Normal–internal resonances in quasi–periodically forced oscillators: a conservative approach

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Abstract

We perform a bifurcation analysis of normal–internal resonances in parametrised families of quasi–periodically forced Hamiltonian oscillators, for small forcing. The unforced system is a one degree of freedom oscillator, called the 'backbone' system; forced, the system is a skew–product flow with a quasi–periodic driving with n basic frequencies. The dynamics of the forced system are simplified by averaging over the orbits of a linearisation of the unforced system. The averaged system turns out to have the same structure as in the well–known case of periodic forcing ($n = 1$); for a real analytic system, the non–integrable part can even be made exponentially small in the forcing strength. We investigate the persistence and the bifurcations of quasi–periodic n–dimensional tori in the averaged system, filling normal–internal resonance 'gaps' that had been excluded in previous analyses. However, these gaps cannot completely be filled up: secondary resonance gaps appear, to which the averaging analysis can be applied again. This phenomenon of 'gaps within gaps' makes the quasi–periodic case more complicated than the periodic case.

Contents

1 Introduction

This article studies families of quasi–periodically forced nonlinear Hamiltonian oscillators at normal–internal resonances. The motivating example, which is quite representative for the general case considered later on, is the quasi–periodically forced mechanical pendulum

$$\ddot{x} + \alpha^2 \sin x = \varepsilon g(t), \tag{1}$$

where $x, t \in \mathbb{R}$, $\alpha > 0$ and $\varepsilon \geq 0$. Quasi–periodicity of the forcing g means that there is a frequency vector $\omega = (\omega_1, \ldots, \omega_n)$ with rationally independent components, and a smooth function $G : \mathbb{T}^n \to \mathbb{R}$ such that

$$g(t) = G(\omega_1 t, \cdots, \omega_n t);$$

here $\mathbb{T}^n = \mathbb{R}^n / 2\pi\mathbb{Z}^n$. The oscillator is said to be at a normal–internal $1 : \ell$ resonance, if α satisfies

$$\langle k, \omega \rangle + \ell\alpha = 0,$$

for some $k \in \mathbb{Z}^n \backslash \{0\}$, some $\ell \in \{1, 2, \cdots\}$, and if ℓ is the smallest positive integer with this property. Throughout this article, the frequency vector ω is fixed and *Diophantine*, while the parameter α ranges over a compact interval not containing 0.

A classical question, which has been posed explicitly by Stoker [80] but which goes back to the acoustical investigations of Helmholtz and Rayleigh [53, 76], asks for solutions of (1) that are quasi–periodic with the same frequency vector as the forcing: so–called *response solutions*. The central question of this article is whether response solutions exist, and if so, how these solutions behave as the parameter α takes values near a normal–internal resonance value.

We reformulate the question in geometric terms by associating to the differential equation (1) a dynamical system on the phase space $\mathbb{T}^n \times \mathbb{R}^2$ by posing

$$\begin{aligned} \dot{\theta} &= \omega, \\ \dot{x} &= y, \\ \dot{y} &= -\alpha^2 \sin x + \varepsilon G(\theta); \end{aligned} \tag{2}$$

here $\theta \in \mathbb{T}^n$ and $x, y \in \mathbb{R}$. Any invariant torus that can be represented as the graph of a function $\tau = (\tau_1, \tau_2) : \mathbb{T}^n \to \mathbb{R}^2$ corresponds to a family of response solutions

$$x(t) = \tau_1(\omega t + \theta_0),$$

parametrised by θ_0. Hence, in geometric terms Stoker's problem asks for invariant tori of the system (2) in graph form. The forcing strength ε will be regarded as a perturbation parameter. For $\varepsilon = 0$, the torus $\tau = 0$ is invariant; we are interested in the fate of this torus if ε is small and positive.

The unperturbed torus is normally elliptic, and its normal frequency is equal to α. Kolmogorov–Arnold–Moser (KAM) theory shows that for small $\varepsilon > 0$, this torus will persist for the parameter α taking values in a Cantor set of positive Lebesgue measure, cf. [8, 19, 59, 60, 69, 75]. The complement of this Cantor set consists of countably many open intervals – usually called 'gaps' in this context – whose union is dense. These gaps correspond to normal–internal resonances; in their presence standard KAM theory cannot be applied directly. Note that in the well–known periodic analogue of this problem, that is in the case $n = 1$, cf. [67], response solutions are known to persist for all α ranging over some compact interval.

The main goal of the present article is hence to analyse what happens in a given normal–internal resonance gap. After averaging at the resonance under consideration, and lifting to a suitable covering space, the system can be written as the sum of an integrable part, which may be reduced to a one degree of freedom system, and a very small non–integrable part. Analysis of the integrable part yields that there are two types of gaps. At a normal–internal $1 : 1$ resonance, the invariant torus is 'pushed away' from the origin, and two more tori are generated in a quasi–periodic centre–saddle bifurcation. At a normal–internal $1 : 2$ resonance the invariant torus changes from being normally elliptic to normally hyperbolic — and back — in two quasi–periodic frequency–halving bifurcations. This second type of gap is proof–generated in classical KAM–schemes, which control the normal behaviour of invariant tori tightly. Consequently this type of gap is not present in [6, 7], where an alternative scheme is used.

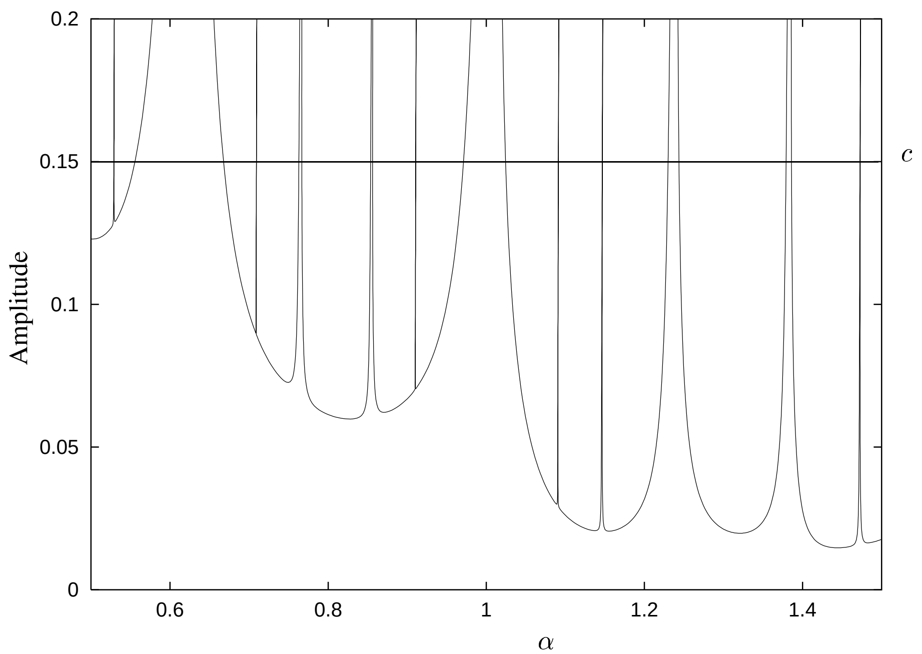

Figure 1: *Response diagram of* (1), (2). *Shown are the parameter* α *on the horizontal axis, against the* L^2 *norm of the response solution on the vertical axis. Classical KAM–theory can only prove the existence of invariant tori whose* L^2*–norm is below some bound c, which is of the same order* $O(\varepsilon)$ *as the forcing, as* $\varepsilon \to 0$. *Moreover the normal dynamics of these invariant tori should be elliptic, which fails to be the case for* $1:2$ *resonances (the corresponding gaps are not shown in this figure).*

1.1 Phenomenology

To illustrate the dynamical phenomena we are interested in, numerical simulations of system (2) have been performed for the case $n = 2$, $\omega = (1, -\frac{1}{2} + \frac{1}{2}\sqrt{5})$, cf. [24]. For G, the trigonometric rational function

$$G(\theta) = (3 + \cos\theta_1 + \cos\theta_2)^{-1}$$

has been taken, whose Fourier coefficients are all non–zero. The perturbation strength has been set to $\varepsilon = 0.05$.

Response solutions have been computed numerically for a range of values of the parameter α. The method works roughly as follows: system (2) induces a Poincaré map of the Poincaré section $\theta_1 = 0$ onto itself. Invariant tori of the system correspond to invariant circles of the Poincaré map, with rotation number $-\pi+\pi\sqrt{5}$. An explicit parametrisation of such an invariant circle is obtained by representing it as a Fourier series and computing its Fourier coefficients numerically. Then, the invariant curve can be continued with respect to the parameter α: for more details, see [31, 56]. The results are summarised in Figure 1, where on the vertical axis the L^2 norm of the Fourier series is plotted against the parameter α.

In the figure, it can be seen clearly how invariant tori are 'pushed away' by $1 : 1$ resonances; however, these resonances do not destroy the tori. As remarked before, 'classical' KAM theory handles only the case that response solutions have an amplitude of order $O(\varepsilon)$: this corresponds to the lower part of the curves. The 'spikes' are cut off, and hence correspond to the 'gaps' in the Cantor sets for which classical KAM theory fails to find invariant tori. These gaps will be almost completely 'filled' by the results of the present paper.

Another case for which gaps occur in the KAM Cantor sets is for $1 : 2$ resonances. Although in that case invariant tori are not pushed away, their normal linear behaviour changes from elliptic to hyperbolic and back again. Classical KAM theory also fails for these tori: this gives rise to another kind of 'gap' in the KAM Cantor sets, which will be filled completely by the present paper.

Let us briefly compare these findings with the results in the case of periodic forcing $n = 1$ (see [67] for a survey). There the resonance gaps are isolated. However, the same amplification of response solutions is observed for the parameter α close to a $1 : 1$ resonance ($k\omega + \alpha = 0$ and $k \neq 0$). At a $1 : 2$ resonance — that is $k\omega + 2\alpha = 0$ with k odd — the response solution changes from elliptic to hyperbolic and back again, through two period doubling (also called frequency halving) bifurcations, cf. [17, 18, 27, 41]. For higher resonances $k\omega + \ell\alpha = 0$ with $\ell \geq 3$, periodic orbits with ℓ times the period branch off from the response solution.

All these periodic resonance phenomena reappear in our quasi–periodic case, but now forming a dense set. In fact, in the case $n = 1$ the parameter values of α for which $k\omega + \ell\alpha = 0$ for some fixed ℓ — the resonances — form a set of isolated points; the corresponding set for $n \geq 2$ is dense.

This denseness of resonances leads us to expect 'secondary' resonances located on the 'primary' resonance curves. To illustrate these, the left–hand picture in figure 2 zooms in on the interval $[0.54, 0.70] \subset [0.5, 1.5]$ of Figure 1. In this way the 'primary' resonance at $\alpha = -\frac{1}{2} + \frac{1}{2}\sqrt{5}$ is magnified and a 'secondary' resonance at $\alpha \approx 0.563813$ becomes visible. Although resonances form a dense set, the width of the associated gaps decreases rapidly (in first approximation, this width is proportional to the corresponding Fourier coefficient of the forcing function $G(\theta)$, see section 5) and the numerical detection of these phenomena becomes difficult.

In order to obtain a two–dimensional impression of the dynamics in phase space, 'quasi–periodic' Poincaré sections of system (2) are displayed in Figure 3. These are constructed by taking the first return map under the flow of (2) to the set $\Sigma = \{\theta_1 = 0,\ \theta_2 \in [a, b]\} \times \mathbb{R}^2$, where $[a, b]$ is a small interval, and then plotting the resulting points only using their $\mathbb{R}^2$–coordinates.

Figure 4 shows three slices of the phase space for the case $k = (0, -3)$, $\ell = 2$, clearly showing a period doubling bifurcation.

1.2 Setting of the problem

In the sequel, the phenomena illustrated in the previous subsection are analysed using a series of well–known techniques, like (efficient) normal form analysis, van der Pol liftings, local equivariant bifurcation theory and KAM theory.

Note that the following Hamiltonian system is equivalent to equation (1)

$$\begin{aligned} \dot{\theta} &= \omega \\ \dot{I} &= \varepsilon \frac{\partial G}{\partial \theta}(\theta) x \end{aligned} \tag{3}$$

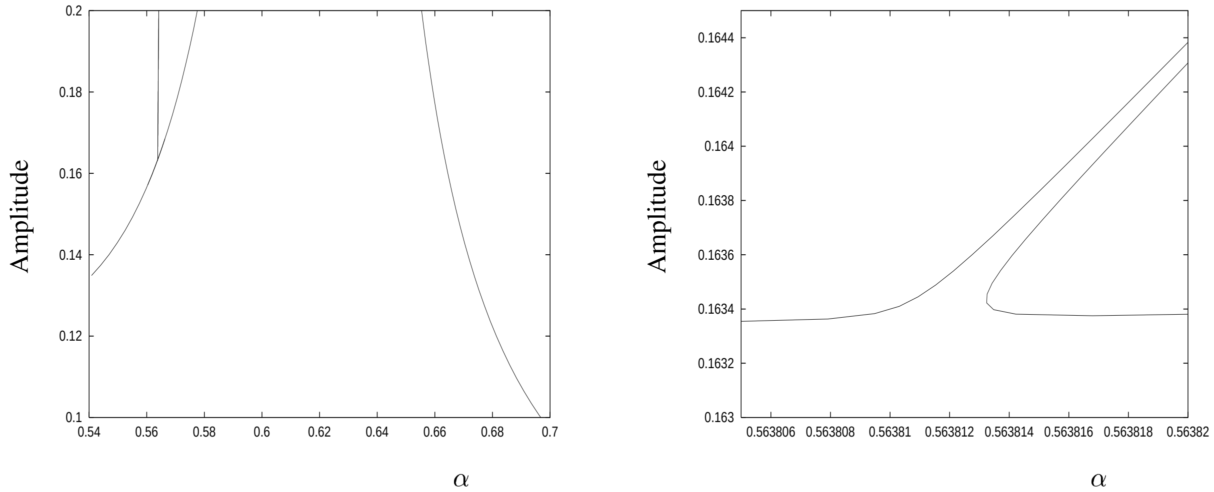

Figure 2: *Two more response diagrams associated to the* $1 : 1$ *normal–internal resonance. The left plot shows the big resonance for* α *near the golden number* $-\frac{1}{2} + \frac{1}{2}\sqrt{5} = 0.618\ldots$*, and the right plot is a magnification of the 'secondary' gap that appears in the left plot.*

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\alpha^2 \sin x + \varepsilon G(\theta) \end{aligned}$$

where the added variable $I \in \mathbb{R}^n$ is canonically conjugate to $\theta \in \mathbb{T}^n$. System (3) is an autonomous Hamiltonian system on the phase space $T^*(\mathbb{T}^n \times \mathbb{R}) = \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} = \{\theta, I, x, y\}$, with symplectic form $d\theta \wedge dI + dx \wedge dy$ and Hamilton function

$$H(\theta, x, I, y) = \langle \omega, I \rangle + \frac{1}{2}y^2 - \alpha^2 \cos x - \varepsilon G(\theta)x.$$

This motivates the introduction of a more general (parametrised) Hamiltonian

$$H = \langle \omega, I \rangle + \alpha \frac{x^2 + y^2}{2} + \tilde{h}(x, y; \alpha) + \varepsilon G(\theta, x, y; \alpha, \varepsilon), \tag{4}$$

where $\tilde{h}$ contains terms of order $O(|(x, y)|^3)$, and where G is an arbitrary real analytic function. For $\varepsilon = 0$, the torus $(x, y) = (0, 0)$ is invariant and a response solution. This response solution is said to be at a normal–internal $1 : \ell$ resonance at $\alpha = \alpha_0$, if

$$\langle k, \omega \rangle + \ell \alpha_0 = 0$$

for some $k \in \mathbb{Z}^n \backslash \{0\}$, some $\ell \in \{1, 2, \cdots\}$, and if ℓ is the smallest positive integer with this property.

Note that system (3) is a special case of this general form. Moreover, note that this general parametrised Hamiltonian is generic in the universe of parametrised families of 'forced' (or 'driven' or 'skew') Hamiltonian systems

$$K = \langle \omega, I \rangle + k(\theta, x, y),$$

which have the property that the 'torus dynamics'

$$\dot{\theta} = \omega$$

are decoupled from the 'normal dynamics'

$$\dot{x} = \frac{\partial k}{\partial y}, \qquad \dot{y} = -\frac{\partial k}{\partial x}.$$

This set–up keeps ω constant, and allows to concentrate on the normal behaviour of invariant tori, while not having to worry about possible internal resonances. For the rest, we want our system to be 'as generic as possible' and 'sufficiently smooth', although in the later parts of the article, the Hamiltonian (4) is assumed to be real analytic in all its arguments.

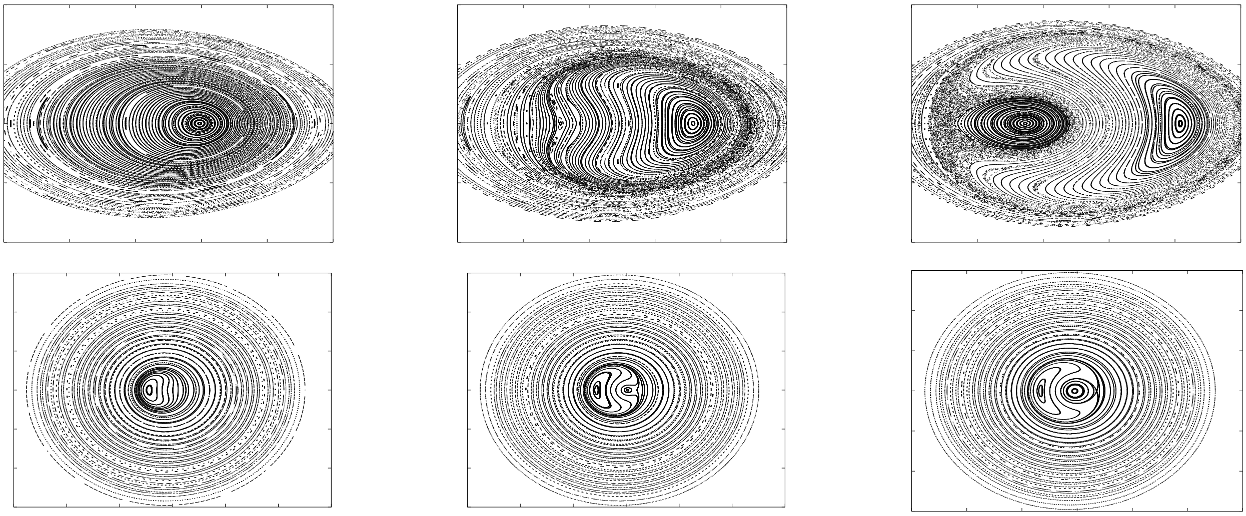

Figure 3: *Six phase portraits associated to the* 1:1*–resonance, the first three for* $\alpha \approx -\frac{1}{2} + \frac{1}{2}\sqrt{5}$*, and the other three for 'a gap within that gap' (*$\alpha \approx 0.563813$*, see Figure* 2*, left).*

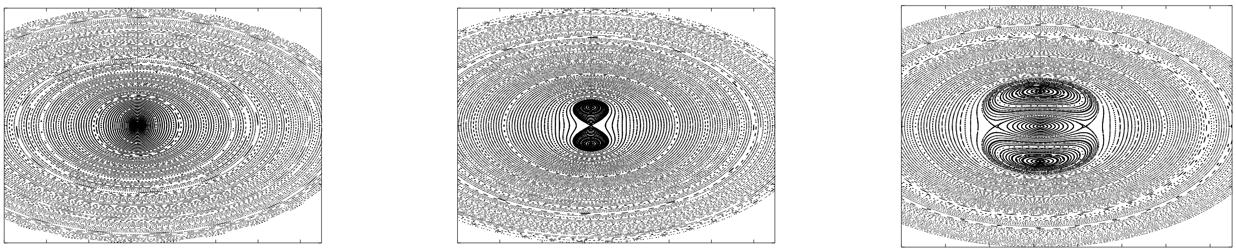

Figure 4: *Three phase portraits associated to a* 1:2*–resonance, for* $-3(-\frac{1}{2} + \frac{1}{2}\sqrt{5}) + 2\alpha \approx 0$.

Remark 1.1

1. We make the general conjecture that the analysis of the present article is, suitably modified, valid for the more general system
$$H \;=\; \langle \omega, I\rangle \;+\; \alpha\frac{x^2+y^2}{2} \;+\; \tilde{h}(x,y;\alpha) \;+\; \varepsilon G(\theta, I, x, y; \alpha, \varepsilon)$$
in the universe of 'ordinary' Hamiltonian systems. Restricting ourselves to the forced case in this article is mainly for convenience, but also because this setting is interesting in its own right.

   Indeed, there are many models in Celestial Mechanics that are written as periodic or quasi-periodic perturbations of autonomous Hamiltonian systems; for instance [31, 37, 45, 46, 47, 48]. The families of invariant tori in such models must present the phenomena studied here. In fact, the classical KAM theory for this case, cf. [59, 60], avoids the normal–internal resonance gaps under study here by excluding the corresponding values in the action and/or parameter spaces.

2. Our restriction to a real analytic universe (endowed with the compact–open topology on holomorphic extensions) partly is for convenience. Large parts of this treatment also work in C^s for $s \in \mathbb{N} \cup \{\infty\}$ large, *e.g.* the normal form transformations, the van der Pol lifting, Kupka–Smale genericity [25, 26], and the quasi–periodic persistence results [20, 75]. Real analyticity is crucial, however, to obtain exponential estimates on the remainder terms in the normal forms, which in turn determine the widths of the gaps [22, 23, 71, 79].

1.3 Outline

We outline the analysis performed in the rest of the article. As mentioned before, we consider the Hamiltonian

$$H \;=\; \langle \omega, I\rangle \;+\; \alpha\frac{x^2+y^2}{2} \;+\; \tilde{h}(x,y;\alpha) \;+\; \varepsilon G(\theta,x,y;\alpha,\varepsilon).$$

Observe that for $\varepsilon = 0$ this Hamiltonian is integrable. Its dynamics leave invariant an elliptic n–dimensional torus $\tau(\theta) = 0$ of graph form and a surrounding family of invariant $(n+1)$–tori. The term εG is regarded as a perturbative forcing term. We are interested in the fate of the n–torus when ε is small and positive, in particular for α in a small neighbourhood of a normal–internal resonance value $\alpha_0 \neq 0$ satisfying

$$\langle k, \omega\rangle \;+\; \ell\alpha_0 \;=\; 0 \tag{5}$$

for certain $\ell \in \{1, 2, \cdots\}$ and $k \in \mathbb{Z}^n\backslash\{0\}$.

If $\ell \in \{1, 2\}$, classical KAM–theory cannot be applied to this perturbation problem directly. However, by restricting to a single normal resonance, the problem may be transformed as follows. For α close to α_0, ε close to 0 and (x, y) in a neighbourhood of 0 (whose size depends on ε), a resonant normal form is computed. By a *van der Pol transformation* the system is lifted to a suitable covering space, where it can be split into an integrable system (that can be reduced to a one degree of freedom 'backbone' system) and a non–integrable remainder term of arbitrary high order in all variables.

The 'backbone' system is equivariant with respect to the deck transformations of the covering space, and hence equivariant bifurcation theory is invoked to analyse its bifurcations with respect to the parameter α. Our main concern, persistence of quasi–periodic response tori of graph form, then reduces to analysing the persistence of response tori of the integrable 'backbone' system under small non–integrable perturbations. For this part of the analysis, we draw on results of quasi–periodic Hamiltonian bifurcation theory already available. From the bifurcation diagrams obtained in this way, the response diagram near the resonance (5) can be reconstructed, using 'classical' KAM–theory, cf. [8, 9, 11, 19, 52, 55, 61, 63, 69, 80]. To this end we use the following Diophantine conditions: the parameter α should satisfy

$$|\langle k', \omega\rangle + \ell'\alpha| \geq \frac{\gamma}{|k'|^\tau}, \tag{6}$$

for all $(k', \ell') \in \mathbb{Z}^n\backslash\{0\} \times \{\pm1, \pm2\}$ that are different from the (k, ℓ) which satisfy (5). These conditions effectively define a Cantor set inside the resonance gap around (5). Though we are mainly interested in the case that $\ell \in \{1, 2\}$, for completeness sake the case $|\ell| \geq 3$ will be included as well. We emphasise that our approach deals with one normal frequency at a time.

We give a brief overview of the effects of the value of ℓ. If $\ell = 1$, the response torus is 'pushed away' by the perturbation and standard KAM theory does not apply. If $\ell = 2$, the torus is not affected but its normal behaviour can change from elliptic to hyperbolic. In this case, KAM theory could be applied directly without including the normal directions (see [7]). If $\ell \geq 3$, the normal stability of the torus is not affected and standard KAM machinery can be applied; in this case the effect of the resonance is the 'birth' of new invariant response tori.

Combining the normal form approach with Neishtadt–Nekhoroshev estimates, which can be obtained in the real–analytic case as in [22, 23, 60, 71, 79], leaves only gaps that are exponentially small in the perturbation strength ε.

Indeed, our analysis shows that normally elliptic tori, known to exist outside these gaps, can be continued inside where they may or may not pass through quasi–periodic bifurcations. However, the continuation is not complete: new gaps show up because of (6), which have to be treated the same way. This 'inductive principle' suggests an infinite repetition of the bifurcation patterns at ever smaller scales. Compare with the 'bubbles inside bubbles' as they occur in dissipative quasi–periodic bifurcation theory [4, 8, 32, 33, 34, 84].

Note that in the formulation of the perturbation problem, there is no essential difference between the periodic case $n = 1$, cf. [17, 18, 27, 38, 67, 80], and the quasi–periodic case $n \geq 2$. In the sequel we shall often compare these two cases; the main difference between them is that normal–internal resonances yield isolated gaps in the periodic scenario, but a dense union of gaps in the quasi–periodic scenario.

Remark 1.2 The gaps associated to $\ell = 0$ in 'classical' KAM–theory concern the internal resonances, which are of a different type. They are in fact excluded in this article, since ω has been fixed at some Diophantine value. Various violations of the corresponding non–resonance conditions are dealt with elsewhere. See for instance [9, 14, 15, 55, 70, 75, 83].

1.4 Organisation

The article is organised as follows. In Section 2 we simplify the Hamiltonian (4) in two steps. First a formal normal form is developped. Second, by a van der Pol transformation, we lift the system to a covering space adapted to the resonance.

After truncation and reduction, a one degree of freedom system is obtained, which is the same as in the periodic case. In Section 3 this one degree of freedom system is investigated for its own sake, since it forms the backbone of our perturbation problem. Our main interest is with the (relative) equilibria, since they correspond to response solutions in the full system. They can be continued by the Implicit Function Theorem. The analysis is simplified by suitable scalings, allowing us to retain only significant lowest order terms of the system.

The corresponding dynamics in $n+1$ degrees of freedom of the integrable part of the normal form is reconstructed in Section 4 using structural stability arguments. Section 5 re–incorporates the non–integrable terms neglected earlier on, and KAM–theory is applied to show persistence of response solutions, solving the original perturbation problem. Actually, by using normal forms with exponentially small remainders, the new gaps in the parameter space are shown to be exponentially small in the perturbation parameter ε. For completeness' sake we present details on how the normal forms are obtained in an appendix.

2 Simplifications of the Hamiltonian

Several canonical transformations are applied to our Hamiltonian system with Hamiltonian (4), simplifying it stepwise. We begin by putting the normal linear part into standard form. After this we put the nonlinear part into a formal resonant normal form, which also is quite standard, cf., e.g., [5, 9, 10, 29, 30, 50, 65, 81]. In this way the dynamics normal to the torus decouples from the 'internal' part (the quasi–periodic forcing), which allows a (formal) reduction to a system with one degree of freedom. Finally, by a van der Pol transformation (also known as 'passing to co–rotating coordinates') the system is lifted to a covering space, see for instance [1, 27, 51].

2.1 Preliminaries

On the phase space $\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^2$ with symplectic structure $\sum \mathrm{d}\theta_i \wedge \mathrm{d}I_i + \mathrm{d}x \wedge \mathrm{d}y$ we have the Hamiltonian (4)

$$H(\theta, I, x, y; \alpha, \varepsilon) \;=\; \langle \omega, I \rangle \;+\; h(x, y; \alpha) \;+\; \varepsilon G(\theta, x, y; \alpha, \varepsilon)\,, \tag{7}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product, and where $h = (\alpha/2)(x^2+y^2) + \tilde{h}$. Consider the one degree of freedom system defined by $h(x, y; \alpha)$. Assuming that the origin is an elliptic equilibrium, it follows for $\varepsilon = 0$ and for fixed I that the n–torus $\mathbb{T}^n \times \{(0, 0)\}$ is a normally elliptic invariant torus for the full system defined by (4). The quadratic part h_2 of h reads as

$$h_2(x, y) \;=\; \alpha \frac{x^2+y^2}{2} \quad .$$

Recall that $\tilde{h} = h - h_2$ contains the higher order terms. For $\varepsilon = 0$, the variables can be interpreted geometrically as follows. Individual tori are given by $I = I_0$ and parametrised by θ . They model the quasi–periodic forcing. We refer to (θ, I) as the *internal* variables and to (x, y) as the *normal* variables of the tori.

Invariant n–tori of (4) give rise to families of response solutions. For $\varepsilon = 0$, the response solution is $x = 0 = y$. From standard KAM–theory, see [19] and references therein, we know that response solutions exist also for small non–zero ε, and that they are of order $O(\varepsilon)$, provided the Diophantine conditions

$$|\langle k, \omega \rangle + \ell \alpha| \geq \frac{\gamma}{|k|^\tau}, \qquad \text{for all } k \in \mathbb{Z}^n \backslash \{0\} \qquad \text{and all } \ell \in \{0, \pm 1, \pm 2\} \tag{8}$$

are met, where $\tau > n-1$ and $\gamma > 0$. We take τ constant, but allow γ to depend on the perturbation parameter ε. Note that for $\ell = 0$ these conditions express that the internal frequency vector ω is Diophantine. The main interest of this paper is however to study the case that a Diophantine condition for $\ell = \pm 1$ or $\ell = \pm 2$ is violated. To this end we investigate the case that α is close to α_0 which is such that for some (fixed) $k \in \mathbb{Z}^n \backslash \{0\}$ and $\ell \in \{1, 2\}$ the equation (5):

$$\langle k, \omega \rangle \;+\; \ell \alpha_0 \;=\; 0$$

holds. For instance, it turns out that for $\ell = 1$ the response solution is at a distance $O(\varepsilon^{\frac{1}{3}})$ of the 'unperturbed torus' $x = 0 = y$. This necessitates a *semi–local* approach, that is we do not view the occurring bifurcations as isolated phenomena, but focus on the total structure of the bifurcation diagram close to the resonance (5). This approach also yields new insights for the resonance $\langle k, \omega\rangle + 3\alpha_0 = 0$, therefore we investigate (5) also in the case $\ell \geq 3$.

2.2 Formal normal form

We begin with a basic observation. Let two pairs $(k_1, \ell_1), (k_2, \ell_2) \in \mathbb{Z}^n \backslash \{0\} \times \mathbb{N}$ both satisfy (5). Since ω is Diophantine, this implies $\ell_2 k_1 = \ell_1 k_2$. It follows that there is at most one pair (k, ℓ) satisfying (5), for which $\ell > 0$ and $\gcd(k) = \gcd(k_1, \ldots, k_n) = 1$; here $\gcd(k)$ of an integer vector $k \in \mathbb{Z}^m \backslash \{0\}$ is the greatest common divisor of its components k_i. Consequently the 'remaining' Diophantine conditions (8) are valid for sufficiently small γ . In the neighbourhood of the resonance (5) we have the following result.

Theorem 2.1 (Formal normal form) *Consider the Hamiltonian* (4) *where for* $\tau > n - 1$ *and* $\gamma > 0$ *the frequency vector* ω *satisfies the Diophantine conditions*

$$|\langle k', \omega\rangle| \geq \frac{\gamma}{|k'|^\tau},$$

for all $k' \in \mathbb{Z}^n \backslash \{0\}$*. For some* $k \in \mathbb{Z}^n \backslash \{0\}$ *and* $\ell \in \{1, 2, \cdots\}$*, let* α_0 *satisfy a resonance condition* $\langle k, \omega\rangle + \ell\alpha_0 = 0$*, and let* α *be sufficiently close to* α_0*. Given* $L \in \mathbb{N}$*, for sufficiently small* ε *there exists a real analytic and canonical change of variables* $\Phi(\cdot; \alpha, \varepsilon)$

$$\begin{array}{rccc} \Phi: & \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^2 & \longrightarrow & \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^2 \\ & (\theta, I, x, y) & \mapsto & (\theta, J, X, Y) \end{array}$$

close to the identity at $(x, y, \alpha, \varepsilon) = (0, 0, \alpha_0, 0)$*, such that the following holds. By the change of variables, the Hamiltonian* (4) *is transformed into normal form*

$$H \circ \Phi^{-1}(\theta, J, X, Y; \alpha, \varepsilon) \;=\; N(\theta, J, X, Y; \alpha, \varepsilon) \;+\; R(\theta, X, Y; \alpha, \varepsilon).$$

The integrable part N *of the normal form reads as*

$$\begin{aligned} N \;=\; & \langle \omega, J\rangle + \alpha_0 \frac{X^2+Y^2}{2} + (\alpha - \alpha_0)\frac{X^2+Y^2}{2} + c\left(\frac{X^2+Y^2}{2}\right)^2 \\ & + a\varepsilon \frac{X^2+Y^2}{2} + A\varepsilon \operatorname{Re}\left((X - \mathrm{i}Y)^\ell \mathrm{e}^{\mathrm{i}(\langle k,\theta\rangle - \psi_0)}\right) \\ & + \sum_{\kappa,m,j,i} F_{\kappa,m,i,j}(\alpha-\alpha_0)^j \varepsilon^i (X^2+Y^2)^m \\ & \qquad \cdot \operatorname{Re}\left((X - \mathrm{i}Y)^{\kappa\ell} \mathrm{e}^{\mathrm{i}(\kappa\langle k,\theta\rangle - \psi_{\kappa,m,i,j})}\right) \end{aligned} \tag{9}$$

with $\kappa \geq 0$*,* $j + i + 2m + \kappa\ell \leq L - 1$*, and* $\psi_{0,m,i,j} = 0$*, where the sum does not contain the terms that are explicitly displayed; the remainder* R *of the normal form satisfies the estimate*

$$R(\theta, X, Y; \alpha, \varepsilon) \;\equiv\; O_L(X, Y, \alpha - \alpha_0, \varepsilon) \quad .$$

The proof is based on the normalisation of the Hamiltonian H with respect to the action of

$$H_2(\theta, J, X, Y) \;=\; \langle \omega, J\rangle + \alpha_0 \frac{X^2+Y^2}{2}. \tag{10}$$

The details are given in the appendix.

Remark 2.1

1. The system is nondegerate whenever $c \neq 0$ and $A \neq 0$, which will be assumed throughout. By reversing the direction of time, if necessary, we can restrict our attention to $c < 0$.

   By an appropriate scaling of the Hamiltonian, the constant c, which gives the second Birkhoff coefficient, can be replaced by -1. This leaves us with two cases, $A > 0$ and $A < 0$. By a rotation over an angle π/ℓ in the normal direction, that is, in (X, Y)–coordinates, these cases are shown to be equivalent. However, for clarity's sake we shall give phase portraits for both.

2. We will see in section 3 that in most cases the higher order part (the last sum term) of (9) can be considered as being part of the non–integrable remainder R of the normal form.

2.3 Reduction to an autonomous system on a covering space

The integrable part N of the normal form (9) is equivariant with respect to the adjoint action of the normal 'linear' part H_2, given in (10), of the Hamiltonian. Since ω is non–resonant, the first term $\langle \omega, J\rangle$ of H_2 generates a free $\mathbb{T}^n$–action. For non–zero α_0 the second term generates a further free $\mathbb{T}^1$–action. As these two actions commute they define a $\mathbb{T}^{n+1}$–action. In the case of normal non–resonance, this $\mathbb{T}^{n+1}$–action is again free; if additionally Diophantine conditions (8) hold, we obtain persisting normally elliptic quasi–periodic response solutions; see again [19, 59, 70, 75].

Given the resonance relation (5), the $\mathbb{T}^{n+1}$–action is not free but reduces to a $\mathbb{T}^n$–action. And even this $\mathbb{T}^n$–action is not free (unless $\ell = 1$), for the n–torus $(x, y) = (0, 0)$ has isotropy $\mathbb{Z}_\ell$: there is a $\mathbb{Z}_\ell$ sub–action of the $\mathbb{T}^n$–action that leaves every point of the n–torus $(x, y) = (0, 0)$ fixed. This situation is most conveniently treated by lifting the normal form (9) to an ℓ–fold covering of $\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^2$ on which the $\mathbb{T}^n$–action is free. Note that there is no point in lifting the complete Hamiltonian $N + R$ to the covering, since the remainder term R is in general not invariant under the $\mathbb{T}^n$–action.

On the covering space the integrable part N of the normal form is invariant under the free $\mathbb{T}^n$–action, that is, it does not depend explicitly on the 'torus variable' θ. Now we proceed as follows. First, a basis of the lattice defining $\mathbb{T}^n$ is chosen that is adapted to the resonance (5), in such a way that N depends only on a single angle. This allows us to continue as if we were in the periodic case.

If for an integer vector $k \in \mathbb{Z}\backslash\{0\}$ its greatest common divisor $\gcd(k)$ is equal to 1, then there exist integer vectors $v_2, \ldots, v_n \in \mathbb{Z}^n$ such that the determinant of the linear map $L_k = (k\, v_2\, \ldots\, v_n)^T$ is equal to 1, cf. [36], chapter 10. This linear map is obviously not unique. The map $L_k \in \mathrm{SL}(n, \mathbb{Z})$ defines a transformation $\hat\theta = L_k\theta$ of the n–torus $\mathbb{T}^n$ to itself, with the property that

$$\hat\theta_1 \;=\; \langle k, \theta\rangle.$$

Note that in these coordinates N is independent of the other angles $\hat\theta_2, \ldots, \hat\theta_n$, which have consequently become cyclic variables.

In the general case that the greatest common divisor κ_0 of the k_i is not necessarily equal to 1, we write k as $\kappa_0\tilde k$, where for $\tilde k \in \mathbb{Z}^n$ we have $\gcd(\tilde k) = 1$. As above, we can find a linear map $L_{\tilde k} \in \mathrm{SL}(n, \mathbb{Z})$ whose first row equals $\tilde k$. By applying $L_{\tilde k}$ to the n–torus, the integrable part N of the normal form becomes a function of $(\hat\theta, \hat J, X, Y; \alpha, \varepsilon)$, where $\hat J = (L_{\tilde k}^{-1})^{\mathrm{T}} J$ is the vector of the new actions J conjugate to $\hat\theta$:

$$\begin{aligned} N \;=\; & \left\langle \hat\omega, \hat J \right\rangle \;+\; \alpha_0 \frac{X^2+Y^2}{2} \;+\; (\alpha-\alpha_0)\frac{X^2+Y^2}{2} \;-\; \left(\frac{X^2+Y^2}{2}\right)^2 \\ & + a\varepsilon \frac{X^2+Y^2}{2} \;+\; A\varepsilon \,\mathrm{Re}\left((X-\mathrm{i}Y)^\ell \mathrm{e}^{\mathrm{i}(\kappa_0\hat\theta_1 - \psi_0)}\right) \\ & + \sum_{\kappa,m,j,i} F_{\kappa,m,i,j}(\alpha-\alpha_0)^j \varepsilon^i (X^2+Y^2)^m \\ & \qquad \cdot \mathrm{Re}\left((X-\mathrm{i}Y)^{\kappa\ell} \mathrm{e}^{\mathrm{i}(\kappa\kappa_0\hat\theta_1 - \psi_{\kappa,m,i,j})}\right) \quad . \end{aligned}$$

Here $\hat\omega = L_{\tilde k}\omega$ is the new frequency vector; in particular, we have that $\hat\omega_1 = \left\langle \tilde k, \omega \right\rangle = \frac{1}{\kappa_0}\langle k, \omega\rangle$. Note that the resonance condition now reads as $\kappa_0\hat\omega_1 + \ell\alpha_0 = 0$.

We are now in the position to mimic the treatment of the periodic case, by introducing co–rotating coordinates through a so–called van der Pol transformation. The system is lifted by this transformation to an ℓ–fold covering space, on which the normal dynamics of N decouple from the quasi–periodic forcing terms. The transformation is given by

$$\left\{\begin{array}{lcl} \begin{pmatrix} X \\ Y \end{pmatrix} & = & \begin{pmatrix} \cos\kappa_0\varphi_1 & -\sin\kappa_0\varphi_1 \\ \sin\kappa_0\varphi_1 & \cos\kappa_0\varphi_1 \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix} \\ \hat{\theta}_1 & = & \ell\varphi_1 + \frac{\psi_0}{\kappa_0} \\ \hat{J}_1 & = & \frac{\kappa_0}{\ell}\frac{P^2+Q^2}{2} + \frac{1}{\ell}I_1 \\ \hat{\theta}_j & = & \varphi_j, \quad \text{for } j = 2,\dots,n \\ \hat{J}_j & = & I_j, \quad \text{for } j = 2,\dots,n \quad . \end{array}\right.$$

After the van der Pol transformation the averaged part N of the Hamiltonian is of the form

$$N(I,Q,P;\alpha,\varepsilon) \;=\; \langle\hat{\omega}_\ell, I\rangle \;+\; F(Q,P;\alpha,\varepsilon) \tag{11}$$

with $\hat{\omega}_\ell = (\frac{\hat{\omega}_1}{\ell}, \hat{\omega}_2, \cdots, \hat{\omega}_n)$, and with F given by

$$\begin{aligned} F \;=\; & (\alpha-\alpha_0)\frac{Q^2+P^2}{2} \;-\; \left(\frac{Q^2+P^2}{2}\right)^2 \\ & + a\varepsilon\frac{Q^2+P^2}{2} \;+\; A\varepsilon\,\mathrm{Re}(Q-\mathrm{i}P)^\ell \\ & + \sum_{\kappa,m,j,i} F_{\kappa,m,i,j}(\alpha-\alpha_0)^j\varepsilon^i(Q^2+P^2)^m \\ & \qquad \cdot \mathrm{Re}\left((Q-\mathrm{i}P)^{\kappa\ell}\mathrm{e}^{\mathrm{i}(\kappa\psi_0-\psi_{\kappa,m,i,j})}\right) \quad . \end{aligned} \tag{12}$$

By construction the lifted dynamics is equivariant with respect to the deck group of the ℓ–fold covering space; this group is isomorphic to $\mathbb{Z}_\ell$. It is generated by

$$\begin{aligned} & (\psi, I, Q, P) \\ & \quad \mapsto \left(\psi_1 + \frac{2\pi}{\ell}, \psi_2, \cdots, \psi_n, I, \begin{pmatrix} \cos\frac{2\pi}{\ell} & \sin\frac{2\pi}{\ell} \\ -\sin\frac{2\pi}{\ell} & \cos\frac{2\pi}{\ell} \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix}\right) . \end{aligned} \tag{13}$$

Furthermore, on the covering space N is independent of the torus variables. By reducing the (now free) $\mathbb{T}^n$–action, or in other words by restricting to the normal dynamics of N, a one degree of freedom system is obtained, which has Hamiltonian F. After this reduction, the $\mathbb{Z}_\ell$–symmetry reads as

$$(Q,P) \mapsto \begin{pmatrix} \cos\frac{2\pi}{\ell} & \sin\frac{2\pi}{\ell} \\ -\sin\frac{2\pi}{\ell} & \cos\frac{2\pi}{\ell} \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix} . \tag{14}$$

Observe that in the present ℓ–fold covering space the resonance condition (5) is trivially satisfied, as k has been mapped to $0 \in \mathbb{Z}^n$ and α_0 to $0 \in \mathbb{R}$.

In the next section we analyse the reduced $\mathbb{Z}_\ell$–symmetric one degree of freedom system F for $\ell = 1, 2, 3, 4, \dots$.

Remark 2.2

1. Notice that for $\ell \neq 1$ further reduction of the $\mathbb{Z}_\ell$–symmetry would give rise to a conical singularity at the origin of the reduced phase space, cf. [40, 41, 62]. It is exactly to avoid this singularity that we work with $\mathbb{Z}_\ell$–equivariant systems on the covering space.

2. We make a short remark on parametrically forced systems. As an example consider the nonlinear Hill–Schrödinger equation
$$\ddot{x} \;+\; (\alpha^2 + \varepsilon\rho(t))\sin x \;=\; 0 \quad ,$$

cf. [17, 18, 24, 27]. For quasi–periodic $\rho(t)$ this fits into the framework of this article if $G(\omega t, x, y) = -\rho(t)\sin x$ is taken. However, the nonlinear Hill–Schrödinger equation is not 'as generic as possible', since $x = \dot{x} = 0$ always is an invariant manifold, and hence if put in the present framework, for $n = 1$ we would have that $A = 0$.

The remaining resonances (5) with $\ell \geq 2$ of the present investigation, however, lead to a $\mathbb{Z}_\ell$–symmetric lifted system where $(x, y) = (0, 0)$ actually is invariant, and hence the classification obtained will be the same as that for parametric resonances, cf. [18, 27].

3 The one degree of freedom 'backbone' system

As a result of the various transformations performed in the previous section, we have obtained the Hamiltonian

$$H = \langle \hat{\omega}_\ell, I\rangle + F + R,$$

which is defined on an ℓ–fold cover of the original phase space. In this section the 'backbone' Hamiltonian $F = F(Q, P; \alpha, \varepsilon)$ is analysed; the next two sections investigate the dynamical consequences of adding to F the integrable term $\langle \hat{\omega}_\ell, I\rangle$ and the non–integrable term R, and of projecting the system back to the base of the covering.

Recall that the backbone system F is $\mathbb{Z}_\ell$–symmetric, where ℓ depends on the type of normal–internal resonance. Below F will be truncated even further: the order of truncation will be the lowest possible, depending on ℓ, such that we maintain a sufficient amount of control over the equilibria and their bifurcations to be able to prove their persistence under the non–integrable perturbation R later on.

Remark 3.1

1. The origin $(Q, P) = (0, 0)$ is always a critical point of $F(Q, P; \alpha, \varepsilon)$ for $\ell \geq 2$; however, generically it is not a critical point when $\ell = 1$.
2. The present backbone system is the same slow system as would be obtained in the periodically forced case $n = 1$. Hence to a large extent the one degree of freedom dynamics is as in [2, 67]. Whenever bifurcations occur, these are governed by the corresponding singularity of the (planar) Hamiltonian function and its universal unfolding. Following [74] simple adjustments have to be made where such a singularity has to be unfolded in a $\mathbb{Z}_\ell$–symmetric context.

3.1 The case $\ell = 1$

In the case $\ell = 1$ the Hamiltonian F of the one degree of freedom backbone system reads as

$$F(Q, P; \alpha, \varepsilon) \;=\; (\alpha - \alpha_0)\frac{Q^2 + P^2}{2} \;-\; \left(\frac{Q^2 + P^2}{2}\right)^2 \;+\; A\varepsilon Q \;+\; \cdots, \tag{15}$$

where the '$\cdots$' indicate those terms in F that will be dropped below. In this case the deck group $\mathbb{Z}_\ell$ is trivial.

Since F is considered for Q, P and ε all three close to 0, a suitable scaling yields the dominant terms of an expansion in ε. For the present case $\ell = 1$ such a scaling is given by

$$\begin{aligned} \varepsilon = \;\; \mu^3, \qquad & Q = \mu q, \\ \alpha - \alpha_0 = \delta\mu^2, \qquad & P = \mu p. \end{aligned} \tag{16}$$

We now drop the higher order terms in μ and divide the truncated $F(P, Q; \alpha, \varepsilon)$ by the remaining common factor μ^4 . This leads to

$$f(q, p; \delta) \;=\; \delta\frac{q^2 + p^2}{2} \;-\; \left(\frac{q^2 + p^2}{2}\right)^2 \;+\; Aq \quad . \tag{17}$$

Note that this amounts to a rescaling of time. Indeed, had we divided by μ^2 instead of μ^4 then the scaling (16) would amount to a *conjugacy* between the flows of the Hamiltonian vector fields defined by f and by the truncation of F , that is, a co–ordinate change that maps orbits into orbits and does respect the time–parametrisation. Since we do rescale time the two flows are no longer conjugate but only *dynamically equivalent*.

Also note that the Hamiltonian f has, unlike F, a reversing symmetry $(q, p) \mapsto (q, -p)$; see subsection 3.5 for more details.

Proposition 3.1 $(\ell = 1)$ *Consider the Hamiltonian system with Hamiltonian f as given by* (17)*, and let $A \neq 0$. This system has at most three equilibria, all on the line $p = 0$. A centre–saddle bifurcation of equilibria occurs at*

$$\delta_{CS} \ := \ 3\left(\frac{A}{2}\right)^{\frac{2}{3}}.$$

For $\delta > \delta_{CS}$, there are three equilibria, two elliptic and one hyperbolic; for $\delta < \delta_{CS}$, there is a single elliptic equilibrium. This is a generic scenario, structurally stable for sufficiently small perturbations.

Proof. We look for the critical points of f, that is, points (q, p) such that the right hand sides of

$$\begin{aligned} \dot q &= \delta p - \left(q^2 + p^2\right) p, \\ \dot p &= -\delta q + \left(q^2 + p^2\right) q - A, \end{aligned}$$

both vanish. As the fixed points have to satisfy $p = 0$, equilibria are given by

$$A + \delta q - q^3 \ = \ 0. \tag{18}$$

Since the scaled system only depends on one parameter, generically centre–saddle bifurcations are expected to occur, where two equilibria collide and disappear. This is expressed by the equations

$$A + \delta q - q^3 = 0, \qquad \delta - 3q^2 = 0.$$

These equations yield a single bifurcation parameter value $\delta_{CS} = 3\left(\frac{A}{2}\right)^{2/3}$, the corresponding bifurcating equilibrium being $(q, p) = (q_{CS}, 0)$ with $q_{CS} = -\left(\frac{A}{2}\right)^{1/3}$. Note that for variable A and δ the left hand side of (18) would be a universal two parameter unfolding of the map $q \mapsto -q^3$. As A is actually fixed to a non–zero value, and only δ is a varying parameter, the resulting parametrised family is a generic one parameter unfolding, corresponding to a non–degenerate centre–saddle bifurcation, cf. [12, 52, 66]. Here we have used standard singularity theory. Note that at bifurcation, the equilibrium $(q_{CS}, 0)$ is parabolic. The system (17) contains no further bifurcations. ∎

The bifurcation diagram at this resonance, which is shown in figure 5, can be described as follows: for increasing δ the amplitude of the elliptic equilibria starts to increase at the resonance. At $\delta = \delta_{CS}$ a centre–saddle bifurcation takes place, generating two additional equilibria, one elliptic and the other hyperbolic. As δ increases further, the amplitude of the hyperbolic branch starts to increase as well and approaches that of the non–bifurcating elliptic branch. The elliptic branch generated at the bifurcation tends to 0 as δ increases past δ_{CS}. See figures 3a–c. Also compare with [80] (chapter IV, section 2), or [2, 67].

Remark 3.2 Equilibria of the system f correspond to response solutions of the system N of the previous section, and their distance to the origin is related to the amplitude of those solutions. Here and in the following, we shall abuse terminology by referring to the 'amplitude' of an equilibrium when we have their distance to the origin in mind. The scaling (16) translates 'amplitudes' of f into 'amplitudes' of F: equilibria of f of order $O(1)$ correspond to responses with amplitudes of order $O(\varepsilon^{\frac{1}{3}})$ in the original system F.

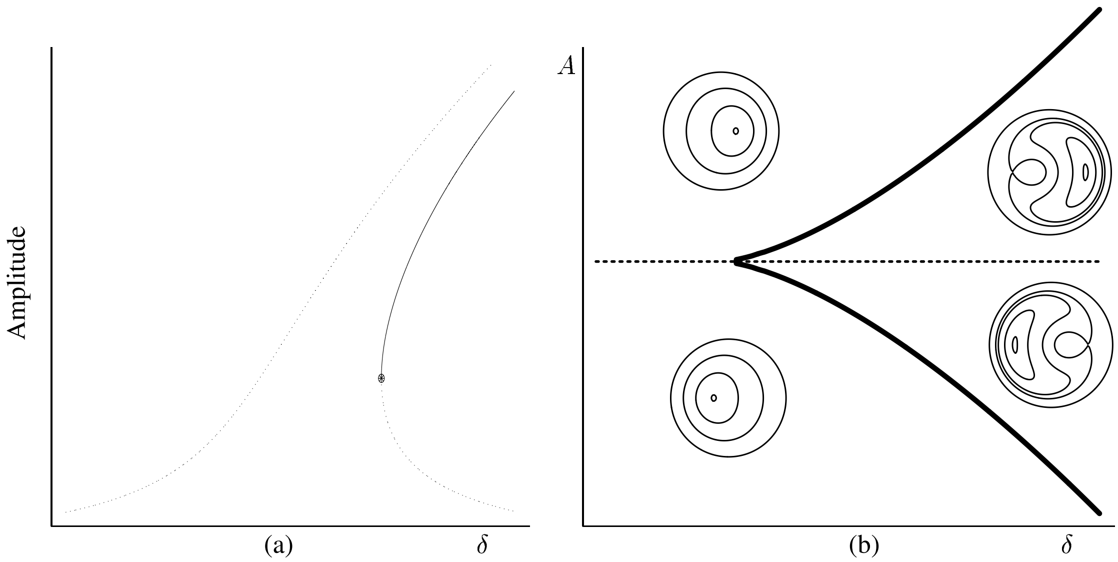

Figure 5: (a) *Response diagram for the case $\ell = 1$: dotted lines indicate the position of elliptic equilibria, solid lines correspond to hyperbolic equilibria. Compare with figure 2, of which the present picture is a scaled version. See also Stoker* [80]. (b) *Structurally stable phase portraits in an (δ, A) diagram: for constant A, the figure shows a one dimensional bifurcation diagram for the bifurcation parameter δ. The dashed horizontal line corresponds to the (excluded) non–generic case $A = 0$. Compare with figure* 3a&c.

3.2 The case $\ell = 2$

For the next case $\ell = 2$, the significant part of the one degree of freedom Hamiltonian reads as

$$\begin{aligned} F(Q, P; \alpha, \varepsilon) &= (\alpha - \alpha_0)\frac{Q^2 + P^2}{2} - \left(\frac{Q^2 + P^2}{2}\right)^2 + a\varepsilon\frac{Q^2 + P^2}{2} \\ &\quad + A\varepsilon(Q^2 - P^2) + \cdots \quad . \end{aligned} \tag{19}$$

Note that this system is $\mathbb{Z}_2$–equivariant, with the symmetry acting by $(Q, P) \mapsto (-Q, -P)$. In this case, an appropriate scaling is

$$\begin{aligned} \varepsilon = \ \mu^2, \qquad & P = \mu p, \\ \alpha - \alpha_0 = \delta\mu^2, \qquad & Q = \mu q. \end{aligned} \tag{20}$$

Again we drop the higher order terms in μ and divide the truncated $F(P, Q; \alpha, \varepsilon)$ by the remaining common factor μ^4 . This leads to

$$f(q, p; \delta) = (\delta + a)\frac{q^2 + p^2}{2} - \left(\frac{q^2 + p^2}{2}\right)^2 + A\,(q^2 - p^2) \quad . \tag{21}$$

As before, the scaled and truncated system f is reversible with respect to $(q, p) \mapsto (q, -p)$.

Proposition 3.2 ($\ell = 2$) *Consider the Hamiltonian system with Hamiltonian f as given by* (21)*, and let $A \neq 0$. The origin $(q, p) = (0, 0)$ is always an equilibrium of this system. Hamiltonian pitchfork bifurcations from the origin occur at $\delta = \delta^1_{PF}$ and $\delta = \delta^2_{PF}$, where*

$$\delta^1_{PF} := -a - 2A, \quad \textit{and} \quad \delta^2_{PF} := -a + 2A.$$

If $A > 0$, then $\delta^1_{PF} < \delta^2_{PF}$, and the origin is hyperbolic for $\delta \in \left]\delta^1_{PF}, \delta^2_{PF}\right[$, elliptic for $\delta < \delta^1_{PF}$ and $\delta > \delta^2_{PF}$. At δ^1_{PF}, a pair of elliptic equilibria on the line $q = 0$ branches off from the origin; at δ^2_{PF}, a pair of hyperbolic equilibria lying on the line $p = 0$ branches off.

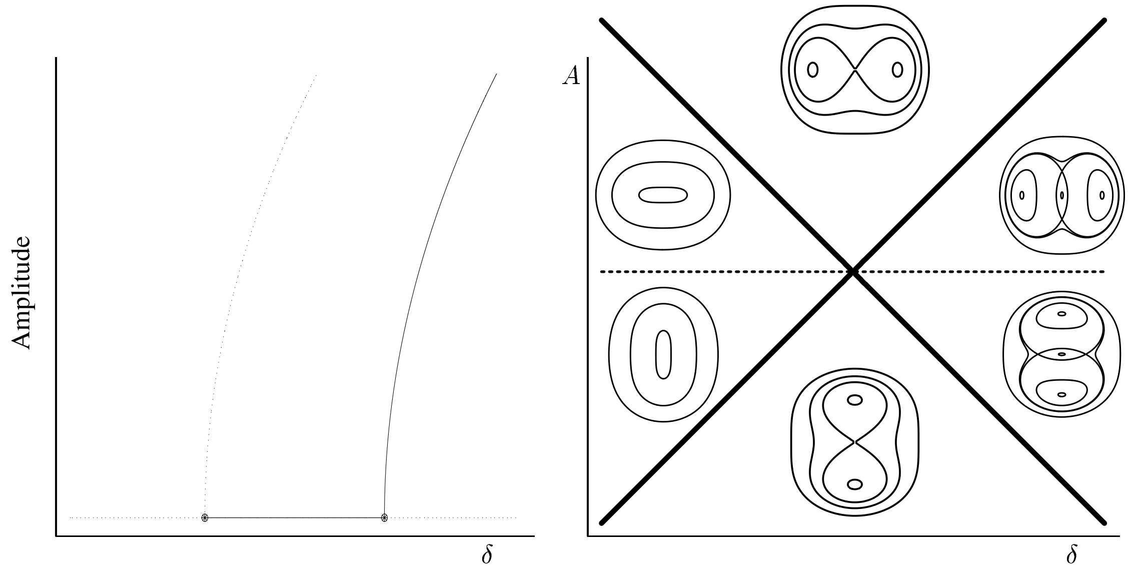

Figure 6: (a) *Response diagram for the case $\ell = 2$: dotted lines indicate elliptic equilibria, solid lines correspond to hyperbolic equilibria.* (b) *Structurally stable phase portraits in an (δ, A)–bifurcation diagram.*

If $A < 0$, we have that $\delta^2_{PF} < \delta^1_{PF}$, and the origin is hyperbolic for $\delta \in]\delta^2_{PF}, \delta^1_{PF}[$ and elliptic for $\delta < \delta^2_{PF}$ and $\delta > \delta^1_{PF}$. At δ^2_{PF}, a pair of elliptic equilibria lying on the line $p = 0$ branches off from the origin; at δ^1_{PF}, a pair of hyperbolic equilibria on the line $q = 0$ branches off.

This is a generic scenario, structurally stable for sufficiently small $\mathbb{Z}_2$–equivariant perturbations.

Proof. We obtain as equations of motion

$$\begin{aligned} \dot{q} &= (\delta + a)p - \left(q^2 + p^2\right) p - 2Ap, \\ \dot{p} &= -(\delta + a)q + \left(q^2 + p^2\right) q - 2Aq. \end{aligned}$$

Since $A \neq 0$, the equilibria (q, p) satisfy either of the following sets of equations:

$$\begin{aligned} &q = 0, \qquad p = 0, \\ &q = 0, \qquad 2A - (\delta + a) + p^2 = 0, \\ &p = 0, \qquad 2A + (\delta + a) - q^2 = 0. \end{aligned}$$

The equilibria are given by

$$(q, p) \in \left\{(0, 0), \left(0, \pm\sqrt{\delta + a - 2A}\right), \left(\pm\sqrt{\delta + a + 2A}, 0\right)\right\} .$$

The equilibrium $(q, p) = (0, 0)$ undergoes two subsequent bifurcations.

If $A > 0$, this equilibrium is elliptic for $\delta < -a - 2A$, parabolic for $\delta = -a - 2A$, and as it turns hyperbolic for $\delta > -a - 2A$, the pair of elliptic equilibria

$$(q, p) = (\pm\sqrt{\delta + a + 2A}, 0) \tag{22}$$

branches off. At $\delta = -a + 2A$ the equilibrium $(q, p) = (0, 0)$ is again parabolic, turning elliptic for even larger values of δ while a second pair of equilibria, this time hyperbolic ones, branches off from the origin.

The obvious modifications for the case $A < 0$ are left to the reader.

In the present $\mathbb{Z}_2$–equivariant context these Hamiltonian pitchfork bifurcations are generic. ∎

The two bifurcations take place in the orthogonal planes $q = 0$ and $p = 0$. Compare this with [27] figures 5 and 8, and again with [80] (chapter IV, section 2) and [2, 38, 67].

Remark 3.3 Since in the scaled system given by f the 'amplitude' of equilibria off the origin is of order $O(1)$, the corresponding equilibria in the original system F have an amplitude of order $O(\varepsilon^{\frac{1}{2}})$.

3.3 Higher order resonances: the cases $\ell \geq 3$

For completeness also the case $\ell \geq 3$ is included, although these resonances will not produce gaps and only give rise to bifurcations, cf., e.g., [38, 19].

Considering (12) in the present context, the suggested scaling is

$$\begin{aligned} \varepsilon &= \mu, & P &= \mu p, \\ \alpha - \alpha_0 &= \delta\mu, & Q &= \mu q, \end{aligned} \tag{23}$$

which leads to

$$\begin{aligned} F \;=\;& \mu^3(\delta + a)\frac{q^2+p^2}{2} \;+\; \mu^4(F_{0,1,2,0} + F_{0,1,1,1}\delta + F_{0,1,0,2}\delta^2)\frac{q^2+p^2}{2} \\ & - \mu^4\left(\frac{q^2+p^2}{2}\right)^2 \;+\; \mu^{\ell+1} A\,\mathrm{Re}(q - \mathrm{i}p)^\ell \\ & + \sum_{5\leq 2m+j+i\leq \ell+1} \mu^{j+i+2m} F_{0,m,i,j}\delta^j (q^2+p^2)^m \;+\; O(\mu^{\ell+2}). \end{aligned}$$

Note that the term $a\mu^3$ can be removed by solving

$$\delta \;=\; -a \;+\; \mu\left(\lambda - F_{0,1,2,0} - F_{0,1,1,1}\delta - F_{0,1,0,2}\delta^2\right)$$

for the new bifurcation parameter λ . This time we truncate F including order $O(\mu^{\ell+1})$ before dividing by μ^4 and obtain

$$\begin{aligned} f(q,p;\lambda,\mu) \;=\;& \lambda\frac{q^2+p^2}{2} \;-\; \left(\frac{q^2+p^2}{2}\right)^2 \\ & + \mu\, C(q^2+p^2;\lambda,\mu) \;+\; A\mu^{\ell-3}\mathrm{Re}(q-\mathrm{i}p)^\ell \quad . \end{aligned}$$

Here the term $\mu\, C(q^2+p^2,\delta,\mu)$ corresponds to the large sum in the expression for F .

Remark 3.4 For $\ell = 3$ the $\mathbb{T}^1$–symmetry breaking term $A\mu^{\ell-3}\mathrm{Re}(q - \mathrm{i}p)^\ell$ is of the same order in μ as the leading order $\mathbb{T}^1$–symmetric terms, while it is of smaller order when $\ell > 3$. Therefore the cases $\ell = 3$ with $\mu = 0$, and $\ell > 3$ with $\mu > 0$ small but nonzero, are analysed separately.

3.3.1 The case $\ell = 3$

In the case $\ell = 3$, which is considered next, the invariant torus $(q,p) = (0,0)$ is normally elliptic for all values of $\lambda \neq 0$. For a certain value λ_{CS} of λ there is a centre–saddle bifurcation of invariant tori outside the origin. The normally hyperbolic torus generated in this bifurcation then approaches the origin, disappears for $\lambda = 0$, and reappears again for $\lambda > 0$. The symmetries of the (truncated) Hamiltonian

$$f(q,p;\lambda,0) \;=\; \lambda\frac{q^2+p^2}{2} \;-\; \left(\frac{q^2+p^2}{2}\right)^2 \;+\; A(q^3 - 3p^2q) \tag{24}$$

are the deck transformations (generated by the $\frac{2\pi}{3}$–rotation about the origin) and the reversing involution $(q,p) \mapsto (q,-p)$ (together with their $\frac{2\pi}{3}$–rotated counterparts). The equations of motion are

$$\begin{aligned} \dot q &= \lambda p \;-\; \left(q^2+p^2\right)p \;-\; 6Apq. \\ \dot p &= -\lambda q \;+\; \left(q^2+p^2\right)q \;-\; 3A(q^2-p^2) \end{aligned} \tag{25}$$

Proposition 3.3 $(\ell = 3)$ *Consider the Hamiltonian systems given by f (see (25)) where $A \neq 0$. Three simultaneous $\mathbb{Z}_3$–related centre–saddle bifurcations occur as λ passes through*

$$\lambda_{CS} \; = \; -\frac{9}{4}A^2 \; < \; 0 \quad .$$

At $\lambda = 0$ the origin has a $\mathbb{Z}_3$–equivariant transcritical bifurcation. This scenario is structurally stable for sufficiently small $\mathbb{Z}_3$–equivariant perturbations.

Proof. First consider the case that $A > 0$. The right hand side of (25) always vanishes at the origin $(q,p) = (0,0)$. For $\lambda > \lambda_{CS}$ one readily computes six further equilibria, three elliptic equilibria at

$$(q,p) \; \in \; \left\{ (\beta,0), \left(-\frac{\beta}{2}, \frac{\beta\sqrt{3}}{2}\right), \left(-\frac{\beta}{2}, -\frac{\beta\sqrt{3}}{2}\right) \right\}$$

with $\beta = \frac{3}{2}A + \frac{1}{2}\sqrt{9A^2+4\lambda}$ and three hyperbolic equilibria at

$$(q,p) \; \in \; \left\{ (\gamma,0), \left(-\frac{\gamma}{2}, \frac{\gamma\sqrt{3}}{2}\right), \left(-\frac{\gamma}{2}, -\frac{\gamma\sqrt{3}}{2}\right) \right\}$$

with $\gamma = \frac{3}{2}A - \frac{1}{2}\sqrt{9A^2+4\lambda}$. Note that these equilibria coalesce ‘pairwise’ for $\lambda = \lambda_{CS}$ and that the latter three coalesce with the origin for $\lambda = 0$. On both sets of equilibria the deck transformations yield even permutations, while the reversing transformations act as the transpositions. These symmetries allow us to concentrate on the equilibria at $(\beta,0)$ and $(\gamma,0)$.

For $\lambda = \lambda_{CS}$ we develop f into a (Taylor) polynomial in the point $(\frac{3}{2}A, 0, \lambda_{CS})$, *i.e.* we write $(q,p;\lambda) = (\frac{3}{2}A + \xi, \eta, \lambda_{CS} + \nu)$ and obtain

$$f(\xi,\eta;\nu) = -\frac{27A^2}{4}\eta^2 - \frac{A}{2}\xi^3 + \frac{3A}{2}\nu\xi + \cdots$$

where ‘$\cdots$’ denotes both constant and higher order terms. This proves that a (non–degenerate) centre–saddle bifurcation takes place, at $(\frac{3}{2}A, 0)$ and hence also at its $\frac{2\pi}{3}$–rotated counterparts, when λ passes through λ_{CS}.

Similarly, we put $p \equiv 0$ and obtain near $(q,\lambda) = (0,0)$

$$f(\xi,0;\nu) \;\; = \;\; A\xi^3 \; + \; \frac{\nu}{2}\xi^2 \; + \; \cdots$$

where ‘$\cdots$’ stands for the sole higher order term $-\frac{\xi^4}{4}$. Consequently, a transcritical bifurcation takes place along the q–axis, and hence also along its $\frac{2\pi}{3}$–rotated counterparts, when λ passes through 0 . Since the only two occurring bifurcations at $\lambda = \lambda_{CS}$ and at $\lambda = 0$ are versally unfolded by the family of Hamiltonian systems defined by (25), we have structural stability.

If $A < 0$, the equilibrium $(\beta,0)$ and its images under the $\mathbb{Z}_3$–action are hyperbolic instead of elliptic, whereas $(\gamma,0)$ and its images are elliptic instead of hyperbolic. The rest of the proof is unchanged. ∎

Remark 3.5

1. An alternative proof of the second part, concerning the $\mathbb{Z}_3$–equivariant transcritical bifurcation, can be given in the following way. First consider the proof of [12] in the general non–symmetric context. This leads to the elliptic umbilic catastrophe, a 3–parameter universal unfolding of the planar singularity[1] at $(q,p;\lambda) = (0,0,0)$. The present $\mathbb{Z}_3$–equivariant 1–parameter unfolding then can be obtained from this by applying [74].

2. The response solutions of ‘amplitude’ of order $O(1)$ and in particular the centre–saddle bifurcations correspond to equilibria with amplitude $O(\varepsilon)$ in the original system determined by F .

[1] In the ‘ADE’ classification this singularity is referred to as D_4^+ , see [3]

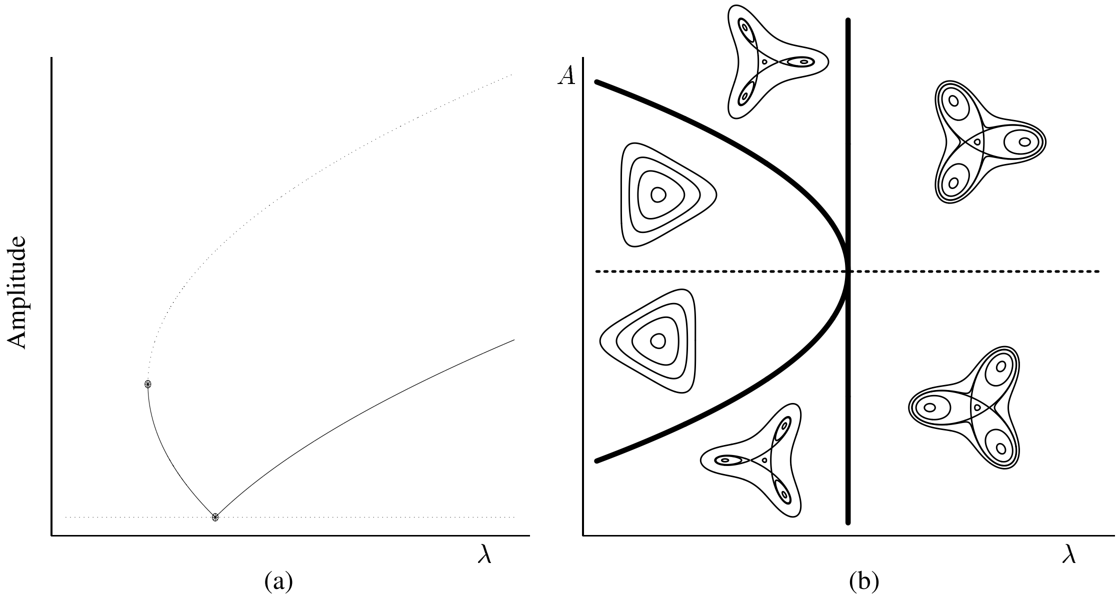

Figure 7: (a) *Response diagram for the case $\ell = 3$: dotted lines indicate elliptic equilibria, solid lines correspond to hyperbolic equilibria.* (b) *Structurally stable phase portraits in a (λ, A)–bifurcation diagram.*

3.3.2 The case $\ell \geq 4$

The remaining case is $\ell \geq 4$. In this case 2ℓ equilibria, ℓ elliptic and ℓ hyperbolic, bifurcate from the central equilibrium at $(q,p) = (0,0)$.

The scaled and truncated Hamilton function f is given by

$$\begin{aligned} f(q,p;\lambda,\mu) \;&=\; \lambda\frac{q^2+p^2}{2} \;-\; \left(\frac{q^2+p^2}{2}\right)^2 \\ &\quad + \mu\, C(q^2+p^2;\lambda,\mu) \;+\; A\mu^{\ell-3}\mathrm{Re}(q-\mathrm{i}p)^\ell \quad , \end{aligned} \tag{26}$$

where as before we assume $A \neq 0$. Recall that $\mathrm{Re}(q-\mathrm{i}p)^\ell$ is shorthand for $q^\ell - \binom{\ell}{2}p^2q^{\ell-2} + \ldots$, and that μ is related to the perturbation strength ε. If $\mu = 0$, the equilibria outside the origin $(p,q) = (0,0)$ lie on a circle

$$q^2 + p^2 \;=\; \lambda\,,$$

that is, they branch off from the origin at $\lambda = 0$. For $\mu \neq 0$ the $\mu^{\ell-3}$–term breaks that $\mathbb{T}^1$–symmetry to the $\mathbb{Z}_\ell$–symmetry of the deck group on the ℓ–fold covering. This is seen most easily in action–angle coordinates (φ, I) introduced by

$$q - \mathrm{i}p \;=\; \sqrt{2I}\,\mathrm{e}^{\mathrm{i}\varphi}.$$

In these coordinates the Hamiltonian takes the form

$$f = \lambda I - I^2 + \mu\, C(I^2;\lambda,\mu) + \mu^{\ell-3}A\,(2I)^{\ell/2}\cos\ell\varphi.$$

The deck group is generated by $(\varphi, I) \mapsto (\varphi + \frac{2\pi}{\ell}, I)$, and it maps elliptic and hyperbolic equilibria on equilibria of the same type. Hence, there are saddles at ℓth roots of $\lambda + O(\lambda^2)$ and centres at those 2ℓth roots that are not ℓth roots.

3.4 On persistence

In all of the above cases we found that the reduced integrable parts of the normal form give rise to a 1–parameter family of one degree of freedom Hamiltonian systems, the so–called backbone systems. These 1–parameter families are structurally stable in appropriate universes of symmetric systems; the corresponding symmetries are the deck symmetries of the

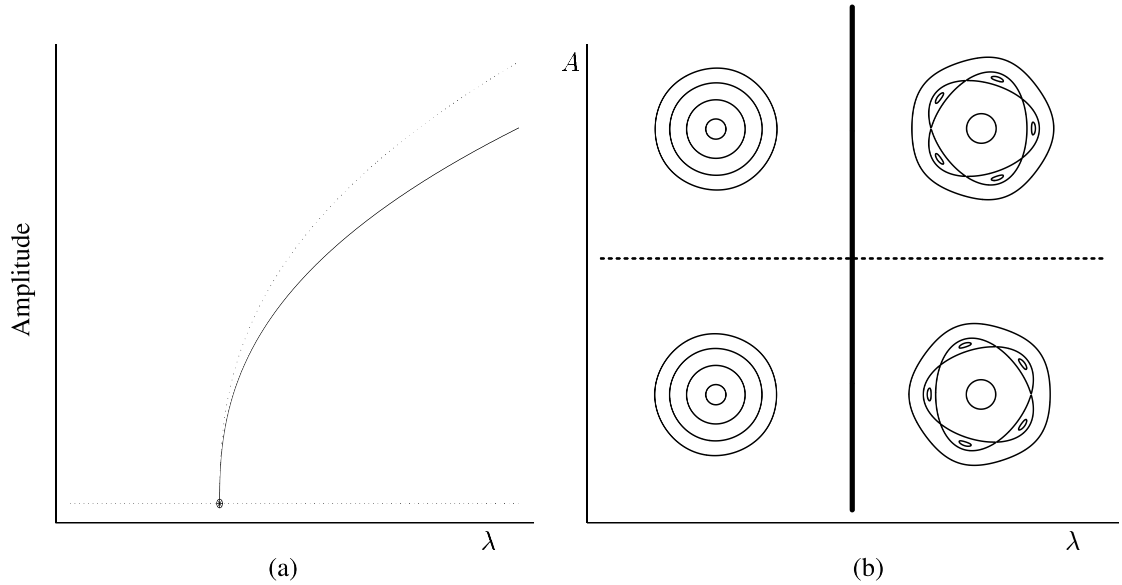

Figure 8: (a) *Response diagram for the case $\ell = 5$: dotted lines indicate elliptic equilibria, solid lines correspond to hyperbolic equilibria.* (b) *Structurally stable phase portraits in an (λ, A)–bifurcation diagram.*

covering. The structural stability implies in particular that the special perturbations obtained by adding terms of higher order do not change the qualitative behaviour of the reduced dynamics, as these terms can be made arbitrarily small by decreasing ε.

In other words: after various transformations, we arrived at a normal form for our system, which consisted of an integrable system and a non–integrable perturbation of arbitrary high order in the perturbation strength. Analysis of the integrable system leads after reduction of the torus symmetry to a planar (equivariant) Hamiltonian system, to which (equivariant) singularity theory of planar functions is applied, compare with [28, 38, 42, 43, 44, 74, 82]; also see [12, 13, 16, 17, 18, 21, 41, 64, 66, 67, 72]. We observe that in our case singularity theory classifies up to smooth equivalences, meaning that addition of normalised higher order terms would only give rise to a near–identity diffeomorphic distortion of the bifurcation diagrams obtained so far.

Note that for $\ell \neq 1$, instead of working in the universe of $\mathbb{Z}_\ell$–symmetric systems, we might have chosen to reduce also the $\mathbb{Z}_\ell$–symmetry (14). This approach gives rise to a conical singularity at the origin of the reduced phase space, which is not a manifold any more, cf. [40, 41, 62]. The singularity reflects that the $\mathbb{T}^n$–action generated by (10) is not free (unless $\ell = 1$), but has isotropy $\mathbb{Z}_\ell$ at the central n–torus. It is exactly to avoid this conical singularity that we lift the system to the covering space (before reducing the $\mathbb{T}^n$–action), always keeping the $\mathbb{Z}_\ell$–symmetry into account. For the alternative approach of also reducing the $\mathbb{Z}_\ell$–symmetry, the bifurcation scenarios can be described as follows.

In the case $\ell = 2$ the two pairs of elliptic and hyperbolic equilibria (that are generated in the two bifurcations) each reduce to one (elliptic resp. hyperbolic) equilibrium on the quotient space $\mathbb{R}^2/\mathbb{Z}_2$, bifurcating off from the singular equilibrium. See [41] for a similar treatment of Hamiltonian flip bifurcations through passage to a 2:1 covering.

In the case $\ell = 3$ the $\mathbb{Z}_3$–equivariant transcritical bifurcation takes place in the singular equilibrium, while the three centre–saddle bifurcations all get reduced to the same centre–saddle bifurcation in one (regular) equilibrium on $\mathbb{R}^2/\mathbb{Z}_3$. Compare this with [2, 12, 38, 67].

For $\ell \geq 4$ one elliptic and one hyperbolic equilibrium bifurcate simultaneously off from the singular equilibrium of $\mathbb{R}^2/\mathbb{Z}_\ell$.

3.5 Reversible systems

In this subsection we outline how the arguments given in the previous sections have to be changed for *reversible* systems.

Let H be a Hamiltonian defined on the phase space $T^*(\mathbb{T}^n \times \mathbb{R}) = \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^2 = \{\theta, I, (x, y)\}$. Introduce the involution

$$\mathcal{I} : (\theta, I, x, y) \;\mapsto\; (-\theta, I, x, -y) \quad . \tag{27}$$

By definition the Hamiltonian H is said to define a reversible Hamiltonian system if $H \circ \mathcal{I} = H$. For instance, the pendulum example (1) is reversible if the function g is even in t. Similarly, a Hamiltonian of the general class (4) is reversible if both $\tilde{h}$ and G are even in y, and if G is 'even' in θ as well. In general, many Hamiltonian systems coming from classical mechanics have this property.

Reversibility is respected by the normalising and covering transformations of the previous section, compare for instance [27]. In particular this means that the integrable truncation N is also reversible, implying that the backbone system F is invariant with respect to the involution $(Q, P) \mapsto (Q, -P)$. This $\mathbb{Z}_2$ reflectional symmetry, combined with the $\mathbb{Z}_\ell$ deck–symmetry (14), gives rise to a $\mathbb{D}_\ell$–symmetry of the backbone system F.

Note that the truncation of F to lowest significant order, which throughout the present section has been denoted by f, *always* displays this extra symmetry, and that therefore it is actually $\mathbb{D}_\ell$–symmetric whether the original system (7) is reversible or not: see equations (17), (21), (24) and (26). This is a quite common artefact of the normalisation process, cf. for instance [18].

In the case of a normal–internal resonance $\langle k, \omega \rangle + \ell\alpha = 0$, the truncation f is structurally stable in the universe of $\mathbb{D}_\ell$–equivariant Hamiltonian functions, and our analysis carries over to the reversible case.

4 Reconstruction of the integrable dynamics

The integrable system $N(I, Q, P; \alpha, \varepsilon)$ is given by (11) and defined on the ℓ–fold covering space of $\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^2$; from it, we have obtained the slow one degree of freedom 'backbone' (12) by reducing out the free $\mathbb{T}^n$–translation action. Hence, reconstructing the dynamics of N on the covering space simply consists in attaching a quasi–periodic n–torus to every point of the reduced system F on the reduced phase space $\mathbb{R}^2$.

For $\ell = 1$ this immediately yields the dynamics of N (as the base is equal to the covering in this case). In the periodic case $n = 1$ the centre–saddle bifurcation of relative equilibria in the 'backbone' system corresponds to a similar bifurcation of periodic solutions in the full system, cf. [12, 13, 66]. In the quasi–periodic case ($n \geq 2$), the centre–saddle bifurcation of relative equilibria corresponds to an (integrable) quasi–periodic centre–saddle bifurcation, cf. [52].

For $\ell \geq 2$ we have to take into account the action (13) of the deck group $\mathbb{Z}_\ell$. The (relative) equilibria of the planar system determined by F give rise to quasi–periodic invariant n–tori of the reconstructed integrable system N on the covering space. These tori have a Diophantine frequency vector ω; in the next section the perturbation analysis of these tori is carried out on the covering space.

The rest of this section is devoted to the description of the structure of the reconstructed flow on the base of the covering in the case $\ell \geq 2$. The key observation to this is the following: tori in the cover which are mapped into each other by the action of the deck group project down to a single torus in the base. Since we are still focusing on a fixed resonance (5) — recall that only then the normal form and its integrable part N are meaningful — this structure strongly resembles that obtained in the case of periodic forcing.

For $\ell = 2$, consider first the periodic case $n = 1$. The equilibrium of the backbone system located at the origin of $\mathbb{R}^2$ gives rise to a periodic orbit of period $T = 2\pi/\omega$ in the base and of period $2T$ in the covering. The two equilibria generated by the Hamiltonian pitchfork bifurcation of the backbone system correspond in the covering to two periodic orbits of period $2T$ generated in a periodic Hamiltonian pitchfork bifurcation. These are mapped onto each other by the action of the deck group, and they project to one and the same periodic orbit of period $2T$ in the base space, which is generated in a Hamiltonian 'period doubling' (also called 'flip' or 'frequency halving') bifurcation, cf. [27, 41, 66].

Completely analogously, we obtain in the quasi–periodic case $n \geq 2$ two quasi–periodic n–tori in the covering which are generated in a quasi–periodic pitchfork bifurcation. These project down to a single n–torus in the base space, which is generated in a quasi–periodic Hamiltonian period doubling bifurcation. We recall that the response solution, while passing through the gap defined by (5), undergoes two such bifurcations : one at the beginning when it turns from elliptic to hyperbolic, and one at the end when it turns back again from hyperbolic to elliptic.

For $\ell = 3$, consider again the periodic case first. The equilibrium at the origin of $\mathbb{R}^2$ gives rise to a periodic solution with period $T = 2\pi/\omega$ in the base and $3T$ in the covering. In a centre–saddle bifurcation away from the origin, two times three equilibria are born. These give rise to two times three period $3T$ periodic solutions in the covering, which project down to two period $3T$ periodic solution in the base, one elliptic, one hyperbolic. The hyperbolic orbit subsequently undergoes a periodic transcritical bifurcation involving the central periodic orbit.

In the quasi–periodic case two times three invariant n–tori are born in an integrable quasi–periodic centre–saddle bifurcation. These correspond in the base to two quasi–periodic response solutions, one elliptic, one hyperbolic, with in each case one of the frequencies divided by 3 (for a well–chosen basis of frequencies). While the amplitude of the elliptic tori is continually growing, the hyperbolic tori first passes through a kind of transcritical bifurcation involving the central ω–quasi–periodic response solution.

5 Quasi–periodic stability and exponentially small gaps

We briefly summarise what happened so far. In Section 2, we started out with a Hamiltonian H defined on $\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^2 = \{\theta, I, (x, y)\}$, of the form

$$H \;=\; \langle \omega, I\rangle \;+\; \alpha\frac{x^2+y^2}{2} \;+\; \tilde{h}(x,y;\alpha) \;+\; \varepsilon G(\theta,x,y;\alpha,\varepsilon),$$

with (x, y) near $(0, 0)$ and for α close to a resonant normal frequency α_0. This means that α_0 satisfies a relation $\langle k, \omega\rangle + \ell\alpha_0 = 0$ for $k \in \mathbb{Z}^n\backslash\{0\}$ and $\ell \neq 0$. We recall that $\omega \in \mathbb{R}^n$ is a fixed, Diophantine frequency vector. By consecutive normal form transformations and a lift to an ℓ–fold covering space, the Hamiltonian has been rewritten into

$$H \;=\; N \;+\; R \;=\; \langle \tilde{\omega}, I\rangle \;+\; F \;+\; R,$$

with $\tilde{\omega} = (\frac{1}{\ell}\hat{\omega}_1, \hat{\omega}_2, \ldots \hat{\omega}_n)$, where F is independent of θ and where R is of arbitrary high order in $(x, y, \alpha - \alpha_0, \varepsilon)$.

Then, in section 3 the one degree of freedom 'backbone' Hamiltonians F were analysed by first truncating to lowest 'significant' order, performing a bifurcation analysis on the truncated part — denoted by f — and finally remarking that, since all bifurcating families obtained are structurally stable, they are equivalent to the full backbone system F as well (see subsection 3.4). As said before, up to this point there is no difference between the cases of periodic and quasi–periodic forcing.

Hence we may consider the behaviour of F, and consequently of the integrable reconstruction N, as known. What remains to do is to analyse the full system H, or in other words, the impact of the quasi–periodic non–integrable terms R. Equilibria of the one degree of freedom system F are invariant quasi–periodic n–tori for N, and their persistence (under the small perturbation R), as well as the persistence of their bifurcations, will be shown by KAM–theory in the present section.

It turns out that the 'periodic' scenarios obtained in section 3 are complicated by quasi–periodic resonance phenomena, leading to a dense collection of resonance gaps in the elliptic branches of the figures 5–8. This means that the smooth branches (continua) in the bifurcation diagrams of the periodic case are replaced by Cantor sets in the quasi–periodic case.

At this point the regularity of the Hamiltonian H becomes an issue. Until now all considerations apply in the world of C^s–systems for $s \in \mathbb{N} \cup \{\infty\}$ sufficiently large. However, in the real analytic setting we can get more information: the size of the resonance gaps is of exponentially small order in ε. To obtain this result we first prove exponentially small estimates on R by Neishtadt–Nekhoroshev techniques, which are then fed into the KAM part.

5.1 Exponentially small estimates of the remainder

From now we restrict to the case where H is real analytic. Notice that any normal form of finite order also can be obtained by a real analytic transformation. Our starting point is the formal normal form as obtained by Theorem 2.1 of Section 2:

$$H \circ \Phi^{-1}(\theta, J, X, Y; \alpha, \varepsilon) \;=\; N(\theta, J, X, Y; \alpha, \varepsilon) \;+\; R(\theta, X, Y; \alpha, \varepsilon).$$

We shall pursue a slightly different strategy than in the first sections, performing first scalings, and only afterwards normal form transformations. Recall from Section 3, that scalings (16), (20) and (23) were applied, which depended on ℓ. Application of these scalings before carrying out the van der Pol transformation yields a Hamiltonian $H_{\text{sc}} = \mu^{-2}H$ of the form

$$H_{\text{sc}} \;=\; \langle \omega, I_{\text{sc}}\rangle \;+\; \alpha_0\frac{x_{\text{sc}}^2+y_{\text{sc}}^2}{2} \;+\; \mu^2 F_{\text{sc}} \;+\; \mu^{L-2}R_{\text{sc}} \tag{28}$$

Here $I_{\text{sc}} = \mu^{-2}I$; moreover L can be chosen as large as desired, and R_{sc} is uniformly bounded in a neighbourhood of $(q, p) = (0, 0)$. Note that the symplectic 2–form $\mathrm{d}\theta \wedge \mathrm{d}I_{\text{sc}} + \mathrm{d}x_{\text{sc}} \wedge \mathrm{d}y_{\text{sc}} = \mu^2(\mathrm{d}\theta \wedge \mathrm{d}I + \mathrm{d}x \wedge \mathrm{d}y)$.

The following result considerably sharpens the bound that can be obtained on R_{sc}, by exploiting the fact that the original Hamiltonian (4) is real analytic. Its proof can be found in appendix 5.3.

Notations. To formulate the result, complex neighbourhoods $\mathcal{D}(\rho, r)$ of the n–dimensional tori $(x_{\rm sc}, y_{\rm sc}) = (0,0)$ are introduced, of the form

$$\begin{aligned}\mathcal{D}(\rho, r) &= \{(\theta, x_{\rm sc}, y_{\rm sc}) \in \mathbb{C}^n \times \mathbb{C} \times \mathbb{C} : \\ &\qquad |\mathrm{Im}\, \theta_j| < \rho \text{ for all } j,\ |x_{\rm sc}|^2 + |y_{\rm sc}|^2 < r^2\}\,.\end{aligned}$$

The supremum norm of any function f defined on $\mathcal{D}(\rho, r)$ is denoted by $\|f\|_{\mathcal{D}(\rho,r)} = \|f\|_{\mathcal{D}}$.

The (constant) frequency vector $\omega \in \mathbb{R}^n$ is assumed to be Diophantine, that is, there are $\gamma > 0$, $\tau > n-1$ such that for all $k' \in \mathbb{Z}^n \backslash \{0\}$:

$$|\langle k', \omega \rangle| \geq \gamma |k'|^{-\tau}.$$

Recall that α_0 is a normal frequency that satisfies the normal resonance relation (5), and that the integrable part of a resonant normal form is a trigonometric polynomial in θ and a polynomial in its other variables, containing only terms in the kernel of the Lie operator defined by (10).

Theorem 5.1 (Normal form with exponentially small remainder)
Consider the Hamiltonian $H_{\rm sc}$ as given by equation (28). *Let $\rho, r > 0$ be such that the remainder $R_{\rm sc}$ is analytic on $\mathcal{D} = \mathcal{D}(\rho, r)$ and $\|R_{\rm sc}\|_{\mathcal{D}} < \infty$.*

Then there is a $\mu_0 > 0$ such that for any $0 \leq \mu < \mu_0$, there is a real analytic canonical transformation

$$\Phi(\theta, I_{\rm sc}, x_{\rm sc}, y_{\rm sc}; \mu) = (\Theta, J, Q, P),$$

such that the following statements hold.

1. *If $\mathcal{D}_* = \mathcal{D}(\rho/2, r\mathrm{e}^{-\rho/2})$, then*
$$\Phi(\mathcal{D}_*) \subset \mathcal{D} \text{ and } \Phi^{-1}(\mathcal{D}_*) \subset \mathcal{D}.$$
2. *There is some constant $c > 0$ not depending on $R_{\rm sc}$ such that*
$$\|\Phi - \mathrm{id}\|_{\mathcal{D}_*}, \|\Phi^{-1} - \mathrm{id}\|_{\mathcal{D}_*} \leq c\mu^{L-4} \|R_{\rm sc}\|_{\mathcal{D}}.$$
where L is as in equation (28). That is, Φ is $\left(C^\omega, \tilde{c}\mu^{L-4}\right)$–close to the identity with $\tilde{c} = c \cdot \|R_{\rm sc}\|_D$.
3. *In the new coordinates (Θ, J, Q, P), the Hamiltonian takes the form*
$$\tilde{H}(\Theta, J, Q, P; \alpha, \mu) = \langle \omega, J \rangle + \alpha_0 \frac{Q^2 + P^2}{2} + \mu^2 \tilde{F} + \mu^{L-2} \tilde{R},$$
where $\tilde{F}(\Theta, Q, P; \alpha, \mu)$ is in resonant normal form,
$$\|F_{\rm sc} - \tilde{F}\|_{\mathcal{D}_*} \leq c\, \mu^{L-4} \|R_{\rm sc}\|_{\mathcal{D}},$$
and $\tilde{R}(\Theta, Q, P; \alpha, \mu)$ satisfies
$$\|\tilde{R}\|_{\mathcal{D}_*} \leq \exp\left(-\frac{c}{\mu^{2/(\tau+2)}}\right) \|R_{\rm sc}\|_{\mathcal{D}}. \qquad (29)$$

Remark 5.1

1. If $H_{\rm sc}$ depends real analytically on additional parameters λ, and $R_{\rm sc}$ is uniformly bounded on some domain with respect to λ, then Φ is also real analytic with respect to these parameters, and the final bounds hold uniformly in λ as well.
2. Assume that $H_{\rm sc}$ depends analytically on μ. From the proof of theorem 5.1, only piecewise analytic dependence of Φ with respect to μ can be inferred, see for instance [23, 22, 71]. Alternatively, the same method of proof yields that if μ_1 is taken small enough, then Φ can be shown to be analytic in μ for $0 \leq \mu \leq \mu_1$, but the exponentially small estimate holds in this case with μ_1 in the exponent.

5.2 Persistence of quasi–periodic response tori

We arrive at the analysis of the persistence of the quasi–periodic response tori, which is one of the main goals of this paper. In the previous subsection, coordinates have been found such that for the remainder term $\tilde{R}$ in the Hamiltonian

$$\tilde{H} \;=\; \langle \omega, J\rangle \;+\; \alpha_0 \frac{Q^2+P^2}{2} \;+\; \mu^2\tilde{F} \;+\; \mu^{L-2}\tilde{R},$$

an exponentially small bound is obtained. By applying the van der Pol transformation, a system of the form

$$H \;=\; \langle \hat{\omega}_\ell, I\rangle \;+\; \mu^2 F \;+\; \mu^{L-2} R \tag{30}$$

is obtained, where $F(q,p;\delta,\mu) = f(q,p;\delta) + O(\mu)$ is independent of θ; recall that $\alpha = \alpha_0 + \mu^2\delta$ in the cases $\ell = 1$ and $\ell = 2$; the other cases are treated similarly. See section 2 for the relationship between ω and $\hat{\omega}_\ell$. Note that we dropped the tildes on H and R.

Recall that in equation (30), the term R is the only non–integrable term in H, which can (and will) be viewed as a small non–integrable perturbation of the integrable Hamiltonian $H_0 = \langle \hat{\omega}_\ell, I\rangle + \mu^2 F$.

Note that H_0 has been investigated in section 3. Depending on the kind of resonance, that is, depending on ℓ in $\langle k,\omega\rangle + \ell\alpha_0 = 0$, parametrised families of elliptic and hyperbolic n–tori have been found, as well as single parabolic n–tori for certain distinguished values of the parameter δ.

In this subsection, the persistence of these response n–tori is investigated under the perturbation $\mu^{L-2}R$, taking into account the quantitative information on the smallness of the remainder R obtained in the previous section.

Hyperbolic tori. This is the simplest case. If the variable I is viewed as a parameter, the phase space of the system is $\mathbb{T}^m \times \mathbb{R}^2$, and hyperbolic equilibria of F correspond to normally hyperbolic invariant n–tori. These persist because of normal hyperbolicity, compare with [35, 49, 54, 78]. Note that the parallel (quasi–periodic) flow on the tori persists trivially in the context of this paper.

In particular, let $\tau(\delta) \in \mathbb{R}^2$ parametrise a normally hyperbolic family of equilibria of F, and let $\lambda(\delta) \in \mathbb{R}$ be the associated positive Floquet exponent — in this case, this exponent is the positive eigenvalue of the corresponding equilibrium of F. Moreover, assume that the Hamiltonian H is analytic in δ on a strip of width σ around some compact interval $A \subset \mathbb{R}$ in the complex plane.

Theorem 5.2 (Persistence of hyperbolic tori)
Let $\mathcal{A} = [\delta_1, \delta_2]$ be an interval such that the Floquet exponent $\lambda(\delta)$ is bounded away from zero on $\mathcal{A}$. Then there is a $\mu_1 > 0$ such that for $0 \le \mu < \mu_1$, there are constants $c > 0$ and

$$\nu = \exp\left(-\frac{c}{\mu^{2/(\tau+2)}}\right),$$

and maps $(\delta,\theta) \mapsto \tilde{\tau}_\delta(\theta) \in \mathbb{R}^2$, $\delta \mapsto \tilde{\lambda}(\delta) \in \mathbb{R}$ such that the following hold.

1. *The map $\tilde{\lambda}$ is analytic on $\mathcal{A}_{\sigma/2} = \{\delta \in \mathbb{C} \,:\, d(\delta,\mathcal{A}) < \sigma/2\}$, and*
$$\sup_{\mathcal{A}_{\sigma/2}} |\tilde{\lambda}(\delta) - \lambda(\delta)| < \nu.$$
2. *The map $\tilde{\tau}$ is analytic for δ taking values in $\mathcal{A}_{\sigma/2}$, and for θ taking values in T, where $T = \{|\operatorname{Im}\theta_j| < 3\rho/8\}$ is a complex neighbourhood of the real n-torus $\mathbb{T}^n$. Moreover,*
$$\sup_{\delta\in\mathcal{A}_{\sigma/2},\theta\in T} |\tilde{\tau}_\delta(\theta) - \tau_\delta| < \nu.$$
3. *For any $\delta \in \mathcal{A}$, the torus $\tilde{\mathcal{T}}_\delta$, given as*
$$\tilde{\mathcal{T}}_\delta = \{(\theta, I, q, p) \,:\, (q,p) = \tilde{\tau}_\delta(\theta)\},$$
is invariant under the flow of H.
4. *The flow on $\tilde{\mathcal{T}}_\delta$ is quasi–periodic with frequency vector $\hat{\omega}_\ell$.*

Elliptic tori. This is the case of the persistence of lower dimensional elliptic tori, treated in (for instance) [20, 39, 57, 75]. Assume that $\delta \mapsto \tau_\delta$ parametrises a family of elliptic integrable n–tori $\mathcal{T}_\delta$, as found in one of the cases of section 3, by

$$\mathcal{T}_\delta = \{(\theta, I, q, p) \, : \, (q,p) = \tau_\delta(\theta)\}.$$

The Floquet exponent $\lambda(\delta)$ of $\mathcal{T}_\delta$ is purely imaginary in this case: therefore, introduce the normal frequency $\Omega(\delta)$ of $\mathcal{T}_\delta$ by setting $\Omega(\delta) = |\mathrm{Im}(\lambda(\delta))| > 0$.

Note that in the present situation the internal frequency ω is assumed constant. Therefore, the normally elliptic n–tori persist on a Cantor set that is defined exclusively by the normal resonances. Modulo these simplifications, the following theorem follows from the results of [20, 39, 57, 75]. To formulate the result, the set $\Lambda(\nu)$ is introduced by

$$\begin{aligned} \Lambda(\nu) \quad = \quad & \big\{\Omega \in \mathbb{R} \, : \, |\langle k', \hat\omega_\ell\rangle + \ell'\Omega| \geq \nu |k'|^{-\tau}, \\ & \quad \text{for all } k' \in \mathbb{Z}^n \backslash \{0\}, 0 < |\ell'| \leq 2\big\} . \end{aligned}$$

Also, introduce the C^s norm $\|f\|_{s,A} = \max_{|\beta|\leq s} \sup_{x\in A} |D_\beta f(x)|$.

Theorem 5.3 (Persistence of elliptic tori)
Fix an interval $\mathcal{A} = [\delta_1, \delta_2]$ such that $\Omega(\delta)$ is bounded away from 0 on $\mathcal{A}$. Then there is a $\mu_1 > 0$ such that for $0 \leq \mu < \mu_1$, there are constants $c > 0$, $c_s > 0$ for every integer $s \in \mathbb{N}$,

$$\nu = \exp\left(-\frac{c}{\mu^{2/(\tau+2)}}\right),$$

and maps $(\delta, \theta) \mapsto \tilde\tau_\delta(\theta) \in \mathbb{R}^2$, $\delta \mapsto \tilde\Omega(\delta) \in \mathbb{R}$ such that the following hold.

1. *The map $\tilde\Omega$ is smooth. Denote by $\mathcal{A}_c$ the inverse image $\mathcal{A} \cap \tilde\Omega^{-1}(\Lambda(\nu))$. On $\mathcal{A}_c$, $\tilde\Omega$ is C^∞–close to Ω, and*

$$\|\tilde\Omega - \Omega\|_{s,A_c} < c_s \nu.$$

2. *For any $\delta \in \mathcal{A}_c$, the map $\theta \mapsto \tilde\tau_\delta(\theta)$ is analytic on T, where T is the following complex neighbourhood of the real n–torus $T = \{\theta \in \mathbb{C}^n \, : \, |\mathrm{Im}\,\theta_j| < 3\rho/8$ for all $j\}$. Moreover, $\tilde\tau_\delta(\theta)$ depends Whitney–smoothly on δ, and*

$$\sup_{\theta\in T} \|\tilde\tau_\delta(\theta) - \tau_\delta\|_{s,\mathcal{A}_c} < c_s\, \nu.$$

3. *For any $\delta \in \mathcal{A}_c$, the torus $\tilde{\mathcal{T}}_\delta$, given as*

$$\tilde{\mathcal{T}}_\delta = \{(\theta, I, q, p) \, : \, (q,p) = \tilde\tau_\delta(\theta)\},$$

   is invariant under the flow of H.

4. *The flow on $\tilde{\mathcal{T}}_\delta$ is quasi–periodic with frequency vector $\hat\omega_\ell$.*

Remark 5.2 Since the gaps in the Cantor set $\mathcal{A}_c$ are of order ν as $\nu \downarrow 0$, it follows that all gap–widths tend to 0 as $\varepsilon \to 0$.

Parabolic tori. There are two occurrences of parabolic tori in the investigations of the one degree of freedom backbone system in section 3: arising from centre–saddle bifurcations, and from Hamiltonian pitchfork bifurcations in a $\mathbb{Z}_2$–symmetric context. We restrict our attention to the former case. This has been treated in [15, 52], the results of which are paraphrased in the following.

If in the integrable system, there is a centre–saddle bifurcation parameter δ_*, then for $\delta = \delta_*$, there exists $\tau_* \in \mathbb{R}^2$ such that

$$\mathcal{T}_* = \{(\theta, q, p) \, : \, (q,p) = \tau_*\},$$

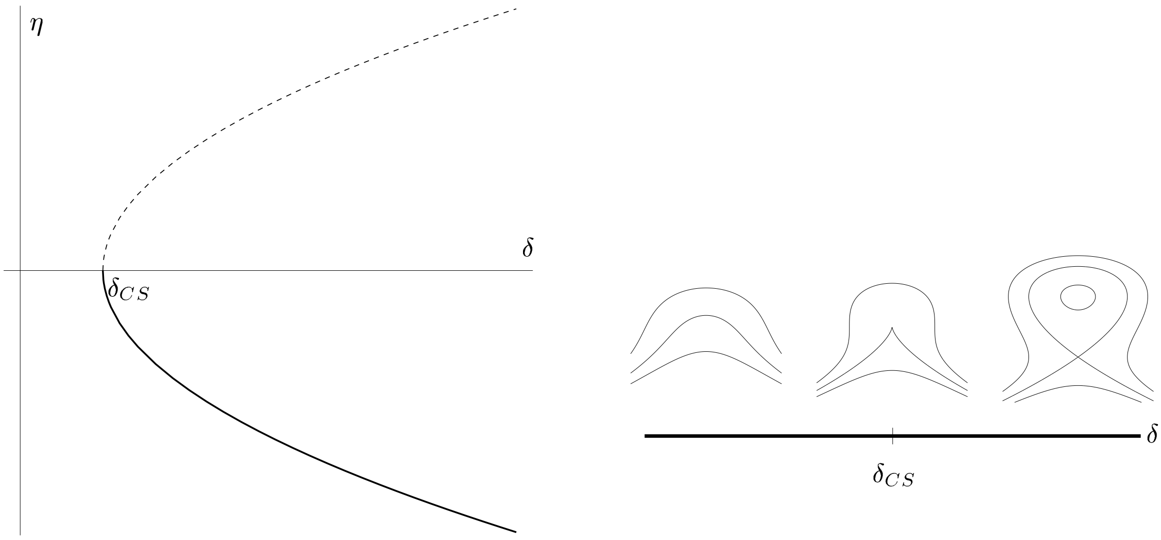

Figure 9: *a). Bifurcation diagram of the centre–saddle bifurcation where δ is the bifurcation parameter and η is some well–chosen auxiliary variable. b). Phase portraits before, at and after the bifurcation.*

is an invariant normally parabolic torus. Introducing an auxiliary parameter $\eta \in \mathbb{R}$, from the fact that δ_* is a centre–saddle bifurcation, it follows that there is a map $\eta \mapsto (\delta(\eta), \tau_\eta) \in \mathbb{R} \times \mathbb{R}^2$, parametrising a family of invariant integrable tori

$$\mathcal{T}_\eta = \{(\theta, I, q, p) \; : \; (q, p) = \tau_\eta\},$$

in the integrable system where $\delta = \delta(\eta)$. This parametrisation has the property that for $\eta = 0$:

$$\delta(0) = \delta_*, \quad \delta'(0) = 0, \quad \delta''(0) \neq 0,$$

and $\tau_0 = \tau_*$. Moreover, the parametrisation can be chosen in such a way that for $\eta < 0$, the tori $\mathcal{T}_\eta$ are normally hyperbolic, while for $\eta > 0$, they are normally elliptic.

Let the Floquet exponent of the invariant torus $\mathcal{T}_\eta$ be denoted by $\lambda(\eta)$, and, for $\eta > 0$, the normal frequency of $\mathcal{T}_\eta$ by $\Omega(\eta) = \mathrm{i}\lambda(\eta)$. For definiteness, it is assumed that $\lambda(\eta) > 0$ if $\eta < 0$, and $\Omega(\eta) > 0$ if $\eta > 0$. Cf. figure 9.

Theorem 5.4 (Persistence of quasi-periodic centre-saddle bifurcation)
For $\eta_0 > 0$ small enough, there is a $\mu_1 > 0$ such that for $0 \leq \mu < \mu_1$, there are constants $c > 0$, $c_s > 0$ for $s \in \mathbb{N}$,

$$\nu = \exp\left(-\frac{c}{\mu^{2/(\tau+2)}}\right),$$

and maps $(\eta, \theta) \mapsto \tilde{\tau}_\eta(\theta) \in \mathbb{R}^2$, $\eta \mapsto \tilde{\delta}(\eta) \in \mathbb{C}$, $\eta \mapsto \tilde{\lambda}(\eta) \in \mathbb{C}$ defined on $\,]-\eta_0, \eta_0[\,$, such that the following hold.

1. *The map $\tilde{\lambda}$ is smooth for $\eta \neq 0$, and as $\eta \to 0$, $|\tilde{\lambda}(\eta)| \sim \mathrm{O}(|\eta|^{\frac{1}{2}})$. Denote by $\mathcal{A}_c \subset [0, \eta_0[$ the set*

$$\mathcal{A}_c = [0, \eta_0[\, \cap \, \tilde{\Omega}^{-1}(\Lambda(\nu)).$$

On $\mathcal{A}_c$, $\tilde{\delta}$ and $\tilde{\Omega}$ are C^∞–close to δ and Ω respectively, and

$$\|\tilde{\delta} - \delta\|_{s,\mathcal{A}_c}, \|\tilde{\Omega} - \Omega\|_{s,\mathcal{A}_c} < c_s \nu,$$

for every $s \geq 0$.

2. *For any $\eta \in \mathcal{A}_c$, the map $\theta \mapsto \tilde{\tau}_\eta(\theta)$ is analytic on*

$$T = \{\theta \in \mathbb{C}^n \; : \; |\mathrm{Im}\, \theta_j| < 3\rho/8 \text{ for all } j\}.$$

Moreover, it depends Whitney–smoothly on η, and

$$\sup_{\theta \in T} \|\tilde{\tau}_\eta(\theta) - \tau_\eta\|_s < c_s \, \nu,$$

for every $s \geq 0$.

3. *For any $\eta \in \,]-\eta_0, 0] \, \cup \mathcal{A}_c$, the torus $\tilde{\mathcal{T}}_\eta$, given as*

$$\tilde{\mathcal{T}}_\eta = \{(\theta, I, q, p) \, : \, (q, p) = \tilde{\tau}_\eta(\theta)\} \, ,$$

is invariant under the flow of $H(\theta, I, q, p; \tilde{\delta}(\eta), \varepsilon)$. Moreover, if $\eta < 0$, the invariant torus is normally hyperbolic, while for $\eta \in \mathcal{A}_c$, it is normally elliptic. For $\eta = 0$ the invariant torus $\mathcal{T}_0$ is normally parabolic.

4. *All invariant tori have quasi–periodic flow with frequency vector $\hat{\omega}_\ell$.*

Remark 5.3

1. As before, since the gaps in the Cantor set $\mathcal{A}_c$ are of order ν as $\nu \downarrow 0$, it follows that all gap–widths tend to 0 as $\varepsilon \to 0$.

2. By the scalings (16), (20), (23), the gap–widths also tend to 0 as $\delta \downarrow \delta_{CS}$.

Base space For the near–integrable system H, theorems 5.2, 5.3 and 5.4 prove the persistence of hyperbolic, elliptic and parabolic response tori on the covering space for certain (large) parameter sets. These response tori correspond to response tori in the base space in exactly the same way as tori of the integrable system N on the cover correspond to tori in the base: the central torus in the cover is mapped ℓ times over the central torus in the base, while tori in the cover that are mapped onto each other by the action of the deck group are mapped onto a single torus in the base, see section 4.

5.3 Conclusions

Let us see how the 'periodic' response diagrams in section 3 have to be modified for the quasi–periodic case in the light of theorems 5.2, 5.3, and 5.4. From theorems 5.2 and 5.4, it follows that each hyperbolic branch of the periodic diagram persists in the quasi–periodic diagrams up to bifurcation points. Theorems 5.3 and 5.4 indicate that each (continuous) elliptic branch of a periodic diagram should be replaced by a Whitney–smooth image of a Cantor subsets of such an elliptic branch.

Concerning the gap–widths in these Cantor sets we observe the following. In the world of C^∞–systems, the only conclusion would be that, the gap–widths decrease more rapidly than any polynomial as $\varepsilon \downarrow 0$; they vanish with infinite flatness for $\varepsilon = 0$. See for instance [19, 25]. For analytic systems, the gaps become exponentially small as $\varepsilon \downarrow 0$. In the figures of the introduction this shows by the very few gaps that can be distinguished at the given level of accuracy.

It is quite common in numerical simulations that only a few of the KAM gaps are visible (the numerical simulations in this paper form a good example of that). One of the goals of this paper is to explain this fact.

The analysis of the backbone systems in section 3 shows that the gap width is proportional to some fractional power of the generic coefficient A, which in turn is proportional to the k'th Fourier coefficient G_k in the Fourier expansion of the forcing $G(\theta, x, y)$. For smooth systems, these coefficients decrease polynomially as $|k|$ increases; for real analytic systems, the order of decrease is exponential, implying very small gaps for high values of $|k|$.

Moreover, as we have seen in this section, the size of the gaps is proportional to the size of the remainder. In general for C^s systems one obtains s–flat remainders, that is, remainders that are smaller than a homogeneous polynomial of degree s. For $s = \infty$, the remainder is smaller than any power, [25]. However, in the present real–analytic case, we have shown that by an optimal choice of the normalising order, the remainder can be made exponentially small, which gives a second reason for the the smallness of the gaps. For more details on these techniques, see [22, 23, 58, 59, 61, 60, 68, 73].

Notice that by normalising to an exponentially small remainder, we move into the world of systems that are only piecewise analytic in the perturbation parameter ε, see remark 5.1. We refer to [22] for a treatment of this.

The strategy of the present paper can be repeated any finite number of times at all of the gaps described in this section. Our result is that there is bifurcation of secondary (and higher order) elliptic and hyperbolic branches at each resonance.

Finally notice that we have simplified our analysis by keeping the internal frequency vector ω constant. As we have mentioned, we conjecture that for the more general perturbations $\varepsilon G(\theta, I, x, y)$ the analysis of this article, suitably modified, would lead to similar results with Cantor sets also in the ω–direction of the parameter space. There is a vast literature on this subject; we only mention [8, 9, 14, 19, 20, 15, 52, 61, 70, 75, 84].

Remark 5.4 In [8, 9], in a dissipative analogue, methods from [35, 54, 78] are exploited to find normally hyperbolic tori nearby the quasi–periodic ones. Also see [19] for further reference.

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References

[1] Arnold, V.I.: *Geometrical methods in the theory of ordinary differential equations*, Springer–Verlag, New York–Berlin, 1983.

[2] Arnold, V.I., Kozlov, V.V. and Neishtadt, A.I.: Mathematical aspects of classical and celestial mechanics, in V.I. Arnold (ed.) *Dynamical Systems III*, Springer–Verlag, 1988.

[3] Arnold, V.I.: Catastrophe Theory, in Arnold, V.I. (ed.) *Dynamical Systems V* , Springer–Verlag, 1994.

[4] Baesens, C., Guckenheimer, J., Kim, S. and MacKay, R.S.: Three coupled oscillators: mode–locking, global bifurcation and toroidal chaos, *Physica D* **49** (1991), 387–475.

[5] Birkhoff, G.D.: *Dynamical systems*, AMS Coll Publ **9**, Amer. Math. Soc., 1927, 1966.

[6] Bourgain, J: Construction of quasi-periodic solutions for Hamiltonian perturbations of linear equations and applications to nonlinear PDE, *Int. Math. Res. Notices* **11** (1994), 475–497.

[7] Bourgain, J.: On Melnikov's persistency problem, *Math. Res. Lett.* **4** (1997), 445–458.

[8] Braaksma, B.L.J. and Broer, H.W.: On a quasi-periodic Hopf bifurcation, *Ann. Institut Henri Poincaré, Analyse non linéaire* **4**(2) (1987), 115–168.

[9] Braaksma, B.L.J., Broer, H.W. and Huitema, G.B.: Towards a quasi-periodic bifurcation theory, *Mem. AMS* **83**(421) (1990), 83–175.

[10] Broer, H.W.: Formal normal form theorems for vector fields and some consequences for bifurcations in the volume preserving case, in D. Rand and L.S. Young (eds.), *Dynamical Systems and Turbulence, Warwick 1980* LNM **898**, Springer-Verlag (1981), 54–74.

[11] Broer, H.W.: Quasi–periodic bifurcations, applications, in *Proceedings of the XI C.E.D.Y.A. (Málaga 1989)*, Universidad de Málaga (1990), 3–22.

[12] Broer, H.W., Chow, S.-N., Kim, Y.-I. and Vegter, G.: A normally elliptic Hamiltonian bifurcation, *ZAMP* **44** (1993), 389–432.

[13] Broer, H.W., Chow, S.-N., Kim, Y.-I. and Vegter, G.: The Hamiltonian double-zero eigenvalue, in: W.F. Langford, W. Nagata (eds.), *Normal Forms and Homoclinic Chaos, Waterloo 1992*, Fields Institute Communications **4** (1995), Amer. Math. Soc., 1–19.

[14] Broer, H.W., Ciocci, M.C. and Vanderbauwhede, A.: Normal 1:1 resonance of invariant tori in reversible systems: a study in quasiperiodic bifurcation theory. *In preparation.*

[15] Broer, H.W., Hanßmann, H. and You, J.; Bifurcations of normally parabolic tori in Hamiltonian systems. *In preparation.*

[16] Broer, H.W., Hoveijn, I., Lunter, G.A. and Vegter, G.: Resonances in a spring–pendulum: algorithms for equivariant singularity theory, *Nonlinearity* **11**(6) (1998), 1569–1605.

[17] Broer, H.W., Hoveijn, I. and Noort, M. van: A reversible bifurcation analysis of the inverted pendulum, *Physica D* **112** (1998), 50–63.

[18] Broer, H.W., Hoveijn, I., Noort, M. van and Vegter, G.: The inverted pendulum: a singularity theory approach, *Journ. Diff. Eqns.* **157** (1999), 120–149.

[19] Broer, H.W., Huitema, G.B. and Sevryuk, M.B.: *Quasi–periodic motions in families of dynamical systems: order amidst chaos*, Lecture Notes in Mathematics, vol. **1645**, Springer–Verlag, Berlin, 1996.

[20] Broer, H.W., Huitema, G.B. and Takens, F.: Unfoldings of quasi-periodic tori, *Mem. AMS* **83**(421) (1990), 1–81.

[21] Broer, H.W., Lunter, G.A. and Vegter, G.: Equivariant singularity theory with distinguished parameters: two case studies of resonant Hamiltonian systems, *Physica D* **112** (1998), 64–80.

[22] Broer, H.W. and Roussarie, R.: Exponential confinement of chaos in the bifurcation set of real analytic diffeomorphisms, In H.W. Broer, B. Krauskopf and G. Vegter (eds.), *Global Analysis of Dynamical Systems, Festschrift dedicated to Floris Takens for his 60th birthday*, Bristol and Philadelphia, IOP (2001), 167–210.

[23] Broer, H.W., Roussarie, R. and Simó, C.: Invariant circles in the Bogdanov-Takens bifurcation for diffeomorphisms, *Ergod. Th. & Dynam. Sys* **16** (1996), 1147–1172.

[24] Broer, H.W. and Simó, C.: Hill's equation with quasi–periodic forcing: resonance tongues, instability pockets and global phenomena, *Bol. Soc. Bras. Mat.* **29** (1998), 253–293.

[25] Broer, H.W. and Takens, F.: Formally symmetric normal forms and genericity, in: U. Kirchgraber and H.-O. Walther (eds.), *Dynamics Reported* **2** (1989), Wiley, 39-59.

[26] Broer, H.W. and Tangerman, F.M.: From a differentiable to a real analytic perturbation theory, applications to the Kupka Smale theorems, *Ergod. Th. & Dynam. Sys.*, **6** (1986), 345–362.

[27] Broer, H.W. and Vegter, G.: Bifurcational aspects of parametric resonance, in: C.K.R.T. Jones, U. Kirchgraber and H.-O. Walther (eds.), *Dynamics Reported (New Series)* **1** (1992), 1–53.

[28] Bröcker, Th.: *Differentiable germs and catastrophes*, London Math. Soc. Soc., LNS **17**, Cambridge University Press, 1975.

[29] Brjuno, A.D.: *Local Methods in Nonlinear Differential Equations*, Springer–Verlag, 1989.

[30] Brjuno, A.D.: *Power geometry in algebraic and differential equations*, North-Holland Mathematical Library **57**, 2000.

[31] Castellà, E. and Jorba, À.: On the vertical families of two-dimensional tori near the triangular points of the bicircular problem, *Cel. Mech. Dynam. Astronom.* **76**(1) (2000), 35–54.

[32] Chenciner, A.: Bifurcations de points fixes elliptiques I. Courbes Invariantes, *Publ. Math. IHES* **61** (1985), 67–127.

[33] Chenciner, A.: Bifurcations de points fixed elliptiques II. Orbites périodiques et ensembles de Cantor invariants, *Invent. Math.* **80**(1) (1985), 81–106.

[34] Chenciner, A.: Bifurcations de points fixes elliptiques III. Orbites périodiques de "petites" périodes et élimination résonnante de couples de courbes invariantes, *Publ. Math. IHES* **66** (1988), 5–91.

[35] Chow, S.-N. and Hale, J.K.: *Methods of bifurcation theory.* Springer–Verlag, New York–Berlin, 1982.

[36] Cohn, P.M.: *Algebra. Vol. 1*, 2nd edition, Wiley, Chichester, 1982.

[37] Díez, C., Jorba, À. and Simó, C.: A dynamical equivalent to the equilateral libration points of the Earth-Moon system, *Cel. Mech.* **50(1)** (1991), 13–29.

[38] Duistermaat, J.J.: Bifurcations of periodic solutions near equilibrium points of Hamiltonian systems, *Bifurcation theory an applications (Montecatini, 1983)*, LNM **1057**, Springer, Berlin (1983), 57–105.

[39] Eliasson, L. H.: Perturbations of stable invariant tori for Hamiltonian systems, *Ann. Sc. Norm. Super. Pisa*, Cl. Sci., Ser. IV, **15** (1988), 115–147.

[40] Elipe, A., Lanchares, V., López–Moratella, T. and Riaguas, A.: Nonlinear stability in resonant cases: a geometrical approach, *J. Nonlinear Sci.* **11** (2001), 211–222.

[41] Ferrer, S., Hanßmann, H., Palacián, J. and Yanguas, P.: On perturbed oscillators in 1-1-1 resonance: the case of axially symmetric cubic potentials, *J. Geom. Phys.* **40** (2002), 320–369.

[42] Golubitsky, M. and Schaeffer, D.G.: *Singularities and groups in bifurcation theory. Volume I*, Appl. Math. Sciences **51**, Springer–Verlag, New York, 1985.

[43] Golubitsky, M, Schaeffer, D.G. and Stewart, I.: *Singularities and groups in bifurcation theory. Volume II*, Appl. Math. Sciences **69**, Springer–Verlag, New York, 1988.

[44] Golubitsky, M., Marsden, J.E., Stewart, I. and Dellnitz, M.: The constrained Liapunov–Schmidt procedure and periodic orbits, in: W.F. Langford, W. Nagata (eds.), *Normal forms and homoclinic chaos, Waterloo 1992*, Fields Institute Communications **4** (1995), Amer. Math. Soc., 81–128.

[45] Gómez, G., Llibre, J., Martínez, R. and Simó, C.: *Dynamics and mission design near libration points - Volume 1: Fundamentals: The case of collinear libration points*, World Scientific Monograph Series in Mathematics **2**, 2001.

[46] Gómez, G., Llibre, J., Martínez, R. and Simó, C.: *Dynamics and mission design near libration points - Volume 2: Fundamentals: The case of triangular libration points*, World Scientific Monograph Series in Mathematics **3**, 2001.

[47] Gómez, G., Jorba, À., Masdemont, J. and Simó, C.: *Dynamics and mission design near libration points - Volume 3: Advanced methods for collinear points*, World Scientific Monograph Series in Mathematics **4**, 2001.

[48] Gómez, G., Jorba, À., Masdemont, J. and Simó, C.: *Dynamics and mission design near libration points - Volume 4: Advanced methods for triangular points*, World Scientific Monograph Series in Mathematics **5**, 2001.

[49] Graff, S.: On the conservation of hyperbolic invariant tori for Hamiltonian systems, *J. Differ. Eq.* **15** (1974), 1–69.

[50] Gustavson, F.G.: On construction formal integrals of a Hamiltonian systems near an equilibrium point, *Astron. J.* **71** (1966), 670–686

[51] Hale, J.K.: *Ordinary Differential Equations*, Wiley 1969, Krieger 1980.

[52] Hanßmann, H.: The quasi–periodic centre–saddle bifurcation, *Journ. Diff. Eqns.* **142**(2) (1998), 305–370.

[53] Helmholtz, H.: *On the Sensations of Tone (Die Lehre von den Tonempfindungen)*. Dover Publications 1954.

[54] Hirsch, M.W., Pugh, C.C. and Shub, M.: *Invariant Manifolds*, Lecture Notes in Mathematics, Vol. **583**, Springer–Verlag, 1977.

[55] Jong, H.H. de: *Quasiperiodic breathers in systems of weakly coupled pendulums*, PhD–Thesis University of Groningen, 1999.

[56] Jorba À.: Numerical computation of the normal behaviour of invariant curves of n-dimensional maps, *Nonlinearity*, **14**(5) (2001), 943-976.

[57] Jorba, À., de la Llave, R. and Zou, M.: Lindstedt series for lower dimensional tori, in: C. Simó (ed.), *Hamiltonian systems with three or more degrees of freedom* ($3\mathcal{DHAM}95$), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., Vol. **533** (1999), Kluwer, Dordrecht, Holland, 628–632.

[58] Jorba A., Ramírez-Ros R. and Villanueva J.: Effective reducibility of quasi-periodic linear equations close to constant coefficients, *SIAM J. Math. Anal.* **28**, (1997), 178–188.

[59] Jorba, À. and Simó, C.: On quasi-periodic perturbations of elliptic equilibrium points, *SIAM J. Math. Anal.*, **27**(6) (1996), 1704–1737.

[60] Jorba, À. and Villanueva, J.: On the persistence of lower dimensional invariant tori under quasi-periodic perturbations, *J. Nonlinear Sci.* **7** (1997), 427–473.

[61] Jorba, À. and Villanueva, J.: On the normal behaviour of partially elliptic lower–dimensional tori of Hamiltonian systems, *Nonlinearity* **10** (1997), 783–822.

[62] Kummer, M.: On resonant Hamiltonian systems with finitely many degrees of freedom, in: A.V. Sáenz, W.W. Zachary and R. Cawley (eds.), *Local and global methods of nonlinear dynamics (Silver Spring 1984)*, Lect. Notes Phys. **252** (1986), 19–31.

[63] Li, Y. and Yi, Y.: A quasi–periodic Poincaré's theorem, *CNSNS preprint* Georgia Tech (1999) submitted.

[64] Lunter, G.A.L.: *Bifurcations in Hamiltonian systems, computing singularities by Gröbner bases*, PhD–Thesis University of Groningen, 1999.

[65] Meer, J.C. van der: *The Hamiltonian Hopf bifurcation*, LNM **1160**, Springer–Verlag, Berlin, 1985.

[66] Meyer, K.: Generic bifurcation of periodic points, *Trans. AMS* **149** (1970), 95–107.

[67] Meyer, K. and Hall, G.: *Introduction to Hamiltonian Dynamical systems and the N–Body Problem*, Applied Mathematical Sciences **90**, Springer–Verlag, New York, 1992.

[68] Morbidelli A. and Giorgilli A.: *Superexponential Stability of KAM Tori*, *J. Statist. Phys.* **78** (1995), 1607–1617.

[69] Moser, J.K.: Combination tones for Duffing's equation, *Com. Pure Appl. Math.* **18** (1965), 167–181.

[70] Moser, J.K.: Convergent series expansions for quasi–periodic motions, *Math. Ann.* **169** (1967), 136–176.

[71] Neishtadt, A.I.: The separation of motions in systems with rapidly rotating phase, *J. Appl. Math. Mech.* **48** (1984), 133–139.

[72] Noort, M. van: *Global coherent dynamics of the parametrically forced pendulum: a case study in one–and–a–half degrees of freedom*, PhD–Thesis University of Groningen, 2001.

[73] Perry, A. D. and Wiggins, S.: *KAM tori are very sticky: rigorous lower bounds on the time to move away from an invariant Lagrangian torus with linear flow*, *Physica D* **71** (1994), 102–121.

[74] Poènaru, V.: *Singularités C^∞ en Présence de Symétrie*, LNM **510**, Springer–Verlag, Berlin–New York, 1976.

[75] Pöschel, J.: On elliptic lower dimensional tori in Hamiltonian systems, *Math. Z.* **202**(4) (1989), 559–608.

[76] John William Strutt, Lord Rayleigh, *The Theory of Sound*, Dover Publications 1945.

[77] Rüssmann, H.: On optimal estimates for the solutions of linear partial differential equations of first order with constant coefficients on the torus, in: J. Moser (ed.), *Dynamical systems, theory and applications*, LNP **38** (1975), 598–624.

[78] Sacker, R.J.: A new approach to the perturbation theory of invariant surfaces, *Comm. Pure Appl. Math.* **18** (1965), 717–732.

[79] Simó, C.: Averaging under fast quasi–periodic forcing, in: J. Seimenis (ed.), *Hamiltonian mechanics,integrability and chaotic behaviour*, NATO Adv. Sci. Inst. Ser. B Phys. Vol. **331** (1994), Plenum Press, New York, 13–34.

[80] Stoker, J.J.: *Nonlinear vibrations in mechanical and electrical systems*, Interscience Publishers 1950; Wiley 1992.

[81] Takens, F.: Singularities of vector fields, *Publ. Math. IHES* **43** (1974), 47–100.

[82] Thom, R.: *Stabilité Structurelle et Morphogénèse*, Benjamin, 1972.

[83] Treschev, D.V.: The mechanism of destruction of resonant tori of Hamiltonian systems, *Math. USSR-Sb.* **68** (1991), 181–203.

[84] Wagener, F.O.O.: Semi–local analysis of the $k : 1$ and $k : 2$ resonances in quasi–periodically forced systems, in H.W. Broer, B. Krauskopf and G. Vegter (eds.), *Global Analysis of Dynamical Systems, Festschrift dedicated to Floris Takens for his 60th birthday*, Bristol and Philadelphia, IOP (2001), 113–129.

Appendix

Formal normal form theorem

In this section the formal normal form Theorem 2.1 is proved, that gives the transformation of

$$H \;=\; \langle \omega, I\rangle \;+\; \alpha\frac{x^2+y^2}{2} \;+\; h(x,y;\alpha) \;+\; \varepsilon G(\theta,x,y;\alpha,\varepsilon) \tag{31}$$

into normal form. Although this is a very well–known result, see [1, 5, 10, 65, 81], we include its proof for completeness' sake and for further use of the notation. The theorem reads as:

Theorem 2.1 *Consider the Hamiltonian* (31) *for α_0 satisfying $\langle k,\omega\rangle + \ell\alpha_0 = 0$. Given $L \in \mathbb{N}$, for sufficiently small ε there exists a real analytic and canonical change of variables $\Phi(\cdot;\alpha,\varepsilon)$*

$$\begin{array}{rccc} \Phi: & \mathbb{T}^n\times\mathbb{R}^n\times\mathbb{R}^2 & \longrightarrow & \mathbb{T}^n\times\mathbb{R}^n\times\mathbb{R}^2 \\ & (\theta,I,x,y) & \mapsto & (\theta,J,X,Y) \end{array}$$

close to the identity at $(x,y,\alpha,\varepsilon) = (0,0,\alpha_0,0)$, such that the following holds. The Hamiltonian (31) *is transformed into normal form*

$$H\circ\Phi^{-1}(\theta,J,X,Y;\alpha,\varepsilon) \;=\; N(\theta,J,X,Y;\alpha,\varepsilon) \;+\; R(\theta,X,Y;\alpha,\varepsilon).$$

The integrable part N of the normal form reads as

$$\begin{aligned} N \;=\; & \langle\omega,J\rangle \;+\; \alpha_0\frac{X^2+Y^2}{2} \;+\; (\alpha-\alpha_0)\frac{X^2+Y^2}{2} \;+\; c\left(\frac{X^2+Y^2}{2}\right)^2 \\ & +\; a\varepsilon\frac{X^2+Y^2}{2} \;+\; A\varepsilon\,\mathrm{Re}\left((X-\mathrm{i}Y)^\ell \mathrm{e}^{\mathrm{i}(\langle k,\theta\rangle-\psi_0)}\right) \\ & +\; \sum_{\kappa,m,j,i} F_{\kappa,m,i,j}\varepsilon^i(\alpha-\alpha_0)^j(X^2+Y^2)^m \\ & \qquad\cdot\,\mathrm{Re}\left((X-\mathrm{i}Y)^{\kappa\ell}\mathrm{e}^{\mathrm{i}(\kappa\langle k,\theta\rangle-\psi_{\kappa,m,i,j})}\right), \end{aligned} \tag{32}$$

with $\kappa \geq 0$, $j+i+2m+\kappa\ell \leq L$, and $\psi_{0,m,i,j} = 0$, where the sum does not contain the terms that are explicitly displayed; the remainder R of the normal form satisfies the estimate

$$R(\theta,X,Y;\alpha,\varepsilon) \;\equiv\; O_L(X,Y,\alpha-\alpha_0,\varepsilon) \quad .$$

Proof. The argument is completely standard. Starting from (31), subsequent coordinate transformations of the form Φ_W are applied to the system, where Φ_W is the time one map of a Hamiltonian vector field X_W with Hamilton function W.

We call a function f be of *order* $\mathcal{O}(n)$ if $f = \mathcal{O}(X^{\beta_1} Y^{\beta_2} (\alpha - \alpha_0)^{\beta_3} \varepsilon^{\beta_4})$, with $\sum_i \beta_i = n$. Assume that H has been normalised with respect to (10) up to order $n-1$, that is:

$$\begin{aligned} H_n(\Theta, J, X, Y; \alpha, \varepsilon) &= H \circ \Psi_{n-1} = H \circ \Phi_{W_{n-1}} \circ \cdots \circ \Phi_{W_3} \\ &= \langle \omega, J \rangle + \sum_{j=2}^{n-1} g_j + h_n + r_{n+1} \end{aligned}$$

where $h_2 = \alpha_0 \left(\frac{X^2+Y^2}{2} \right)$, and, more generally, where $g_j(X, Y; \alpha - \alpha_0, \varepsilon)$ is a homogeneous polynomial of order j in its variables already in (resonant) normal form, $h_n = \mathcal{O}(n)$, and $r_{n+1} = \mathcal{O}(n+1)$.

The Hamilton function $W = W_n$ of the next normalising transformation is taken as a homogeneous polynomial of order n in the variables $(X, Y, \alpha - \alpha_0, \varepsilon)$. We have

$$\begin{aligned} H_n \circ \Phi_W &= H_n + \{H, W\} + \mathcal{O}(n+1) \\ &= \langle \omega, J \rangle + \sum_{j=2}^{n-1} g_j + \{H_2, W\} + h_n + \mathcal{O}(n+1), \end{aligned}$$

where $H_2 = \langle \omega, J \rangle + h_2$, and where $\{H, W\}$ denotes the Poisson bracket of H and W. We will solve the *homological equation*

$$\{H_2, W\} + h_n = g_n,$$

such that the contribution g_n of h_n to the normal form has as few terms as possible.

The action of H_2 on W is given as

$$\{H_2, W\} = -\omega \frac{\partial W}{\partial \Theta} + \alpha_0 X \frac{\partial W}{\partial Y} - \alpha_0 Y \frac{\partial W}{\partial X}.$$

This action diagonalises in suitable complex coordinates $Z = X + \mathrm{i}Y$; introduce the derivative $\partial/\partial Z = \frac{1}{2}(\partial/\partial X - \mathrm{i}\partial/\partial Y)$. Then $\{H_2, W\}$ reads

$$\{H_2, W\} = -\omega \frac{\partial W}{\partial \Theta} + \mathrm{i}\alpha_0 Z \frac{\partial W}{\partial Z} - \mathrm{i}\alpha_0 \bar{Z} \frac{\partial W}{\partial \bar{Z}}.$$

Hence the Lie operator $\mathrm{ad}_{H_2}(W) = \{H_2, W\}$ is semi–simple, and its kernel can be chosen as complement to its image. Split correspondingly $h_n = b_n + g_n$, with b_n in the image and g_n in the kernel of ad_{H_2}. A complex diagonalising basis of ad_{H_2} is given by $v_{s,m} = \mathrm{e}^{\mathrm{i}\langle s, \theta \rangle} Z^{m_1} \bar{Z}^{m_2}$, and $\mathrm{ad}_{H_2}(v_{s,m}) = 0$ if $s = \kappa k$ and $m_2 = m + \kappa \ell$, $m_1 = m$, for some integer κ with $m + \kappa \ell \geq 0$ for $m \geq 0$. Consequently, g_n consists of a sum of monomials of the kind

$$\begin{aligned} &\mathrm{Re} \left(\mathrm{e}^{\mathrm{i}(\langle \kappa s, \theta \rangle + \psi_{\kappa, m, i, j})} |Z|^{2m} \bar{Z}^{\kappa \ell} \right) \varepsilon^i (\alpha - \alpha_0)^j \\ &= (X^2 + Y^2)^m \varepsilon^i (\alpha - \alpha_0)^j \mathrm{Re} \left((X - \mathrm{i}Y)^{\kappa \ell} \mathrm{e}^{\mathrm{i}\langle \kappa s, \theta \rangle} \right), \end{aligned}$$

where $2m + \kappa \ell + i + j = n$ and $\kappa \geq 0$. The equation $\mathrm{ad}_{H_2}(W) = b_n$ is then solved as in the next section of this appendix, yielding an analytic function W.

The normal form given in the theorem is then obtained by performing L of these normalising steps. This finishes the proof of the normal form theorem. ∎

Normal form with exponentially small remainder

Here, theorem 5.1 of section 5 concerning the quantitative estimates of the normal form is proved. The following notations are recalled from section 5, since they will be needed in the proof below.

By $\mathcal{D}(\rho, r)$, complex neighbourhoods of the n–dimensional tori $(x_{\mathrm{sc}}, y_{\mathrm{sc}}) = (0, 0)$ are denoted, where

$$\begin{aligned} \mathcal{D}(\rho, r) = \{ &(\theta, x_{\mathrm{sc}}, y_{\mathrm{sc}}) \in \mathbb{C}^n \times \mathbb{C} \times \mathbb{C} : \\ &|\mathrm{Im}\, \theta_j| < \rho \text{ for all } j, \ |x_{\mathrm{sc}}|^2 + |y_{\mathrm{sc}}|^2 < r^2 \}. \end{aligned}$$

For a function $f : \mathcal{D}(\rho, r) \to \mathbb{C}^n$, the supremum norm is denoted by $\|f\| = \|f\|_{\mathcal{D}(\rho,r)}$.

Let $H_{\rm sc}$ be a Hamiltonian given by

$$H_{\rm sc} \;=\; \langle \omega, I_{\rm sc}\rangle \;+\; \alpha_0 \frac{x_{\rm sc}^2 + y_{\rm sc}^2}{2} \;+\; \mu^2 F_{\rm sc} \;+\; \mu^{L-2} R_{\rm sc}. \tag{33}$$

It is assumed that $H_{\rm sc}$ can be extended analytically to $\mathcal{D}(\rho, r)$, and that the frequency vector $\omega \in \mathbb{R}^n$ is Diophantine: there are $\gamma > 0$, $\tau > n - 1$ such that for all $k' \in \mathbb{Z}^n \backslash \{0\}$:

$$|\langle k', \omega\rangle| \geq \gamma |k'|^{-\tau}.$$

The normal frequency α_0 satisfies a normal resonance relation $\langle k, \omega\rangle + \ell\alpha_0 = 0$.

Theorem 5.1 *Consider the Hamiltonian $H_{\rm sc}$ as given by equation (33). Let $\rho, r > 0$ be such that the remainder $R_{\rm sc}$ is analytic on $\mathcal{D} = \mathcal{D}(\rho, r)$, and $\|R_{\rm sc}\|_{\mathcal{D}} < \infty$.*

Then there is a $\mu_0 > 0$ such that for any $0 \leq \mu < \mu_0$, there is a real analytic canonical transformation

$$\Phi(\theta, I, x_{\rm sc}, y_{\rm sc}; \mu) = (\Theta, J, Q, P),$$

such that the following statements hold.

1. *If $\mathcal{D}_* = \mathcal{D}(\rho/2, r{\rm e}^{-\rho/2})$, then*
$$\Phi(\mathcal{D}_*) \subset \mathcal{D} \text{ and } \Phi^{-1}(\mathcal{D}_*) \subset \mathcal{D}.$$
2. *There is some constant $c > 0$ not depending on $R_{\rm sc}$ such that*
$$\|\Phi - {\rm id}\|_{\mathcal{D}_*}, \|\Phi^{-1} - {\rm id}\|_{\mathcal{D}_*} \leq c\mu^{L-4} \|R_{\rm sc}\|_{\mathcal{D}}.$$
That is, Φ is $\left(C^\omega, c\mu^{L-4}\right)$–close to the identity.
3. *In the new coordinates (Θ, J, Q, P), the Hamiltonian takes the form*
$$H(\Theta, J, Q, P; \alpha, \mu) \;=\; \langle \omega, J\rangle + \alpha_0 \frac{Q^2 + P^2}{2} + \mu^2 \tilde{F} + \mu^{L-2} \tilde{R},$$
where $\tilde{F}(\Theta, Q, P; \alpha, \mu)$ is in resonant normal form,
$$\|F_{\rm sc} - \tilde{F}\|_{\mathcal{D}_2} \leq c\,\mu^{L-4} \|R_{\rm sc}\|_{\mathcal{D}},$$
and $\tilde{R}(\Theta, Q, P; \alpha, \mu)$ satisfies
$$\|\tilde{R}\|_{\mathcal{D}_2} \leq \exp\left(-\frac{c}{\mu^{2/(\tau+2)}}\right) \|R_{\rm sc}\|_{\mathcal{D}}. \tag{34}$$

Proof. This proof can be seen as an expansion of the proof of the formal normal form, given in section 5.3. As in that proof, a (finite) series $\{H^j\}$ of different Hamiltonians is constructed iteratively, which are all of the form:

$$H^j = \langle \omega, I\rangle + \alpha_0 \frac{x^2 + y^2}{2} + \mu^2 F^j + \mu^{L-2} R^j. \tag{35}$$

For $j = 0$, we have that $H^0 = H_{\rm sc}$ *etc*. Moreover, for all $k' \in \mathbb{Z}^n \backslash \{0\}$:

$$|\langle k', \omega\rangle| \geq \gamma |k'|^{-\tau}.$$

The functions H^j, F^j and R^j are analytic and bounded on the domain $\mathcal{D}_j = \mathcal{D}(\rho_j, r_j)$. Moreover, it is assumed that F^j consists only of resonance terms, while R^j contains only non–resonant terms. In order to improve readability of the following discussion, for the next couple of paragraphs $\mu^2 F^j$ and $\mu^{L-2} R^j$ are replaced by F^j and R^j.

Since our main concern will be with a single iteration step, the superscript j will be dropped in the following, and $j+1$ will be replaced by $+$ (so–called $+$–notation). Also, in the analysis of the iteration step, a host of different constants appear in the estimates, which do not depend on the index j of the step, nor on the 'stepsize' δ, which will be introduced below.

These constants will collectively be denoted by a dot following an inequality sign. For instance, $a <\cdot\,(x+y)$ will be taken to mean $a < c(x+y)$, where c is some constant not depending on j or δ.

As before, then, the new Hamiltonian H^+ is obtained from the given Hamiltonian H by a variable transformation Φ_W, which is the time–one map of the Hamiltonian vector field X_W associated to a function W. This function is determined by the homological equation

$$\{H_2, W\} + R = 0, \tag{36}$$

and the requirement that W itself contains no resonant terms. Note that in the present context $H_2 = \langle\omega, I\rangle + \alpha_0(x^2+y^2)/2$. Introducing complex coordinates $z = x + \mathrm{i}y$ as above, the function W can be expanded in a Taylor–Fourier series:

$$W(\theta, z, \bar{z}) = \sum_{k' \in \mathbb{Z}^n} \sum_{|\beta| \geq 0} W_{k'\beta}\, \mathrm{e}^{\mathrm{i}\langle k', \theta\rangle} z^{\beta_1} \bar{z}^{\beta_2},$$

where the notation $W(\theta, z, \bar{z})$ indicates that $W(\theta, z, w)$ is an analytic function in all its variables. Note that $\beta = (\beta_1, \beta_2)$ is a multi–index. Also R is expanded in a Taylor–Fourier series, with coefficients $R_{k'\beta}$. By equating coefficients of $\mathrm{e}^{\mathrm{i}\langle k', \omega\rangle} z^{\beta_1} \bar{z}^{\beta_2}$ in the homological equation (36), the following formal expression for W is derived:

$$W = \sum_{k' \in \mathbb{Z}^n} \sum_{|\beta| \geq 0} \frac{R_{k'\beta}}{\mathrm{i}\,(\langle k', \omega\rangle - \alpha_0(\beta_1 - \beta_2))} \mathrm{e}^{\mathrm{i}\langle k', \theta\rangle} z^{\beta_1} \bar{z}^{\beta_2}.$$

Convergence. The formal expression obtained for W will be shown to converge uniformly on some domain $\mathcal{D}_\delta = \mathcal{D}(\rho - \delta, r\mathrm{e}^{-\delta})$, where the 'stepsize' δ will be determined below. For this, note first that from the Cauchy inequalities it follows that

$$|R_{k'\beta}| \leq \|R\|_{\mathcal{D}} \mathrm{e}^{-\rho|k'|} r^{-|\beta|}.$$

This implies, together with the Diophantine inequalities, the following estimate for W :

$$\|W\|_{\mathcal{D}_\delta} \quad <\cdot \quad \frac{\|R\|_{\mathcal{D}}}{\gamma\delta^\tau}.$$

Remark A.1 We here used the optimal estimates obtained by Rüssmann [77], obtained by fine tuning the Diophantine non-resonance estimates.

Estimates. Let X_W be the Hamiltonian vector field associated with W, and let Φ_W^s denote its time-s map. The new Hamiltonian H^+ can be written as

$$H^+ = H + \{H, W\} + \int_0^1 (1-s)\{\{H, W\}, W\} \circ \Phi_W^s\, \mathrm{d}s. \tag{37}$$

This expression is defined on any domain $\mathcal{D}_*$ which is mapped by Φ_W^s, for all s, into $\mathcal{D}_\delta$. In order to find a suitable $\mathcal{D}_*$, note that

$$\|X_W\|_{\mathcal{D}_{2\delta}} = \|\mathrm{grad}\, W\|_{\mathcal{D}_{2\delta}} <\cdot\, \frac{\|W\|_{\mathcal{D}_\delta}}{\delta} \leq c \frac{\|R\|_{\mathcal{D}}}{\gamma\delta^{\tau+1}}.$$

Since $\|\Phi_W^s - \mathrm{id}\|_{\mathcal{D}_{2\delta}} \leq \|X_W\|_{\mathcal{D}_{2\delta}}$ for all $s \in [0,1]$, the domain $\mathcal{D}_{2\delta}$ will be mapped into $\mathcal{D}_\delta$, if

$$\|R\|_{\mathcal{D}} \leq c^{-1}\gamma\delta^{\tau+2}. \tag{38}$$

Note that the constant c does neither depend on j, nor on δ. This inequality is verified below, after the constant δ is determined.

To obtain expressions for F^+ and R^+, equation (35) is substituted into (37). Using the homological equation (36), this yields for H^+

$$\begin{aligned} H^+ &= H_2 + F + R + \{H_2, W\} + \{F, W\} + \{R, W\} + S \\ &= H_2 + F + \{F, W\} + \{R, W\} + S. \end{aligned} \tag{39}$$

Note that the integral in (37) is indicated by S.

By applying Cauchy's estimate to all derivatives in the Poisson brackets, successively the following estimates are obtained:

$$\begin{aligned}
\|\{F,W\}\|_{\mathcal{D}_{2\delta}} &<\cdot \frac{\|F\|_{\mathcal{D}_\delta}}{\delta}\frac{\|W\|_{\mathcal{D}_\delta}}{\delta} <\cdot \frac{\|R\|_{\mathcal{D}}}{\gamma\delta^{\tau+2}}, &(40)\\
\|\{R,W\}\|_{\mathcal{D}_{2\delta}} &<\cdot \frac{\|R\|_{\mathcal{D}_\delta}}{\delta}\frac{\|W\|_{\mathcal{D}_\delta}}{\delta} <\cdot \frac{\|R\|^2_{\mathcal{D}}}{\gamma\delta^{\tau+2}}, &(41)\\
\|S\|_{\mathcal{D}_{3\delta}} \leq \|\{\{H,W\},W\}\|_{\mathcal{D}_{2\delta}} &<\cdot \frac{\|H\|_{\mathcal{D}_\delta}\|W\|^2_{\mathcal{D}_\delta}}{\delta^4} <\cdot \frac{\|R\|^2_{\mathcal{D}}}{\gamma\delta^{\tau+4}}. &(42)
\end{aligned}$$

Here it has been used that F (and consequently H) can be bounded by a constant that is independent of j or δ, a fact which will be verified below.

Incorporation of the perturbation parameter. At this point, the parameter μ is incorporated again, and the functions F, R and W are replaced by μ^2F, $\mu^{L-2}R$ and $\mu^{L-2}W$. Hence, from (39), we obtain for F^+ and R^+ from:

$$\mu^2F^+ + \mu^{L-2}R^+ = \mu^2F + \mu^L\{F,W\} + \mu^{2L-4}\{R,W\} + \mu^{2L-4}S,$$

where F^+ consists only of resonant terms and R^+ only of nonresonant ones. Choosing $\rho_+ = \rho - 3\delta$ and $r_+ = r\mathrm{e}^{-3\delta}$, the domain $\mathcal{D}_+$ equals $\mathcal{D}_{3\delta}$. Using estimates (40)–(42) obtained above, the following are obtained:

$$\begin{aligned}
\|R^+\|_{\mathcal{D}_+} &\leq \frac{C}{\gamma}\|R\|_{\mathcal{D}}\left(\frac{\mu^2}{\delta^{\tau+2}} + \frac{\mu^{L-2}}{\delta^{\tau+4}}\right),\\
\|F^+ - F\|_{\mathcal{D}_+} &\leq \frac{C\mu^{L-4}}{\gamma}\|R\|_{\mathcal{D}}\left(\frac{\mu^2}{\delta^{\tau+2}} + \frac{\mu^{L}}{\delta^{\tau+4}}\right),
\end{aligned}$$

where C does not depend on the iteration index j.

Exponentially small estimates. The constant δ is now determined by setting

$$\frac{C\mu^2}{\gamma\delta^{\tau+2}} = \frac{1}{2\mathrm{e}}, \quad \text{that is} \quad \delta = \left(\frac{2C\mathrm{e}\mu^2}{\gamma}\right)^{\frac{1}{\tau+2}}.$$

Then it follows that for μ sufficiently small,

$$\|R^+\|_{\mathcal{D}_+} \leq \mathrm{e}^{-1}\|R\|_{\mathcal{D}}, \qquad \|F - F^+\|_{\mathcal{D}_+} \leq c\mu^{L-4}\|R\|_{\mathcal{D}}.$$

Note that as a consequence of this $\|F^j\|_{\mathcal{D}_j} \leq \frac{\mathrm{e}}{\mathrm{e}-1}\|R^0\|_{\mathcal{D}_0}$, a fact that was used in the estimates above. Also, the estimate (38) holds for μ small enough, since then

$$\frac{\gamma}{c}\delta^{\tau+2} \geq \tilde{c}\mu^2,$$

and $\mu^{L-2}\|R\|_{\mathcal{D}}$ is smaller than $\tilde{c}\mu^2$ for μ small enough (recall that the factor μ^{L-2} is absent from (38) by convention).

As a single iteration step decreases the width of a domain by 3δ in the Imθ–direction, the total number of steps j_* should be such such that $j_* \cdot 3\delta < \rho_0/2$, yielding:

$$j_* = \left[\frac{\rho_0}{2}\delta^{-1}\right] = \left[\frac{c}{\mu^{\frac{2}{\tau+2}}}\right].$$

The remainder term R^{j_*} can now be estimated by

$$\|R^{j_*}\|_{\mathcal{D}_{j_*}} \leq \mathrm{e}^{-j_*}\|R^0\|_{\mathcal{D}_0},$$

which yields the exponentially small estimate.

Finally, the transformation Φ of the theorem is given by the concatenation of the Φ_{G^j}'s:

$$\Phi = \Phi_{W^1} \circ \cdots \circ \Phi_{W^{j_*}}.$$

The statements of the theorem now follow easily. ∎