredient of the proof of Theorem 1, 2 is the theory developed in [2, 8], namely to consider the Hilbert space (called $\mathcal{L}^2$) which consists of matrix elements of operators, and consider the Liouville operator of $H_\omega$. In section 2, we define $\mathcal{L}^2$, and study some basic properties. We provide proofs even for what is elementary, because it would be better to be careful in the treatment of Liouville operators for unbounded Hamiltonians. In section 3, 4, we prove Theorem 1, 2, respectively. The main part of proofs is to study $\mathcal{E} \downarrow 0$ limit: $\lim_{\mathcal{E}\downarrow 0} \frac{1}{\mathcal{E}}$. In order for that, we prove resolvent equation (for Theorem 1), and Duhamel formula (for Theorem 2), respectively.

## 2 Preliminaries

In this section, we define a Hilbert space $\mathcal{L}^2$, study its basic properties, and consider some operators on it. First of all, we notice that, the operator $H_\omega$ is covariant w.r.t. the translation in the following sense

$$U(a)H_\omega U(a)^* = H_{T^a\omega}, \quad a \in \mathbb{Z}^d,\ \omega \in \Omega, \tag{2.1}$$

where $U(a)$ is the translation operator on $l^2(\mathbb{Z}^d)$: $(U(a)\varphi)(x) := \varphi(x - a)$, $U(a)^*$ is the adjoint operator of $U(a)$, and $T^a : \Omega \to \Omega$ is a measure preserving map defined by $\omega = \{V_\omega(x)\}_{x\in\mathbb{Z}^d} \mapsto T^a\omega = \{V_\omega(x-a)\}_{x\in\mathbb{Z}^d}$. It is well-known that $T^a$ is ergodic. Because of (2.1), the matrix element of $H_\omega$ satisfies the following relation

$$H(\omega\,; x, y) = H(T^{-a}\omega\,; x-a, y-a), \quad a, x, y \in \mathbb{Z}^d,\ \omega \in \Omega. \tag{2.2}$$

We say that a function $H(\omega\,; x, y)$ on $\Omega \times \mathbb{Z}^d \times \mathbb{Z}^d$ satisfies (CR) if (2.2) holds. We define $\mathcal{L}^2$ as the space of functions on $\Omega \times \mathbb{Z}^d \times \mathbb{Z}^d$ which satisfies (CR) and has a certain decaying property.

$$\begin{aligned} \mathcal{L}^2 := \{A(\omega\,; x, y) :\ & \text{function on } \Omega \times \mathbb{Z}^d \times \mathbb{Z}^d \text{ which is finite a.e.}, \\ & (1) \quad A(\omega\,; x, y) \text{ satisfies (CR)}, \\ & (2) \quad \|A\|^2_{\mathcal{L}^2} := \int_\Omega d\mathbf{P} \sum_{x\in\mathbb{Z}^d} |A(\omega\,; x, 0)|^2 < \infty\}. \end{aligned}$$

**Remark 2.1** *In the above definition, "a.e." is in the sense of $\tilde{\mathbf{P}}$-a.e., where $\tilde{\mathbf{P}}$ is the product measure of $\mathbf{P}$ and the counting measure on $\mathbb{Z}^d \times \mathbb{Z}^d$. Since $\mathbb{Z}^d \times \mathbb{Z}^d$ is countable, "$A(\omega\,; x, y)$ is finite for $\tilde{\mathbf{P}}$-a.e." is equivalent to the* *statement that "there exists a subset* $\Omega_0 \subset \Omega$ *with* $\mathbf{P}(\Omega_0) = 1$*, such that* $A(\omega\,; x, y)$ *is finite for* $\omega \in \Omega_0$, $x$, $y \in \mathbb{Z}^d$*".*

We shall see there are many functions which belong to $\mathcal{L}^2$. For this purpose, we consider a subspace of $\mathcal{L}^2$.

$$\begin{aligned} \mathcal{A} := \{\{A_\omega\}_{\omega\in\Omega} \ : \quad & \text{family of bounded operators on } l^2(\mathbb{Z}^d), \\ & (1)\, U(a) A_\omega U(a)^* = A_{T^a\omega}, \quad \omega \in \Omega,\ a \in \mathbb{Z}^d, \\ & (2) \sup_{\omega\in\Omega} \|A_\omega\|_{op} < \infty\}. \end{aligned}$$

$\|A\|_{op}$ is the operator norm on $l^2(\mathbb{Z}^d)$.

**Remark 2.2** *If* $A_\omega \in \mathcal{A}$*, then its matrix element* $A(\omega; x, y)$ *belongs to* $\mathcal{L}^2$*. In fact, since* $A_\omega$ *is bounded on* $l^2(\mathbb{Z}^d)$*,* $A(\omega\,; x, y)$ *is finite for any* $(\omega, x, y) \in \Omega \times \mathbb{Z}^d \times \mathbb{Z}^d$*. The condition (1) in the definition of* $\mathcal{A}$ *implies that* $A(\omega\,; x, y)$ *satisfies (CR). To check the condition (2) in the definition of* $\mathcal{L}^2$*, we write* $A(\omega\,; x, 0) = (A_\omega \delta_0)(x)$*, and since* $A_\omega$ *is bounded on* $l^2(\mathbb{Z}^d)$*, and* $\|\delta_0\|_{l^2(\mathbb{Z}^d)} = 1$ *(*$\|\cdot\|_{l^2(\mathbb{Z}^d)}$ *is* $l^2(\mathbb{Z}^d)$*-norm),*

$$\begin{aligned} \sum_{x\in\mathbb{Z}^d} |A(\omega\,; x, 0)|^2 &= \|(A_\omega\delta_0)(x)\|^2_{l^2(\mathbb{Z}^d)} \\ &\le \|A_\omega\|^2_{op}. \end{aligned}$$

*The condition (2) in the definition of* $\mathcal{A}$ *now implies* $\|A\|_{\mathcal{L}^2} < \infty$*, and thus,* $A(\omega\,; x, y) \in \mathcal{L}^2$*. Therefore, we will sometimes abuse the notation, and regard* $\mathcal{A}$ *as a subspace of* $\mathcal{L}^2$*, though the former is a space of operators while the latter is that of functions.*

To introduce a Hilbert space structure on $\mathcal{L}^2$, we consider $\tilde{\mathcal{L}}^2 := L^2(\Omega \times \mathbb{Z}^d)$.

**Proposition 2.1** *We define a map* $\pi : \mathcal{L}^2 \to \tilde{\mathcal{L}}^2$ *by*

$$(\pi A)(\omega, x) := A(\omega\,; x, 0), \quad A \in \mathcal{L}^2.$$

*Then,* $\pi$ *is a bijection, and*

$$(\pi^{-1} a)(\omega\,; x, y) = a(T^{-y}\omega, x - y), \quad a \in \tilde{\mathcal{L}}^2.$$

*Proof.* It is clear that $\pi A \in \tilde{\mathcal{L}}^2$ for $A \in \mathcal{L}^2$. Conversely, let $(\pi^{-1}a)(\omega\,; x, y) := a(T^{-y}\omega, x-y)$, $a \in \tilde{\mathcal{L}}^2$. By Remark 2.1, $(\pi^{-1}a)(\omega\,; x, y)$ is finite a.e.

$$\begin{aligned}(\pi^{-1}a)(T^{-b}\omega\,; x-b, y-b) &= a(T^{-b}T^{-(y-b)}\omega, (x-b)-(y-b)) \\ &= a(T^{-y}\omega, x-y) \\ &= (\pi^{-1}a)(\omega\,; x, y), \quad x, y, b \in \mathbb{Z}^d.\end{aligned}$$

Hence $(\pi^{-1}a)(\omega\,; x, y)$ satisfies (CR). $\|\pi^{-1}a\|_{\mathcal{L}^2} = \|a\|_{\tilde{\mathcal{L}}^2} < \infty$. Therefore, $\pi^{-1}a \in \mathcal{L}^2$ for $a \in \tilde{\mathcal{L}}^2$. Then, it suffices to show $\pi^{-1} \circ \pi$ and $\pi \circ \pi^{-1}$ are equal to identities on $\mathcal{L}^2$ and $\tilde{\mathcal{L}}^2$ respectively, which follows from direct computations. $\Box$

We introduce an inner product on $\mathcal{L}^2$:

$$< A, B >_{\mathcal{L}^2} := < \pi(A), \pi(B) >_{\tilde{\mathcal{L}}^2}$$

$\mathcal{L}^2$ becomes a Hilbert space under the above inner product.

**Remark 2.3** *To see $\mathcal{L}^2$ is complete, we let $\{A_n\}_{n=1}^{\infty} \subset \mathcal{L}^2$ be a Cauchy sequence. Then $\pi(A_n)(\omega, x) = A_n(\omega\,; x, 0)$ is a Cauchy sequence in $\tilde{\mathcal{L}}^2$, and thus, there exists an element $a(\omega, x) \in \tilde{\mathcal{L}}^2$, such that $\pi(A_n) \to a$ in $\tilde{\mathcal{L}}^2$ as $n \to \infty$. We define $A := \pi^{-1}(a) \in \mathcal{L}^2$, then we have $A_n \to A$ in $\mathcal{L}^2$ as $n \to \infty$. We regard $\mathcal{L}^2$ as a complete metric space in this sense.*

For $A$, $B \in \mathcal{L}^2$, we define a product $AB$ which is a function on $\Omega \times \mathbb{Z}^d \times \mathbb{Z}^d$:

$$(AB)(\omega\,; x, y) := \sum_{z \in \mathbb{Z}^d} A(\omega\,; x, z) B(\omega\,; z, y). \tag{2.3}$$

If $a = \pi(A)$, $b = \pi(B)$, it is written as

$$\begin{aligned}\pi(AB)(\omega, x) &= (AB)(\omega\,; x, 0) \\ &= \sum_{z \in \mathbb{Z}^d} a(T^{-z}\omega, x-z) b(\omega, z).\end{aligned} \tag{2.4}$$

**Lemma 2.1** *The sum in (2.3) absolutely converges for a.e. $\omega$. Moreover, for any fixed $(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d$, $|A(\omega\,; x, z)||B(\omega\,; z, y)| \in L^1(\Omega \times \mathbb{Z}^d)$ as a function of $(\omega, z) \in \Omega \times \mathbb{Z}^d$, and $(AB)(\omega\,; x, y) \in L^1(\Omega)$.*

*Proof.*

$$|A(\omega\,;x,z)||B(\omega\,;z,y)| \leq \frac{1}{2}\Big(|A(\omega\,;x,z)|^2 + |B(\omega\,;z,y)|^2\Big). \tag{2.5}$$

On the other hand,

$$\begin{aligned} \int_\Omega d\mathbf{P} \sum_{z\in\mathbb{Z}^d} |A(\omega\,;x,z)|^2 &= \int_\Omega d\mathbf{P} \sum_{z\in\mathbb{Z}^d} |A(T^{-z}\omega\,;x-z,0)|^2 \\ &= \int_\Omega d\mathbf{P} \sum_{z\in\mathbb{Z}^d} |A(\omega\,;x-z,0)|^2 \\ &= \int_\Omega d\mathbf{P} \sum_{z\in\mathbb{Z}^d} |A(\omega\,;z,0)|^2 \\ &= \|A\|^2_{\mathcal{L}^2}. \end{aligned} \tag{2.6}$$

We used (CR) property of $A$ and Fubini's theorem. Similarly, we have $\int_\Omega d\mathbf{P} \sum_{z\in\mathbb{Z}^d} |B(\omega\,;z,y)|^2 = \|B\|^2_{\mathcal{L}^2} < \infty$, which together imply the second statement of Lemma 2.1. Moreover, (2.6) implies $\sum_{z\in\mathbb{Z}^d} |A(\omega\,;x,z)|^2 < \infty$, and similarly, $\sum_{z\in\mathbb{Z}^d} |B(\omega\,;z,y)|^2 < \infty$, for a.e. $\omega$. For such $\omega \in \Omega$, the sum of the RHS of (2.5) w.r.t. $z \in \mathbb{Z}^d$ converges, and

$$\sum_{z\in\mathbb{Z}^d} |A(\omega\,;x,z)||B(\omega\,;z,y)| \leq \frac{1}{2} \sum_{z\in\mathbb{Z}^d} \Big(|A(\omega\,;x,z)|^2 + |B(\omega\,;z,y)|^2\Big). \tag{2.7}$$

The RHS of (2.7) $\in L^1(\Omega)$, and thus, $(AB)(\omega\,;x,y) \in L^1(\Omega)$. □

By using Lemma 2.1, we can show that the trace per volume $\mathcal{T}(AB)$ exists almost surely if $A$, $B \in \mathcal{L}^2$.

**Proposition 2.2** *(1) If $A$, $B \in \mathcal{L}^2$,*

$$\mathcal{T}(AB) = \int_\Omega d\mathbf{P}(AB)(\omega\,;0,0), \quad \text{for a.e. } \omega.$$

*In particular, $\mathcal{T}(AB)$ exists and finite for a.e. $\omega$.*
*(2) $\mathcal{T}(AB) = \mathcal{T}(BA)$, for a.e. $\omega$.*

*Proof.*
(1) By Lemma 2.1, $(AB)(\omega\,;x,y)$ is finite for a.e. $\omega$. For such $\omega \in \Omega$, it is clear that trace$(\chi_{\Lambda_L} AB \chi_{\Lambda_L})$ makes sense which is simply the usual trace

of $(2L+1)^d \times (2L+1)^d$ matrix $\{(AB)(\omega\,;x,y)\}_{x,y\in\Lambda_L}$.

$$\begin{aligned}\mathcal{T}(AB) &= \lim_{L\uparrow\infty}\frac{1}{|\Lambda_L|}\mathrm{trace}(\chi_{\Lambda_L}AB\chi_{\Lambda_L}) \\ &= \lim_{L\uparrow\infty}\frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_L}(AB)(\omega\,;x,x).\end{aligned} \tag{2.8}$$

On the other hand, by using (CR) property of $\mathcal{L}^2$,

$$\begin{aligned}(AB)(\omega\,;x,x) &= \sum_{y\in\mathbb{Z}^d} A(\omega\,;x,y)B(\omega\,;y,x) \\ &= \sum_{y\in\mathbb{Z}^d} A(T^{-x}\omega\,;0,y-x)B(T^{-x}\omega\,;y-x,0) \\ &= (AB)(T^{-x}\omega\,;0,0).\end{aligned} \tag{2.9}$$

By Lemma 2.1, the RHS of (2.9)$\in L^1(\Omega)$. Therefore, by Birkhoff's ergodic theorem,

$$\mathcal{T}(AB) = \int_\Omega d\mathbf{P}(AB)(\omega\,;0,0), \quad \text{for a.e. } \omega.$$

(2) For a.e. $\omega$,

$$\begin{aligned}\mathcal{T}(AB) &= \int_\Omega d\mathbf{P}(AB)(\omega\,;0,0) \\ &= \int_\Omega d\mathbf{P}\sum_{y\in\mathbb{Z}^d} A(\omega\,;0,y)B(\omega\,;y,0) \\ &= \int_\Omega d\mathbf{P}\sum_{y\in\mathbb{Z}^d} B(T^y\omega\,;0,y)A(T^y\omega\,;y,0).\end{aligned} \tag{2.10}$$

By Lemma 2.1, $|B(\omega\,;0,y)||A(\omega\,;y,0)| \in L^1(\Omega\times\mathbb{Z}^d)$, so that we can use Fubini's theorem.

$$\begin{aligned}\text{RHS of (2.10)} &= \sum_{y\in\mathbb{Z}^d}\int_\Omega d\mathbf{P}B(\omega\,;0,y)A(\omega\,;y,0) \\ &= \int_\Omega d\mathbf{P}\sum_{y\in\mathbb{Z}^d} B(\omega\,;0,y)A(\omega\,;y,0) \\ &= \mathcal{T}(BA).\end{aligned}$$

$\square$

We have the following fact, which will make it easier to study unbounded operators on $\mathcal{L}^2$.

**Proposition 2.3** *$\mathcal{A}$ is dense in $\mathcal{L}^2$.*

*Proof.* Let $A \in \mathcal{L}^2$, $a := \pi(A) \in \tilde{\mathcal{L}}^2$. Define

$$a_n(\omega, x) := \begin{cases} a(\omega, x), & (\text{if } |x| \le n \text{ and } |a(\omega, x)| \le n), \\ 0, & (\text{otherwise}), \end{cases}$$

$$A_n := \pi^{-1}(a_n).$$

Then $a_n \to a$, and $A_n \to A$ in $\tilde{\mathcal{L}}^2$ and $\mathcal{L}^2$ respectively as $n \to \infty$. We will show $A_n \in \mathcal{A}$ which completes the proof. For $\varphi$, $\psi \in l^2(\mathbb{Z}^d)$, we compute, by Schwarz inequality and the definition of $\pi^{-1}$,

$$(2.11) \quad | < \varphi, A_n \psi >_{l^2(\mathbb{Z}^d)} | \le \sum_{x,y \in \mathbb{Z}^d} |\varphi(x)||A_n(\omega\,; x, y)||\psi(y)|$$

$$\le \Big( \sum_{x,y \in \mathbb{Z}^d} |\varphi(x)|^2 |A(\omega\,; x, y)| \Big)^{1/2} \Big( \sum_{x,y \in \mathbb{Z}^d} |A(\omega\,; x, y)||\psi(y)|^2 \Big)^{1/2}$$

$$= \Big( \sum_{x \in \mathbb{Z}^d} \sum_{y \in \mathbb{Z}^d} |\varphi(x)|^2 |a_n(T^{-y}\omega, x - y)| \Big)^{1/2} \times$$

$$\times \Big( \sum_{y \in \mathbb{Z}^d} \sum_{x \in \mathbb{Z}^d} |a_n(T^{-y}\omega, x - y)||\psi(y)|^2 \Big)^{1/2}.$$

We notice that, $|a_n(T^{-y}\omega, x - y)| \le n$, and $a_n(T^{-y}\omega, x - y) \ne 0$ only if $|x - y| \le n$. Therefore,

RHS of (2.11)

$$\le \Big( n(2n+1)^d \sum_{x \in \mathbb{Z}^d} |\varphi(x)|^2 \Big)^{1/2} \Big( n(2n+1)^d \sum_{y \in \mathbb{Z}^d} |\psi(y)|^2 \Big)^{1/2}$$

$$= C_n \|\varphi\|_{l^2(\mathbb{Z}^d)} \|\psi\|_{l^2(\mathbb{Z}^d)},$$

where the constant $C_n > 0$ is independent of $\omega \in \Omega$. Therefore, $A_n$ is bounded on $l^2(\mathbb{Z}^d)$, and $\sup_{\omega \in \Omega} \|A_n\|_{op} < \infty$. Because $A_n = \pi^{-1}(a_n)$, $A_n$ satisfies (CR). Hence $A_n \in \mathcal{A}$. □

It will be convenient to introduce another subspace of $\mathcal{L}^2$.

$$\begin{aligned} \mathcal{A}_0 := \{ A(\omega\,; x, y) \ : \quad & \text{function on } \Omega \times \mathbb{Z}^d \times \mathbb{Z}^d, \\ & (1)\ A \text{ satisfies (CR)}, \quad (2)\ |A(\omega\,; x, y)| \text{ is bounded}, \\ & (3) \text{ there exists a constant } K > 0 \text{ such that} \\ & \qquad A(\omega\,; x, y) = 0, \ \text{if } |x - y| > K \}. \end{aligned}$$

Then the proof of Proposition 2.3 also shows $\mathcal{A}_0 \subset \mathcal{A}$, and $\mathcal{A}_0$ is dense in $\mathcal{L}^2$.

From now on, we consider some operations on $\mathcal{L}^2$. For a function $A(\omega\,; x, y)$ on $\Omega \times \mathbb{Z}^d \times \mathbb{Z}^d$, we define $A^*(\omega\,; x, y)$ by

$$A^*(\omega\,; x, y) := \overline{A(\omega\,; y, x)},$$

which corresponds to the matrix element of the adjoint operator if $A \in \mathcal{A}$. The corresponding element in $\tilde{\mathcal{L}}^2$ is

$$a^* := \pi(A^*) = \overline{a(T^{-x}\omega, -x)},$$

where $a = \pi(A)$.

**Lemma 2.2** *If $A \in \mathcal{L}^2$, then $A^* \in \mathcal{L}^2$, and $\|A^*\|_{\mathcal{L}^2} = \|A\|_{\mathcal{L}^2}$.*

*Proof.* It is clear that $A^*(\omega\,; x, y)$ is finite a.e. That $A^*(\omega\,; x, y)$ satisfies (CR) follows from direct computation. The equality $\|A^*\|_{\mathcal{L}^2} = \|A\|_{\mathcal{L}^2} < \infty$ follows from (CR) and Fubini's theorem, which is the same argument in (2.6) in the proof of Lemma 2.1. $\square$

Due to Birkhoff's ergodic theorem, trace per volume is related to the inner product on $\mathcal{L}^2$.

**Proposition 2.4** *Let $A$, $B \in \mathcal{L}^2$.*

*(1) As functions on $\Omega \times \mathbb{Z}^d \times \mathbb{Z}^d$,*

$$(BA)^*(\omega\,; x, y) = (A^*B^*)(\omega\,; x, y), \quad \textit{for a.e. } \omega.$$

*(2) $\mathcal{T}(A^*B) =< A, B >_{\mathcal{L}^2}$ .*

*Proof.* (1) By definition,

$$\begin{aligned}(BA)^*(\omega\,; x, y) &= \overline{(BA)(\omega\,; y, x)} \\ &= \sum_{z \in \mathbb{Z}^d} \overline{B(\omega\,; y, z)}\,\overline{A(\omega\,; z, x)}.\end{aligned}$$

On the other hand,

$$A^*B^*(\omega\,; x, y) = \sum_{z \in \mathbb{Z}^d} A^*(\omega\,; x, z) B^*(\omega\,; z, y)$$

$$
\begin{aligned}
&= \sum_{z\in\mathbb{Z}^d} \overline{A(\omega\,;z,x)}\,\overline{B(\omega\,;y,z)} \\
&= (BA)^*(\omega\,;x,y).
\end{aligned}
$$

By Lemma 2.1, the above sums absolutely converge for a.e. $\omega$.

(2) By Proposition 2.2,

$$
\begin{aligned}
\mathcal{T}(A^*B) &= \int_\Omega d\mathbf{P} \sum_{x\in\mathbb{Z}^d} A^*(\omega\,;0,x)B(\omega\,;x,0) \\
&= \int_\Omega d\mathbf{P} \sum_{x\in\mathbb{Z}^d} \overline{A(\omega\,;x,0)}\,B(\omega\,;x,0),
\end{aligned}
$$

for a.e. $\omega$. Therefore, since $A(\omega\,;x,0) = \pi(A)$,

$$\mathcal{T}(A^*B) =< \pi(A), \pi(B) >_{\tilde{\mathcal{L}}^2} .$$

□

The following ideal property of $\mathcal{L}^2$ will be used frequently in this paper.

**Proposition 2.5** *(1) Let $A \in \mathcal{L}^2$, $B \in \mathcal{A}$. Since $B \in \mathcal{A} \subset \mathcal{L}^2$ (Remark 2.2), we can define products $BA$, $AB$ as functions on $\Omega \times \mathbb{Z}^d \times \mathbb{Z}^d$. Then $BA$, $AB \in \mathcal{L}^2$, and*

$$\|BA\|_{\mathcal{L}^2} = \|AB\|_{\mathcal{L}^2} \leq \sup_{\omega\in\Omega} \|B\|_{op}\|A\|_{\mathcal{L}^2}.$$

*Hence the multiplication operators on $\mathcal{L}^2$ defined by $A \mapsto BA$, $A \mapsto AB$ are linear and bounded.*

*(2) The adjoint operators of $A \mapsto BA$, $A \mapsto AB$ are $A \mapsto B^*A$, and $A \mapsto AB^*$ respectively.*

*Proof.* (1) By Lemma 2.1, $(BA)(\omega\,;x,y)$ and $(AB)(\omega\,;x,y)$ are finite a.e. It is easy to see $(BA)(\omega\,;x,y)$ and $(AB)(\omega\,;x,y)$ satisfy (CR). Since $\int_\Omega d\mathbf{P} \sum_{x\in\mathbb{Z}^d} |A(\omega\,;x,0)|^2 = \|A\|^2_{\mathcal{L}^2} < \infty$, $\sum_{x\in\mathbb{Z}^d} |A(\omega\,;x,0)|^2 < \infty$, for a.e. $\omega$, which implies $(A\delta_0)(x) = A(\omega\,;x,0) \in l^2(\mathbb{Z}^d)$ w.r.t. $x$ for a.e. $\omega$. For such $\omega \in \Omega$, $(BA)(\omega\,;x,0) = (BA\delta_0)(x) \in l^2(\mathbb{Z}^d)$, and

$$
\begin{aligned}
\sum_{x\in\mathbb{Z}^d} |BA(\omega\,;x,0)|^2 &= \|(BA\delta_0)(x)\|^2_{l^2(\mathbb{Z}^d)} \\
&\leq \|B\|^2_{op}\|(A\delta_0)(x)\|^2_{l^2(\mathbb{Z}^d)} \\
&\leq \sup_{\omega\in\Omega} \|B\|^2_{op} \sum_{x\in\mathbb{Z}^d} |A(\omega\,;x,0)|^2.
\end{aligned}
\tag{2.12}
$$

In (2.12), we recall that $B = \{B_\omega\}_{\omega\in\Omega}$, where $B_\omega$ is a bounded operator on $l^2(\mathbb{Z}^d)$ for any fixed $\omega \in \Omega$. Since the RHS of (2.12)$\in L^1(\Omega)$, we have $BA \in \mathcal{L}^2$, and $\|BA\|_{\mathcal{L}^2} \le \sup_{\omega\in\Omega}\|B\|_{op}\|A\|_{\mathcal{L}^2}$. By Lemma 2.2, $A^* \in \mathcal{L}^2$, and $\|A^*\|_{\mathcal{L}^2} = \|A\|_{\mathcal{L}^2}$. Since $B \in \mathcal{A}$, $B^*$ is the matrix element of the adjoint operator of $B$, if we regard $B$ as a bounded operator on $l^2(\mathbb{Z}^d)$. Moreover, $B^* \in \mathcal{A}$. By Proposition 2.4, $(AB)^*(\omega\,; x, y) = B^*A^*(\omega\,; x, y)$, as functions on $\Omega \times \mathbb{Z}^d \times \mathbb{Z}^d$. By what we have just shown, $B^*A^* \in \mathcal{L}^2$. Therefore, $(AB)^*$, and hence, $AB \in \mathcal{L}^2$. Since $\|B\|_{op} = \|B^*\|_{op}$, and $\|A\|_{\mathcal{L}^2} = \|A^*\|_{\mathcal{L}^2}$, we have

$$\begin{aligned}\|AB\|_{\mathcal{L}^2} &= \|(AB)^*\|_{\mathcal{L}^2}\\ &= \|B^*A^*\|_{\mathcal{L}^2}\\ &\le \sup_{\omega\in\Omega}\|B^*\|_{op}\|A^*\|_{\mathcal{L}^2}\\ &= \sup_{\omega\in\Omega}\|B\|_{op}\|A\|_{\mathcal{L}^2}.\end{aligned}$$

To prove (2) of Proposition 2.5, we prepare the following Lemma, which is not *a priori* trivial.

**Lemma 2.3** *Let $A$, $C \in \mathcal{L}^2$, and $B \in \mathcal{A}$. Then, as functions on $\Omega\times\mathbb{Z}^d\times\mathbb{Z}^d$,*

$$\Big((AB)C\Big)(\omega\,; x, y) = \Big(A(BC)\Big)(\omega\,; x, y), \quad a.e.\ \omega.$$

*Proof.* By Proposition 2.3, we can take sequence of bounded operators $\{A_n\}_{n=1}^\infty$, $\{C_n\}_{n=1}^\infty \subset \mathcal{A}$ which satisfy $A_n \to A$, $C_n \to C$ in $\mathcal{L}^2$ as $n \to \infty$. Then, by Proposition 2.5 (1), $A_nB \to AB$, $BC_n \to BC$, in $\mathcal{L}^2$ as $n \to \infty$. Moreover, by Schwarz inequality and (2.6),

$$\begin{aligned}\int_\Omega d\mathbf{P}\Big|\Big((AB)C\Big)(\omega\,; x, y)\Big| &\le \int_\Omega d\mathbf{P}\sum_{z\in\mathbb{Z}^d}|(AB)(\omega\,; x, z)||C(\omega\,; z, y)|\\ &\le \Big(\int_\Omega d\mathbf{P}\sum_{z\in\mathbb{Z}^d}|(AB)(\omega\,; x, z)|^2\Big)^{1/2}\times\\ &\qquad\times\Big(\int_\Omega d\mathbf{P}\sum_{z\in\mathbb{Z}^d}|C(\omega\,; z, y)|^2\Big)^{1/2}\\ &= \|AB\|_{\mathcal{L}^2}\|C\|_{\mathcal{L}^2}\\ &\le \sup_{\omega\in\Omega}\|B\|_{op}\|A\|_{\mathcal{L}^2}\|C\|_{\mathcal{L}^2}.\end{aligned}$$

Therefore, for any fixed $(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d$,

$$\Big((A_n B)C_n\Big)(\omega\,; x, y) \to \Big((AB)C\Big)(\omega\,; x, y),$$

as $n \to \infty$ in $L^1(\Omega)$. Since $\mathbb{Z}^d \times \mathbb{Z}^d$ is countable, by diagonal trick, we can take a subsequence $n' = n'(k)$ such that

$$\Big((A_n B)C_n\Big)(\omega\,; x, y) \to \Big((AB)C\Big)(\omega\,; x, y),$$

pointwise for a.e. $\omega$, as $k \to \infty$, for any $x$, $y \in \mathbb{Z}^d$. Similarly, by taking a subsequence further: $n'' = n''(n'(k))$,

$$\Big(A_n(BC_n)\Big)(\omega\,; x, y) \to \Big(A(BC)\Big)(\omega\,; x, y),$$

pointwise for a.e. $\omega$, as $k \to \infty$. Therefore, it is sufficient to show

$$\Big((A_n B)C_n\Big)(\omega\,; x, y) = \Big(A_n(BC_n)\Big)(\omega\,; x, y),$$

pointwise, which is clear. In fact, since $A_n$, $B$, $C_n$ are bounded on $l^2(\mathbb{Z}^d)$,

$$\begin{aligned}\Big((A_n B)C_n\Big)(\omega\,; x, y) &= <\delta_x, A_n B C_n \delta_y>_{l^2(\mathbb{Z}^d)} \\ &= \Big(A_n(BC_n)\Big)(\omega\,; x, y).\end{aligned}$$

$\square$

*Proof of Proposition 2.5 (2)*

Let $A$, $C \in \mathcal{L}^2$, and $B \in \mathcal{A}$. By Proposition 2.4,

$$\begin{aligned}<BA, C>_{\mathcal{L}^2} &= \mathcal{T}((BA)^* C) \\ &= \mathcal{T}((A^* B^*)C).\end{aligned}$$

By Proposition 2.2, and Lemma 2.3,

$$\begin{aligned}\mathcal{T}((A^* B^*)C) &= \int_\Omega d\mathbf{P}\Big((A^* B^*)C\Big)(\omega\,; 0, 0) \\ &= \int_\Omega d\mathbf{P}\Big(A^*(B^* C)\Big)(\omega\,; 0, 0) \\ &= \mathcal{T}(A^*(B^* C)) \\ &= <A, B^* C>_{\mathcal{L}^2}.\end{aligned}$$

Therefore, the adjoint of the linear bounded operator $A \mapsto BA$ on $\mathcal{L}^2$ is the map $A \mapsto B^*A$. The adjoint operator of $A \mapsto AB$ can be derived similarly. $\square$

We are going to study unbounded operators on $\mathcal{L}^2$. The following operator corresponds to taking commutator with $x_1$.

**Proposition 2.6** *For $A \in \mathcal{L}^2$, we define*

$$(\partial_1 A)(\omega\,; x, y) := (x_1 - y_1)A(\omega\,; x, y).$$

*Then, the operator $A \mapsto \partial_1 A$ is self-adjoint with domain*

$$\mathcal{D}(\partial_1) := \{A \in \mathcal{L}^2 \,:\, \partial_1 A \in \mathcal{L}^2\}.$$

Therefore, Assumption (1.6) is equivalent to the statement that $P_{\epsilon_F} \in \mathcal{D}(\partial_1)$.

*Proof.* Let $a := \pi(A)$. Then, $(\pi(\partial_1 A))(\omega, x) = x_1 a(\omega, x)$. Thus, the operator $A \mapsto \partial_1 A$ corresponds to the multiplication operator of $x_1$ on $\tilde{\mathcal{L}}^2$, which is self-adjoint with domain $\{a \in \tilde{\mathcal{L}}^2 \,:\, x_1 a \in \tilde{\mathcal{L}}^2\}$. $\square$

Next, we consider an operator on $\mathcal{L}^2$ which corresponds to taking commutator with $H_{\omega,\mathcal{E}}$. The matrix element of $H_{\omega,\mathcal{E}}$ is given by

$$H_{\omega,\mathcal{E}}(\omega\,; x, y) = \begin{cases} 1, & (\text{if } |x-y| = 1), \\ \lambda V_\omega(x) + \mathcal{E}x_1, & (\text{if } x = y), \\ 0, & (\text{otherwise}). \end{cases}$$

For $A \in \mathcal{L}^2$, we define $(H_{\omega,\mathcal{E}}A)(\omega\,; x, y)$ and $(AH_{\omega,\mathcal{E}})(\omega\,; x, y)$ as (2.3)

$$\begin{aligned}(H_{\omega,\mathcal{E}}A)(\omega\,; x, y) &:= \sum_{|z-x|=1} A(\omega\,; z, y) + (\lambda V_\omega(x) + \mathcal{E}x_1)A(\omega\,; x, y), \\ (AH_{\omega,\mathcal{E}})(\omega\,; x, y) &:= \sum_{|z-y|=1} A(\omega\,; x, z) + (\lambda V_\omega(y) + \mathcal{E}y_1)A(\omega\,; x, y).\end{aligned}$$

We define

$$(\mathcal{L}_{H_{\omega,\mathcal{E}}}A)(\omega\,; x, y) := (H_{\omega,\mathcal{E}}A)(\omega\,; x, y) - (AH_{\omega,\mathcal{E}})(\omega\,; x, y), \tag{2.13}$$

which makes sense as a function on $\Omega\times\mathbb{Z}^d\times\mathbb{Z}^d$. We note $(\mathcal{L}_{H_{\omega,\mathcal{E}}}A)(\omega\,;x,y)$ satisfies (CR), though $(H_{\omega,\mathcal{E}}A)(\omega\,;x,y)$ and $(AH_{\omega,\mathcal{E}})(\omega\,;x,y)$ do not. The corresponding element in $\tilde{\mathcal{L}}^2$ is

$$
\begin{aligned}
(2.14)\quad \pi\Big(\mathcal{L}_{H_{\omega,\mathcal{E}}}A\Big)(\omega,x) = & \sum_{|z-x|=1} a(\omega,z) + (\lambda V_\omega(x)+\mathcal{E}x_1)a(\omega,x) \\
& - \Big\{\sum_{|z|=1} a(T^{-z}\omega,x-z) + \lambda V_\omega(0)a(\omega,x)\Big\}.
\end{aligned}
$$

Since $\mathbf{E}|V_\omega(0)|^2<\infty$, $\mathcal{L}_{H_{\omega,\mathcal{E}}}A\in\mathcal{L}^2$ if $A\in\mathcal{A}_0$. Hence $\mathcal{L}_{H_{\omega,\mathcal{E}}}$ is densely defined on $\mathcal{L}^2$.

**Proposition 2.7** *(1) $\mathcal{L}_{H_{\omega,\mathcal{E}}}$ is self-adjoint with domain*

$$
\begin{aligned}
\mathcal{D}(\mathcal{L}_{H_{\omega,\mathcal{E}}}) := \Big\{ & A\in\mathcal{L}^2: \\
& \Big(\lambda(V_\omega(x)-V_\omega(y))+\mathcal{E}(x_1-y_1)\Big)A(\omega\,;x,y)\in\mathcal{L}^2\Big\}.
\end{aligned}
$$

*(2)* $e^{it\mathcal{L}_{H_{\omega,\mathcal{E}}}}A = e^{itH_{\omega,\mathcal{E}}}Ae^{-itH_{\omega,\mathcal{E}}}$, $A\in\mathcal{L}^2$.

**Remark 2.4** *$H_{\omega,\mathcal{E}}$ has the following property*

$$(2.15)\qquad U(a)H_{\omega,\mathcal{E}}U(a)^* = H_{T^a\omega,\mathcal{E}}-\mathcal{E}a_1,\quad \omega\in\Omega,\ a\in\mathbb{Z}^d,$$

*which implies*

$$(2.16)\qquad U(a)e^{itH_{\omega,\mathcal{E}}}U(a)^* = e^{itH_{T^a\omega,\mathcal{E}}}e^{-it\mathcal{E}a_1}.$$

*However, by using (2.16), we can show $(e^{itH_{\omega,\mathcal{E}}}Ae^{-itH_{\omega,\mathcal{E}}})(\omega\,;x,y)$ satisfies (CR) ( it is straightforward if $A\in\mathcal{A}$. For $A\in\mathcal{L}^2$, we use Proposition 2.3). By using the argument in the proof of Proposition 2.5, $e^{itH_{\omega,\mathcal{E}}}Ae^{-itH_{\omega,\mathcal{E}}}\in\mathcal{L}^2$, for $A\in\mathcal{L}^2$.*

*Proof of Proposition 2.7(1)* It is easy to see the operators $a(\omega,x)\mapsto\sum_{|z-x|=1}a(\omega,z)$, and $a(\omega,x)\mapsto\sum_{|z|=1}a(T^{-z}\omega,x-z)$, which appear in (2.14), are bounded on $\tilde{\mathcal{L}}^2$. The multiplication operator $a(\omega,x)\mapsto\Big(\lambda(V_\omega(x)-V_\omega(0))+\mathcal{E}x_1\Big)a(\omega,x)$ is clearly self-adjoint with domain

$$\mathcal{D}:=\Big\{a\in\tilde{\mathcal{L}}^2:\Big(\lambda(V_\omega(x)-V_\omega(0))+\mathcal{E}x_1\Big)a(\omega,x)\in\tilde{\mathcal{L}}^2\Big\}.$$

Therefore, $\mathcal{L}_{H_{\omega,\mathcal{E}}}$ is self-adjoint with domain $\pi^{-1}\mathcal{D} = \mathcal{D}(\mathcal{L}_{H_{\omega,\mathcal{E}}})$. $\square$

To prove Proposition 2.7(2), we prepare the following Lemma. Let $\mathcal{D}(H_{\omega,\mathcal{E}})$ be the domain of the self-adjoint operator $H_{\omega,\mathcal{E}}$ on $l^2(\mathbb{Z}^d)$ given by

$$\mathcal{D}(H_{\omega,\mathcal{E}}) := \{\varphi \in l^2(\mathbb{Z}^d) : (\lambda V_\omega(x) + \mathcal{E}x_1)\varphi \in l^2(\mathbb{Z}^d)\}.$$

**Lemma 2.4** *Suppose $\varphi \in \mathcal{D}(H_{\omega,\mathcal{E}})$ for any $\omega \in \Omega$, and $A \in \mathcal{A}_0$. Then, $A\varphi \in \mathcal{D}(H_{\omega,\mathcal{E}})$ for any $\omega \in \Omega$.*

*Proof.* We have only to show

$$\begin{aligned}(\lambda V_\omega(x) + \mathcal{E}x_1)(A\varphi)(x) = (\lambda V_\omega(x) + \mathcal{E}x_1) \sum_{y\in\mathbb{Z}^d} A(\omega\,; x, y)\varphi(y) \\ \in l^2(\mathbb{Z}^d),\end{aligned}$$

for any $\omega \in \Omega$. Since $A \in \mathcal{A}_0$, there exist constants $C > 0$, $K > 0$, such that $|A(\omega\,; x, y)| \le C$, and $A(\omega\,; x, y) = 0$ if $|x - y| > K$. Then,

$$\begin{aligned}&|(\lambda V_\omega(x) + \mathcal{E}x_1) \sum_{y\in\mathbb{Z}^d} A(\omega\,; x, y)\varphi(y)| \\ &\qquad \le \ C|(\lambda V_\omega(x) + \mathcal{E}x_1)| \sum_{|z|\le K} |\varphi(x + z)|.\end{aligned}$$

We will show $|(\lambda V_\omega(x) + \mathcal{E}x_1)\varphi(x + z)| \in l^2(\mathbb{Z}^d)$, w.r.t. $x$ for any fixed $z$ ($|z| \le K$) which completes the proof. In fact, letting $y = x + z$,

$$\begin{aligned}&|(\lambda V_\omega(x) + \mathcal{E}x_1\ )\ \varphi(x + z)| \\ &\qquad = |\lambda V_{T^{-x+y}\omega}(y) + \mathcal{E}y_1 + \mathcal{E}(x_1 - y_1)||\varphi(x + z)| \\ &\qquad \le |\lambda V_{T^z\omega}(y) + \mathcal{E}y_1||\varphi(y)| + |\mathcal{E}z_1||\varphi(y)|.\end{aligned}$$

By assumption, the first term of the RHS in the above expression belongs to $l^2(\mathbb{Z}^d)$, for any $\omega \in \Omega$. $\square$

*Proof of Proposition 2.7 (2)* Let $U(t)A := e^{itH_{\omega,\mathcal{E}}}Ae^{-itH_{\omega,\mathcal{E}}}$, $A \in \mathcal{L}^2$. By Proposition 2.5 and Remark 2.4, $U(t)$ is uniformly bounded on $\mathcal{L}^2$ w.r.t. $t \in \mathbb{R}$. Thus, it suffices to show $e^{it\mathcal{L}_{H_{\omega,\mathcal{E}}}}A = U(t)A$, for $A \in \mathcal{A}_0$. Let

$\varphi \in \mathcal{D}(H_{\omega,\mathcal{E}})$ for any $\omega \in \Omega$, and $A \in \mathcal{A}_0$. Then, by Lemma 2.4,

$$\begin{aligned} (2.17) \qquad & U(t)A\varphi - A\varphi \\ & \qquad = \int_0^t ds\; e^{isH_{\omega,\mathcal{E}}} i[H_{\omega,\mathcal{E}}, A] e^{-isH_{\omega,\mathcal{E}}} \varphi \end{aligned}$$

as elements of $l^2(\mathbb{Z}^d)$. We let $\varphi = \delta_y \in \mathcal{D}(H_{\omega,\mathcal{E}})$, and take the $l^2(\mathbb{Z}^d)$-inner product of (2.17) with $\delta_x$ ($x,\, y \in \mathbb{Z}^d$). Then, we have

$$(2.18) \qquad U(t)A - A = \int_0^t ds\; e^{isH_{\omega,\mathcal{E}}} i[H_{\omega,\mathcal{E}}, A] e^{-isH_{\omega,\mathcal{E}}}$$

as elements of $\mathcal{L}^2$, for LHS of (2.18) belongs to $\mathcal{L}^2$. We will show $U(t)$ is strongly continuous on $\mathcal{L}^2$. Let $A \in \mathcal{A}_0$. By (2.18),

$$\begin{aligned} (2.19)\; & < (U(t)-I)A, (U(t)-I)A >_{\mathcal{L}^2} \\ &= \; < \int_0^t ds\; e^{isH_{\omega,\mathcal{E}}} i[H_{\omega,\mathcal{E}}, A] e^{-isH_{\omega,\mathcal{E}}}, \\ & \qquad\qquad \int_0^t du\; e^{iuH_{\omega,\mathcal{E}}} i[H_{\omega,\mathcal{E}}, A] e^{-iuH_{\omega,\mathcal{E}}} >_{\mathcal{L}^2} \\ &= \; \int_\Omega d\mathbf{P} \sum_{x \in \mathbb{Z}^d} \int_0^t ds \int_0^t du\; \overline{(V(s))(\omega\,; x, 0)} (V(u))(\omega\,; x, 0), \end{aligned}$$

where

$$(V(s))(\omega\,; x, y) := \Big( e^{isH_{\omega,\mathcal{E}}} i[H_{\omega,\mathcal{E}}, A] e^{-isH_{\omega,\mathcal{E}}} \Big)(\omega\,; x, y).$$

By Proposition 2.5, $V(s)$ is bounded in $\mathcal{L}^2$ uniformly w.r.t. $s \in \mathbb{R}$. Thus, we use Fubini's theorem

$$\begin{aligned} & \text{the RHS of (2.19)} \\ &= \int_0^t ds \int_0^t du \int_\Omega d\mathbf{P} \sum_{x \in \mathbb{Z}^d} \overline{(V(s))(\omega\,; x, 0)} (V(u))(\omega\,; x, 0) \\ &= \int_0^t ds \int_0^t du\; < V(s), V(u) >_{\mathcal{L}^2} . \end{aligned}$$

Since $\mathcal{A}_0$ is dense in $\mathcal{L}^2$, we conclude that $U(t)$ is strongly continuous on $\mathcal{L}^2$. By (2.18),

$$U(t)A - A = \int_0^t ds\; U(s)\Big( i[H_{\omega,\mathcal{E}}, A] \Big).$$

If $A \in \mathcal{A}_0$, then $i[H_{\omega,\mathcal{E}}, A] \in \mathcal{L}^2$ so that the integrand in RHS in the above expression is continuous in $\mathcal{L}^2$ w.r.t. $s \in \mathbb{R}$. Hence, for $A \in \mathcal{A}_0$, $U(t)A$ is differentiable, and we have

$$\frac{d}{dt}U(t)A = U(t)\Big(i[H_{\omega,\mathcal{E}}, A]\Big).$$

By Lemma 2.4, it is easy to see

$$i\mathcal{L}_{H_{\omega,\mathcal{E}}}\Big(U(t)A\Big)(\omega\,; x, y) = U(t)\Big(i[H_{\omega,\mathcal{E}}, A]\Big)(\omega\,; x, y) \in \mathcal{L}^2,$$

for $A \in \mathcal{A}_0$, and hence, we have $U(t)A \in \mathcal{D}(\mathcal{L}_{H_{\omega,\mathcal{E}}})$, and

$$\frac{d}{dt}U(t)A = i\mathcal{L}_{H_{\omega,\mathcal{E}}}\Big(U(t)A\Big), \quad A \in \mathcal{A}_0.$$

Therefore, by uniqueness of solution of time-independent Schrödinger equation,

$$U(t)A = e^{it\mathcal{L}_{H_{\omega,\mathcal{E}}}}A, \quad A \in \mathcal{A}_0,$$

as desired. □

**Proposition 2.8** *For* $f(\lambda) = f_{\beta,\mu}(\lambda)$, *or* $\chi_{(-\infty,\mu]}(\lambda)$,

$$\mathcal{T}(f(H_\omega)\partial_1 H_\omega) = 0, \quad a.e.\ \omega. \tag{2.20}$$

*Proof.* We divide the proof into some steps.

*Step 1.* We will show

$$\mathcal{T}(e^{-itH_{\omega,\mathcal{E}}}\partial_1 H_\omega) = 0, \quad \text{a.e. } \omega. \tag{2.21}$$

*proof of (2.21)*

Let $\nabla_n$ be a bounded operator on $\tilde{\mathcal{L}}^2$ defined by

$$(\nabla_n a)(\omega, x) := \begin{cases} x_1 a(\omega, x), & (\text{if } |x_1| \le n), \\ (\text{sgn} x_1) n\, a(\omega, x), & (\text{otherwise}). \end{cases}$$

In $\mathcal{L}^2$, $\nabla_n$ corresponds to the following operator which we also denote by $\nabla_n$.

$$(\nabla_n A)(\omega\,; x, y) = \begin{cases} (x_1 - y_1)A(\omega\,; x, y), & (\text{if } |x_1 - y_1| \le n), \\ (\text{sgn}(x_1 - y_1)) n\, A(\omega\,; x, y), & (\text{otherwise}). \end{cases}$$

$e^{isH_\omega}\nabla_n(e^{-isH_\omega}) \in \mathcal{L}^2$, and

$$\Big(e^{isH_\omega}\nabla_n(e^{-isH_\omega})\Big)(\omega\,;x,y) =< \delta_x, e^{isH_\omega}\nabla_n(e^{-isH_\omega})\delta_y >_{l^2(\mathbb{Z}^d)} .$$

Since $\nabla_n$ corresponds to a multiplication operator by a bounded function, and $\delta_y \in \mathcal{D}(H_\omega)$, for any $\omega \in \Omega$, it is easy to show $(\nabla_n(e^{-isH_{\omega,\varepsilon}})\delta_y)(x) \in \mathcal{D}(H_\omega)$, for any $\omega \in \Omega$. Therefore,

$$\begin{aligned} &\Big(e^{itH_\omega}\nabla_n(e^{-itH_\omega})\Big)(\omega\,;x,y) \\ &= i\int_0^t ds\ < \delta_x, e^{isH_\omega}\Big\{H_\omega(\nabla_n(e^{-isH_\omega})) - \nabla_n(H_\omega e^{-isH_\omega})\Big\}\delta_y >_{l^2(\mathbb{Z}^d)} . \end{aligned}$$

Because both sides in the above equality belong to $l^2(\mathbb{Z}^d)$ w.r.t. $x$, we have

$$\begin{aligned} &\nabla_n(e^{-itH_\omega}) \\ &= e^{-itH_\omega}\int_0^t ds\ e^{isH_\omega}\Big\{H_\omega\nabla_n(e^{-isH_\omega}) - \nabla_n(H_\omega e^{-isH_\omega})\Big\}. \end{aligned} \tag{2.22}$$

We take the trace per volume on both sides. By Proposition 2.2,

$$\mathcal{T}(\nabla_n(e^{-itH_\omega})) = 0, \quad \text{a.e. } \omega. \tag{2.23}$$

On the other hand, in the proof of Lemma 3.1 in section 3, we will show that,

(1) $\mathcal{L}^2$ norm of $\Big\{H_\omega\nabla_n(e^{-isH_\omega}) - \nabla_n(H_\omega e^{-isH_\omega})\Big\}$ is bounded uniformly in $s$ (3.5).

(2) $\Big\{H_\omega\nabla_n(e^{-isH_\omega}) - \nabla_n(H_\omega e^{-isH_\omega})\Big\} \to -(\partial_1 H_\omega)e^{-isH_\omega}$ in $\mathcal{L}^2$ as $n \to \infty$ (Remark 3.1).

By (1), we can use Fubini's theorem and conclude that

$$\begin{aligned} &\mathcal{T}\ \Big(e^{-itH_\omega}\int_0^t ds\ e^{isH_\omega}\Big\{H_\omega\nabla_n(e^{-isH_\omega}) - \nabla_n(H_\omega e^{-isH_\omega})\Big\}\Big) \\ &= \int_0^t ds\ \mathcal{T}\Big(e^{-i(t-s)H_\omega}\Big\{H_\omega\nabla_n(e^{-isH_\omega}) - \nabla_n(H_\omega e^{-isH_\omega})\Big\}\Big) \\ &\to -\int_0^t ds\ \mathcal{T}(e^{-itH_\omega}(\partial_1 H_\omega)), \quad \text{as } n \to \infty. \end{aligned} \tag{2.24}$$

In the last step, we used (2) above, and Proposition 2.2. By (2.22), (2.23), and (2.24), we have (2.21).

*Step 2.* For $z \in \mathbb{C}$, Im $z \neq 0$, we will show

$$\mathcal{T}((H_\omega - z)^{-1}\partial_1 H_\omega) = 0, \quad \text{a.e. } \omega. \tag{2.25}$$

*Proof of (2.25).*

If $z = a + i\lambda$, for $\lambda > 0$,

$$\begin{aligned}(2.26)\ \ & \mathcal{T}\left((H_\omega - a - i\lambda)^{-1}\partial_1 H_\omega\right) \\ & = i\int_\Omega d\mathbf{P} \sum_{x\in\mathbb{Z}^d} \int_0^\infty dt\ e^{-\lambda t} e^{ita} (e^{-itH_\omega})(\omega\,; 0, x)(\partial_1 H_\omega)(\omega\,; x, 0).\end{aligned}$$

Since $e^{-itH_\omega} \in \mathcal{A}$ and $\partial_1 H_\omega \in \mathcal{A}_0$, we can use Fubini's theorem to exchange $\int_\Omega d\mathbf{P} \sum_{x\in\mathbb{Z}^d}$ with $\int_0^\infty dt$.

$$\begin{aligned}\text{RHS of (2.26)} &= i\int_0^\infty dt\ e^{-\lambda t} e^{ita} \mathcal{T}(e^{-itH_\omega}\partial_1 H_\omega) \\ &= 0.\end{aligned}$$

The proof of (2.25) for $\lambda < 0$ is similar.

*Step 3.* (completion) We use the following formula which is norm convergent.

$$f_{\beta,\mu}(H_\omega) = \frac{1}{2\pi i}\int_{\mathbb{C}} (\partial_{\bar z}\tilde f_{\beta,\mu}(z))(H_\omega - z)^{-1} dz d\bar z, \tag{2.27}$$

where $\tilde f_{\beta,\mu}(z)$ is the almost analytic continuation of $f_{\beta,\mu}$, $(\partial_{\bar z} f)(x+iy) = \frac{1}{2}(\partial_x f + i\partial_y f)(x+iy)$, for $z = x+iy$ ([11]). Then, by (2.25),

$$\mathcal{T}(f_{\beta,\mu}(H_\omega)\partial_1 H_\omega) = 0, \quad \text{a.e. } \omega,$$

and thus, we have (2.20) for $f(\lambda) = f_{\beta,\mu}(\lambda)$. Since $f_{\beta,\mu}(\lambda) \to \chi_{(-\infty,\mu]}(\lambda)$, pointwise if $\lambda \neq \mu$, as $\beta \to \infty$, and the integrated density of states is continuous,

$$\begin{aligned}\mathcal{T}\Big((f_{\beta,\mu}(H_\omega) - P_\mu)^2\Big) &= \int_{\mathbb{R}} (f_{\beta,\mu}(\lambda) - \chi_{(-\infty,\mu]}(\lambda))^2 d\mathcal{N}(\lambda) \\ &\to 0, \quad \text{as } \beta\uparrow\infty,\end{aligned}$$

where $d\mathcal{N}(\lambda)$ is the density of states measure. By Schwarz inequality,

$$0 = \mathcal{T}(f_{\beta,\mu}(H_\omega)\partial_1 H_\omega) \to \mathcal{T}(P_\mu \partial_1 H_\omega),$$

as $\beta \to \infty$, and we obtain (2.20) for $\chi_{(-\infty,\mu]}(\lambda)$. $\square$

**Remark 2.5** *When we impose constant magnetic field, the analysis in this paper goes through with suitable modifications in the definitions of some* *notations described below. Let $\mathbf{B} := \{B_{ij}\}_{1\le i,j\le d}$ be $d \times d$ antisymmetric matrix whose components are given by $B_{ij} = -B_{ij} \in \mathbb{R}$, if $i < j$, and $0$ if $i = j$. $B_{ij}$ corresponds to the magnetic flux penetrating plaquettes in $(i,j)$-planes.*

*(1) The translation operator $U(a)$ defined after (2.1) should be replaced by the magnetic translation given by*

$$(U_{\mathbf{B}}(a)\psi)(x) := e^{\frac{i}{2}<a,\mathbf{B}x>_{\mathbb{R}^d}}\psi(x-a), \quad x, a \in \mathbb{Z}^d.$$

*(2) The definition of (CR) in (2.2) should be replaced by*

$$H(\omega\,; x, y) = e^{\frac{i}{2}<a,\mathbf{B}(x-y)>_{\mathbb{R}^d}}H(T^{-a}\omega\,; x-a, y-a), \quad x, y, a \in \mathbb{Z}^d.$$

*(3) $\pi^{-1} : \mathcal{L}^2 \to \tilde{\mathcal{L}}^2$ in Proposition 2.1, and the product of $A$, $B \in \mathcal{L}^2$ represented in $\tilde{\mathcal{L}}^2$ in (2.4) should be given by the following quantities respectively:*

$$\begin{aligned}(\pi^{-1}a)(\omega\,; x, y) &= e^{\frac{i}{2}<y,\mathbf{B}x>_{\mathbb{R}^d}}a(T^{-y}\omega\,; x-y),\\ \pi(AB)(\omega, x) &= \sum_{z\in\mathbb{Z}^d} e^{\frac{i}{2}<z,\mathbf{B}x>_{\mathbb{R}^d}}a(T^{-z}\omega, x-z)b(\omega, z).\end{aligned}$$

## 3 Proof of Theorem 1

In this section, we prove Theorem 1. The following proposition is important to study $\mathcal{E} \downarrow 0$ limit.

**Proposition 3.1** *Let $A \in \mathcal{D}(\partial_1)$,*

$$\mathcal{D}(\partial_1) := \{A \in \mathcal{L}^2 \;:\; \int_\Omega d\mathbf{P} \sum_{x\in\mathbb{Z}^d} |x_1|^2 |A(\omega\,; x, 0)|^2 < \infty\}.$$

*Then, we have the following resolvent equation.*

$$\begin{aligned}(3.1)\quad &(z-\mathcal{L}_{H_{\omega,\mathcal{E}}})^{-1}A - (z-\mathcal{L}_{H_\omega})^{-1}A\\ &\qquad = (z-\mathcal{L}_{H_{\omega,\mathcal{E}}})^{-1}\mathcal{E}\partial_1(z-\mathcal{L}_{H_\omega})^{-1}A, \quad Im\ z \neq 0.\end{aligned}$$

Before we prove Proposition 3.1, we prepare the following lemma.

**Lemma 3.1** *(1) Let $C \in \mathcal{L}^2$. Then, $\mathcal{L}^2$-norm of $[\mathcal{L}_{H_\omega}, \nabla_n]C$ which is defined by*

$$[\mathcal{L}_{H_\omega}, \nabla_n]C := [H_\omega, \nabla_n C] - \nabla_n([H_\omega, C])$$

*is bounded uniformly w.r.t. $n$ ($\nabla_n$ is defined in the proof of Proposition 2.8).*

*(2) $\partial_1(z - \mathcal{L}_{H_\omega})^{-1}A \in \mathcal{L}^2$, if $A \in \mathcal{D}(\partial_1)$, $Im\ z \neq 0$.*

*Proof of Lemma 3.1*

(1) Let $c := \pi(C)$, $h := \pi(H_\omega)$. Then, $h(\omega, x)$ is given by

$$h(\omega, x) = \begin{cases} 1, & (\text{if } |x| = 1), \\ V_\omega(0), & (\text{if } x = 0), \\ 0, & (\text{otherwise}). \end{cases}$$

By (2.4), the product is given by

$$hc = \pi(H_\omega C) = \sum_{z \in \mathbb{Z}^d} h(T^{-z}\omega, x - z)c(\omega, z).$$

First of all, we compute $(h\nabla_n c)(\omega, x) - (\nabla_n(hc))(\omega, x)$. By definition of $\nabla_n$, we have

$$(h\nabla_n c)(\omega, x) = \sum_{z \in \mathbb{Z}^d, |z_1| \leq n} h(T^{-z}\omega, x - z)\, z_1\, c(\omega, z) + \sum_{z \in \mathbb{Z}^d, |z_1| > n} h(T^{-z}\omega, x - z)(\mathrm{sgn} z_1)\, n\, c(\omega, z), \tag{3.2}$$

$$(\nabla_n(hc))(\omega, x) = \begin{cases} x_1 \sum_{z \in \mathbb{Z}^d} h(T^{-z}\omega, x - z)c(\omega, z), & (\text{if } |x_1| \leq n), \\ (\mathrm{sgn}\ x_1)n \sum_{z \in \mathbb{Z}^d} h(T^{-z}\omega, x - z)c(\omega, z), & (\text{if } |x_1| > n). \end{cases}$$

(3.3)

$h(T^{-z}\omega, x - z) \neq 0$ only if $|x - z| \leq 1$, and thus the sums in (3.2), (3.3) are finite sums and they make sense for each $(\omega, x) \in \Omega \times \mathbb{Z}^d$. We divide into several cases.

*Case A* ($|x_1| \leq n$)

In this case, we have

$$(h\nabla_n c)(\omega,x) - (\nabla_n(hc))(\omega,x) \\ = \sum_{|z_1|\le n} (z_1 - x_1)h(T^{-z}\omega, x-z)c(\omega,z) \\ + \sum_{|z_1|>n} ((\mathrm{sgn}\ z_1)n - x_1)h(T^{-z}\omega, x-z)c(\omega,z) \tag{3.4}$$

The second term of RHS is nonzero only if $x_1 = n$, $z_1 = n+1$, or $x_1 = -n$, $z_1 = -(n+1)$. In either case, $(\mathrm{sgn}\ z_1)n - x_1 = 0$. Therefore, if $|x_1| \le n$, we have

$$|(h\nabla_n c)(\omega,x) - (\nabla_n(hc))(\omega,x)| \\ \le |c(\omega, x+\mathbf{e}_1)| + |c(\omega, x-\mathbf{e}_1)|, \tag{3.5}$$

where $\mathbf{e}_1 := (1,0,\cdots,0) \in \mathbb{Z}^d$.

*Case B* ($|x_1| > n$)

In this case, we have

$$(h\nabla_n c)(\omega,x) - (\nabla_n(hc))(\omega,x) \\ = \sum_{z\in\mathbb{Z}^d, |z_1|\le n} (z_1 - (\mathrm{sgn}\ x_1)n)h(T^{-z}\omega, x-z)c(\omega,z) \\ + \sum_{z\in\mathbb{Z}^d, |z_1|>n} ((\mathrm{sgn}\ z_1) - (\mathrm{sgn}\ x_1))\, n\, h(T^{-z}\omega, x-z)c(\omega,z).$$

The first term of RHS in the above expression vanishes by the same reason as in Case A. The second term is nonzero only if $|x_1| > n$, $|z_1| > n$, and $|x - z| \le 1$, in which case $\mathrm{sgn}\ x_1 = \mathrm{sgn}\ z_1$. Thus, the second term also vanishes.

Next, we compute $((\nabla_n c)h)(\omega,x) - (\nabla_n(ch))(\omega,x)$. Each term is given by

$$((\nabla_n c)h)(\omega,x) = \sum_{z\in\mathbb{Z}^d, |x_1-z_1|\le n} (x_1 - z_1)\, c(T^{-z}\omega, x-z)h(\omega,z) \\ + \sum_{z\in\mathbb{Z}^d, |x_1-z_1|>n} n\,\mathrm{sgn}\ (x_1-z_1)c(T^{-z}\omega, x-z)h(\omega,z),$$

$$(\nabla_n(ch))(\omega,x) = \begin{cases} \sum_{z\in\mathbb{Z}^d} x_1 c(T^{-z}\omega, x-z)h(\omega,z), & (\text{if } |x_1|\le n), \\ \sum_{z\in\mathbb{Z}^d} n(\mathrm{sgn}\ x_1)c(T^{-z}\omega, x-z)h(\omega,z), & (\text{if } |x_1| > n). \end{cases}$$

*Case C* $(|x_1| \leq n)$

In this case, we have

$$\begin{aligned}((\nabla_n c)h)(\omega,x) &- (\nabla_n(ch))(\omega,x) \\ &= \sum_{z\in\mathbb{Z}^d, |x_1-z_1|\leq n} (-z_1)c(T^{-z}\omega, x-z)h(\omega,z) \\ &+ \sum_{z\in\mathbb{Z}^d, |x_1-z_1|> n} (n \,\mathrm{sgn}\,(x_1-z_1) - x_1)c(T^{-z}\omega, x-z)h(\omega,z).\end{aligned}$$

The second term is nonzero only if $x_1 = -n$, $z_1 = 1$, or $x_1 = n$, $z_1 = -1$. In either case, the second term vanishes. Hence, if $|x_1| \leq n$, we have

$$\Big|((\nabla_n c)h)(\omega,x) - (\nabla_n(ch))(\omega,x)\Big| \leq \sum_{|z|=1} |c(T^{-z}\omega, x-z)|. \tag{3.6}$$

*Case D* $(|x_1| > n)$

In this case, we have

$$\begin{aligned}&((\nabla_n c)h)(\omega,x) - (\nabla_n(ch))(\omega,x) \\ &= \sum_{z\in\mathbb{Z}^d, |x_1-z_1|\leq n} ((x_1-z_1) - (\mathrm{sgn}\; x_1)\, n)\, c(T^{-z}\omega, x-z)h(\omega,z) \\ &+ \sum_{z\in\mathbb{Z}^d, |x_1-z_1|> n} n\,(\mathrm{sgn}\;(x_1-z_1) - \mathrm{sgn}\; x_1)\, c(T^{-z}\omega, x-z)h(\omega,z).\end{aligned}$$

By the similar argument as in Case B, C, we see that the RHS in the above expression vanishes. By (3.5), (3.6), we conclude that $[\mathcal{L}_{H_\omega}, \nabla_n]C$ is uniformly bounded in $\mathcal{L}^2$ w.r.t. $n$.

(2) Let $C := (z-\mathcal{L}_{H_\omega})^{-1}A$, for $A \in \mathcal{D}(\partial_1)$, $\mathrm{Im}\, z \neq 0$. Then, $C \in \mathcal{D}(\mathcal{L}_{H_\omega})$, and as discussed in the proof of Proposition 2.8, $\nabla_n C \in \mathcal{D}(\mathcal{L}_{H_\omega})$. Then,

$$\begin{aligned}\nabla_n(z-\mathcal{L}_{H_\omega})^{-1}A &= \nabla_n C \\ &= (z-\mathcal{L}_{H_\omega})^{-1}(z-\mathcal{L}_{H_\omega})\nabla_n C \\ &= (z-\mathcal{L}_{H_\omega})^{-1}\nabla_n((z-\mathcal{L}_{H_\omega})C) \\ &\qquad + (z-\mathcal{L}_{H_\omega})^{-1}[(z-\mathcal{L}_{H_\omega}), \nabla_n]C \\ &= (z-\mathcal{L}_{H_\omega})^{-1}\nabla_n A - (z-\mathcal{L}_{H_\omega})^{-1}[\mathcal{L}_{H_\omega}, \nabla_n]C.\end{aligned}$$

By (1) of Lemma 3.1, the RHS in the above expression is uniformly bounded in $\mathcal{L}^2$ w.r.t. $n$. On the other hand, by monotone convergence theorem,

$$\|\partial_1 C\|^2_{\mathcal{L}^2} = \lim_{n\to\infty} \|\nabla_n C\|^2_{\mathcal{L}^2} < \infty.$$

Therefore, we have $\|\partial_1(z-\mathcal{L}_{H_\omega})^{-1}A\|_{\mathcal{L}^2}<\infty$, and $\partial_1(z-\mathcal{L}_{H_\omega})^{-1}A\in\mathcal{L}^2$. Lemma 3.1 is proved. □

**Remark 3.1** *(3.4) shows*

$$h(\nabla_n c)-\nabla_n(hc)\to-(\partial_1 h)\,c,$$

*in $\tilde{\mathcal{L}}^2$ as $n\to\infty$, which we needed in the proof of Proposition 2.8.*

*Proof of Proposition 3.1*
Let $A\in\mathcal{D}(\partial_1)$, Im $z\neq 0$. Then,

$$\begin{aligned}
&(z-\mathcal{L}_{H_{\omega,\mathcal{E}}})^{-1}A-(z-\mathcal{L}_{H_\omega})^{-1}A\\
&\quad=(z-\mathcal{L}_{H_{\omega,\mathcal{E}}})^{-1}(z-\mathcal{L}_{H_\omega})(z-\mathcal{L}_{H_\omega})^{-1}A\\
&\qquad\quad-(z-\mathcal{L}_{H_{\omega,\mathcal{E}}})^{-1}(z-\mathcal{L}_{H_\omega}-\mathcal{E}\partial_1)(z-\mathcal{L}_{H_\omega})^{-1}A\\
&\quad=(z-\mathcal{L}_{H_{\omega,\mathcal{E}}})^{-1}\mathcal{E}\partial_1(z-\mathcal{L}_{H_\omega})^{-1}A.
\end{aligned}$$

In the second step, we used Lemma 3.1. □

From now on, we will carry out each operations in the definition of $\sigma_2(\omega,\epsilon_F)$.

**Lemma 3.2** *If $P_{\epsilon_F}$ satisfies Assumption (1.6),*

$$\begin{aligned}
&\lim_{\mathcal{E}\downarrow 0}\frac{1}{\mathcal{E}}\int_0^\infty dt\ \delta e^{-t\delta}\mathcal{T}(e^{itH_{\omega,\mathcal{E}}}i[H_\omega,x_1]e^{-itH_{\omega,\mathcal{E}}}P_{\epsilon_F})\\
&\qquad\qquad=\ i<(-i\delta-\mathcal{L}_{H_\omega})^{-1}\partial_1 H_\omega,\partial_1 P_{\epsilon_F}>_{\mathcal{L}^2}.
\end{aligned}$$

*Proof.* By Proposition 2.7, $e^{itH_{\omega,\mathcal{E}}}i[H_\omega,x_1]e^{-itH_{\omega,\mathcal{E}}}=-e^{it\mathcal{L}_{H_{\omega,\mathcal{E}}}}(i\partial_1 H_\omega)$, and by Proposition 2.5 and the fact that identity operator on $l^2(\mathbb{Z}^d)$ belongs to $\mathcal{L}^2$, $P_{\epsilon_F}\in\mathcal{L}^2$. Then,

$$\begin{aligned}
&\int_0^\infty dt\ \delta e^{-t\delta}\mathcal{T}(e^{itH_{\omega,\mathcal{E}}}i[H_\omega,x_1]e^{-itH_{\omega,\mathcal{E}}}P_{\epsilon_F})\\
&\qquad=\ -\int_0^\infty dt\ \delta e^{-t\delta}<e^{it\mathcal{L}_{H_{\omega,\mathcal{E}}}}(i\partial_1 H_\omega),P_{\epsilon_F}>_{\mathcal{L}^2}\\
&\qquad=\ (-\delta)<(-i\delta-\mathcal{L}_{H_{\omega,\mathcal{E}}})^{-1}\partial_1 H_\omega,P_{\epsilon_F}>_{\mathcal{L}^2}\\
&\qquad=\ (-\delta)<\partial_1 H_\omega,(i\delta-\mathcal{L}_{H_{\omega,\mathcal{E}}})^{-1}P_{\epsilon_F}>_{\mathcal{L}^2}.
\end{aligned}$$

By Assumption (1.6), $P_{\epsilon_F} \in \mathcal{D}(\partial_1)$. Then, by Proposition 3.1,

$$\begin{aligned} &= (-\delta) < \partial_1 H_\omega, (i\delta - \mathcal{L}_{H_\omega})^{-1} P_{\epsilon_F} >_{\mathcal{L}^2} \\ &\qquad +(-\delta)\mathcal{E} < \partial_1 H_\omega, (i\delta - \mathcal{L}_{H_{\omega,\mathcal{E}}})^{-1} \partial_1 (i\delta - \mathcal{L}_{H_\omega})^{-1} P_{\epsilon_F} >_{\mathcal{L}^2} \\ &=: I + II. \end{aligned}$$

For bounded Borel measurable function $f(\lambda)$ on $\mathbb{R}$, by definition, it is easy to see that $H_\omega f(H_\omega)(\omega\,; x, y) = f(H_\omega) H_\omega(\omega\,; x, y)$, and thus $\mathcal{L}_{H_\omega} f(H_\omega) = 0$. Hence,

$$P_{\epsilon_F} = (i\delta)^{-1}(i\delta - \mathcal{L}_{H_\omega}) P_{\epsilon_F}. \tag{3.7}$$

By Proposition 2.8 and (3.7),

$$\begin{aligned} I &= i < \partial_1 H_\omega, P_{\epsilon_F} >_{\mathcal{L}^2} \\ &= 0, \\ II &= i\mathcal{E} < \partial_1 H_\omega, (i\delta - \mathcal{L}_{H_{\omega,\mathcal{E}}})^{-1} \partial_1 P_{\epsilon_F} >_{\mathcal{L}^2} \\ &= i\mathcal{E} < (-i\delta - \mathcal{L}_{H_{\omega,\mathcal{E}}})^{-1} \partial_1 H_\omega, \partial_1 P_{\epsilon_F} >_{\mathcal{L}^2} . \end{aligned}$$

Therefore,

$$\begin{aligned} &\frac{1}{\mathcal{E}} \int_0^\infty \delta e^{-t\delta} \mathcal{T}(e^{itH_{\omega,\mathcal{E}}} i[H_\omega, x_1] e^{-itH_{\omega,\mathcal{E}}} P_{\epsilon_F}) \\ &\qquad = i < (-i\delta - \mathcal{L}_{H_{\omega,\mathcal{E}}})^{-1} \partial_1 H_\omega, \partial_1 P_{\epsilon_F} >_{\mathcal{L}^2} . \end{aligned} \tag{3.8}$$

Since $\partial_1 H_\omega \in \mathcal{A}_0 \subset \mathcal{D}(\partial_1)$, we use Proposition 3.1 again.

$$\begin{aligned} &\text{RHS of (3.8)} \\ &= i < (-i\delta - \mathcal{L}_{H_\omega})^{-1} \partial_1 H_\omega, \partial_1 P_{\epsilon_F} >_{\mathcal{L}^2} \\ &+ i\mathcal{E} < (-i\delta - \mathcal{L}_{H_{\omega,\mathcal{E}}})^{-1} \partial_1 (-i\delta - \mathcal{L}_{H_\omega})^{-1} \partial_1 H_\omega, \partial_1 P_{\epsilon_F} >_{\mathcal{L}^2} . \end{aligned}$$

For fixed $\delta > 0$, the second term goes to zero as $\mathcal{E} \downarrow 0$. Hence, we have Lemma 3.2. □

**Remark 3.2** *The equality $I = 0$ corresponds to the fact that $\sigma_2(\omega, \epsilon_F) = 0$ if $\mathcal{E} = 0$.*

The following lemma would be useful to study finite temperature properties of charge transport, though we will not use it in the proof of Theorem 1, 2.

**Lemma 3.3** *If $f_{\beta,\epsilon_F}(H_\omega)$ satisfies the following decay estimate*

$$\mathbf{E}|f_{\beta,\epsilon_F}(H_\omega)(\omega\,;x,y)| \leq C < x-y >^{-\gamma}, \tag{3.9}$$

*for some constant $C>0$ independent of $x$, $y \in \mathbb{Z}^d$, $\gamma > d+2$, and $\beta > 0$ ($< x > := 1+|x|$), then we have*

$$f_{\beta,\epsilon_F}(H_\omega) \to P_{\epsilon_F},$$
$$\partial_1 f_{\beta,\epsilon_F}(H_\omega) \to \partial_1 P_{\epsilon_F},$$

*in $\mathcal{L}^2$, as $\beta \to \infty$.*

*Proof.* Since $\mathbf{P}\{\epsilon_F$ is the eigenvalue of $H_\omega\} = 0$,

$$f_{\beta,\epsilon_F}(H_\omega)(\omega\,;x,y) \to P_{\epsilon_F}(\omega\,;x,y),$$

as $\beta \to \infty$, for a.e. $\omega$. Then, by dominated convergence theorem,

$$\int_\Omega d\mathbf{P}\left|f_{\beta,\epsilon_F}(H_\omega)(\omega\,;x,0) - P_{\epsilon_F}(\omega\,;x,0)\right|^2 \to 0,$$

as $\beta \to \infty$, for any $x \in \mathbb{Z}^d$. Then, by (3.9) and dominated convergence theorem again, we can easily arrive at the conclusion. □

To study $\delta \downarrow 0$ limit, we prepare a series of lemmas, by borrowing the arguments in [2]. For the sake of simplicity, we write $P$ instead of $P_{\epsilon_F}$.

**Lemma 3.4** *If $P \in \mathcal{D}(\partial_1)$, we have*

$$\partial_1 P = P(\partial_1 P)Q + Q(\partial_1 P)P,$$

*where $Q := 1 - P$.*

*Proof.* Since $P^2 = P$, we have

$$\begin{aligned}(\partial_1 P)(\omega\,;x,y) = &\sum_{z\in\mathbb{Z}^d}(x_1 - z_1)P(\omega\,;x,z)P(\omega\,;z,y)\\ &+ \sum_{z\in\mathbb{Z}^d}(z_1-y_1)P(\omega\,;x,z)P(\omega\,;z,y).\end{aligned}$$

Since $\partial_1 P \in \mathcal{L}^2$, the sums in the above expression are absolutely convergent (Lemma 2.1). Hence, as elements of $\mathcal{L}^2$, $\partial_1 P = (\partial_1 P)P + P(\partial_1 P)$. Multiplying $P$ and $Q$ from both sides, we have $P(\partial_1 P)P = 0$, $Q(\partial_1 P)Q = 0$. Therefore,

$$\begin{aligned}\partial_1 P &= (P+Q)(\partial_1 P)(P+Q)\\ &= P(\partial_1 P)Q + Q(\partial_1 P)P.\end{aligned}$$

$\square$

**Lemma 3.5** *If $A \in \mathcal{D}(\mathcal{L}_{H_\omega})$, then $PAQ \in \mathcal{D}(\mathcal{L}_{H_\omega})$, and we have*

$$\mathcal{L}_{H_\omega}(PAQ) = P\mathcal{L}_{H_\omega}(A)Q.$$

**Remark 3.3** *By Proposition 2.5, $PAQ \in \mathcal{L}^2$. $\mathcal{L}_{H_\omega}(PAQ)$ is defined as*

$$\mathcal{L}_{H_\omega}(PAQ) :=< \delta_x, [H_\omega, PAQ]\delta_y >_{l^2(\mathbb{Z}^d)} .$$

*$h := \pi(H_\omega)$ satisfies $h(\omega, x) = 0$, if $|x| \geq 2$, and $h(\omega, x) < \infty$. Therefore, $< \delta_x, [H_\omega, PAQ]\delta_y >_{l^2(\mathbb{Z}^d)}$ makes sense. We note that $A \in \mathcal{D}(\mathcal{L}_{H_\omega})$ is equivalent to the statement that $\mathcal{L}_{V_\omega}(A)(\omega\,; x, y) :=< \delta_x, [V_\omega, A]\delta_y >_{l^2(\mathbb{Z}^d)} \in \mathcal{L}^2$.*

*Proof.* Let $a := \pi(A)$, and define

$$a_n(\omega, x) := \begin{cases} a(\omega, x), & (\text{if } |x| \leq n, \text{ and } |a(\omega, x)| \leq n), \\ 0, & (\text{otherwise}). \end{cases}$$

Then, $A_n := \pi^{-1}(a_n) \to A$ in $\mathcal{L}^2$. Since $\mathcal{L}_{V_\omega}(A) \in \mathcal{L}^2$, $\mathcal{L}_{V_\omega}(A_n) \to \mathcal{L}_{V_\omega}(A)$ in $\mathcal{L}^2$ as $n \to \infty$. We write $H_\omega = H_0 + V_\omega$. Since $H_0$, $P$, $Q$, $A_n$ are all bounded on $l^2(\mathbb{Z}^d)$,

$$\begin{aligned}(3.10)\qquad & \mathcal{L}_{H_0}(PA_nQ)(\omega\,; x, y)\\ &= (\mathcal{L}_{H_0}(P)A_nQ)(\omega\,; x, y) + (P\mathcal{L}_{H_0}(A_n)Q)(\omega\,; x, y)\\ &\quad +(PA_n\mathcal{L}_{H_0}(Q))(\omega\,; x, y).\end{aligned}$$

On the other hand,

$$
\begin{aligned}
(3.11)\quad & \mathcal{L}_{V_\omega}(PA_nQ)(\omega\,;x,y) \\
&= \sum_{z_1,z_2\in\mathbb{Z}^d} P(\omega\,;x,z_1)A_n(\omega\,;z_1,z_2)Q(\omega\,;z_2,y)\times \\
&\quad \times(V_\omega(x)-V_\omega(z_1)+V_\omega(z_1)-V_\omega(z_2)+V_\omega(z_2)-V_\omega(y)) \\
&= (\mathcal{L}_{V_\omega}(P)A_nQ)(\omega\,;x,y)+(P\mathcal{L}_{V_\omega}(A_n)Q)(\omega\,;x,y) \\
&\quad +(PA_n\mathcal{L}_{V_\omega}(Q))(\omega\,;x,y).
\end{aligned}
$$

Since $A$, $P\in\mathcal{D}(\mathcal{L}_{H_\omega})$, the sums appearing in the above expression absolutely converge. By (3.10), (3.11), and the equality $\mathcal{L}_{H_\omega}(P)=\mathcal{L}_{H_0}(P)+\mathcal{L}_{V_\omega}(P)=0$, we have

$$\mathcal{L}_{H_\omega}(PA_nQ)=P\mathcal{L}_{H_\omega}(A_n)Q. \tag{3.12}$$

$P\mathcal{L}_{H_\omega}(A_n)Q\to P\mathcal{L}_{H_\omega}(A)Q$ in $\mathcal{L}^2$ $(n\to\infty)$, and by taking subsequence,

$$(P\mathcal{L}_{H_\omega}(A_n)Q)(\omega\,;x,y)\to(P\mathcal{L}_{H_\omega}(A)Q)(\omega\,;x,y), \tag{3.13}$$

as $n\to\infty$ for a.e. $\omega$, and for any $x$, $y\in\mathbb{Z}^d$. On the other hand, we have $(PA_nQ)(\omega\,;x,y)\to(PAQ)(\omega\,;x,y)$, as $n\to\infty$ for a.e. $\omega$, by the same reason. Since $h(\omega,x)=0$ if $|x|\geq 2$, and $|h(\omega,x)|<\infty$ for a.e. $\omega$,

$$\mathcal{L}_{H_\omega}(PA_nQ)(\omega\,;x,y)\to\mathcal{L}_{H_\omega}(PAQ)(\omega\,;x,y) \tag{3.14}$$

for a.e. $\omega$, as $n\to\infty$. By (3.12), (3.13), and (3.14), we arrive at the conclusion. □

**Lemma 3.6**

$$Q\{(-i\delta-\mathcal{L}_{H_\omega})^{-1}\partial_1H_\omega\}P=(-i\delta-\mathcal{L}_{H_\omega})^{-1}(Q\partial_1H_\omega P), \tag{3.15}$$

$$P\{(-i\delta-\mathcal{L}_{H_\omega})^{-1}\partial_1H_\omega\}Q=(-i\delta-\mathcal{L}_{H_\omega})^{-1}(P\partial_1H_\omega Q). \tag{3.16}$$

*Proof.* Let

$$A:=Q\{(-i\delta-\mathcal{L}_{H_\omega})^{-1}\partial_1H_\omega\}P-(-i\delta-\mathcal{L}_{H_\omega})^{-1}(Q\partial_1H_\omega P).$$

The first term of RHS belongs to $\mathcal{D}(\mathcal{L}_{H_\omega})$ by Lemma 3.5. Moreover,

$$
\begin{aligned}
(-i\delta-\mathcal{L}_{H_\omega})A &= Q(\partial_1H_\omega)P-Q(\partial_1H_\omega)P \\
&= 0.
\end{aligned}
$$

Therefore, $A = 0$. The proof of (3.16) is similar. $\square$

**Lemma 3.7** *If $P \in \mathcal{D}(\partial_1)$, then $(Qx_1P)(\omega\,; x, y)$ is finite for a.e. $\omega$.*

*Proof.* By Schwarz inequality, we have

$$
\begin{aligned}
(3.17)\quad & \int_\Omega d\mathbf{P}|(Qx_1P)(\omega\,; x, y)| \\
&= \sum_{z \in \mathbb{Z}^d} \int_\Omega d\mathbf{P}|Q(\omega\,; x, z)||z_1||P(\omega\,; z, y)| \\
&\le \sum_{z \in \mathbb{Z}^d} \Big(\int_\Omega d\mathbf{P}|Q(\omega\,; x, z)|^2|z_1|\Big)^{1/2} \Big(\int_\Omega d\mathbf{P}|P(\omega\,; z, y)|^2|z_1|\Big)^{1/2}.
\end{aligned}
$$

Since $P$, $Q \in \mathcal{D}(\partial_1)$,

$$
\begin{aligned}
&\text{RHS of (3.17)} \\
&\le \frac{1}{2} \sum_{z \in \mathbb{Z}^d} \int_\Omega d\mathbf{P}|Q(\omega\,; x, z)|^2(|x_1 - z_1| + |x_1|) \\
&\quad + \frac{1}{2} \sum_{z \in \mathbb{Z}^d} \int_\Omega d\mathbf{P}|P(\omega\,; z, y)|^2(|z_1 - y_1| + |y_1|) \\
&\le C\Big(\|\partial_1 Q\|^2_{\mathcal{L}^2} + |x_1|\|Q\|^2_{\mathcal{L}^2} \\
&+ \|\partial_1 P\|^2_{\mathcal{L}^2} + |y_1|\|P\|^2_{\mathcal{L}^2}\Big) \\
&\le C_{x,y} < \infty,
\end{aligned}
$$

for some constants $C$, $C_{x,y}$ depending on $x$, $y \in \mathbb{Z}^d$. $\square$

**Lemma 3.8** *If $P \in \mathcal{D}(\partial_1)$, we have*

$$\mathcal{L}_{H_\omega}(Qx_1P) = Q\mathcal{L}_{H_\omega}(x_1)P.$$

*In particular, $\mathcal{L}_{H_\omega}(Qx_1P) \in \mathcal{L}^2$.*

*Proof.* By Lemma 3.7, $(Qx_1P)(\omega\,; x, y)$ is finite for a.e. $\omega$. Thus, by Remark 3.3, $\mathcal{L}_{H_\omega}(Qx_1P)$ makes sense as a function on $\Omega \times \mathbb{Z}^d \times \mathbb{Z}^d$ for a.e. $\omega$. We will show

$$(3.18)\qquad \mathcal{L}_{V_\omega}(Qx_1P) = \mathcal{L}_{V_\omega}(Q)x_1P + Qx_1\mathcal{L}_{V_\omega}(P),$$

$$(3.19)\qquad \mathcal{L}_{H_0}(Qx_1P) = \mathcal{L}_{H_0}(Q)x_1P + Q\mathcal{L}_{H_0}(x_1)P + Qx_1\mathcal{L}_{H_0}(P),$$

as functions on $\Omega \times \mathbb{Z}^d \times \mathbb{Z}^d$. Then, Lemma 3.8 follows by using $\mathcal{L}_{H_\omega}(P) = \mathcal{L}_{H_\omega}(Q) = 0$. To prove (3.18), we compute

$$(\mathcal{L}_{V_\omega}(Qx_1P))(\omega\,;x,y) = III + IV,$$

where

$$\begin{aligned}
III &:= \sum_{z\in\mathbb{Z}^d} Q(\omega\,;x,z)\,z_1\,P(\omega\,;z,y)(V_\omega(x) - V_\omega(z)),\\
IV &:= \sum_{z\in\mathbb{Z}^d} Q(\omega\,;x,z)\,z_1\,P(\omega\,;z,y)(V_\omega(z) - V_\omega(y)).
\end{aligned}$$

We will show that, $III$, $IV$ make sense for a.e. $\omega$. In fact, by Schwarz inequality,

$$\begin{aligned}
\int_\Omega d\mathbf{P}|III| \le & \Big(\int_\Omega d\mathbf{P} \sum_{z\in\mathbb{Z}^d} |Q(\omega\,;x,z)|^2|V_\omega(x) - V_\omega(z)|^2\Big)^{1/2} \times\\
&\times\Big(\int_\Omega d\mathbf{P} \sum_{z\in\mathbb{Z}^d} |z_1|^2|P(\omega\,;z,y)|^2\Big)^{1/2}.
\end{aligned}$$

Since $P$ satisfies Assumption (1.6),

$$\begin{aligned}
&\int_\Omega d\mathbf{P} \sum_{z\in\mathbb{Z}^d} |z_1|^2|P(\omega\,;z,y)|^2\\
&\qquad\le 2\int_\Omega d\mathbf{P} \sum_{z\in\mathbb{Z}^d} |z_1 - y_1|^2|P(\omega\,;z,y)|^2\\
&\qquad\quad +2|y_1|^2 \int_\Omega d\mathbf{P} \sum_{z\in\mathbb{Z}^d} |P(\omega\,;z,y)|^2\\
&\qquad\le C_y < \infty,
\end{aligned}$$

for some constant $C_y$ depending only on $y \in \mathbb{Z}^d$. Since $Q \in \mathcal{D}(\mathcal{L}_{H_\omega})$,

$$\begin{aligned}
&\int_\Omega d\mathbf{P} \sum_{z\in\mathbb{Z}^d} |Q(\omega\,;x,z)|^2|V_\omega(x) - V_\omega(z)|^2\\
&= \quad \|\mathcal{L}_{V_\omega}(Q)\|^2_{\mathcal{L}^2} < \infty.
\end{aligned}$$

Therefore, $III$ makes sense for a.e. $\omega$, and so does $IV$ by the same argument. Since

$$III + IV = (\mathcal{L}_{V_\omega}(Q)x_1P)(\omega\,;x,y) + (Qx_1\mathcal{L}_{V_\omega}(P))(\omega\,;x,y),$$

we proved (3.18).

To prove (3.19), we notice that, by Remark 3.3, $(H_0Qx_1P)(\omega\,;x,y)$, $(QH_0x_1P)(\omega\,;x,y)$, $(Qx_1H_0P)(\omega\,;x,y)$, and $(Qx_1PH_0)(\omega\,;x,y)$ are all finite for a.e. $\omega$. Then,

$$\begin{aligned}
&(\mathcal{L}_{H_0}(Qx_1P))(\omega\,;x,y)\\
&\qquad = (H_0Qx_1P)(\omega\,;x,y) - (Qx_1PH_0)(\omega\,;x,y)\\
&\qquad = (H_0Qx_1P)(\omega\,;x,y) - (QH_0x_1P)(\omega\,;x,y)\\
&\qquad\quad +(QH_0x_1P)(\omega\,;x,y) - (Qx_1H_0P)(\omega\,;x,y)\\
&\qquad\quad +(Qx_1H_0P)(\omega\,;x,y) - (Qx_1PH_0)(\omega\,;x,y)\\
&\qquad = (\mathcal{L}_{H_0}(Q)x_1P)(\omega\,;x,y) + (Q\mathcal{L}_{H_0}(x_1)P)(\omega\,;x,y)\\
&\qquad\quad +(Qx_1\mathcal{L}_{H_0}(P))(\omega\,;x,y).
\end{aligned}$$

Thus (3.19) is proved. $\square$

The following lemma follows by simple argument of functional calculus.

**Lemma 3.9**

$$\lim_{\delta\downarrow 0}(-i\delta - \mathcal{L}_{H_\omega})^{-1}\mathcal{L}_{H_\omega}A = -A, \quad \textit{for } A \in (Ker\ \mathcal{L}_{H_\omega})^{\perp}.$$

**Lemma 3.10**

$$\Big[[x_1,P],P\Big] \in (Ker\ \mathcal{L}_{H_\omega})^{\perp}.$$

*Proof.* By Proposition 2.5,

$$< A, \Big[[x_1,P],P\Big] >_{\mathcal{L}^2} = < [A,P],[x_1,P] >_{\mathcal{L}^2} .$$

Thus, it suffices to show

$$[A,P] = 0, \quad \text{if } A \in \mathrm{Ker}\ \mathcal{L}_{H_\omega}. \tag{3.20}$$

Let $A_n \in \mathcal{A}_0$ so that $A_n \to A$ in $\mathcal{L}^2$ as $n \to \infty$ (Proposition 2.3). Since $A_n \in \mathcal{A}_0$, $(z-H_\omega)^{-1}(z-H_\omega)A_n = A_n$ ($\mathrm{Im}\ z \neq 0$) makes sense as elements of $\mathcal{L}^2$. Then,

$$[A_n,(z-H_\omega)^{-1}] = (z-H_\omega)^{-1}[A_n,H_\omega](z-H_\omega)^{-1}.$$

Since $A_n \to A$ in $\mathcal{L}^2$, and $(z - H_\omega)^{-1}$ is bounded on $l^2(\mathbb{Z}^d)$, the LHS of the above expression converges to $[A, (z - H_\omega)^{-1}]$ in $\mathcal{L}^2$ as $n \to \infty$. On the other hand, trivially, $[A_n, H_0] \to [A, H_0]$, and by the construction of $A_n$, $[A_n, V_\omega] = -\mathcal{L}_{V_\omega}(A_n) \to -\mathcal{L}_{V_\omega}(A) = [A, V_\omega]$. Therefore,

$$\begin{aligned}[A, (z - H_\omega)^{-1}] &= (z - H_\omega)^{-1}[A, H_\omega](z - H_\omega)^{-1} \\ &= 0.\end{aligned}$$

By using the almost analytic continuation of $f_{\beta,\epsilon_F}(\lambda)$ (2.27) which is a norm convergent integral, we have

$$[A, f_{\beta,\epsilon_F}(H_\omega)] = 0.$$

Let $B \in \mathcal{A}_0$. By Proposition 2.5,

$$< [A, P], B >_{\mathcal{L}^2} = < A, [B, P] >_{\mathcal{L}^2} .$$

In step 3 in the proof of Proposition 2.8, we saw $f_{\beta,\epsilon_F}(H_\omega) \to P$ in $\mathcal{L}^2$ as $\beta \uparrow \infty$. Thus

$$\begin{aligned}< A, [B, P] >_{\mathcal{L}^2} &= \lim_{\beta\uparrow\infty} < A, [B, f_{\beta,\epsilon_F}(H_\omega)] >_{\mathcal{L}^2} \\ &= \lim_{\beta\uparrow\infty} < [A, f_{\beta,\epsilon_F}(H_\omega)], B >_{\mathcal{L}^2} \\ &= 0.\end{aligned}$$

Therefore, $[A, P] = 0$. $\square$

We are ready to prove Theorem 1.

*Proof of Theorem 1*

By Lemma 3.2, we have

$$\sigma_2(\omega, \epsilon_F) = \lim_{\delta\downarrow 0} i < (-i\delta - \mathcal{L}_{H_\omega})^{-1}\partial_1 H_\omega, \partial_1 P >_{\mathcal{L}^2} .$$

By Lemma 3.4, Proposition 2.5, and Lemma 3.6,

$$\begin{aligned}&i \ < (-i\delta - \mathcal{L}_{H_\omega})^{-1}\partial_1 H_\omega, \partial_1 P >_{\mathcal{L}^2} \\ &= i < (-i\delta - \mathcal{L}_{H_\omega})^{-1}\partial_1 H_\omega, P(\partial_1 P)Q + Q(\partial_1 P)P >_{\mathcal{L}^2} \\ &= i < P\{(-i\delta - \mathcal{L}_{H_\omega})^{-1}\partial_1 H_\omega\}Q + Q\{(-i\delta - \mathcal{L}_{H_\omega})^{-1}\partial_1 H_\omega\}P, \partial_1 P >_{\mathcal{L}^2} \\ &= i < (-i\delta - \mathcal{L}_{H_\omega})^{-1}\Big(P(\partial_1 H_\omega)Q + Q(\partial_1 H_\omega)P\Big), \partial_1 P >_{\mathcal{L}^2} .\end{aligned}$$

As functions on $\Omega \times \mathbb{Z}^d \times \mathbb{Z}^d$, we have $(\partial_1 H_\omega)(\omega\,; x, y) = -(\mathcal{L}_{H_\omega}(x_1))(\omega\,; x, y)$. Together with Lemma 3.8,

$$\begin{aligned} & i\ < (-i\delta - \mathcal{L}_{H_\omega})^{-1}\Big(P(\partial_1 H_\omega)Q + Q(\partial_1 H_\omega)P\Big), \partial_1 P >_{\mathcal{L}^2} \\ & = (-i) < (-i\delta - \mathcal{L}_{H_\omega})^{-1}\mathcal{L}_{H_\omega}(Qx_1P + Px_1Q), \partial_1 P >_{\mathcal{L}^2} . \end{aligned}$$

By direct computation, $\Big[[x_1, P], P\Big] \in \mathcal{D}(\mathcal{L}_{H_\omega})$, and $Qx_1P + Px_1Q = \Big[[x_1, P], P\Big]$. By Lemma 3.9 and (3.10), we compute

$$\begin{aligned} & i\ < (-i\delta - \mathcal{L}_{H_\omega})^{-1}\partial_1 H_\omega, \partial_1 P >_{\mathcal{L}^2} \\ & = (-i) < (-i\delta - \mathcal{L}_{H_\omega})^{-1}\mathcal{L}_{H_\omega}(\Big[[x_1, P], P\Big]), \partial_1 P >_{\mathcal{L}^2} \\ & \quad \longrightarrow i < \Big[[x_1, P], P\Big], \partial_1 P >_{\mathcal{L}^2} \quad (\delta \downarrow 0) \\ & = i < [\partial_1 P, P], \partial_1 P >_{\mathcal{L}^2} \\ & = i < (\partial_1 P)P, \partial_1 P >_{\mathcal{L}^2} - i < P(\partial_1 P), \partial_1 P >_{\mathcal{L}^2} . \end{aligned}$$

By Propositions 2.2, 2.4,

$$\begin{aligned} < (\partial_1 P)P, \partial_1 P >_{\mathcal{L}^2} & = -\mathcal{T}(P(\partial_1 P)(\partial_1 P)) \\ & = -\mathcal{T}((\partial_1 P)P(\partial_1 P)) \\ & = < P(\partial_1 P), (\partial_1 P) >_{\mathcal{L}^2} . \end{aligned}$$

Therefore, we proved $\sigma_2(\omega, \epsilon_F) = 0$, for a.e. $\omega$. $\square$

# 4 Proof of Theorem 2

The following proposition is important to study $\mathcal{E} \downarrow 0$ limit in the proof of Theorem 2.

**Proposition 4.1** *If $A \in \mathcal{L}^2$ satisfies*

$$A \in \mathcal{D}(\mathcal{L}_{H_\omega}), \tag{4.1}$$

$$e^{it\mathcal{L}_{H_\omega}} A \in \mathcal{D}(\partial_1), \quad t \in \mathbb{R}, \tag{4.2}$$

*Then, we have the following "Duhamel" formula for $e^{it\mathcal{L}_{H_{\omega,\mathcal{E}}}}$ and $e^{it\mathcal{L}_{H_\omega}}$*

$$e^{it\mathcal{L}_{H_{\omega,\mathcal{E}}}} A - e^{it\mathcal{L}_{H_\omega}} A = i\mathcal{E} \int_0^t ds\ e^{i(t-s)\mathcal{L}_{H_{\omega,\mathcal{E}}}}[x_1, e^{is\mathcal{L}_{H_\omega}} A].$$

*Proof.* By (4.1), (4.2), $e^{is\mathcal{L}_{H_\omega}}A \in \mathcal{D}(\mathcal{L}_{H_\omega})$, $e^{is\mathcal{L}_{H_\omega}}A \in \mathcal{D}(\partial_1)$. Thus $e^{is\mathcal{L}_{H_\omega}}A \in \mathcal{D}(\mathcal{L}_{H_{\omega,\mathcal{E}}})$, and $e^{-is\mathcal{L}_{H_{\omega,\mathcal{E}}}}e^{is\mathcal{L}_{H_\omega}}A$ is differentiable w.r.t. $s$.

$$\begin{aligned} e^{-it\mathcal{L}_{H_{\omega,\mathcal{E}}}}e^{it\mathcal{L}_{H_\omega}}A - A &= \int_0^t ds\, \frac{d}{ds}\Big(e^{-is\mathcal{L}_{H_{\omega,\mathcal{E}}}}e^{is\mathcal{L}_{H_\omega}}A\Big) \\ &= -i\mathcal{E}\int_0^t ds\, e^{-is\mathcal{L}_{H_{\omega,\mathcal{E}}}}\partial_1\Big(e^{is\mathcal{L}_{H_\omega}}A\Big). \end{aligned}$$

By multiplying $e^{it\mathcal{L}_{H_{\omega,\mathcal{E}}}}$ from the right on both sides, we arrive at the conclusion. □

**Lemma 4.1** *(1) $\partial_1 H_\omega$ satisfies (4.1), (4.2),*
*(2) $[x_1, e^{is\mathcal{L}_{H_\omega}}(\partial_1 H_\omega)]$ satisfies (4.1), (4.2).*

To prove Lemma 4.1, we use the following estimate.

**Theorem 4.1** *(Theorem 3 in [2]) Let $F$ be a function analytic and bounded on a strip $S_\eta := \{z \in \mathbb{C} : |Im\ z| < \eta\}$. Then,*

$$|F(H_\omega)(\omega\,;x,y)| \le 18\sqrt{2}\|F\|_\infty e^{-\mu|x-y|}, \tag{4.3}$$

*for any $\mu > 0$ such that the inequality $\sum_{|x|=1}(e^{\mu|x|}-1) \le \eta/2$ holds, and $\|F\|_\infty := \sup_{z\in S_\eta}|F(z)|$.*

We will make use of the fact that the bound in (4.3) holds uniformly w.r.t. $\omega \in \Omega$.

*Proof of Lemma 4.1* (1) Since $\partial_1 H_\omega \in \mathcal{A}_0$, and $H_\omega \in \mathcal{L}^2$, we have $\mathcal{L}_{H_\omega}(\partial_1 H_\omega) \in \mathcal{L}^2$. Hence $\partial_1 H_\omega \in \mathcal{D}(\mathcal{L}_{H_\omega})$. By (4.3), for any fixed $\eta > 0$, we have

$$|e^{itH_\omega}(\omega\,;x,y)| \le 18\sqrt{2}e^{\eta|t|}e^{-\mu|x-y|}, \tag{4.4}$$

where $\mu > 0$ is taken such that $\sum_{|x|=1}(e^{\mu|x|}-1) \le \frac{\eta}{2}$. By (4.4) and the fact that $\partial_1 H_\omega \in \mathcal{A}_0$, we have

$$\begin{aligned} &|e^{it\mathcal{L}_{H_{\omega,\mathcal{E}}}}(\partial_1 H_\omega)(\omega\,;x,y)| \\ &= \Big|\Big(e^{itH_\omega}(\partial_1 H_\omega)e^{-itH_\omega}\Big)(\omega\,;x,y)\Big| \le C_t e^{-\mu'|x-y|}, \end{aligned} \tag{4.5}$$

for some constants $C_t > 0$, $\mu' > 0$, uniformly in $(\omega\,;x,y) \in \Omega \times \mathbb{Z}^d \times \mathbb{Z}^d$. $C_t$ is uniformly bounded w.r.t. $t$, if $|t| \le T$ for some $T > 0$. Thus, $[x_1, e^{it\mathcal{L}_{H_\omega}}(\partial_1 H_\omega)] \in \mathcal{A} \subset \mathcal{L}^2$, because $[x_1, (e^{it\mathcal{L}_{H_\omega}}(\partial_1 H_\omega))](\omega\,;x,y) = (x_1 -$

$y_1)(e^{it\mathcal{L}_{H_\omega}}(\partial_1 H_\omega))(\omega\,;x,y)$ decays exponentially by (4.5). Hence $\partial_1 H_\omega$ satisfies (4.1), (4.2).

(2) Because $[x_1, e^{is\mathcal{L}_{H_\omega}}(\partial_1 H_\omega)] \in \mathcal{A}$, and $H_\omega \in \mathcal{L}^2$ which follows from the assumption that $\mathbf{E}|V_\omega(0)|^2 < \infty$, we have $\mathcal{L}_{H_\omega}[x_1, e^{is\mathcal{L}_{H_\omega}}(\partial_1 H_\omega)] \in \mathcal{L}^2$, and therefore, $[x_1, e^{is\mathcal{L}_{H_\omega}}(\partial_1 H_\omega)] \in \mathcal{D}(\mathcal{L}_{H_\omega})$. By (4.4), $(e^{itH_\omega})(\omega\,;x,y)$ decays exponentially as well as $[x_1, e^{is\mathcal{L}_{H_\omega}}(\partial_1 H_\omega)](\omega\,;x,y)$, and so does $\left(e^{it\mathcal{L}_{H_\omega}}[x_1, e^{is\mathcal{L}_{H_\omega}}(\partial_1 H_\omega)]\right)(\omega\,;x,y)$. Hence $e^{it\mathcal{L}_{H_\omega}}\left([x_1, e^{is\mathcal{L}_{H_\omega}}(\partial_1 H_\omega)]\right) \in \mathcal{D}(\partial_1)$. $\square$

*Proof of Theorem 2* By the same argument as in the proof of Lemma 3.2, we have

$$\mathcal{T}(e^{itH_{\omega,\varepsilon}} i[H_\omega, x_1] e^{-itH_{\omega,\varepsilon}} P) = -i < e^{it\mathcal{L}_{H_{\omega,\varepsilon}}}(\partial_1 H_\omega), P >_{\mathcal{L}^2} .$$

By Lemma 4.1, $\partial_1 H_\omega$ satisfies the condition to use Proposition 4.1.

$$\begin{aligned}
-i <& e^{it\mathcal{L}_{H_{\omega,\varepsilon}}}(\partial_1 H_\omega), P >_{\mathcal{L}^2} \\
&= -i < e^{it\mathcal{L}_{H_\omega}}(\partial_1 H_\omega), P >_{\mathcal{L}^2} \\
&\quad + \mathcal{E}\int_0^t ds \ < e^{i(t-s)\mathcal{L}_{H_{\omega,\varepsilon}}}[x_1, e^{is\mathcal{L}_{H_\omega}}(\partial_1 H_\omega)], P >_{\mathcal{L}^2} \\
&=: V + VI.
\end{aligned}$$

By Proposition 2.2, 2.4, and 2.8,

$$\begin{aligned}
V &= -i < e^{itH_\omega}(\partial_1 H_\omega) e^{-itH_\omega}, P >_{\mathcal{L}^2} \\
&= -i\mathcal{T}(e^{itH_\omega}(\partial_1 H_\omega) e^{-itH_\omega} P) \\
&= -i\mathcal{T}((\partial_1 H_\omega) P) \\
&= 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{1}{\mathcal{E}}\mathcal{T}(e^{itH_{\omega,\varepsilon}} i[H_\omega, x_1] e^{-itH_{\omega,\varepsilon}} P) \\
&= \int_0^t ds \ < e^{i(t-s)\mathcal{L}_{H_{\omega,\varepsilon}}}[x_1, e^{is\mathcal{L}_{H_\omega}}(\partial_1 H_\omega)], P >_{\mathcal{L}^2} .
\end{aligned}$$

By Lemma 4.1, we can use Proposition 4.1 for $[x_1, e^{is\mathcal{L}_{H_\omega}}(\partial_1 H_\omega)]$.

$$\int_0^t ds \ < e^{i(t-s)\mathcal{L}_{H_{\omega,\varepsilon}}}[x_1, e^{is\mathcal{L}_{H_\omega}}(\partial_1 H_\omega)], P >_{\mathcal{L}^2}$$

$$= \int_0^t ds \ < e^{i(t-s)\mathcal{L}_{H_\omega}}[x_1, e^{is\mathcal{L}_{H_\omega}}(\partial_1 H_\omega)], P >_{\mathcal{L}^2}$$
$$+ i\mathcal{E} \int_0^t ds \int_0^{t-s} du \ < e^{i(t-s-u)\mathcal{L}_{H_{\omega,\mathcal{E}}}} \Big[x_1, e^{iu\mathcal{L}_{H_\omega}}[x_1, e^{is\mathcal{L}_{H_\omega}}(\partial_1 H_\omega)]\Big], P >_{\mathcal{L}^2} .$$

By (4.4), $\Big[x_1, e^{iuH_{\omega,\mathcal{E}}}[x_1, e^{isH_{\omega,\mathcal{E}}}(\partial_1 H_\omega)]\Big]$ is uniformly bounded in $\mathcal{L}^2$ for $s \in [0,t]$, $u \in [0,t]$. Therefore, for fixed $t$, the second term of RHS in the above expression goes to zero as $\mathcal{E} \downarrow 0$. Then, we compute, by using the fact that $e^{it\mathcal{L}_{H_\omega}} P = P$, and $P \in \mathcal{D}(\partial_1)$,

$$\begin{aligned}
&\lim_{\mathcal{E}\downarrow 0} \int_0^t ds < e^{i(t-s)\mathcal{L}_{H_{\omega,\mathcal{E}}}}[x_1, e^{is\mathcal{L}_{H_\omega}}(\partial_1 H_\omega)], P >_{\mathcal{L}^2} \\
&\qquad = \int_0^t ds \ < e^{i(t-s)\mathcal{L}_{H_\omega}}[x_1, e^{is\mathcal{L}_{H_\omega}}(\partial_1 H_\omega)], P >_{\mathcal{L}^2} \\
&\qquad = \int_0^t ds \ < [x_1, e^{is\mathcal{L}_{H_\omega}}(\partial_1 H_\omega)], P >_{\mathcal{L}^2} \\
&\qquad = \int_0^t ds \ < e^{is\mathcal{L}_{H_\omega}}(\partial_1 H_\omega), \partial_1 P >_{\mathcal{L}^2} .
\end{aligned}$$

We proceed as in the proof of Theorem 1, using Lemma 3.4, 3.8,

$$\begin{aligned}
&\int_0^t ds \ < e^{is\mathcal{L}_{H_\omega}}(\partial_1 H_\omega), \partial_1 P >_{\mathcal{L}^2} \\
&= \int_0^t ds \ < e^{is\mathcal{L}_{H_\omega}} \mathcal{L}_{H_\omega}\Big(\Big[[x_1, P], P\Big]\Big), \partial_1 P >_{\mathcal{L}^2} \\
&= -i \int_0^t ds \frac{d}{ds} < e^{is\mathcal{L}_{H_\omega}}\Big(\Big[[x_1, P], P\Big]\Big), \partial_1 P >_{\mathcal{L}^2} \\
&= -i < e^{it\mathcal{L}_{H_\omega}}\Big(\Big[[x_1, P], P\Big]\Big), \partial_1 P >_{\mathcal{L}^2} + i < \Big[[x_1, P], P\Big], \partial_1 P >_{\mathcal{L}^2} .
\end{aligned}$$

The second term in the above expression vanishes as we saw in the proof of Theorem 1. On the other hand, by spectral theorem,

$$\frac{1}{T}\int_0^T dt \ e^{it\mathcal{L}_{H_\omega}} \to P_{\{0\}},$$

strongly in $\mathcal{L}^2$ as $T \uparrow \infty$, where $P_{\{0\}}$ is the orthogonal projection onto Ker $(\mathcal{L}_{H_\omega})$. Thus, by Lemma 3.10,

$$\lim_{T\uparrow\infty} \frac{1}{T}\int_0^T dt < e^{it\mathcal{L}_{H_\omega}}\Big(\Big[[x_1, P], P\Big]\Big), \partial_1 P >_{\mathcal{L}^2} = 0,$$

and we prove $\sigma_1(\omega, \epsilon_F) = 0$, for a.e. $\omega$. $\square$

**Remark 4.1** *If $\lambda$ is sufficiently large, we have the following dynamical localization estimate*

$$\mathbf{E}|e^{itH_\omega}(\omega\,;x,y)| \le Ce^{-\mu|x-y|}, \tag{4.6}$$

*for some constants $C>0$, $\mu>0$, independent of $t\in\mathbb{R}$, $x$, $y\in\mathbb{Z}^d$ ([1, 2, 4]). In this case, operations $\lim_{\mathcal{E}\downarrow 0}$ and $\int_0^\infty dt\ e^{-\delta t}$ in the definition of $\sigma_2(\omega,\epsilon_F)$ can be exchanged. In fact, we can show*

$$\frac{1}{\mathcal{E}}\mathcal{T}(e^{itH_{\omega,\mathcal{E}}}i[H_\omega,x_1]e^{-itH_{\omega,\mathcal{E}}}P) = O(t), \tag{4.7}$$

*as $t\uparrow\infty$, uniformly in $\mathcal{E}$. To show (4.7), we use Proposition 4.1,*

$$\begin{aligned}&\frac{1}{\mathcal{E}}\mathcal{T}(e^{itH_{\omega,\mathcal{E}}}i[H_\omega,x_1]e^{-itH_{\omega,\mathcal{E}}}P)\\ &= \int_0^t ds\ < e^{i(t-s)\mathcal{L}_{H_{\omega,\mathcal{E}}}}[x_1, e^{is\mathcal{L}_{H_\omega}}(\partial_1 H_\omega)], P>_{\mathcal{L}^2}.\end{aligned}$$

*(4.6) implies that $[x_1, e^{is\mathcal{L}_{H_\omega}}(\partial_1 H_\omega)]$ is uniformly bounded in $\mathcal{L}^2$. In fact,*

$$\begin{aligned}&|e^{is\mathcal{L}_{H_\omega}}(\partial_1 H_\omega)(\omega\,;x,y)|^2\\ &\le 2d|e^{is\mathcal{L}_{H_\omega}}(\partial_1 H_\omega)(\omega\,;x,y)|\\ &\le 2d\sum_{z_1,z_2\in\mathbb{Z}^d}|e^{isH_\omega}(\omega\,;x,z_1)||\partial_1 H_\omega(\omega\,;z_1,z_2)||e^{-isH_\omega}(\omega\,;z_2,y)|.\end{aligned} \tag{4.8}$$

*We used $|(\partial_1 H_\omega)(\omega\,;x,y)|\le\|\partial_1 H_\omega\|_{op}\le 2d$. We take expectation, and use Schwarz inequality w.r.t. $\mathbf{E}$.*

$$\begin{aligned}&\mathbf{E}(\textit{RHS of (4.8)})\\ &\le 2d\sum_{z_1,z_2\in\mathbb{Z}^d}\Big(\mathbf{E}|e^{isH_\omega}(\omega\,;x,z_1)|^3\Big)^{1/3}\times\\ &\qquad\times\Big(\mathbf{E}|\partial_1 H_\omega(\omega\,;z_1,z_2)|^3\Big)^{1/3}\Big(\mathbf{E}|e^{-isH_\omega}(\omega\,;z_2,y)|^3\Big)^{1/3}.\end{aligned}$$

*By (4.6) and the fact that $|e^{isH_\omega}(\omega\,;x,y)|\le\|e^{isH_\omega}\|_{op}=1$, we have*

$$\mathbf{E}|e^{is\mathcal{L}_{H_\omega}}(\partial_1 H_\omega)(\omega\,;x,y)|^2 \le Ce^{-\mu|x-y|},\quad x,y\in\mathbb{Z}^d,$$

*for some constants $C>0$, $\mu>0$. Therefore, $[x_1, e^{is\mathcal{L}_{H_\omega}}(\partial_1 H_\omega)]$ is uniformly bounded in $\mathcal{L}^2$ w.r.t. $s\in\mathbb{R}$, and we conclude (4.7).*

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