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p-MECHANICS AS A PHYSICAL THEORY. AN INTRODUCTION

VLADIMIR V. KISIL

ABSTRACT. The paper provides an introduction into p-mechanics, which is a consistent physical theory suitable for a simultaneous description of classical and quantum mechanics. p-Mechanics naturally provides a common ground for several different approaches to quantisation (geometric, Weyl, coherent states, Berezin, deformation, Moyal, etc.) and has a potential for expansions into field and string theories. The backbone of p-mechanics is solely the representation theory of the Heisenberg group.

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On leave from the Odessa University.

> It is a common defect of philosophers to like "grand" examples rather than such as come from ordinary daily life.
> Bertrand Russell; *Sceptical Essays,* Chap. V

1. Introduction

Starting from its origin the quantum theory poses serious challenges for a rigour mathematical description. Notably it resists to subdue under mathematical models with unphysical conditions.

1.1. **Could Observables Form an Algebra?** We all were told during our school years that physical theories are scrutiny tested by experiments. Before we could check our formulae by laboratory measurements we have to fix certain units for physical quantities, e.g. second, meter, and gramme. We are allowed to multiply and divide (almost?) any two units, e.g. dividing meters by seconds we get a new unit to measure velocities. However there isn't a reasonable way to add physical quantities measured in different units, say mass and length. Any negligence of the following Rule was severely punished during exams.

Rule 1.1. Only physical quantities of the *same dimension* can be added together or subtracted one from another.

Since then we were trained much more. On a certain stage of abstraction we readily accept that "observables of physical quantities form an algebra", i.e. can be freely multiplied by scalars, multiplied each other and (sic!) added together as well. However the last operation is simply meaningless: a knowledge of certain value, say 2, for the sum $q+m$ of a length q measured in meters and a mass m measured in grammes can *not* provide us with a value of $q+m$ if q is measured in centimetres and m is measured in kilogrammes. By the same reason the simple mathematical expression $u+u^2$ is physically invalid unless u is a dimensionless quantity.

Remark 1.2. Note that mathematical functions like the exponent $\exp(u) = 1 + u + u^2/2! + \ldots$ or $\sin(u)$ can be naturally constructed out of a dimensionless number u only. Thus Fourier dual variables, say x and q, should posses reciprocal dimensions because they have to form the expression $e^{\mathrm{i}xq}$.

There is a temptation to dismiss the above problem saying that observables form a *graded* algebra. However the grading of observables with physical units is just another elaborated form of the initial statement that there is no a meaningful sum for two observables with different nature. Thus only *dimensionless* pure number can be freely multiplied and added together in a reasonable physical theory. The most affected by this comment theory is probably the *quantisation* which is defined [4, 9] as a some sort of correspondence between "algebras" of classical and quantum observables. *Could it survive if the* first *arithmetic operation of addition is not freely permitted?* Our answer is "Yes" *if* "algebra" is replaced by "symmetric space".

The common habit of thinking about observables in terms of algebra infiltrated in physics probably from the pure mathematical field of Banach algebras. The Gelfand-Naimark [15, § 4.3] theorem states that any *commutative* Banach algebra is completely defined by the "commutative" topological space of its maximal ideals. This mathematical statement was transformed by an association into a mystical believe that any "non-commutative space" is described by a certain non-commutative algebra, for a further discussion and references see [19, 21].

There is also a *psychological* reason (besides the esoteric power of numbers in the Western philosophy declared by Pythagoras) why the Rule 1.1 is so oftenly neglected. Since our primary school years we accustomed to think that multiplications is a more advanced operation than addition or even that a product is necessarily defined through multiple sums. This habit push us to add any physical quantities on the basis that "we can even multiply them".

1.2. **Analysis of Dimensions.** Let M be a unit of mass, L—of length, T—of time. Then coordinates of a point are measured by L, its momentums by ML/T. Derivatives $\frac{\partial}{\partial q}$, $\frac{\partial}{\partial p}$, $\frac{\partial}{\partial t}$ of an observables with respect to coordinates, momentum, and time are measured by $1/L$, $T/(ML)$, and $1/T$ respectively. Corresponding differentials dq, dp, and dt have measure of the according units: L, ML/T, and T in order to make the inner product of vectors and 1-forms a dimensionless pure number.

Energy is measured by ML^2/T^2 and action, i.e. "energy $\times$ time", is measured by ML^2/T. Poisson brackets, which are made of the "products" of $\frac{\partial}{\partial q}$ and $\frac{\partial}{\partial p}$ are "measured" by $T/(ML^2)$, i.e. their dimension is inverse to the dimension of action. Then the right hand side of the classical Hamilton equation

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial H}{\partial q}\frac{\partial f}{\partial p}. \tag{1.1}$$

is made out of a Hamiltonian function (energy units) and Poisson brackets has the dimension $1/T$. This matches the dimension of the time derivative in the left hand side of this equation.

The Heisenberg equations in quantum mechanics are based on the commutator $[H, \cdot]$ with Hamiltonian H (energy units). Thus to get the units $1/T$ of time derivative we need to divide by the units of actions ML^2/T. This is done by division by the Planck constant:

$$\frac{\partial f}{\partial t} = \frac{1}{i\hbar}[H, f]. \tag{1.2}$$

One could ask the following question

Problem 1.3. Is it only a coincidence that the Poisson brackets (1.1) and the inverse $1/h$ of the Planck constant *have the same dimensionality and play the similar rôle in two fundamental dynamic equations*? (Our answer is provided in Section 4.1.)

The purpose of this paper is to demonstrated that the quantisation is well defined without the assumption that observables form an algebra. Moreover the both classical and quantum mechanics are naturally united in the same construction originated in the Heisenberg group $\mathbb{H}^n$. There is an equation (4.9) on $\mathbb{H}^n$ which generates both Hamilton and Heisenberg equations and corresponding classical and quantum dynamics. The whole consideration will be in the complete agreement with the fundamental Rule 1.1 and the above analysis of dimensions.

The paper outline is as follows. In the next Section we present the representation theory of the Heisenberg group based on the *orbit method* of Kirillov [17] and utilising Fock–Segal–Bargmann spaces [11, 13]. We emphasise the existence *and* usability of the family one-dimensional representations: they play for classical mechanics exactly the same rôle as infinite dimensional representations—for quantum. In Section 3 we introduce a concept of observable in p-mechanics and describe their relations with quantum and classical observables. These links are provided by the representations of the Heisenberg group and *wavelet* transforms. In Section 4 we study p-mechanical brackets and the associated dynamic equation together with its classical and quantum representations and their symplectic invariance.

2. The Heisenberg Group and Its Representations

We start from the representation theory of the Heisenberg group $\mathbb{H}^n$ based on the *orbit method* of Kirillov. Analysis of the unitary dual of $\mathbb{H}^n$ in Subsection 2.2 suggests that the family of one-dimensional representations of $\mathbb{H}^n$ forms the phase space of a classical system. Infinite dimensional representations in the Fock type space are described in Subsection 2.3.

2.1. Representations $\mathbb{H}^n$ and Method of Orbit.

Let (s, x, y), where x, $y \in \mathbb{R}^n$ and $s \in \mathbb{R}$, be an element of the Heisenberg group $\mathbb{H}^n$ [11, 13].

Convention 2.1. We assign to x and y components of (s, x, y) physical units $1/L$ and $T/(LM)$ respectively.

The above Convention is the only *a priori* assumption which we made about physical dimensions and it will be justified *a posteriori* as follows. We need (see Remark 1.2) dimensionless products qx and py in order to get the exponent in (2.15), where according to the main observation in p-mechanics q and p represent the classical coordinates and momenta. All other dimensions will be assigned strictly in agreement with the Rule 1.1 and Remark 1.2.

The group law on $\mathbb{H}^n$ is given as follows:

$$(s, x, y) * (s', x', y') = (s + s' + \frac{1}{2}\omega(x, y; x', y'), x + x', y + y'), \tag{2.1}$$

where the non-commutativity is solely due to ω—the *symplectic form* [2, § 37] on the Euclidean space $\mathbb{R}^{2n}$:

$$\omega(x, y; x', y') = xy' - x'y. \tag{2.2}$$

Consequently the parameters s should be measured in $T/(L^2M)$—the product of units of x and y. The Lie algebra $\mathfrak{h}^n$ of $\mathbb{H}^n$ is spanned by a basis S, X_j, Y_j, $j = 1, \ldots, n$ which may be represented by either left- or right-invariant vector fields on $\mathbb{H}^n$:

$$S^{l(r)} = \pm\frac{\partial}{\partial s}, \qquad X_j^{l(r)} = \pm\frac{\partial}{\partial x_j} - \frac{y_j}{2}\frac{\partial}{\partial s}, \qquad Y_j^{l(r)} = \pm\frac{\partial}{\partial y_j} + \frac{x_j}{2}\frac{\partial}{\partial s} \tag{2.3}$$

with the Heisenberg *commutator relations*

$$[X_i^{l(r)}, Y_j^{l(r)}] = \delta_{i,j} S^{l(r)} \tag{2.4}$$

and all other commutators (including any between a left and a right fields) vanishing. Units to measure $S^{l(r)}$, $X_j^{l(r)}$, and $Y_j^{l(r)}$ are inverse to s, x, y—i.e. L^2M/T, L, and LM/T respectively—which are obviously compatible with (2.4).

The exponential map $\exp : \mathfrak{h}^n \to \mathbb{H}^n$ respecting the multiplication (2.1) and Heisenberg commutators (2.4) is provided by the formula:

$$\exp : sS + \sum_{j=1}^{n}(x_j X_j + y_j Y_j) \mapsto (s, x_1, \ldots, x_n, y_1, \ldots, y_n).$$

The composition of the exponential map with representations (2.3) of $\mathfrak{h}^n$ by the left(right)-invariant vector fields produces the right (left) regular representation $\lambda_{r(l)}$ of $\mathbb{H}^n$ by right (left) shifts. Linearised [15, § 7.1] to $L_2(\mathbb{H}^n)$ they are:

$$\lambda_r(g) : f(h) \mapsto f(hg), \quad \lambda_l(g) : f(h) \mapsto f(g^{-1}h), \qquad \text{where } f(h) \in L_2(\mathbb{H}^n). \quad (2.5)$$

As any group $\mathbb{H}^n$ acts on itself by the conjugation automorphisms $\mathrm{A}(g)h = g^{-1}hg$, which fix the unit $e \in \mathbb{H}^n$. The differential $\mathrm{Ad} : \mathfrak{h}^n \to \mathfrak{h}^n$ of A at e is a linear map which can be differentiated again to the representation ad of the Lie algebra $\mathfrak{h}^n$ by the commutator: $\mathrm{ad}\,(A) : B \mapsto [B, A]$. The adjoint space $\mathfrak{h}^*_n$ of the Lie algebra $\mathfrak{h}^n$ can be realised by the left invariant first order differential forms on $\mathbb{H}^n$. By the duality between $\mathfrak{h}^n$ and $\mathfrak{h}^*_n$ the map Ad generates the *co-adjoint representation* [15, § 15.1] $\mathrm{Ad}^* : \mathfrak{h}^*_n \to \mathfrak{h}^*_n$:

$$\mathrm{Ad}^*(s, x, y) : (h, q, p) \mapsto (h, q + hy, p - hx), \quad \text{where } (s, x, y) \in \mathbb{H}^n \qquad (2.6)$$

and $(h, q, p) \in \mathfrak{h}^*_n$ in bi-orthonormal coordinates to the exponential ones on $\mathfrak{h}^n$. These coordinates h, q, p should have units of an *action* ML^2/T, *coordinates* L, and *momenta* LM/T correspondingly. Again nothing in (2.6) violates the Rule 1.1.

There are two types of orbits in (2.6) for Ad^*: the Euclidean spaces $\mathbb{R}^{2n}$ and single points:

$$\begin{aligned} \mathcal{O}_h &= \{(h, q, p) : \text{ for a fixed } h \neq 0 \text{ and all } (q, p) \in \mathbb{R}^{2n}\}, && (2.7)\\ \mathcal{O}_{(q,p)} &= \{(0, q, p) : \text{ for a fixed } (q, p) \in \mathbb{R}^{2n}\}. && (2.8) \end{aligned}$$

The *orbit method* of Kirillov [15, § 15], [17] starts from the observation that the above orbits parametrise all irreducible unitary representations of $\mathbb{H}^n$. All representations are *induced* [15, § 13] by a character $\chi_h(s, 0, 0) = e^{2\pi \mathrm{i} hs}$ of the centre of $\mathbb{H}^n$ generated by $(h, 0, 0) \in \mathfrak{h}^*_n$ and shifts (2.6) from the "left hand side" (i.e. by g^{-1}) on orbits. Using [15, § 13.2, Prob. 5] we get a neat formula, which (unlike some other in literature, e.g. [28, Chap. 1, (2.23)]) respects the Rule 1.1 for all physical units:

$$\rho_h(s, x, y) : f_h(q, p) \mapsto e^{-2\pi \mathrm{i}(hs + qx + py)} f_h\left(q - \frac{h}{2}y, p + \frac{h}{2}x\right). \qquad (2.9)$$

Exactly the same formula is obtained if we apply the Fourier transform $\hat{\ } : L_2(\mathbb{H}^n) \to L_2(\mathfrak{h}^*_n)$ given by:

$$\hat{\phi}(F) = \int_{\mathfrak{h}^n} \phi(\exp X) e^{-2\pi \mathrm{i}\langle X, F\rangle}\, dX \qquad \text{where } X \in \mathfrak{h}^n,\ F \in \mathfrak{h}^*_n \qquad (2.10)$$

to the left regular action (2.5), see [17, § 2.3] for relations of the Fourier transform (2.10) and the orbit method.

The derived representation $d\rho_h$ of the Lie algebra $\mathfrak{h}^n$ defined on the vector fields (2.3) is:

$$d\rho_h(S) = -2\pi \mathrm{i} hI,\ d\rho_h(X_j) = \frac{h}{2}\partial_{p_j} - 2\pi \mathrm{i} q_j I,\ d\rho_h(Y_j) = -\frac{h}{2}\partial_{q_j} - 2\pi \mathrm{i} p_j I, \quad (2.11)$$

which clearly represent the commutation rules (2.4). The representation ρ_h (2.9) is reducible on whole $L_2(\mathcal{O}_h)$ as can be seen from the existence of the set of "right-invariant", i.e. commuting with (2.11), differential operators:

$$d\rho_h^r(S) = 2\pi\mathrm{i}hI, \quad d\rho_h^r(X_j) = -\frac{h}{2}\partial_{p_j} - 2\pi\mathrm{i}q_jI, \quad d\rho_h^r(Y_j) = \frac{h}{2}\partial_{q_j} - 2\pi\mathrm{i}p_jI, \tag{2.12}$$

which also represent the commutation rules (2.4).

To obtain an irreducible representation defined by (2.9) we need to restrict it to a subspace of $L_2(\mathcal{O}_h)$ where operators (2.12) acts as scalars, e.g. use a *polarisation* from *geometric quantisation* [30]. Consider for $h > 0$ the vector field $-X_j + \mathrm{i}c_\mathrm{i}Y_j$ from the complexification of $\mathfrak{h}^n$, where a constant c_i has the dimension T/M in order to satisfy the Rule 1.1, the numerical value of c_i in given units can be assumed 1. We introduce operators D_h^j, $1 \le j \le n$ representing vectors $-X_j + \mathrm{i}c_\mathrm{i}Y_j$:

$$D_h^j = d\rho_h^r(-X_j + \mathrm{i}c_\mathrm{i}Y_j) = \frac{h}{2}(\partial_{p_j} + c_\mathrm{i}\mathrm{i}\partial_{q_j}) + 2\pi(c_\mathrm{i}p_j + \mathrm{i}q_j)I = h\partial_{\bar{z}_j} + 2\pi z_jI \tag{2.13}$$

where $z_j = c_\mathrm{i}p_j + \mathrm{i}q_j$. For $h < 0$ we define $D_h^j = d\rho_h^r(-c_\mathrm{i}Y_j + \mathrm{i}X_j)$. Operators (2.13) are used to give the following classical result in terms of orbits:

Theorem 2.2 (Stone–von Neumann, cf. [11, Chap. 1, § 5], [15, § 18.4])**.** *All unitary irreducible representations of $\mathbb{H}^n$ are parametrised up to equivalence by two classes of orbits* (2.7) *and* (2.8) *of co-adjoint representation* (2.6) *in $\mathfrak{h}_n^*$:*

(1) *The infinite dimensional representations by transformation ρ_h* (2.9) *for $h \neq 0$ in Fock* [11, 13] *space $F_2(\mathcal{O}_h) \subset L_2(\mathcal{O}_h)$ of null solutions to the operators D_h^j* (2.13)*:*

$$F_2(\mathcal{O}_h) = \{f_h(q,p) \in L_2(\mathcal{O}_h) \mid D_h^j f_h = 0,\ 1 \le j \le n\}. \tag{2.14}$$

(2) *The one-dimensional representations as multiplication by a constant on $\mathbb{C} = L_2(\mathcal{O}_{(q,p)})$ which drops out from* (2.9) *for $h = 0$:*

$$\rho_{(q,p)}(s,x,y) : c \mapsto e^{-2\pi\mathrm{i}(qx+py)}c, \tag{2.15}$$

with the corresponding derived representation

$$d\rho_{(q,p)}(S) = 0, \qquad d\rho_{(q,p)}(X_j) = -2\pi\mathrm{i}q_j, \qquad d\rho_{(q,p)}(Y_j) = -2\pi\mathrm{i}p_j. \tag{2.16}$$

2.2. **Structure and Topology of the Unitary Dual of $\mathbb{H}^n$.** The structure of the unitary dual object to $\mathbb{H}^n$—the collection of all different classes of unitary irreducible representations—as it appears from the method of orbits is illustrated by the Figure 1, cf. with [16, Chap. 7, Fig. 6 and 7]. The adjoint space $\mathfrak{h}_n^*$ is sliced into "horizontal" hyperplanes. A plane with a parameter $h \neq 0$ forms a single orbits (2.7) and correspond to a particular class of unitary irreducible representation (2.9). The plane with parameter $h = 0$ is a family of one-point orbits $(0, q, p)$ (2.8), which produce one-dimensional representations (2.15). The topology on the dual object is the factor topology inherited from the adjoint space $\mathfrak{h}_n^*$ under the above identification, see [17, § 2.2].

Example 2.3. A set of representations ρ_h (2.9) with $h \to 0$ is dense in the whole family of one-dimensional representations (2.15), as can be seen either from the Figure 1 or the analytic expressions (2.9) and (2.15) for those representations.

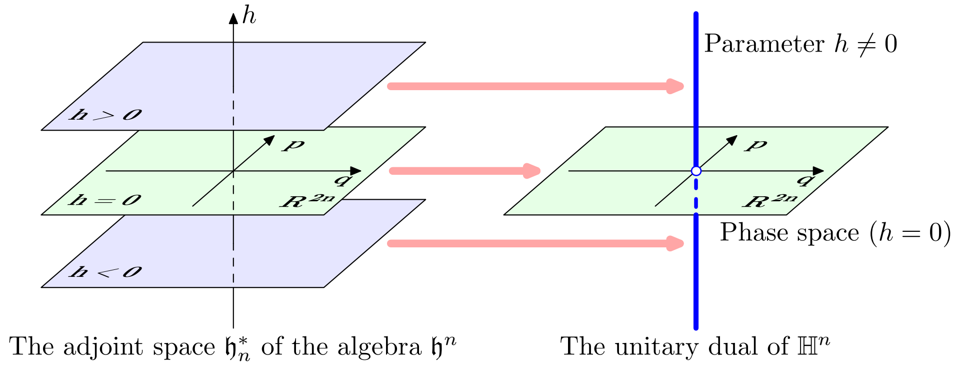

FIGURE 1. The structure of unitary dual object to $\mathbb{H}^n$ appearing from the method of orbits. The space $\mathfrak{h}_n^*$ is sliced into "horizontal" hyperplanes. Planes with $h \neq 0$ form single orbits and correspond to different classes of unitary irreducible representation. The plane $h = 0$ is a family of one-point orbits $(0, q, p)$, which produce one-dimensional representations. The topology on the dual object is the factor topology inherited from the $\mathfrak{h}_n^*$ [17, § 2.2].

Non-commutative representations ρ_h, $h \neq 0$ (2.9) are known from the very beginning to be connected with quantum mechanics [11], this explains, for example, the name of the Heisenberg group. In the contrast commutative representations (2.15) are always neglected and only mentioned for completeness in mathematical formulation of the Stone–von Neumann theorem. The development of *p-mechanics* starts [18] from the observation that the union of all representations $\rho_{(q,p)}$, $(q,p) \in \mathbb{R}^{2n}$ naturally acts as the classical *phase space*. The sensibleness of the single union

$$\mathcal{O}_0 = \bigcup_{(q,p)\in\mathbb{R}^{2n}} \mathcal{O}_{(q,p)} \tag{2.17}$$

rather than unrelated set of disconnected orbits manifests itself in several ways:

(1) The topological position of $\mathcal{O}_0$ as the limiting case (cf. Example 2.3) of quantum mechanics for $h \to 0$ realises the *correspondence principle* between the quantum and classical mechanics.
(2) Symplectic automorphisms of the Heisenberg group (see Subsection 4.3) produce the metaplectic representation in quantum mechanics and *transitively* acts by linear symplectomorphisms on the whole set $\mathcal{O}_0 \setminus \{0\}$.
(3) We got the Poisson brackets (4.7) on $\mathcal{O}_0$ from the same source (4.2) which leads to the correct Heisenberg equation in quantum mechanics.

The identification of $\mathcal{O}_0$ with the classical phase space justifies that q and p are measured by units of length and momentum respectively, that supports our choice of units for x and y in Convention 2.1.

Remark 2.4. Since unitary representations are classified up to a unitary equivalence one may think that their explicit realisations in particular Hilbert spaces are "the same". However a suitable form of a representation can give many technical advantages. The classical illustration is the paper [13], where the comparison of (unitary equivalent!) Schrödinger and Fock representations of $\mathbb{H}^n$ is the principal tool of investigation.

Our form (2.9) of representations of $\mathbb{H}^n$ given in Theorem 2.2 has at least two following advantages which are rarely combined together:

(1) There is the explicit physical meaning of all entries in (2.9) as will be seen bellow. In the contrast the formula (2.23) in [28, Chap. 1] contains terms $\sqrt{h}$ (in our notations) which could be hardly justified from a physical point of view.
(2) The one-dimension representations (2.15) explicitly correspond to the case $h = 0$ in (2.9). The Schrödinger representation (the most used in quantum mechanics!) is handicapped in this sense: the transition for $h \to 0$ from ρ_h in the Schödinger form to $\rho_{(q,p)}$ requires a long discussion [16, Ex. 7.11].

We finish the discussion of the unitary dual of $\mathbb{H}^n$ by a remark about negative value of h. Since its position in the Heisenberg equation (1.2) the negative value of $\hbar$ revert the flow of time. Thus representations ρ_h with $h < 0$ seems to be suitable for description of anti-particle with the explicit (cf. Figure 1) mirror symmetry between matter and anti-matter through the classical mechanics. In this paper however we will consider only the case of $h > 0$.

2.3. **Fock Spaces $F_2(\mathcal{O}_h)$ and Coherent States.** Our Fock type spaces (2.14) are not very different [20, Ex. 4.3] from the standard Segal–Bargmann spaces.

Definition 2.5. [11, 13] The Segal–Bargmann space (with a parameter $h > 0$) consists of functions on $\mathbb{C}^n$ which are holomorphic in z, i.e. $\partial_{\bar{z}_j} f(z) = 0$, and square integrable with respect to the measure $e^{-2|z|^2/h} dz$ on $\mathbb{C}^n$:

$$\int_{\mathbb{C}^n} |f(z)|^2 \, e^{-2|z|^2/h} dz < \infty.$$

Noticing the $\partial_{\bar{z}_j}$ component in the operator D^j_h (2.13) we obviously obtain

Proposition 2.6. *A function $f_h(q,p)$ is in $F_2(\mathcal{O}_h)$ (2.14) for $h > 0$ if and only if the function $f_h(z)e^{|z|^2/h}$, $z = p + \mathrm{i}q$ is in the classical Segal–Bargmann space.*

The space $F_2(\mathcal{O}_h)$ can be also described in the language of *coherent states*, (also known as *wavelets*, *matrix elements* of representation, *Berezin transform*, etc., see [1]). Since the representation ρ_h is irreducible any vector v_0 in $F_2(\mathcal{O}_h)$ is *cyclic*, i.e. vectors $\rho_h(g)v_0$ for all $g \in G$ span the space $F_2(\mathcal{O}_h)$. However even if all vectors are equally good in principle, some of them are more equal for particular purposes (cf. Remark 2.4). For the harmonic oscillator the preferred vector is the dimensionless *vacuum state*:

$$v_0(q,p) = \exp\left(-\frac{2\pi}{h}\left(c_{\mathrm{i}}^{-1} q^2 + c_{\mathrm{i}} p^2\right)\right), \tag{2.18}$$

which corresponds to the minimum level of energy. Here c_{i} as was defined before (2.13) has the dimensionality T/M. One can check directly the validity of the both equation (2.14) and Rule 1.1 for (2.18), particularly the exponent is taken from a dimensionless pure number. Note also that $v_0(q,p)$ is destroyed by the *annihilation operators* (sf. (2.11) and (2.13)):

$$A^j_h = d\rho_h(X_j + \mathrm{i}c_{\mathrm{i}} Y_j) = \frac{h}{2}(\partial_{p_j} - \mathrm{i}c_{\mathrm{i}}\partial_{q_j}) + 2\pi(c_{\mathrm{i}} p_j - \mathrm{i}q_j)I. \tag{2.19}$$

We introduce a dimensionless inner product on $F(\mathcal{O}_h)$ by the formula:

$$\langle f_1, f_2 \rangle = \left(\frac{4}{h}\right)^n \int_{\mathbb{R}^{2n}} f_1(q,p)\, \bar{f}_2(q,p)\, dq\, dp \tag{2.20}$$

With respect to this product the vacuum vector (2.18) is normalised: $\|v_0\| = 1$. For any observable A the formula

$$\langle A v_0, v_0 \rangle = \left(\frac{4}{h}\right)^n \int_{\mathbb{R}^{2n}} A v_0(q,p)\, \bar{v}_0(q,p)\, dq\, dp$$

gives an expectation in the units of A since both the vacuum vector $v_0(q,p)$ and the inner product (2.20) are dimensionless. The term h^{-n} in (2.20) not only normalises the vacuum and fixes the dimensionality of the inner product; it is also related to the *Plancherel measure* [11, (1.61)], [28, Chap. 1, Th. 2.6] on the unitary dual of $\mathbb{H}^n$.

Elements $(s,0,0)$ of the centre of $\mathbb{H}^n$ trivially act in the representation ρ_h (2.9) as multiplication by scalars, e.g. any function is a common eigenvector of all operators $\rho_h(s,0,0)$. Thus the essential part of the operator $\rho_h(s,x,y)$ is determined solely by $(x,y) \in \mathbb{R}^{2n}$. The *coherent states* $v_{(x,y)}(q,p)$ are "left shifts" of the vacuum vector $v_0(q,p)$ by operators (2.9):

$$\begin{aligned} v_{(x,y)}(q,p) &= \rho_h(0,x,y) v_0(q,p) \qquad\qquad (2.21)\\ &= \exp\left(-2\pi \mathrm{i}(qx+py) - \frac{2\pi}{h}\left(c_{\mathrm{i}}^{-1}\left(q - \frac{h}{2}y\right)^2 + c_{\mathrm{i}}\left(p + \frac{h}{2}x\right)^2\right)\right). \end{aligned}$$

Now any function from the space $F_2(\mathcal{O}_h)$ can be represented [20, Ex. 4.3] as a linear superposition of coherent states:

$$\begin{aligned} f(q,p) = [\mathcal{M}_h \breve{f}](q,p) &= h^n \int_{\mathbb{R}^{2n}} \breve{f}(x,y) v_{(x,y)}(q,p)\, dx\, dy \qquad (2.22)\\ &= h^n \int_{\mathbb{R}^{2n}} \breve{f}(x,y) \rho_h(x,y)\, dx\, dy\, v_{(0,0)}(q,p) \end{aligned}$$

where $\breve{f}(x,y)$ is the *wavelet transform* [1, 20] of $f(q,p)$:

$$\begin{aligned} \breve{f}(x,y) = [\mathcal{W}_h f](x,y) &= \left\langle f, v_{(x,y)} \right\rangle_{F_2(\mathcal{O}_h)} \qquad (2.23)\\ &= \left(\frac{4}{h}\right)^n \int_{\mathbb{R}^{2n}} f(q,p) \bar{v}_{(x,y)}(q,p)\, dq\, dp. \end{aligned}$$

The formula (2.22) can be regarded [20] as the *inverse wavelet transform* $\mathcal{M}$ of $\breve{f}(x,y)$. Note that all above integrals are dimensionless and both the wavelet transform and its inverse are measured in the same units.

The straightforward use of the basic formula:

$$\int_{-\infty}^{\infty} \exp(-ax^2 + bx + c)\, dx = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{4a} + c\right), \qquad \text{where } a > 0. \tag{2.24}$$

for the wavelet transform (2.22) leads to:

$$\breve{v}_0(s,x,y) = \exp 2\pi \left(\mathrm{i}hs - \frac{h}{4}\left(c_{\mathrm{i}} x^2 + c_{\mathrm{i}}^{-1} y^2\right)\right). \tag{2.25}$$

Since the wavelet transform $\mathcal{W}_h$ (2.22) intertwines [20] ρ_h (2.9) with the left regular representation λ_l (2.5):

$$\mathcal{W}_h \circ \rho_h(g) = \lambda_l(g) \circ \mathcal{W}_h, \qquad \text{for all } g \in \mathbb{H}^n$$

the image of an arbitrary coherent state is:

$$\breve{v}_{(s',x',y')}(s,x,y) = \tag{2.26}$$
$$\exp 2\pi \left(\mathrm{i}h \left(s - s' - \frac{1}{2}(x'y - xy') \right) - \frac{h}{4} \left(c_{\mathrm{i}}(x-x')^2 + c_{\mathrm{i}}^{-1}(y-y')^2 \right) \right).$$

Needless to say that this functions are obeying Rule 1.1 and Remark 1.2.

We should mention however a problem related to coherent states (2.21): all their "classical limit" for $h \to 0$ are functions with a support in the neighbourhood of $(0,0)$.In the contrast we may them be supported around different classical states (q,p). This difficulty can be resolved through a replacement of the group action of $\mathbb{H}^n$ in (2.21) by the "shifts" (4.8) generated by the p-mechanical brackets (4.3).

3. p-Mechanics: Statics

We define p-mechanical observables to be convolutions on the Heisenberg group. The next Subsection describes their multiplication and commutator as well as quantum and classical representations. The Berezin quantisation in form of wavelet transform is considered in Subsection 3.2. This is developed in Subsection 3.3 into a construction of p-observables out of either quantum or classical ones.

3.1. Observables in p-Mechanics, Convolutions and Commutators.

In line with the standard quantum theory we give the following definition:

Definition 3.1. *Observables* in p-mechanics (p-observables) are presented by operators on $L_2(\mathbb{H}^n)$.

Actually we will need here[1] only operators generated by convolutions on $L_2(\mathbb{H}^n)$. Let dg be a left invariant measure [15, § 7.1] on $\mathbb{H}^n$, which coincides with the standard Lebesgue measure on $\mathbb{R}^{2n+1}$ in the exponential coordinates (s,x,y). Then a function k_1 from the linear space $L_1(\mathbb{H}^n, dg)$ acts on $k_2 \in L_2(\mathbb{H}^n, dg)$ by a convolution as follows:

$$\begin{aligned} (k_1 * k_2)(g) &= c_h^{n+1} \int_{\mathbb{H}^n} k_1(g_1)\, k_2(g_1^{-1}g)\, dg_1 \\ &= c_h^{n+1} \int_{\mathbb{H}^n} k_1(gg_1^{-1})\, k_2(g_1)\, dg_1. \end{aligned} \tag{3.1}$$

where the constant c_h has the value 1 in the units of action. Then c_h^{n+1} has units inverse to dg. Thus the convolution $k_1 * k_2$ is measured in units which are product of units for k_1 and k_2. The composition of two convolution operators K_1 and K_2 with kernels k_1 and k_2 has the kernel defined by the same formula (3.1). Clearly two products K_1K_2 and K_2K_1 could have a different value due to non-commutativity of $\mathbb{H}^n$ but always are measured in the same units. Thus we can find out how distinct

[1] More general operators are in use for a string-like version of p-mechanics, see Subsection 5.2.3.

they are from the difference $K_1K_2 - K_2K_1$ which does not violate the Rule 1.1. This also produces *inner derivations* D_k of $L_1(\mathbb{H}^n)$ by the *commutator*:

$$\begin{aligned} D_k : f \mapsto [k, f] &= k * f - f * k \\ &= c_h^{n+1} \int_{\mathbb{H}^n} k(g_1) \left(f(g_1^{-1}g) - f(gg_1^{-1})\right) dg_1. \end{aligned} \tag{3.2}$$

Because we only consider observables which are convolutions on $\mathbb{H}^n$ we can extend a unitary representation ρ_h of $\mathbb{H}^n$ to a $*$-representation $L_1(\mathbb{H}^n, dg)$ by the formula:

$$\begin{aligned} [\rho_h(k)f](q,p) &= c_h^{n+1} \int_{\mathbb{H}^n} k(g)\rho_h(g)f(q,p)\,dg \\ &= c_h^{n} \int_{\mathbb{R}^{2n}} \left(c_h \int_{\mathbb{R}} k(s,x,y)e^{-2\pi \mathrm{i} hs}\,ds\right) \\ &\qquad \times e^{-2\pi\mathrm{i}(qx+py)} f\left(q - \frac{h}{2}y, p + \frac{h}{2}x\right) dx\,dy. \end{aligned} \tag{3.3}$$

The last formula in the Schrödinger representation defines for $h \neq 0$ an *pseudodifferential operator* [11, 13, 27] on $L_2(\mathbb{R}^n)$ (2.14), which are known to be *quantum observables* in the Weyl quantisation. For representations $\rho_{(q,p)}$ (2.15) the expression analogous to (3.3) defines an operator of multiplication on $\mathcal{O}_0$ (2.17) by the Fourier transform of $k(s, x, y)$:

$$\rho_{(q,p)}(k) = \hat{k}\,(0, q, p) = c_h^{n+1} \int_{\mathbb{H}^n} k(s,x,y)e^{-2\pi\mathrm{i}(qx+py)}\,ds\,dx\,dy, \tag{3.4}$$

where the direct $\hat{}$ and inverse $\check{}$ Fourier transforms are defined by the formulae:

$$\hat{f}(v) = \int_{\mathbb{R}^m} f(u)e^{-2\pi\mathrm{i}uv}\,du \qquad \text{and} \qquad f(u) = (\hat{f})\check{}\,(u) = \int_{\mathbb{R}^m} \hat{f}(v)e^{2\pi\mathrm{i}vu}\,dv.$$

For reasons discussed in subsections 2.2 and 4.1 we regard the functions (3.4) on $\mathcal{O}_0$ as *classical observables*. Again the both representations $\rho_h(k)$ and $\rho_{(q,p)}k$ are measured in the same units as function k.

From (3.3) follows that $\rho_h(k)$ for a fixed $h \neq 0$ depends only from $\hat{k}_s(h, x, y)$—the partial Fourier transform $s \to h$ of $k(s, x, y)$. Then the representation of the composition of two convolutions depends only from

$$\begin{aligned} (k' * k)\hat{}_s &= c_h \int_{\mathbb{R}} e^{-2\pi\mathrm{i}hs}\, c_h^{n+1} \int_{\mathbb{H}^n} k'(s',x',y') \\ &\qquad \times k(s - s' + \frac{1}{2}(xy' - yx'), x - x', y - y')\,ds'dx'dy'ds \\ &= c_h^n \int_{\mathbb{R}^{2n}} e^{\pi\mathrm{i}h(xy'-yx')} \cdot c_h \int_{\mathbb{R}} e^{-2\pi\mathrm{i}hs'} k'(s',x',y')\,ds' \\ &\qquad \times c_h \int_{\mathbb{R}} e^{-2\pi\mathrm{i}h(s-s'+\frac{1}{2}(xy'-yx'))} \\ &\qquad \times k(s - s' + \frac{1}{2}(xy' - yx'), x - x', y - y')\,ds\,dx'dy' \\ &= c_h^n \int_{\mathbb{R}^{2n}} e^{\pi\mathrm{i}h(xy'-yx')} \hat{k}'_s(h,x',y')\hat{k}_s(h, x - x', y - y')\,dx'dy'. \end{aligned} \tag{3.5}$$

Note that if we apply the Fourier transform $(x,y) \to (q,p)$ to the last expression (3.5) then we get the *star product* of $\hat{k}'$ and $\hat{k}$ known in *deformation* quantisation, cf. [31, (9)–(13)]. Consequently the representation $\rho_h([k',k])$ of commutator (3.2) depends only from:

$$\begin{aligned}[k',k]\hat{}_s = c_h^n \int_{\mathbb{R}^{2n}} \left(e^{\mathrm{i}\pi h(xy'-yx')} - e^{-\mathrm{i}\pi h(xy'-yx')}\right) \\ \times \hat{k}'_s(-h,x',y')\hat{k}_s(-h,x-x',y-y')\,dx'dy' \\ = \quad 2\mathrm{i}c_h^n \int_{\mathbb{R}^{2n}} \sin\left(\pi h(xy'-yx')\right) \hat{k}'_s(h,x',y')\hat{k}_s(h,x-x',y-y')\,dx'dy'.\end{aligned} \tag{3.6}$$

The integral (3.6) turns to be equivalent to the *Moyal brackets* [31] for the (full) Fourier transforms of k' and k. It is commonly accepted that method of orbit is a mathematical side of *geometric* quantisation [30]. Our derivation of the Moyal brackets in terms of orbits shows that deformation and geometric quantisations are closely connected and both are not very far from the original quantisation of Heisenberg and Schrödinger. Yet one more their close relative can be identified as the Berezin quantisation [5], see the next Subsection.

Remark 3.2. The expression (3.6) vanishes for $h = 0$ as can be expected from the commutativity of representations (2.15). Thus it does not produce anything interesting on $\mathcal{O}_0$, that supports the common negligence to this set.

Summing up, p-mechanical observables, i.e. convolutions on $L_2(\mathbb{H}^n)$ are transformed

(1) by representations ρ_h (2.9) into quantum observables (3.3) with the Moyal bracket (3.6) between them;
(2) by representations $\rho_{(q,p)}$ (2.15) into classical observables (3.4).

We did not get a meaningful brackets on classical observables yet, this will be done in Section 4.1.

3.2. **Berezin Quantisation and Wavelet Transform.** There is the following construction, known as the *Berezin quantisation*, which allows to assign a function to an operator (observable) and an operator to a function. The scheme is based on the construction of the *coherent states*, which can be derived from different sources [24, 25]. We prefer the group-theoretic origin of Perelomov coherent states [25] which is realised in (2.21). Following [5] we introduce a *covariant* symbol $a(g)$ of an operator A on $F_2(\mathcal{O}_h)$ by the simple expression:

$$a(g) = \langle Av_g, v_g \rangle, \tag{3.7}$$

i.e. we get a map from the linear space of operators on $F_2(\mathcal{O}_h)$ to a linear space of function on $\mathbb{H}^n$. A map in the opposite direction assigns to a function $\breve{a}(g)$ on $\mathbb{H}^n$ a linear operator A on $F_2(\mathcal{O}_h)$ by the formula

$$A = c_h^{n+1} \int_{\mathbb{H}^n} \mathring{a}(g) P_g\, dg, \qquad \text{where } P_g \text{ is the projection } P_g v = \langle v, v_g \rangle v_g. \tag{3.8}$$

The function $\mathring{a}(g)$ is called the *contravariant* symbol of the operator A (3.8).

The co- and contravariant symbols of operators are defined through the coherent states, in fact both types of symbols are realisations [20, § 3.1] of the direct (2.23) and inverse (2.22) wavelet transforms. Let us define a representation ρ_{bh} of the group $\mathbb{H}^n \times \mathbb{H}^n$ in the space $\mathcal{B}(F_2(\mathcal{O}_h))$ of operators on $F_2(\mathcal{O}_h)$ by the formula:

$$\rho_{bh}(g_1, g_2) : A \mapsto \rho_h(g_1^{-1})A\rho_h(g_2), \qquad \text{where } g_1, g_2 \in \mathbb{H}^n. \tag{3.9}$$

According to the scheme from [20] for any state f_0 on $\mathcal{B}(F_2(\mathcal{O}_h))$ we get a wavelet transform $\mathcal{W}_{f_0} : \mathcal{B}(F_2(\mathcal{O}_h)) \to C(\mathbb{H}^n \times \mathbb{H}^n)$:

$$\mathcal{W}_{f_0} : A \mapsto \breve{a}(g_1, g_2) = \langle \rho_{bh}(g_1, g_2)A, f_0 \rangle . \tag{3.10}$$

The important particular case is given by f_0 defined through the vacuum vector v_0 (2.18) by the formula $\langle A, f_0 \rangle = \langle Av_0, v_0 \rangle$. Then the wavelet transform (3.10) produce the *covariant presymbol* $\breve{a}(g_1, g_2)$ of operator A. Its restriction $a(g) = \breve{a}(g, g)$ to the diagonal D of $\mathbb{H}^n \times \mathbb{H}^n$ is exactly [20] the Berezin covariant symbol (3.7) of A. Such a restriction to the diagonal is done without a lost of information due to holomorphic properties of $\breve{a}(g_1, g_2)$ [3].

Another important example of the state f_0 is given by the trace:

$$\langle A, f_0 \rangle = \operatorname{tr} A = h^n \int_{\mathbb{R}^{2n}} \left\langle Av_{(x,y)}, v_{(x,y)} \right\rangle_{F_2(\mathcal{O}_h)} \, dx\, dy, \tag{3.11}$$

where coherent states $v_{(x,y)}$ are again defined in (2.21). The operators $\rho_{bh}(g, g)$ from the diagonal D of $\mathbb{H}^n \times \mathbb{H}^n$ trivially acts on the wavelet transform (3.10) generated by the trace (3.11) since the trace is invariant under $\rho_{bh}(g, g)$. According to the general scheme we can consider *reduced wavelet transform* [20] to the homogeneous space $\mathbb{H}^n \times \mathbb{H}^n/D$ instead of the entire group $\mathbb{H}^n \times \mathbb{H}^n$. The space $\mathbb{H}^n \times \mathbb{H}^n/D$ is isomorphic to $\mathbb{H}^n$ with an embedding $\mathbb{H}^n \to \mathbb{H}^n \times \mathbb{H}^n$ given by $g \mapsto (g; 0)$. Furthermore the centre Z of $\mathbb{H}^n$ acts trivially in the representation ρ_{bh} as usual. Thus the only essential part of $\mathbb{H}^n \times \mathbb{H}^n/D$ in the wavelet transform is the homogeneous space $\Omega = \mathbb{H}^n/Z$. A Borel section $\mathbf{s} : \Omega \to \mathbb{H}^n \times \mathbb{H}^n$ in the principal bundle $G \to \Omega$ can be defined as $\mathbf{s}(x, y) \mapsto ((0, x, y); (0, 0, 0))$. We got the reduced realisation $\mathcal{W}_r$ of the wavelet transform (3.10) in the form:

$$\begin{aligned}
\mathcal{W}_r : A \mapsto \breve{a}_r(x, y) &= \langle \rho_{bh}(\mathbf{s}(x, y))A, f_0 \rangle \\
&= \operatorname{tr} (\rho_h((0, x, y)^{-1})A) \qquad (3.12)\\
&= h^n \int_{\mathbb{R}^{2n}} \left\langle \rho_h((0, x, y)^{-1})Av_{(x',y')}, v_{(x',y')} \right\rangle_{F_2(\mathcal{O}_h)} \, dx'\, dy' \\
&= h^n \int_{\mathbb{R}^{2n}} \left\langle Av_{(x',y')}, v_{(x,y)\cdot(x',y')} \right\rangle_{F_2(\mathcal{O}_h)} \, dx'\, dy'. \qquad (3.13)
\end{aligned}$$

The formula (3.12) is the principal ingredient of the *inversion formula* for the Heisenberg group [11, Chap. 1, (1.60)], [28, Chap. 1, Th. 2.7], which reconstructs the kernel of convolution $k(g)$ out of operators $\rho_h(k)$. Therefore if we define the mother wavelet to be the identity operator I the inverse wavelet transform (cf. (2.22)) will be

$$\begin{aligned}
\mathcal{M}_r \breve{a}_r &= h^n \int_{\mathbb{R}^{2n}} a(x, y)\rho_{bh}(\mathbf{s}((0, x, y)^{-1}))I \, dx\, dy \qquad (3.14)\\
&= h^n \int_{\mathbb{R}^{2n}} a(x, y)\rho_h(0, x, y) \, dx\, dy.
\end{aligned}$$

The inversion formula for $\mathbb{H}^n$ insures that

Proposition 3.3. *The composition $\mathcal{M}_r \circ \mathcal{W}_r$ is the identity map on the representations $\rho_h(k)$ of convolution operators on $\mathcal{O}_h$.*

Example 3.4. The wavelet transform $\mathcal{W}_r$ (3.13) applied to the quantum coordinate $Q = d\rho_h(X)$, momentum $P = d\rho_h(Y)$ (see (2.11)), and the energy function of the harmonic oscillator $(c_1Q^2 + c_2P^2)/2$ produces the distributions on $\mathbb{R}^{2n}$:

$$\begin{aligned} Q &\mapsto \frac{1}{2\pi\mathrm{i}}\delta^{(1)}(x)\delta(y), \\ P &\mapsto \frac{1}{2\pi\mathrm{i}}\delta(x)\delta^{(1)}(y), \\ \frac{1}{2}\left(c_1Q^2 + c_2P^2\right) &\mapsto -\frac{1}{8\pi^2}\left(c_1\delta^{(2)}(x)\delta(y) + c_2\delta(x)\delta^{(2)}(y)\right), \end{aligned}$$

where $\delta^{(1)}$ and $\delta^{(2)}$ are the first and second derivatives of the Dirac *delta function* δ respectively. The constants c_1 and c_2 have units M/T^2 and $1/M$ correspondingly. We will use them later in Example 3.7.

3.3. **From Classical and Quantum Observables to p-Mechanics.** It is commonly accepted that we can not deal with quantum mechanics directly and thus the classical dynamics serves as an unavoidable intermediate step. The passage from classical observables to quantum ones—known as a *quantisation*—is the huge field with many concurring approaches (geometric, deformation, Weyl, Berezin, etc. quantisations) each having its own merits and demerits. Similarly one has to construct p-mechanical observables starting from classical or quantum ones by some procedure (should it be named "p-mechanisation"?), which we about to describe now.

The transition from a p-mechanical observable to a classical one is given by the formula (3.4), which in a turn is a realisation of inverse wavelet transform (2.22):

$$\rho_{(q,p)}k = \hat{k}\,(0,q,p) = c_h^{n+1} \int_{\mathbb{H}^n} k(s,x,y) e^{-2\pi\mathrm{i}(qx+py)}\, ds\, dx\, dy. \tag{3.15}$$

Just like in the case of quantisation the classical image $\rho_{(q,p)}k$ (3.15) contains only a partial information about p-observable k unless we will make some additional assumptions. Let us start from a classical observable $c(q,p)$ and try to construct the corresponding p-observable. As follows from general consideration (see [20] and Section 2.3) we can partially invert the formula (3.15) by the wavelet transform (2.23):

$$\check{c}(x,y) = [\mathcal{W}_0 c](x,y) = \left\langle c v_{(0,0)}, v_{(x,y)} \right\rangle = c_h^n \int_{\mathbb{R}^{2n}} c(q,p) e^{2\pi\mathrm{i}(qx+py)}\, dq\, dp, \tag{3.16}$$

where $v_{(x,y)} = \rho_{(q,p)} v_{(0,0)} = e^{-2\pi\mathrm{i}(qx+py)}$.

However the function $\check{c}(x,y)$ (3.16) is not defined on the entire $\mathbb{H}^n$. The natural domain of $\check{c}(x,y)$ according to the construction of the reduced wavelet transform [20] is the homogeneous space $\Omega = G/Z$, where $G = \mathbb{H}^n$ and Z is its normal subgroup of central elements $(s,0,0)$. Let $\mathbf{s} : \Omega \to G$ be a Borel section in the principal bundle $G \to \Omega$, which is used in the construction of induced representation, see [15, § 13.1]. For the Heisenberg group [20, Ex. 4.3] it can be simply defined as $\mathbf{s} : (x,y) \in \Omega \mapsto (0,x,y) \in \mathbb{H}^n$. One can naturally transfer functions from Ω to the image $\mathbf{s}(\Omega)$ of the map $\mathbf{s}$ in G. However the range $\mathbf{s}(\Omega)$ of $\mathbf{s}$ has oftenly (particularly for $\mathbb{H}^n$) a zero Haar measure in G. Probably two simplest possible ways out are:

(1) To increase the "weight" of function $\tilde{c}(s,x,y)$ vanishing outside of the range $\mathbf{s}(\Omega)$ of $\mathbf{s}$ by a suitable Dirac delta function on the subgroup Z. For the Heisenberg group this can be done, for example, by the map:

$$\mathcal{E}:\check{c}(x,y)\mapsto\tilde{c}(s,x,y)=\delta(s)\check{c}(x,y), \tag{3.17}$$

where $\check{c}(x,y)$ is given by the inverse wavelet (Fourier) transform (3.16). As we will see in Proposition 3.6 this is related to the Weyl quantisation and the Moyal brackets.

(2) To extend the function $\check{c}(x,y)$ to the entire group G by a tensor product with a suitable function on Z, for example e^{-s^2}:

$$\check{c}(x,y)\mapsto\tilde{c}(s,x,y)=e^{-s^2}\check{c}(x,y).$$

In order to get the *correspondence principle* between classical and quantum mechanics (cf. Example 2.3) the function on Z has to satisfy some additional requirements. For $\mathbb{H}^n$ it should vanish for $s\to\pm\infty$, which fulfils for both e^{-s^2} and $\delta(s)$ from the previous item. In this way we get infinitely many essentially different quantisations with non-equivalent *deformed* Moyal brackets between observables.

There are other more complicated possibilities not mentioned here, which can be of some use if some additional information or assumptions are used to extend functions from Ω to G. We will focus here only on the first "minimalistic" approach from the two listed above.

Example 3.5. The composition of the wavelet transform $\mathcal{W}_0$ (3.16) and the map $\mathcal{E}$ (3.17) applied to the classical coordinate, momentum, and the energy function of the harmonic oscillator produces the distributions on $\mathbb{H}^n$:

$$\begin{aligned}
q &\mapsto \frac{1}{2\pi\mathrm{i}}\delta(s)\delta^{(1)}(x)\delta(y), && (3.18)\\
p &\mapsto \frac{1}{2\pi\mathrm{i}}\delta(s)\delta(x)\delta^{(1)}(y), && (3.19)\\
\frac{1}{2}\left(c_1q^2+c_2p^2\right) &\mapsto -\frac{1}{8\pi^2}\left(c_1\delta(s)\delta^{(2)}(x)\delta(y)+c_2\delta(s)\delta(x)\delta^{(2)}(y)\right), && (3.20)
\end{aligned}$$

where $\delta^{(1)}$, $\delta^{(2)}$, c_1, and c_2 are defined in Example 3.4. We will use them later in the Example 4.3.

If we apply the representation ρ_h (3.3) to the function $\tilde{c}(s,x,y)$ (3.17) we will get the operator on $F_2(\mathcal{O}_h)$:

$$\begin{aligned}
\mathcal{Q}_h(c) &= c_h^{n+1}\int_{\mathbb{H}^n}\tilde{c}(s,x,y)\rho_h(s,x,y)\,ds\,dx\,dy\\
&= c_h^{n+1}\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}}\delta(s)\check{c}(x,y)e^{s\cdot d\rho_h(S)+x\cdot d\rho_h(X)+y\cdot d\rho_h(Y)}\,ds\,dx\,dy\\
&= c_h^{n+1}\int_{\mathbb{R}}\delta(s)e^{-2\pi\mathrm{i}sh}ds\int_{\mathbb{R}^{2n}}\check{c}(x,y)e^{x\cdot d\rho_h(X)+y\cdot d\rho_h(Y)}\,dx\,dy\\
&= c_h^{n}\int_{\mathbb{R}^{2n}}\check{c}(x,y)e^{x\cdot d\rho_h(X)+y\cdot d\rho_h(Y)}\,dx\,dy, && (3.21)
\end{aligned}$$

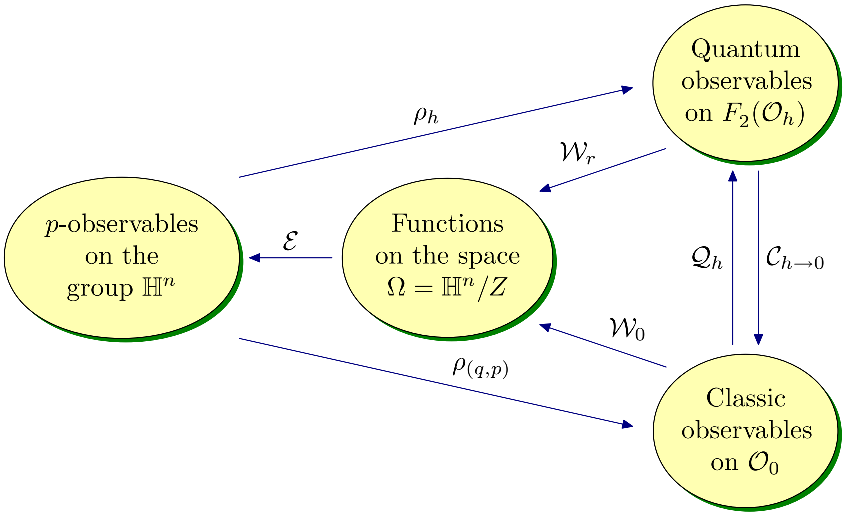

FIGURE 2. The relations between:
$\mathcal{Q}_h$—the Weyl quantisation from classical mechanics to quantum;
$\mathcal{C}_{h\to 0}$—the classical limit $h \to 0$ of quantum mechanics;
ρ_h and $\rho_{(q,p)}$—unitary representations of Heisenberg group $\mathbb{H}^n$;
$\mathcal{W}_r$ and $\mathcal{W}_0$—wavelet transforms defined in (3.12) and (3.16);
$\mathcal{E}$—extension of functions from $\Omega = \mathbb{H}^n/Z$ to the whole group $\mathbb{H}^n$.
Note the relations $\mathcal{Q}_h = \rho_h \circ \mathcal{E} \circ \mathcal{W}_0$ and $C_{h\to 0} = \rho_{(q,p)} \circ \mathcal{E} \circ \mathcal{W}_r$.

where the last expression is exactly the *Weyl quantisation* (the *Weyl correspondence* [11, § 2.1]) if the Schrödinger realisation with $d\rho_h(X) = q$ and $d\rho_h(Y) = \mathrm{i}h\partial_q$ on $L_2(\mathbb{R}^n)$ is chosen for ρ_h. Thus we demonstrate that

Proposition 3.6. *The* Weyl quantisation $\mathcal{Q}_h$ (3.21) *is the composition of the wavelet transform* (3.16)*, the extension* $\mathcal{E}$ (3.17)*, and the representation* ρ_h (2.9)*:*

$$\mathcal{Q}_h = \rho_h \circ \mathcal{E} \circ \mathcal{W}_0. \tag{3.22}$$

The similar construction can be carried out if we have a quantum observable A and wish to recover a related p-mechanical object. The wavelet transform $\mathcal{W}_r$ (3.12) maps A into the function $a(x,y)$ defined on Ω and we again face the problem of extension of $a(x,y)$ to the entire group $\mathbb{H}^n$. If it will be once more solved as in the classical case by tensor product with the delta function $\delta(s)$ then we get the following formula:

$$A \mapsto a(s,x,y) = \mathcal{E} \circ \mathcal{W}_r(A) = h^n \delta(s) \int_{\mathbb{R}^{2n}} \left\langle A v_{(x',y')}, v_{(x,y)\cdot(x',y')} \right\rangle_{F_2(\mathcal{O}_h)} \, dx'\,dy'.$$

We can apply to this function $a(s,x,y)$ the representation $\rho_{(q,p)}$ and obtain a classical observables $\rho_{(q,p)}(a)$. For a reasonable quantum observable A its classical image $\rho_{(q,p)} \circ \mathcal{E} \circ \mathcal{W}_r(A)$ will coincide with its classical limit $C_{h\to 0}A$:

$$C_{h\to 0} = \rho_{(q,p)} \circ \mathcal{E} \circ \mathcal{W}_r, \tag{3.23}$$

which is expressed here through integral transformations and does not explicitly use the limit $h \to 0$. The Figure 2 illustrates various transformations between quantum, classical, and p-observables. Besides the mentioned decompositions (3.22) and (3.23) there are presentations of identity maps on classical and quantum spaces correspondingly:

$$\mathcal{I}_c = \rho_{(q,p)} \circ \mathcal{E} \circ \mathcal{W}_0, \qquad \mathcal{I}_h = \rho_h \circ \mathcal{E} \circ \mathcal{W}_h.$$

Example 3.7. The wavelet transform $\mathcal{W}_r$ applied to the quantum coordinate Q, momentum P, and the energy function of the harmonic oscillator $(c_1Q^2 + c_2P^2)/2$ was calculated in Example 3.4. A composition with the above map $\mathcal{E}$ yields the distributions:

$$\begin{aligned} Q &\mapsto \frac{1}{2\pi \mathrm{i}}\delta(s)\delta^{(1)}(x)\delta(y), \\ P &\mapsto \frac{1}{2\pi \mathrm{i}}\delta(s)\delta^{(1)}(x)\delta(y), \\ \frac{1}{2}\left(c_1Q^2 + c_2P^2\right) &\mapsto -\frac{1}{2\pi^2}\left(c_1\delta(s)\delta^{(2)}(x)\delta(y) + c_2\delta(s)\delta(x)\delta^{(2)}(y)\right), \end{aligned}$$

which are exactly the same as in the Example 3.5.

4. p-Mechanics: Dynamics

We introduce the p-mechanical brackets which suits to all essential physical requirements and have a non-trivial classical representation coinciding with the Poisson brackets. A consistent p-mechanical dynamic equations is given in Subsection 4.2 and is analysed for a harmonic oscillator. Symplectic automorphisms of the Heisenberg groups produce symplectic symmetries of p-mechanical, quantum, and classical dynamics in Subsection 4.3.

4.1. **p-Mechanical Brackets on $\mathbb{H}^n$.** Having observables as convolutions on $\mathbb{H}^n$ we need a dynamic equation for their evolution. To this end we seek a time derivative generated by the observable associated with energy.

Remark 4.1. The first candidate is the derivation coming from commutator (3.2). However the straight commutator has at least two failures:

(1) It can not produce any dynamics on $\mathcal{O}_0$ (2.17), see Remark 3.2.
(2) It violate the Rule 1.1 as indicated bellow.

As mentioned in Section 1.2 the classical energy is measured in ML^2/T^2 so does the p-mechanical energy E. Consequently the commutator $[E,\cdot]$ (3.2) with the p-energy has units ML^2/T^2 whereas the time derivative should be measured in $1/T$, i.e. the mismatch is in units of action ML^2/T.

Fortunately, there is a possibility to fix the both above defects of the straight commutator at the same time. Let us define a multiple $\mathcal{A}$ of a right inverse operator to the vector field S (2.3) on $\mathbb{H}^n$ by its actions on exponents—characters of the centre $Z \in \mathbb{H}^n$:

$$S\mathcal{A} = 4\pi^2 I, \qquad \text{where} \quad \mathcal{A}e^{2\pi \mathrm{i} hs} = \begin{cases} \dfrac{2\pi}{\mathrm{i}h}e^{2\pi \mathrm{i} hs}, & \text{if } h \neq 0, \\ 4\pi^2 s, & \text{if } h = 0. \end{cases} \tag{4.1}$$

An alternative definition of $\mathcal{A}$ as a convolution with a distribution is given in [22].

We can extend $\mathcal{A}$ by the linearity to the entire space $L_1(\mathbb{H}^n)$. As a multiplier of a right inverse to S the operator $\mathcal{A}$ is measured in $T/(ML^2)$—exactly that we need to correct the second of above mentioned defect of the straight commutator. Thus we introduce [22] a modified convolution operation $\star$ on $L_1(\mathbb{H}^n)$:

$$k' \star k = (k' * k)\mathcal{A} \tag{4.2}$$

and the associated modified commutator (p-mechanical brackets):

$$\{[k', k]\} = [k', k]\mathcal{A} = k' \star k - k \star k'. \tag{4.3}$$

Obviously (4.3) is a bilinear antisymmetric form on the convolution kernels. It was also demonstrated in [22] that p-mechanical brackets satisfy to the Leibniz and Jacoby identities. They are all important for a consistent dynamics [7] along with the dimensionality condition given in the beginning of this Subsection.

From (3.3) one gets $\rho_h(\mathcal{A}k) = \frac{2\pi}{ih}\rho_h(k)$ for $h \neq 0$. Consequently the modification of the commutator for $h \neq 0$ is only slightly different from the original one:

$$\rho_h \{[k', k]\} = \frac{1}{\mathrm{i}\hbar}[\rho_h(k'), \rho_h(k)], \qquad \text{where } \hbar = \frac{h}{2\pi} \neq 0. \tag{4.4}$$

The integral representation of modified commutator kernel become (cf. (3.6)):

$$\{[k', k]\}\hat{}_s = c_h^n \int_{\mathbb{R}^{2n}} \frac{4\pi}{h} \sin\left(\pi h(xy' - yx')\right) \hat{k}'_s(h, x', y')\hat{k}_s(h, x - x', y - y')\, dx'dy', \tag{4.5}$$

where we may understand the expression under the integral as

$$\frac{4\pi}{h} \sin\left(\pi h(xy' - yx')\right) = 4\pi^2 \sum_{k=1}^{\infty} (-1)^{k+1} (\pi h)^{2(k-1)} \frac{(xy' - yx')^{2k-1}}{(2k-1)!} \tag{4.6}$$

This makes the operation (4.5) for $h = 0$ significantly distinct from the vanishing integral (3.6). Indeed it is natural to assign the value $4\pi^2(xy' - yx')$ to (4.6) for $h = 0$. Then the integral in (4.5) becomes the Poisson brackets for the Fourier transforms of k' and k defined on $\mathcal{O}_0$ (2.17):

$$\rho_{(q,p)} \{[k', k]\} = \frac{\partial \hat{k}'(0, q, p)}{\partial q} \frac{\partial \hat{k}(0, q, p)}{\partial p} - \frac{\partial \hat{k}'(0, q, p)}{\partial p} \frac{\partial \hat{k}(0, q, p)}{\partial q}. \tag{4.7}$$

The same formula is obtained [22, Prop. 3.5] if we directly calculate $\rho_{(q,p)} \{[k', k]\}$ rather than resolve the indeterminacy for $h = 0$ in (4.6). This means the continuity of our construction at $h = 0$ and represents the *correspondence principle* between quantum and classical mechanic.

We saw that the remedy of the second failure of commutator in Remark 4.1 (which was our duty according to Rule 1.1) by the antiderivative (4.1) improves the first defect as well (which is a very pleasant and surprising bonus). There are probably much simpler ways to fix the dimensionality of commutator "by hands". However not all of them obviously will produce the Poisson brackets on $\mathcal{O}_0$ as the antiderivative (4.1).

We arrived to the following answer to the Problem 1.3: *Poisson brackets and inverse of the Planck constant* $1/h$ *have the same dimensionality because they are image of the same object (anti-derivative* (4.1)*) under different representations* (2.9) *and* (2.15) *of the Heisenberg group.*

Note that functions $X = \delta(s)\delta^{(1)}(x)\delta(y)$ and $Y = \delta(s)\delta(x)\delta^{(1)}(y)$ (see (3.18) and (3.19)) on $\mathbb{H}^n$ are measured in units L and ML^2/T (inverse to x and y) correspondingly as respective derivatives of the dimensionless function $\delta(s)\delta(x)\delta(y)$.

Then the p-mechanical brackets $\{[X,\cdot]\}$ and $\{[Y,\cdot]\}$ with those functions have dimensionality of $T/(ML^2)$ and $1/L$ correspondingly. Their representation $\rho_* \{[X,\cdot]\}$ and $\rho_* \{[Y,\cdot]\}$ (for both type of representations ρ_h and $\rho_{(q,p)}$) are measured by L and ML^2/T and are just derivatives:

$$\rho_* \{[X,\cdot]\} = \frac{\partial}{\partial p}, \qquad \rho_* \{[Y,\cdot]\} = \frac{\partial}{\partial q}. \tag{4.8}$$

Thus $\rho_* \{[X,\cdot]\}$ and $\rho_* \{[Y,\cdot]\}$ are generators of shits on both types of orbits $\mathcal{O}_h$ and $\mathcal{O}_0$ independent from h.

4.2. **p-Mechanical Dynamic Equation.** Since the modified commutator (4.3) with a p-mechanical energy has the dimensionality $1/T$—the same as the time derivative—we introduce the dynamic equation for an observable $f(s,x,y)$ on $\mathbb{H}^n$ based on that modified commutator as follows

$$\frac{df}{dt} = \{[f,E]\}. \tag{4.9}$$

Remark 4.2. It is a general tendency to make a Poisson brackets or quantum commutator out of any two observables and say that they form a Lie algebra. However there is a physical meaning to do that if at least one of two observables is an energy, coordinate or momentum: in these cases the bracket produce the time derivative (4.9) or corresponding shift generators (4.8) of the other observable.

A simple consequence of the previous consideration is that the p-dynamic equation (4.9) is reduced

(1) by the representation ρ_h, $h \neq 0$ (2.9) on $F_2(\mathcal{O}_h)$ (2.7) to Moyal's form of Heisenberg equation [31, (8)] based on the formulae (4.4) and (4.5):

$$\frac{d\rho_h(f)}{dt} = \frac{1}{i\hbar}[\rho_h(f), H_h], \qquad \text{where the operator } H_h = \rho_h(E); \tag{4.10}$$

(2) by the representations $\rho_{(q,p)}$ (2.15) on $\mathcal{O}_0$ (2.17) to Poisson's equation [2, § 39] based on the formula (4.7):

$$\frac{d\hat{f}}{dt} = \{\hat{f}, H\} \qquad \text{where the function } H(q,p) = \rho_{(q,p)}E = \hat{E}\,(0,q,p)\,. \tag{4.11}$$

The same connections are true for the solutions of the three equations (4.9)–(4.11).

Example 4.3 (harmonic oscillator, of course ;-)**.** [22] Let the p-mechanical energy function of a harmonic oscillator be as obtained in Examples 3.5 and 3.7:

$$E(s,x,y) = -\frac{1}{8\pi^2}\left(c_1\delta(s)\delta^{(2)}(x)\delta(y) + c_2\delta(s)\delta(x)\delta^{(2)}(y)\right), \tag{4.12}$$

Then the p-dynamic equation (4.9) on $\mathbb{H}^n$ obeying the Rule 1.1 is

$$\frac{d}{dt}f(t;s,x,y) = \sum_{j=1}^{n}\left(c_2 x_j\frac{\partial}{\partial y_j} - c_1 y_j \frac{\partial}{\partial x_j}\right) f(t;s,x,y). \tag{4.13}$$

Solutions to the above equations is well known to be rotations in each of (x_j, y_j) planes given by:

$$\begin{aligned} f(t;s,x,y) \quad = \quad & f_0\left(s, x\cos(\sqrt{c_1c_2}t) - \sqrt{\frac{c_1}{c_2}}y\sin(\sqrt{c_1c_2}t),\right. \\ & \left.\sqrt{\frac{c_2}{c_1}}x\sin(\sqrt{c_1c_2}t) + y\cos(\sqrt{c_1c_2}t)\right). \end{aligned} \tag{4.14}$$

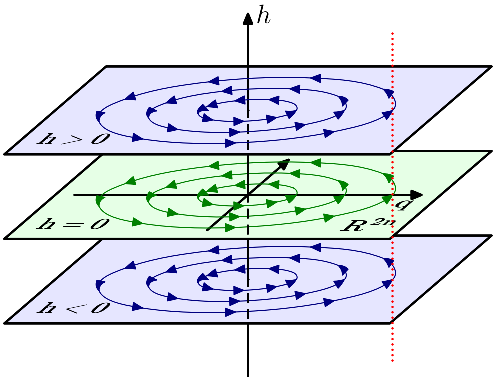

FIGURE 3. Dynamics of the harmonic oscillator in the adjoint space $\mathfrak{h}_n^*$ is given by the identical linear symplectomorphisms of all orbits $\mathcal{O}_h$ and $\mathcal{O}_0$. The vertical dotted string is uniformly rotating in the "horisontal" plane around the h-axis without any dynamics along the "vertical" direction.

This expression respects the Rule 1.1 and Remark 1.2. Since the dynamics on $L_2(\mathbb{H}^n)$ is given by a symplectic linear transformation of $\mathbb{H}^n$ its Fourier transform (2.10) to $L_2(\mathfrak{h}_n^*)$ will be the adjoint symplectic linear transformations of orbits $\mathcal{O}_h$ and $\mathcal{O}_0$ in $\mathfrak{h}_n^*$, see Figure 3.

The representations ρ_h transform the energy function E (4.12) into the operator

$$H_h = -\frac{1}{8\pi^2}(c_1 Q^2 + c_2 P^2), \tag{4.15}$$

where $Q = d\rho_h(X)$ and $P = d\rho_h(Y)$ are defined in (2.11). The representation $\rho_{(q,p)}$ transforms E into the classical Hamiltonian

$$H(q,p) = \frac{c_1}{2}q^2 + \frac{c_2}{2}p^2. \tag{4.16}$$

The p-dynamic equation (4.9) in form (4.13) is transformed by the representations ρ_h into the Heisenberg equation

$$\frac{d}{dt}f(t;Q,P) = \frac{1}{i\hbar}[f,H_h], \qquad \text{where} \qquad \frac{1}{i\hbar}[f,H_h] = c_1 p\frac{\partial f}{\partial q} - c_2 q\frac{\partial f}{\partial p}, \tag{4.17}$$

defined by the operator H_h (4.15). The representation $\rho_{(q,p)}$ produces the Hamilton equation

$$\frac{d}{dt}f(t;q,p) = c_1 p\frac{\partial f}{\partial q} - c_2 q\frac{\partial f}{\partial p} \tag{4.18}$$

defined by the Hamiltonian $H(q,p)$ (4.16). Finally, to get the solution for equations (4.17) and (4.18) it is enough to apply representations ρ_h and $\rho_{(q,p)}$ to the solution (4.14) of p-dynamic equation (4.13).

Summing up we can rephrase the title of [31]: *quantum and classical mechanics live and work* ***together*** *on the Heisenberg group and are separated only in irreducible representations of* $\mathbb{H}^n$.

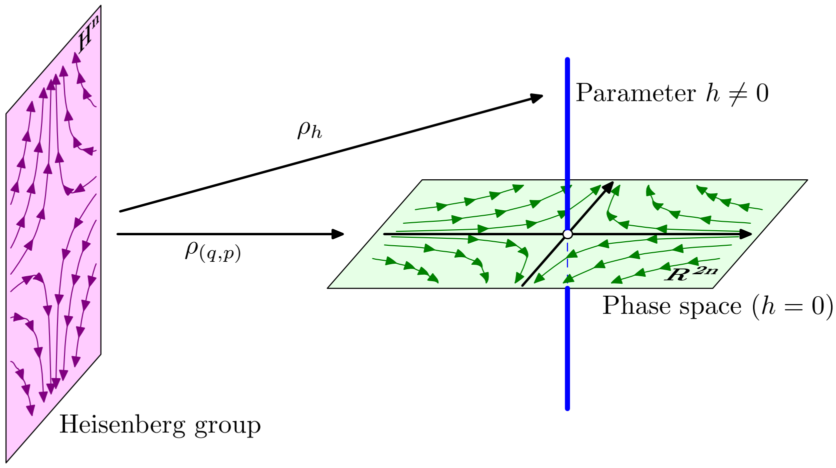

FIGURE 4. Automorphisms of $\mathbb{H}^n$ generated by the symplectic group $Sp(n)$ do not mix representations ρ_h with different Planck constants h and act by the metaplectic representation inside each of them. In the contrast those automorphisms of $\mathbb{H}^n$ act transitively on the set of one-dimensional representations $\rho_{(q,p)}$ joining them into the tangent space of the classical phase space $\mathbb{R}^{2n}$.

4.3. **Symplectic Invariance from Automorphisms of $\mathbb{H}^n$.** Let $A : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a linear *symplectomorphism* [2, § 41], [11, § 4.1], i.e. a map defined by $2n \times 2n$ matrix:

$$A : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

preserving the symplectic form (2.2):

$$\omega\left(A(x, y); A(x', y')\right) = \omega(x, y; x', y'). \tag{4.19}$$

All such transformations form the symplectic group $Sp(n)$. The Rule 1.1 implies that sub-blocks a and d of A have to be dimensionless while b and c have to be of reciprocal dimensions M/T and T/M respectively.

It is follows from the identities (4.19) and (2.1) that the the linear transformation $\alpha : \mathbb{H}^n \to \mathbb{H}^n$ such that $\alpha(s, x, y) = (s, A(x, y))$ is an automorphism of $\mathbb{H}^n$. Let us also denote by $\tilde{\alpha} = \tilde{\alpha}_A$ a unitary transformation of $L_2(\mathbb{H}^n)$ in the form

$$\tilde{\alpha}(f)(s, x, y) = \sqrt{\det a}\, f(s, A(x, y)),$$

which is well defined [11, § 4.2] on the double cover $\widetilde{Sp}(n)$ of the group $Sp(n)$. The correspondence $A \mapsto \tilde{\alpha}_A$ is a linear unitary representation of the symplectic group in $L_2(\mathbb{H}^n)$. One can also check the intertwining property

$$\lambda_{l(r)}(g) \circ \tilde{\alpha} = \tilde{\alpha} \circ \lambda_{l(r)}(\alpha(g)) \tag{4.20}$$

for the left (right) regular representations (2.5) of $\mathbb{H}^n$.

Because α is an automorphism of $\mathbb{H}^n$ the map $\alpha^* : k(g) \mapsto k(\alpha(g))$ is an automorphism of the convolution algebra $L_1(\mathbb{H}^n)$ with the multiplication $*$ (3.1), i.e. $\alpha^*(k_1) * \alpha^*(k_2) = \alpha^*(k_1 * k_2)$. Moreover α^* commutes with the antiderivative $\mathcal{A}$ (4.1), thus $\tilde{\alpha}$ is an automorphism of $L_1(\mathbb{H}^n)$ with the modified multiplication $\star$ (4.2) as well, i.e. $\alpha^*(k_1) \star \alpha^*(k_2) = \alpha^*(k_1 \star k_2)$. By the linearity we can extend the intertwining property (4.20) to the convolution operator K as follows:

$$\alpha^* K \circ \tilde{\alpha} = \tilde{\alpha} \circ K. \tag{4.21}$$

Since α is automorphism of $\mathbb{H}^n$ it fixes the unit e of $\mathbb{H}^n$ and its differential $d\alpha : \mathfrak{h}^n \to \mathfrak{h}^n$ at e is given by the same matrix as α in the exponential coordinates. Obviously $d\alpha$ is an automorphism of the Lie algebra $\mathfrak{h}^n$. By the duality between $\mathfrak{h}^n$ and $\mathfrak{h}^*_n$ we obtain the adjoint map $d\alpha^* : \mathfrak{h}^*_n \to \mathfrak{h}^*_n$ defined by the expression

$$d\alpha^* : (h, q, p) \mapsto (h, A^t(q, p)), \tag{4.22}$$

where A^t is the transpose of A. Obviously $d\alpha^*$ preserves any orbit $\mathcal{O}_h$ (2.7) and maps the orbit $\mathcal{O}_{(q,p)}$ (2.8) to $\mathcal{O}_{A^t(q,p)}$.

Identity (4.22) indicates that both representations ρ_h and $(\rho_h \circ \alpha)(s, x, y) = \rho_h(s, A(x, y))$ for $h \neq 0$ correspond to the same orbit $\mathcal{O}_h$. Thus they should be equivalent, i.e. there is an intertwining operator $U_A : F_2(\mathcal{O}_h) \to F_2(\mathcal{O}_h)$ such that $U_A^{-1} \rho_h U_A = \rho_h \circ \alpha$. Then the correspondence $\sigma : A \mapsto U_A$ is a linear unitary representation of the double cover $\widetilde{Sp}(n)$ of the symplectic group called *metaplectic representation* [11, § 4.2].

Thus we have

Proposition 4.4. *The p-mechanical brackets are invariant under the symplectic automorphisms of* $\mathbb{H}^n$*:* $\{[\tilde{\alpha} k_1, \tilde{\alpha} k_2]\} = \tilde{\alpha}\,\{[k_1, k_2]\}$*. Consequently the dynamic equation* (4.9) *has symplectic symmetries which are reduced*

(1) *by* ρ_h*,* $h \neq 0$ *on* $\mathcal{O}_h$ (2.7) *to the metaplectic representation in quantum mechanics;*
(2) *by* $\rho_{(q,p)}$ *on* $\mathcal{O}_0$ (2.17) *to the symplectic symmetries of classical mechanics* [2, § 38].

Combining intertwining properties of all three components (3.22) in the Weyl quantisation we get

Corollary 4.5. *The Weyl quantisation* $\mathcal{Q}_h$ (3.21) *is the* intertwining operator *between classical and metaplectic representations.*

5. Conclusions

5.1. **Discussion.** Our intention is to demonstrate that p-mechanics is not a speculation of a purely mathematical mind remote from the base physical principles. It is rooted in the common objects of mathematical physics and exploits their full potential. The complete representation theory of the Heisenberg group $\mathbb{H}^n$, which includes one-dimensional commutative representations, is a sufficient language for *both* classical and quantum theory.

It is naturally to describe the complete set of unitary irreducible representations by orbit method of Kirillov. The analysis carried out in Subsection 2.2 and illustrated in Figure 1 shows that the position of one-dimensional representations $\rho_{(q,p)}$ within the unitary dual of $\mathbb{H}^n$ relates them to classical mechanics. Various connections of infinite dimensional representations ρ_h of $\mathbb{H}^n$ to quantum mechanics is known for a long time.

The convolutions operators on $\mathbb{H}^n$ is a natural class to be associated with physical observables. They are reduced by infinite dimensional representations ρ_h to the pseudodifferential operators, which are observables in Weyl quantisation. The one-dimensional representations $\rho_{(q,p)}$ map convolutions into classical observables—functions on the phase space. The wavelets technique allows transform these three types of observables into each other, which is illustrated by Figure 2.

The nontrivial dynamics in the phase space—the space of one-dimensional representations of $\mathbb{H}^n$—could be obtained from the commutator on $\mathbb{H}^n$ with a help of the anti-derivative operator $\mathcal{A}$ (4.1). The p-mechanical dynamic equation (4.9) based on the operator $\mathcal{A}$ posses all desirable properties for description of a physical time evolution and its solutions gives both classic and quantum dynamics. See Figure 3 for the familiar dynamics of a harmonic oscillator.

Finally, the symplectic automorphisms of the Heisenberg group preserve the dynamic equation (4.9) and all its solutions. In representations of the Heisenberg group this reduces to the symplectic invariance of classical mechanics and the metaplectic invariance of quantum description. Moreover the symplectic transformations act transitively on the set $\mathcal{O}_0$ (2.17) of one-dimensional representations supporting its p-mechanical interpretation as the classical phase space, see Figure 4.

5.2. **Further Developments.** The present paper deals only with elementary aspects of p-mechanics. However this approach can (and should) be extended to many different situations. We will outline here only three of them:

5.2.1. *Quantum-Classical Interaction.* The long standing discussion [7, 26] about quantum-classical interaction can be approached as follows. Let $\mathbb{B}$ be a nilpotent step two Heisenberg-like group of elements $(s_1, s_2; x_1, y_1; x_2, y_2)$ with the only non-trivial commutators in the Lie algebra (cf. (2.4)) as follows:

$$[X_i, Y_j] = \delta_{ij} S_i.$$

Thus $\mathbb{B}$ has the two dimensional centre $(s_1, s_2, 0, 0, 0, 0)$ and the adjoint space of characters of $\mathbb{B}$ is also two dimensional. We can regard it as being spanned by two different Planck constants h_1 and h_2. There is a possibility to study the case $h_1 \neq 0$ and $h_2 = 0$ which correspond to a quantum behaviour of coordinates (x_1, y_1) and a classical dynamics in (x_2, y_2). This study was initiated in [26] but oversaw some homological aspects of the construction and is not satisfactory completed yet.

5.2.2. *Quantum Field Theory.* Mathematical formalism of quantum mechanics uses complex numbers in order to provide a unitary infinite dimensional representation of the Heisenberg group $\mathbb{H}^n$. The De Donder–Weyl formalism for classical fields theories [14] in a similar way requires *Clifford numbers* [12] for their quantisation. It was recently realised [8] that the appearance of Clifford algebras is induced by the Galilean group—a nilpotent step two Lie group with multidimensional centre. In the one-dimensional case an element of the Galilean group is $(s_1, \ldots, s_n, x, y_1, \ldots, y_n)$ with corresponding Lie algebra described by the non-vanishing commutators:

$$[X, Y_j] = S_j.$$

For the field theory it worth to consider Clifford valued representations induced by the Clifford valued "characters" $e^{2\pi(e_1 h_1 s_1 + \cdots + e_n h_n s_n)}$ of the centre, where $e_1, \ldots e_n$ are imaginary units spanning the Clifford algebra. The associated Fock spaces was described in [8]. There are important mathematical and physical questions related to the construction, which deserve careful considerations.

5.2.3. *String Theory.* There is a possibility to use p-mechanical picture for a string-like theory. Indeed the p-dynamics of a harmonic oscillator as presented in the Example 4.3 and Figure 3 consists of uniform rotations around the h-axis of lines—one can say *strings*—with the same (q, p) coordinates but different values of h.

In case of a more general energy, which is still however given by a *convolution* on $\mathbb{H}^n$, the dynamics could be more complicated. For example, it may not correspond to a point transformation of the adjoint space $\mathfrak{h}_n^*$; or in case of a point transformation it may transform a straight line (h, q_0, p_0) with fixed $(q_0, p_0) \in \mathbb{R}^{2n}$ and variable h into a generic curve transversal to all (q, p)-planes. However all spaces $F_2(\mathcal{O}_h)$ are invariant under any p-dynamics generated by a convolution on $\mathbb{H}^n$.

However if an energy is given by an arbitrary operator on $L_2(\mathbb{H}^n)$ [10, 23] then spaces $F_2(\mathcal{O}_h)$ for different h are no longer invariant during the evolution and could be mixed together. This opens a possibility of longitudinal dynamics of strings along the h-axis as well. It may seem strange to have a dynamics along h which is a *constant*, not a variable. However there is a duality [29] between the "Planck constant" h and the "tension of string" α'. Dualities and symmetries between h and α' could be reflected in a dynamics which mixes spaces $F_2(\mathcal{O}_h)$ with different h.

p-Mechanics could be also suitable for an encoding of *many-world* interpretation of quantum mechanics but we do not discuss this possibility now.

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School of Mathematics, University of Leeds, Leeds LS2 9JT, UK
E-mail address: `kisilv@maths.leeds.ac.uk`
URL: `http://maths.leeds.ac.uk/~kisilv/`