Geometry of contours and Peierls estimates in d=1 Ising models with long range interactions

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Abstract. Following Fröhlich and Spencer, [8], we study one dimensional Ising spin systems with ferromagnetic, long range interactions which decay as $|x-y|^{-2+\alpha}$, $0 \le \alpha \le 1/2$. We introduce a geometric description of the spin configurations in terms of triangles which play the role of contours and for which we establish Peierls bounds. This in particular yields a direct proof of the well known result by Dyson about phase transitions at low temperatures.

Key words: Ferromagnetic, long range interactions, Phase transitions, Contours, Peierls estimates

1. Introdution

A rigorous proof of liquid-vapor phase transitions is a long standing challenge for mathematical physicists. A clear understanding of the phenomenon goes back to van der Waals, but a mathematically consistent theory is still lacking. Lebowitz, Mazel and Presutti, [**14**], have tried to capture van der Waals ideas by considering an Hamiltonian which has a term given by an attractive, two body Kac potential. The effort was to study the model without taking the Kac scaling parameter $\gamma \to 0$, as in the original works of Kac, Uhlenbeck and Hemmer, [**13**], and Lebowitz and Penrose, [**15**]. Technically, the idea was to study the system as a perturbation of mean field, which corresponds to the limit case $\gamma = 0$, and to adapt to such a context the Pirogov-Sinai theory of finite temperature perturbations of ground states. To carry through the program, one needs a good control of an approximate model where the Kac potential term in the hamiltonian is replaced by a self-consistent, external one body field, whose intensity depends on the true value of the order parameter [the particles density] at equilibrium. In the continuum, the hamiltonian cannot consist of just the attractive Kac potential (as in Ising models with Kac potentials) and a repulsive force is needed to prevent a collapse of matter. The natural choice (as proposed originally by Kac et al.) is then to add a hard core interaction, but, at the required values of the particles density, the cluster expansion results for the system with only hard cores are not valid and the implementation of the Pirogov-Sinai methods collapses. In [**14**] the problem has been avoided by using repulsive forces which are also given by Kac potentials, in particular four-body positive interactions.

1991 *Mathematics Subject Classification.* 82B26, 82B05, 82B20 .

The escamotage is physically not totally satisfactory, as the phase transition should arise from a competition between the short range repulsive and the much longer range attractive inter-molecular forces. Several efforts to extend [**14**] to such a context and in particular to the model with hard core plus attractive two body Kac potentials have failed.

There is however some margin left if we restrict to one dimensions, because the pure hard rods system is isomorphic to an ideal gas. Unfortunately, there is a price to pay: to have a phase transition in $d = 1$, we need to consider long range forces (potentials which decay as $|x - y|^{-2+\alpha}$, $\alpha \in [0, 1)$) which are not covered by the traditional Pirogov Sinai theory. Prior to [**14**], the problem of phase transition in the continuum in $d = 1$ with such long range interactions had already been considered by Johansson, [**11, 12**], who studied the system in the canonical ensemble, proving phase transition for the thermodynamic potentials. The existence of distinct DLR measures at the proper values of chemical potential and temperature remains however open.

The Pirogov-Sinai theory seems the natural way to answer these questions, as it provides powerful tools for investigating phase transitions at low temperatures and at low effective temperatures as well, with a quite satisfactory description of systems in dimensions larger or equal to two. In view of the desired applications to continuum particle models, our mid-term program is to extend Pirogov-Sinai to one dimensional spin systems with long range interactions. The content of this paper will be the definition of contours and the establishment of Peierls estimates, as a preliminary step in this direction. After the papers by Dyson, [**6**], [**7**], on a model with hierarchical interactions (which, by ferromagnetic inequalities, prove phase transitions in Ising systems as well), we find in the literature the fundamental paper by Fröhlich and Spencer, [**8**], where the critical case $\alpha = 0$ is studied by deriving Peierls estimates for suitably defined contours. A further step forward has then been done by Imbrie, [**9**], who proved the validity of the cluster expansion for this gas of contours. A different approach, based on inequalities, has instead been followed by Dümcke and Spohn, [**5**], and Spohn, [**18**], to prove phase transitions for systems of ± 1 spins on $\mathbb{R}$ with long range interactions, $\alpha \in [0, 1)$. The results were used in the analysis of ground states for some quantum systems.

In this paper we revisit Fröhlich and Spencer [**8**] and extend it to the case $\alpha \in (0, 1/2]$. In particular we prove that the probability of occurrence of a droplet of the opposite phase is depressed at least by $c \exp\{-\beta\zeta L^{\alpha}\}$, c and ζ positive constants, L the length of the droplet. The analogy with $d > 1$ where the bound goes as $c \exp\{-\beta\zeta L^{(d-1)/d}\}$, is evident (our proof applies essentially unchanged through $\alpha = 0$, where it yields the bound $c \exp\{-\beta\zeta \ln L\}$, loosing however the analogy with $d > 1$). Comforted by these results and the analogy with $d > 1$, we plan, in the future, to extend the analysis to Ising systems with Kac potentials and then, hopefully, to prove phase transitions for hard rods with attractive Kac potentials, at least for $\alpha > 0$.

The bibliography on the subject should also include the papers, [**1, 2, 16, 10, 17**], which refer to $d = 1$, long range percolation. In fact, using the FK representation, the results can be transferred to Ising systems, but it is not clear whether the approach could extend to the continuum particle systems where ferromagnetic inequalities are absent.

Thus, the model we consider here is an Ising ferromagnet on a one dimensional lattice, with total energy

$$h(\sigma) = \frac{1}{2} \sum_{x,y \in \mathbb{Z}} J(|x-y|) \mathbf{1}_{\sigma(x) \neq \sigma(y)} \tag{1.1}$$

$$J(n) = \begin{cases} J(1) >> 1 & \\ \\ \dfrac{1}{n^{2-\alpha}} & \text{if } n > 1. \end{cases} \qquad \text{with} \quad \alpha \in [0, 1/2] \tag{1.2}$$

which will be studied at equilibrium with $\beta \gg 1$. In the sequel, for notational convenience, we restrict $\alpha \in (0, 1/2]$, the analysis of the case $\alpha = 0$ is analogous and treated in Appendix A, E and F.

In this paper we will show that the equilibrium configurations for the system associated to the hamiltonian (1.1) can be described in terms of contours whose weights satisfy a Peierls bound. These contours (as in Pirogov-Sinai) are defined as regions which collect close-by deviations from the ground states. The Peierls bound follows from the fact that the excess energy of the associated interfaces is bounded from below proportionally to the size of the region to a positive power. To illustrate this point consider the simple case of three contiguous intervals B^-, A and B^+. Let A be of size L and $B^\pm$ of size larger or equal to L and call $\mathcal{C}$ the set of configurations s.t. $\sigma = +1$ for all sites belonging to A and $\sigma = -1$ for all sites belonging to $B^\pm$. An explicit calculation, see Appendix A, shows that for all configurations in $\mathcal{C}$ the variation of energy obtained by flipping the spins inside A (thus getting all spins equal to $+1$ in $A \sqcup B^+ \sqcup B^-$) is bounded from below by $\zeta_\alpha L^\alpha$, with $\zeta_\alpha > 0$ for $\alpha \in (0, 1/2]$ and if $J(1)$ is large enough.

In Section 2 we give a graphical description of a spin configuration in terms of a configuration of triangles, which allows to introduce the notion of internal and external interfaces (like in $d > 1$ dimensions), see Fig. 1 in Section 2.

In Section 3 we introduce the notion of contours as clusters of nearby triangles and prove Peierls bounds for their energy. Our definition is very similar to that in [**8**], but our aim is to get a geometric representation of the contours more explicit and better suited for further generalizations.

In Section 4 we prove that for β large enough the Peierls estimates on the energy of contours enable to control their entropy.

The approach we use can be generalized to a larger class of long range attractive forces where the assumption $J(1) \gg 1$ is dropped and $\alpha \in (0, 1)$. We will discuss this point in a forthcoming paper together with a characterization of the typical configurations for slow decreasing ferromagnetic Kac potentials. As mentioned the ultimate goal is the extension to a one dimensional system of hard-core particles interacting via such long range attractive forces, but at the moment we have not yet concrete results in this direction.

2. Spin and triangle configurations

We will consider in this paper homogeneous boundary conditions, i.e. the spins in the boundary conditions are either all $+1$ or all -1. By the spin flip symmetry, we may and will restrict to the former, so that we will only study configurations $\sigma = \{\sigma_x, x \in \mathbb{Z}\} \in \mathcal{X}_+$, namely such that $\sigma_x = 1$ for all $|x|$ large enough. Our aim here is to recover a picture as in $d > 1$, where the configurations are described by a collection of interfaces. In one dimensions, an interface at $(x, x+1)$ means $\sigma_x \sigma_{x+1} = -1$. The precise location of the interface in the interval is immaterial and we will use it to our advantage by choosing a point in each interval $(x + 1/2) \pm 1/100$, $x \in \mathbb{Z}$, with the property that for any four distinct points r_i, $i = 1, .., 4$, $|r_1 - r_2| \neq |r_3 - r_4|$. We suppose the choice done once for all, so that hereafter an interface point between x and $x + 1$ is uniquely fixed.

Any interface point, by its definition, represents a change of phase so that after the first interface point (coming from the left), the second one corresponds to a reestablishing of the original phase, and so on. However, this is not the most convenient way to look at the spin configurations. Our construction is similar to that in [**8**] (where interface points were called spin flip points) and it is based on suitably coupling together pairs of interface points. To this end we will use the criterion of minimal distance, which will be made geometrically intuitive by using a graphical representation where each spin configuration is mapped into a set of triangles. The endpoints of the triangles will be the pairs of coupled interface points.

Due to the above choice of the boundary conditions, any $\sigma \in \mathcal{X}_+$ has a finite, even number of interface points. We then let each interface point evolve into two trajectories represented in the (r, t) plane by the two lines $r \pm t, t \geq 0$. We have thus a bunch of growing v-lines each one emanating from an interface point. Once two v-lines meet, they are frozen and stop their growth, while the others are undisturbed and keep growing. Our choice of the location of the interface points ensure that collisions occur one at a time so that the above definition is unambiguous.

The collision of two points is represented graphically in the (r, t) plane by a triangle whose basis is the line joining the two interface points and whose sides are the two arms of the v-lines which enter into contact at the time of collision. Triangles will be usually denoted by T and we will write

$$|T| = \text{ cardinality of } T \sqcap \mathbb{Z}, \qquad \text{dist}(T, T') = \text{cardinality of } I \sqcap \mathbb{Z}, \tag{2.1}$$

where I is the interval between T and T' if T and T' are disjoint; if T and T' are one contained in the other (no other possibility may arise in the above construction) then I denotes the minimal interval between the two.

We have thus represented a configuration $\sigma \in \mathcal{X}_+$ as a collection $\underline{T} = (T_1, .., T_n)$ of triangles in the (r, t) plane. The set of configurations of triangles obtained in this way are denoted by $\{\underline{T}\}$, and the above construction defines a one to one map from $\mathcal{X}_+$ onto $\{\underline{T}\}$. It is easy to see that a triangle configuration $\underline{T}$ belongs to $\{\underline{T}\}$ iff for any pair T and T' in $\underline{T}$

$$\text{dist}(T, T') \geq \min\big\{|T|, |T'|\big\} \tag{2.2}$$

The two endpoints of a triangle play the role which has the interface in higher dimensions and we thus have, also in $d = 1$, a notion of external and internal interfaces. (see Fig. 1)

The above construction is taken from a model for $d = 1$ coarsening, see Derrida, [**4**], Carr and Pego, [**3**] and also some old, unpublished notes of two of us (P.F and E.P). Coarsening in $d = 1$ is extremely slow and it is often a good approximation to say that in a given sequence of intervals of alternating phases, the shortest one disappears first, while all the others are

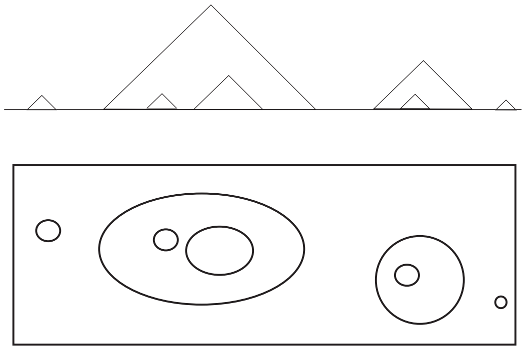

FIGURE 1

unchanged. The dynamics is then described in terms of triangles by calling the first interval which disappears as the basis of the smallest triangle and then iterating the procedure. The interesting setup when studying coarsening is to have initially infinitely many phase changes and one of the aims is to understand if and which are the self similar structures which emerge from the triangles picture. Here our task is simpler, we have finitely many phase changes and want to prove that each one of them has a small Gibbs weight.

Writing

$$H(\underline{T}) = h(\sigma), \qquad \sigma \in \mathcal{X}_+ \Leftrightarrow \underline{T} \in \{\underline{T}\} \tag{2.3}$$

and calling $\underline{T} = (T_1, .., T_n)$ with $|T_i| \le |T_{i+1}|$, we have

$$H(\underline{T}) = H\big(T_1 \,\big|\, \underline{T} \setminus T_1\big) + H\big(\underline{T} \setminus T_1\big), \qquad H\big(\underline{S} \,\big|\, \underline{T}\big) := H(\underline{S} \sqcup \underline{T}) - H(\underline{T}) \tag{2.4}$$

In fact if $\underline{T} \in \{\underline{T}\}$, and $T \in \underline{T}$, then $\underline{T} \setminus T$ obviously satisfies (2.2) and therefore it is in $\{\underline{T}\}$. $\underline{T} \setminus T$ is obtained from the configuration σ corresponding to $\underline{T}$ by flipping all the spins inside the basis of T. By iteration,

$$H(\underline{T}) = \sum_{i=1}^{n} H\big(T_i \,\big|\, \underline{T} \setminus [T_1 \sqcup \cdots \sqcup T_i]\big) \tag{2.5}$$

Lemma 2.1. *For any i,*

$$H\big(T_i \,\big|\, \underline{T} \setminus [T_1 \sqcup \cdots \sqcup T_i]\big) \ge W(|T_i|) \tag{2.6}$$

where

$$W(L) = \sum_{x=1}^{L} \left(\sum_{\substack{y \in [L+1, 2L] \\ y \in [-L+1, 0]}} J(|x-y|) - \sum_{\substack{y \in [2L+1, \infty] \\ y \in [-\infty, -L]}} J(|x-y|) \right) \tag{2.7}$$

Proof. Call $I_i^{\pm}$ the two intervals in $\mathbb{Z}$ which are to the right and to the left of T_i, each one consisting of $|T_i|$ sites. There is no interface point inside T_i, and inside $I_i^{\pm}$ as well, because $|T_i|$ is the minimal length in $\underline{T} \setminus [T_1 \sqcup \cdots \sqcup T_{i-1}]$ and all T_j, $j > i$, have distance from T_i which is $\ge |T_i|$. Then, if σ corresponds to $\underline{T} \setminus [T_1 \sqcup \cdots \sqcup T_{i-1}]$, the spins in $T_i \sqcap \mathbb{Z}$ are all equal to each other and opposite to those in $I_i^{\pm}$. Instead, in the configuration σ' which corresponds to $\underline{T} \setminus [T_1 \sqcup \cdots \sqcup T_i]$, the spins are all the same in $(T_i \sqcap \mathbb{Z}) \sqcup I_i^+ \sqcup I_i^-$. By (2.4), $H\big(T_i \,\big|\, \underline{T} \setminus [T_1 \sqcup \cdots \sqcup T_i]\big) = h(\sigma) - h(\sigma')$, so that (2.6) and the lemma are proved.

$\square$

In Lemma A.1 it is proved that for $J(1)$ large enough, there is $\zeta > 0$ so that

$$W(L) \geq \zeta h_\alpha(L) \qquad \text{where} \qquad h_\alpha(L) := \begin{cases} L^\alpha & \alpha \in (0, 1/2] \\ \ln L + 4 & \alpha = 0. \end{cases} \tag{2.8}$$

in the sequel we fix our attention on the case $\alpha \in (0, 1/2]$, and discuss the case $\alpha = 0$ in Appendix A, E and F. Thus

$$H(\underline{T}) \geq \zeta \sum_{i=1}^{n} |T_i|^\alpha, \qquad \underline{T} = (T_1, .., T_n) \tag{2.9}$$

The inequality must be seen as an analogue of the Peierls estimate in $d > 1$ where the excess energy of a configuration of interfaces is bounded from below proportionally to the surface area of such interfaces. Since $|T_i|$ is the volume surrounded by the interface, α is identified to the ratio $(d-1)/d$, with d an "effective dimension" of the system.

This is however only an analogy. To really implement a Peierls bound in our setup, we need to "localize the estimates", being able to compute the weight of a given triangle in a generic configuration. The previous bound was easy, because we could estimate successively the weights of the triangles in the same order as their lengths. If we want to bound the energy of a generic triangle T in configuration $\underline{T}$, $|T|$ may not be the smallest length so that we are confronted with cases where there are other triangles in $T \sqcup I^+ \sqcup I^-$ (see Lemma 2.1 for notation). Indeed, we could add to T smaller triangles T' in $T \sqcup I^+ \sqcup I^-$ without violating (2.2). Our approach will be • to "connect" triangles if they are "dangerously close" to each other, • to define contours as "connected clusters" of triangles and • to compute probabilities of contours rather than of single triangles. To compute the probability of a contour, we first order increasingly the triangles in the contour, according to their lengths. Then the previous argument can be generalized, exploiting the fact that the triangles which are not in the contour are "sufficiently far away" (by the way contours are defined). In the next section we will see how triangles can be clustered into contours and then extend Lemma 2.1 to contours, thus concluding the analysis of the energy of contours; in Section 4 we will prove entropy bounds (on the number of contours), which show that for β large enough, energy wins against entropy.

3. Contours and Peierls estimates

In Subsection 3.1 we will define a function $\mathcal{R}$ which associates to any configuration $\underline{T} \in \{\underline{T}\}$ a configuration $\{\Gamma_j\}$ of contours, each Γ_j being a subset of triangles in $\underline{T}$. The crucial point in the definition is that the triangles in a contour are "close to each other", while all the other triangles are "far away"; using such a property we will be able to extend to contours the energy estimate of the previous section, thus deriving the Peierls estimates of Subsection 3.2. In Subsection 3.3 we will recall the classical argument for existence of a phase transition, using the Peierls bound proved in Subsection 3.2 and the entropy estimates which will be proved in Section 4.

3.1. Contours. A contour Γ is a collection $\underline{T}$ of triangles ($\underline{T}$ in this Section will always, and sometimes tacitly, denote an element in $\{\underline{T}\}$) joined together by a hierarchical network of connections, under which all the triangles of a contour become mutually connected. The structure has a self similar property which we will exploit when counting the contours. The coarsest picture of a contour Γ is the pair $\{T(\Gamma), |\Gamma|\}$, $T(\Gamma)$ a triangle, $|\Gamma|$ its mass. $T(\Gamma)$ is the triangle whose basis is the smallest interval which contains all the triangles of the contour, the right and left endpoints of $T(\Gamma) \sqcap \mathbb{Z}$ are denoted by $x_\pm(\Gamma)$.$|\Gamma|$, the mass of the contour, is the sum of the masses of all the triangles in Γ, the mass $|T_i|$ of a triangle being defined in (2.1).

Our aim is to define an algorithm $\mathcal{R}(\underline{T})$ on $\{\underline{T}\}$, which associates to any configuration $\underline{T}$ a configuration $\{\Gamma_j\}$ of contours with the following properties.

P.0 *Let* $\mathcal{R}(\underline{T}) = (\Gamma_1, .., \Gamma_n)$, $\Gamma_i = \{T_{j,i}, 1 \le j \le k_i\}$, *then* $\underline{T} = \{T_{j,i}, 1 \le i \le n,\ 1 \le j \le k_i\}$.

P.1 *Contours are well separated from each other.* Any pair $\Gamma \neq \Gamma'$ in $\mathcal{R}(\underline{T})$ verifies one of the following two alternatives. (i): $T(\Gamma) \sqcap T(\Gamma') = \emptyset$, in which case

$$\operatorname{dist}(\Gamma, \Gamma') > c \min\left\{|\Gamma|^3, |\Gamma'|^3\right\} \tag{3.1}$$

where c is as in (3.15) below and $\operatorname{dist}(\cdot,\cdot)$ means distance as defined in (2.1) between the set of all the triangles in Γ from the corresponding set in Γ':

$$\operatorname{dist}(\Gamma, \Gamma') := \min_{\substack{T \in \Gamma \\ T' \in \Gamma'}} \operatorname{dist}(T, T')$$

(which in the present case is equal to the distance between the two triangles $T(\Gamma)$ and $T(\Gamma')$)

(ii): $T(\Gamma) \sqcap T(\Gamma') \neq \emptyset$, then either $T(\Gamma) \sqsubset T(\Gamma')$ or $T(\Gamma') \sqsubset T(\Gamma)$; moreover, supposing for instance that the former case is verified, (in which case we call Γ an inner contour) then for any triangle $T'_i \in \Gamma'$, either $T(\Gamma) \sqsubset T'_i$ or $T(\Gamma) \sqcap T'_i = \emptyset$; and

$$\operatorname{dist}(\Gamma, \Gamma') > c|\Gamma|^3, \quad \text{if } T(\Gamma) \sqsubset T(\Gamma') \tag{3.2}$$

P.2 *Independence.* Let $\{\underline{T}^{(1)}, .., \underline{T}^{(k)}\}$, be $k > 1$ configurations of triangles; $\mathcal{R}(\underline{T}^{(i)}) = \{\Gamma_j^{(i)}, j = 1, .., n_i\}$ the contours of the configuration $\underline{T}^{(i)}$. Then, if any distinct pair $\Gamma_j^{(i)}$ and $\Gamma_{j'}^{(i')}$ satisfies P.1,

$$\mathcal{R}\big(\underline{T}^{(1)}, .., \underline{T}^{(k)}\big) = \{\Gamma_j^{(i)}, j = 1, .., n_i; i = 1, .., k\} \tag{3.3}$$

It is a nice fact of life that not only P.0, P.1 and P.2 can be actually implemented by some algorithm $\mathcal{R}$, but also that such an algorithm is unique. In Appendix B we will prove the following theorem:

Theorem 3.1 (Existence and uniqueness)**.** *There is a unique algorithm $\mathcal{R}(\underline{T})$ which satisfies P.0, P.1 and P.2.*

3.2. Peierls estimates. The idea behind the proof of the Peierls estimates, Theorem 3.2 below, is that the property P.1 will ensure that the triangles which do not belong to a contour are so far away that, to leading order, they can be neglected and the bond (2.6) can be extended to contours.

Theorem 3.2. *Let the constant c in the definition of the contours (see P.1) be so large that* (3.15) *below holds. For any $\underline{T} \in \{\underline{T}\}$, let $\Gamma_0 \in \mathcal{R}(\underline{T})$, $\underline{T}^{(0)}$ the triangles in Γ_0, $\zeta > 0$ as in* (2.8)*. Then*

$$H\big(\underline{T}^{(0)}\,\big|\,\underline{T}\setminus\underline{T}^{(0)}\big) \ge \frac{\zeta}{2}\sum_{T\in\underline{T}^{(0)}} |T|^{\alpha} \tag{3.4}$$

(for $\alpha = 0$, (3.4) *holds with $|T|^\alpha$ replaced by $\log|T| + 4$).*

Proof. Calling $\underline{T}^{(0)} = (T_1, .., T_k)$, $|T_i| \le |T_{i+1}|$, $i = 1, .., k-1$,

$$H\big(\underline{T}^{(0)}\,\big|\,\underline{T}\setminus\underline{T}^{(0)}\big) = \sum_{i=1}^{k} H\big(T_i\,\big|\,\underline{T}\setminus\{T_1,\dots,T_i\}\big) \tag{3.5}$$

As a difference with Section 2, here we may have triangles in $I_i^{\pm}$, but, by the argument after (2.4),

$$(I_i^+ \sqcup I_i^-) \sqcap T_j = \emptyset, \qquad \text{for all } j > i\ T_j \not\supset T_i \tag{3.6}$$

We also have, calling $\{\Gamma_j, j \ge 1\}$, the other contours of $\underline{T}$, different from Γ_0,

$$(I_i^+ \sqcup I_i^-) \sqcap T = \emptyset, \qquad \text{for all } T \in \Gamma_j,\ T \not\supset T_i,\ j \ge 1 \text{ such that } |\Gamma_j| \ge |\Gamma_0| \tag{3.7}$$

because, by property P.1 of Section 4, $\mathrm{dist}(T_i, \Gamma_j) \ge \mathrm{dist}(\Gamma_0, \Gamma_j) \ge c|\Gamma_0|^3 \ge |T_i|$.

Finally, using again P.1,

$$\mathrm{dist}(T_i, T) > c|\Gamma_j|^3, \qquad \text{for all } T \in \Gamma_j,\ j \ge 1 \text{ and such that } |\Gamma_j| < |\Gamma_0| \tag{3.8}$$

With the notation introduced after (2.6), and with

$$A(T_i;\Gamma_j) = \bigsqcup_{T\in\Gamma_j, T_i\not\subset T} T \sqcap \mathbb{Z}; \qquad |\Gamma| = \sum_{T\in\Gamma} |T| \tag{3.9}$$

we claim that

$$\begin{aligned} &H\big(T_i\,\big|\,\underline{T}\setminus[T_1\sqcup\cdots\sqcup T_i]\big) \ge W(|T_i|) \\ &\quad -2\sum_{M}\sum_{j=1}^{n} \mathbf{1}_{|\Gamma_j|=M} \sum_{x\in T_i\sqcap\mathbb{Z}}\sum_{y\in I_i^{\pm}} J(|x-y|)\Big(\mathbf{1}_{y\in A(T_i;\Gamma_j)} + \mathbf{1}_{x\in A(T_i;\Gamma_j)}\Big) \end{aligned} \tag{3.10}$$

To prove (3.10), we observe that the contribution of σ_x and σ_y, (x and y as in (3.10)) is the same as in $W(|T_i|)$ whenever $\sigma_x\sigma_y = -1$; on the other hand, if $\sigma_x = \sigma_y$ then there must exist a triangle distinct from T_i which contains one site and not the other one. We thus automatically exclude the triangles which contain T_i, as, by (2.2), they will also contain $I_i^{\pm}$; by (3.6), $(T_{i+1}, .., T_k)$ are also excluded. Then (3.10) follows after noticing that if $\sigma_x = \sigma_y$, the pair σ_x, σ_y contributes with the opposite sign to the energy as for $W(|T_i|)$, hence the factor 2 in the second term on the r.h.s. of (3.10). In (3.10) we have also split the sum over all contours putting together contours with same mass, the mass of a contour Γ being defined in (3.9).

Call y_0 the rightmost point of $\mathbb{Z}$ in T_i, $y_1 \in I_i^+$ the point, if it exists, separated from y_0 by $[cM^3]$ sites, $[\cdot]$ the integer part of $\cdot$. By (3.7)-(3.8) the following holds: any Γ_j with $|\Gamma_j| = M$ is such that all its triangles which do not contain T_i and are to its right, have their left endpoint to the right of y_1. After changing labels, let Γ_1 be the contour of mass M with the closest triangle to y_1 (and to its right). The triangles in Γ_j, $j > 1$, with mass M, cannot be closer than $y_2 \in I_i^+$, where y_2 (if it exists) is separated from y_1 by $[cM^3]$ sites. By iteration we define y_j, $j > 2$, and have that the n-th closest contour to T_i of mass M and to its right, is to the right of y_j. Calling y_n the last of such points in I_i^+, we have, for any $x \in T_i$,

$$\sum_{j=1}^{n} \mathbf{1}_{|\Gamma_j|=M} \sum_{y \in I_i^+} J(|x-y|)\mathbf{1}_{y \in A(T;\Gamma_j)} \le M \sum_{k=1}^{n} J(|x-y_k|) \tag{3.11}$$

because $J(|x-y|) = J(y-x)$ is a decreasing function of y and the total number of sites in the triangles of a contour Γ is not larger than $|\Gamma|$ (not necessarily equal because a triangle might be contained in another one). Moreover, by monotonicity,

$$J(|x-y_k|) \le \frac{1}{[cM^3]} \sum_{y \in (y_{k-1}, y_k)} J(|x-y|) \tag{3.12}$$

so that

$$\sum_{j=1}^{n} \mathbf{1}_{|\Gamma_j|=M} \sum_{y \in I_i^+} J(|x-y|)\mathbf{1}_{y \in A(T;\Gamma_j)} \le \frac{M}{[cM^3]} \sum_{y \in I_i^+} J(|x-y|) \tag{3.13}$$

The sum is the same as in $W(|T_i|)$. Repeating the same procedure for T_i and I_i^- we finally get

$$H\big(T_i \,\big|\, \underline{T} \setminus [T_1 \sqcup \cdots \sqcup T_i]\big) \ge W(|T_i|)\Big(1 - \sum_M \frac{4M}{[cM^3]}\Big) \tag{3.14}$$

By choosing c so large that

$$\sum_M \frac{4M}{[cM^3]} \le \frac{1}{2} \tag{3.15}$$

and recalling (2.8) we then prove the theorem. □

3.3. Phase transitions. To prove phase transitions we follow the well known argument for $d > 1$. Let Λ be an interval containing the origin, μ_Λ^+ the Gibbs measure in Λ with $+$

boundary conditions. Then

$$\mu_\Lambda^+(\sigma_0 = -1) \le \mu_\Lambda^+\Big(\{0 \in \Gamma\}\Big) \tag{3.16}$$

where $\{0 \in \Gamma\}$ denotes the event that there is a contour Γ which has a triangle T which contains the origin. Then

$$\mu_\Lambda^+\Big(\{0 \in \Gamma\}\Big) = \frac{1}{Z_\Lambda^+} \sum_{\Gamma \ni 0} \sum_{\underline{T}:\Gamma \in \mathcal{R}(\underline{T})} e^{-\beta H(\underline{T})}$$

Calling $\underline{T}^{(0)}$ the collection of triangles in Γ, $\mathcal{R}(\underline{T}^{(0)}) = \Gamma$, by Theorem 3.2,

$$e^{-\beta H(\underline{T})} \le e^{-\beta H\left(\underline{T} \setminus \underline{T}^{(0)}\right)} w_{\zeta\beta/2}(\Gamma) \tag{3.17}$$

where, for $b > 0$,

$$w_b(\Gamma) := \prod_{T \in \Gamma} e^{-b|T|^\alpha} \tag{3.18}$$

$w_b(\Gamma)$ is called the b-weight of the contour Γ. Then, using (3.17),

$$\mu_\Lambda^+\Big(\{0 \in \Gamma\}\Big) \le \sum_{\Gamma \ni 0} w_{\zeta\beta/2}(\Gamma) = \sum_m \sum_{\Gamma:|\Gamma|=m, 0 \in \Gamma} w_{\zeta\beta/2}(\Gamma) \tag{3.19}$$

and, by (4.1) below, valid for β large enough,

$$\mu_\Lambda^+\Big(\{0 \in \Gamma\}\Big) \le 2 \sum_m m e^{-\zeta\beta m^\alpha/2} \tag{3.20}$$

Since the sum starts from $m \ge 1$, the r.h.s. is $< 1/2$ if β is large enough, hence the spin flip symmetry is broken and there is a phase transition.

4. Entropy of contours

The main result in this section is Theorem 4.1 below, where we prove (3.20), and hence that, for β large, entropy is controlled by energy and a phase transition occurs.

Theorem 4.1. *For any b large enough and any $m > 0$*

$$\sum_{\Gamma:|\Gamma|=m, 0 \in \Gamma} w_b(\Gamma) \le 2 m e^{-b m^\alpha} \tag{4.1}$$

where $w_b(\Gamma)$ has been defined in (3.18).

The theorem is proved in Subsection 4.3, by exploiting a self-similarity property of the contours which is the argument of the next two subsections.

4.1. An auxiliary branching process. Contours can be described in terms of trees with a self-similar, hierarchical structure. We will first describe abstractly the trees and then relate them to the contours.

The nodes of the tree are "individuals" of two species: heavy triangles, h-triangles in short, and spheres; the h-triangles can be either black or white. Only black triangles can procreate and their offsprings contain at least two h-triangles. The offsprings in a branching are ordered, the h-triangles are drawn sequentially, the spheres, also drawn sequentially, can lie in each one of the intervals in between two consecutive h-triangles, but also "inside" the white triangles, the latter will be called "attached" to the white triangle in which they are contained.

Finally the tree has a root which consists either of a single black triangle or of a single white triangle with possibly spheres inside the white triangle. In the second alternative the tree consists of only its root, as white triangles and spheres cannot procreate. An example of tree is drawn in Figure 2.

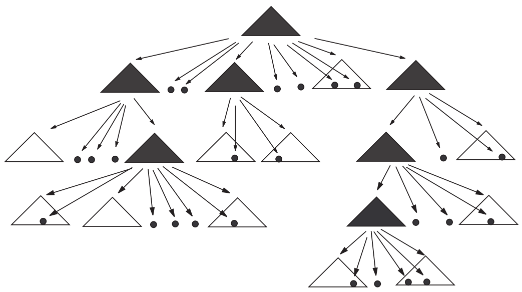

FIGURE 2

We will construct an algorithm which associates to any contour a tree with the above properties and later use such a correspondence to prove (4.1). We will in fact organize the sum over contours in (4.1) by summing over trees after having summed over all contours which produce the same tree. The identification of the nodes of the tree in terms of contours will allow for an inductive procedure which greatly reduces the complexity of the computation. We describe here the the main features of the algorithm, which are those used in the proof of (4.1), while the existence of the algorithm itself will be proved in Subsection 4.2, by exploiting a graphical representation of contours.

We will restrict in the sequel to configurations $\underline{T}$ such that $\mathcal{R}(\underline{T})$ is a singleton. As mentioned, the basic property of the algorithm which associates a tree to $\underline{T}$, is that each node of the tree is representative of a subset $\underline{T}'$ of $\underline{T}$ such that $\mathcal{R}(\underline{T}')$ is a singleton.

The root corresponds to the full $\underline{T}$. Moreover the collection of all the triangles associated to all the individuals of an offspring is the same as the set of triangles associated to the parent, so that a branching is nothing else than a partition of the triangles present in the branching node. In particular mass is conserved in a branching.

White triangles are associated to contours consisting of a single triangle, such triangle must be maximal, i.e. not contained in any other triangle of $\underline{T}$. Take notice, however, that the converse may not be true, as it may happen that a maximal triangle is one of the triangles associated to a black triangle or a sphere. If a maximal triangle T corresponds to a white triangle, then the spheres in the white triangle are associated to the contours of the configuration $\underline{T}'$ made of all the triangles of $\underline{T}$ which are contained in T.

The next properties we mention establish a quantitative relation between the ordering of the offsprings in the tree and the location of the corresponding triangles. If a black triangle generates $n \ge 2$ h-triangles labelled consecutively, call Γ_i, $1 \le i \le n$, the triangle associated to the i-th h-triangle (by itself each Γ_i is a contour, hence the notation). Then the triangles $\{T(\Gamma_i)\}$ (recall that $T(\Gamma)$ is the minimal triangle which contains all the triangles forming Γ) are consecutive from left to right and

$$\mathrm{dist}\big(T(\Gamma_i), T(\Gamma_{i+1})\big) \le c \min\big\{|\Gamma_i|^3, |\Gamma_{i+1}|^3\big\}, \qquad i = 1, .., n-1 \tag{4.2}$$

Moreover, if there are k_i spheres between the i-th and $(i+1)$-th h-triangle, call $\Gamma_j^{(i)}$, $j = 1, .., k_i$, the contours associated to these spheres. Then all $T(\Gamma_j^{(i)})$ are in between $T(\Gamma_i)$ and $T(\Gamma_{i+1})$, $\{T(\Gamma_j^{(i)})\}$ is sequential and the following constraint on their mutual distances holds. Letting $x_\pm(\Gamma)$ as in the beginning of Subsection 3.1, the set of endpoints $\{x_\pm(\Gamma_j^{(i)}), a = x_+(\Gamma_i), b = x_-(\Gamma_{i+1})\}$ is such that there is $p_i : 0 \le p_i \le k_i$ so that

$$\begin{aligned}
&0 \le x_-(\Gamma_1^{(i)}) - a \le c|\Gamma_1^{(i)}|^3 + 1, \quad 0 \le x_-(\Gamma_2^{(i)}) - x_+(\Gamma_1^{(i)}) \le c|\Gamma_2^{(i)}|^3 + 1, \dots \\
&\qquad \dots, x_-(\Gamma_{p_i}^{(i)}) - x_+(\Gamma_{p_i-1}^{(i)}) \le c|\Gamma_{p_i}^{(i)}|^3 + 1 \\
&0 \le b - x_+(\Gamma_{k_i}^{(i)}) \le c|\Gamma_{k_i}^{(i)}|^3 + 1, \quad 0 \le x_+(\Gamma_{k_i}^{(i)}) - x_-(\Gamma_{k_i-1}^{(i)}) \le c|\Gamma_{k_i-1}^{(i)}|^3 + 1, \dots \\
&\qquad \dots, x_+(\Gamma_{p_i+1}^{(i)}) - x_-(\Gamma_{p_i+2}^{(i)}) \le c|\Gamma_{p_i+1}^{(i)}|^3 + 1
\end{aligned} \tag{4.3}$$

Finally if Γ_j, $j = 1, .., k$, are the sets of triangles associated to the spheres inside a white triangle, represented by T, then the triangles $T(\Gamma_j)$ satisfy the analogue of (4.3) with $a = x_-(T) + 1$ and $b = x_+(T) - 1$. These are the only properties on the structure of contours that we will use in the proof of (4.1) in Subsection 4.3, next subsection is only an existence proof of the algorithm for associating a tree to a contour with the properties we have been describing so far, and, to a first reading, it may be skipped.

4.2. Graphical construction. We will construct here an algorithm which associates a tree (with the properties described in the previous subsection) to any $\underline{T}$ such that $\mathcal{R}(\underline{T})$ is a single contour. The algorithm is obtained via a graphical representation of $\underline{T}$, where we draw at any integer time $t \in \{0, 1, ..\}$ a configuration of mutually disjoint squares with a side in $\mathbb{R}$, called the basis of the square. Each square S is representative of a cluster $\{\underline{T}\}_S$ of triangles in $\underline{T}$, with the property that $\mathcal{R}(\{\underline{T}\}_S)$ is a singleton; the name "squares" is just to avoid confusion with the original triangles and the h-triangles of the tree. The mass of a square S equals the sum of the masses of the triangles in $\{\underline{T}\}_S$ (the mass of a triangle being the number of integers contained in its basis). The configurations of squares at the different times will be viewed as the successive applications of a renormalization group transformation.

The time $t = 0$ configuration

This is obtained by associating to each "maximal" triangle of $\underline{T}$ a square with same basis: $T \in \underline{T}$ is maximal if it is not contained in any other triangle of $\underline{T}$. By definition of maximality the set of maximal triangles, hence of squares, is sequential. The cluster of triangles $\{\underline{T}\}_S$ represented by the square S consists of a maximal triangle T and of all the triangles contained in T. The mass of S, according to the general rule, is then the sum of all the masses in $\{\underline{T}\}_S$. Statement (ii) in Lemma 4.2 below, proves that these squares verify the property that the sets $\{\underline{T}\}_S$ form a single contour, thus our definition of the square configuration at $t = 0$ is well posed. In Appendix C we will prove:

Lemma 4.2. *Let S be a square corresponding to a maximal triangle T. Then: (i), $\mathcal{R}(\{\underline{T}\}_S \setminus T) = \{\Gamma_j\}$ is sequential and the sequence $\{T(\Gamma_j)\}$ satisfies the analogue of* (4.3) *with $a = x_-(T)$ and $b = x_+(T)$; moreover, (ii), $\mathcal{R}(\{\underline{T}\}_S)$ consists of a single contour.*

By (i) the sequence $\{T(\Gamma_j)\}$ satisfies the same properties as the sequence of triangles obtained from the spheres attached to a white triangle, as described in the previous subsection. Together with (ii), this shows that each one of the squares at time $t = 0$ is a candidate for being a white triangle. Whether this will really happen, does depend in a complex way on the relative positions of the other triangles of $\underline{T}$, as we will see after completing the construction of the square process.

The next time-step configuration

The construction of the configuration of squares at time $t = n + 1$ only depends on the configuration at time $t = n$, namely on the location of the squares in the configuration and on their masses. Like at time $t = 0$, each square S is representative of a collection $\{\underline{T}\}_S$ of triangles in $\underline{T}$, more and more complex as time increases, but, as said, the construction of the configurations at the successive time will only depend on locations and masses of the squares, the latter being the sum of all the masses of the triangles represented. The rule for constructing the configuration at time $t = n+1$ given the one at time $t = n$, defines the action of the renormalization group transformation mentioned at the beginning of the subsection.

We start by drawing oriented arrows between pairs of squares: we put an arrow (S, S') from S to S' if $|S| \le |S'|$, $|S|$ the mass of the square S (in case of equality if S is before S', going from left to right) and if the distance between S and S' is $\le c|S|^3$. Arrows define a connection, which will be referred to as a-connection (a for arrow), to distinguish it from the connection used in the definition of contours. Two squares are a-connected if they can be joined via a chain of pairs of squares, each pair linked by an arrow (independently of the direction of the arrow).

To each a-connected component we associate a proto-square, which is the minimal square which contains all the squares in that component, we call them proto-squares because some of the proto-squares will become a square in the configuration at time $t = n + 1$. We will prove below that any two such proto-squares are either disjoint or one contained in the other. We call maximal those which are not contained in any other one. *The maximal proto-squares are the squares at time $t = n + 1$.* The set $\{\underline{T}\}_S$ represented by a maximal proto-square S, is the collection of all $\{\underline{T}\}_{S'}$, with S' running over all the squares at time $t = n$ which are contained in S. By maximality the new squares at time $t = n + 1$ are sequential. In Lemma C.2, we will prove that $\mathcal{R}(\{\underline{T}\}_S)$ consists of a single contour, thus legitimating the present definition of the square configuration at time $t = n + 1$.

We next state some features of the construction needed later for the identification of a tree structure. To this end it is convenient to erase some arrows, thus we will call "old arrows" the arrows defined so far and (old a)-connected squares connected by old arrows. Old arrows are erased with the following rule: if there are several arrows emanating from a same square, all in a same direction (i.e. right or left), we keep only the minimal one and erase all the others. This is done for all squares and all directions. The arrows which are left are the new arrows, and we will call (new a)-connected, squares connected by the new arrows. In Lemma D.1, it is proved that a set is (new a)-connected iff it is (old a)-connected. We will hereafter in this section call arrows the new arrows and a-connected, (new a)-connected sets.

The *"shadow"* of the arrow (S, S') is the interval between the two endpoints of S and S' which face each other. If two shadows have non empty intersection, they must be one contained in the other, Lemma D.2; such a statement proves the above property about the fact that the proto-squares are either disjoint or one contained in the other.

We then call *primary* an arrow with maximal shadow, i.e. which is not contained in any other shadow, and *primary* the two squares connected by a *primary* arrow. The set $\{\underline{T}'\}$ of all triangles in $\underline{T}$ whose basis are contained in the shadow (a, b) of a primary arrow (S_1, S_2) is such that:

Lemma 4.3. *With the above notation, (i),* $\mathcal{R}(\underline{T}') = \{\Gamma_j\}$ *and* $\{T(\Gamma_j)\}$ *is a sequence which satisfies the analogue of* (4.3) *with* $a = x_+(S_1)$ *and* $b = x_-(S_2)$ *supposing for instance that* S_1 *is before* S_2*; moreover, (ii),* $\mathcal{R}(\underline{T}'')$ *consists of a single contour,* $\underline{T}''$ *being the union of all triangles in* $\underline{T}'$ *and those associated to* S_1 *and* S_2.

Lemma 4.3 is proved in Appendix C. The primary squares may thus become h-triangles, as they form a sequence which satisfy (4.2), while all the squares in a shadow, called *secondary*, are eligible for being the spheres which lie between two h-triangles in the tree.

We finally observe that after a finite number of iterations the process stabilizes, the final configuration consisting of a single square S, $\{\underline{T}\}_S = \underline{T}$, its mass therefore being the sum of all $|T|$ over $\underline{T}$; the basis of S is $T(\Gamma)$, the triangle representative of the contour $\Gamma = \mathcal{R}(\underline{T})$. Any other final state would in fact contradict the assumption that $\mathcal{R}(\underline{T})$ consists of a single contour.

The tree structure

We have constructed so far, for any contour, a process, called the square process, evolving at integer times, whose state space is a square configuration, each square with its own mass. The evolution consists of a clustering mechanism, for which a cluster of squares at time t becomes a single square at time $t + 1$. We have also distinguished in a forming cluster some squares which are primary, the others being called secondary. Our purpose now is to identify a tree structure from $\underline{T}$ via the realization of the square process. To this end, let t_f be the first time when the final configuration, consisting of a single square, is reached. This is identified to the root of the tree we are going to construct. If $t_f = 0$ the root is a white triangle, otherwise it is black. In the former case, the configuration at time 0 has only one square, S, which, recalling the definition, means that there is a unique maximal triangle, T, in $\underline{T}$, and S has same basis as T. The spheres attached to the white triangle root of the tree, are identified to the contours $\mathcal{R}(\underline{T} \setminus T)$, by Lemma 4.2 such an identification respects the requests of Subsection 4.1. Notice that the identification of the spheres attached to a white triangle requires the knowledge of $\underline{T}$ and cannot be read only from the square process, which, as we will see, only identifies white and black triangles and spheres between h-triangles, all with their masses, but it does not give any information on the structure of the spheres inside the white triangles, except for their masses.

If $t_f > 0$, the root is a black triangle and its offspring is the configuration at time $t_f - 1$, identifying primary squares with h-triangles and secondary squares with spheres, consistently with the properties of such objects, by Lemma 4.3. Let S be one of the primary squares and $t_S < t_f - 1$ the time in the square process when there is a cluster of more than one square which at time $t_S + 1$ becomes S; if such a time does not exist, then S was present also at time 0, and it is identified to a white triangle with same procedure as above. Otherwise S is identified to a black triangle, whose offspring is determined by the configuration of squares at time t_S which merge into S at time $t_S + 1$, with same rules as those described for the branching of the root. By iterating the procedure we complete the identification of the tree.

4.3. Proof of Theorem 4.1. Since the number of translates of a contour with the property tha $0 \in \Gamma$, is bounded by $|\Gamma|$, the proof of Theorem 4.1 reduces to proving that for any $b > 0$ large enough and any $m > 0$,

$$G_m := \sum_{\Gamma:|\Gamma|=m, x_-(\Gamma)=0} w_b(\Gamma) \le 2e^{-bm^\alpha} \tag{4.4}$$

The proof is by induction on the mass m of the contour, recall that the mass of a contour is necessarily an integer. We thus suppose (4.4) proved whenever $m \le M - 1$ and want to prove it for M. We have

$$G_M = G'_M + G''_M \tag{4.5}$$

$$G'_M = \sum_{\substack{|\Gamma|=M, x_-(\Gamma)=0 \\ \text{root of } \Gamma \text{ is white}}} w_b(\Gamma), \qquad G''_M = \sum_{\substack{|\Gamma|=M, x_-(\Gamma)=0 \\ \text{root of } \Gamma \text{ is black}}} w_b(\Gamma) \tag{4.6}$$

We start by bounding G'_M. We call $\ell = |T(\Gamma)|$, n the number of contours (spheres) attached to the white triangle; $m_1, .., m_n$ their masses. These variables are not independent, as, for instance, we must have $m_1 + \cdots + m_n + \ell = M$. We organize the sum in (4.6) by fixing ℓ, n, $m_1, .., m_n$, then summing over all the contours compatible with such specifications and with (4.3) and finally summing over the specifications ℓ, n, $m_1, .., m_n$.

Let $\Gamma_1, .., \Gamma_n$ be n contours whose masses are $m_1, .., m_n$ and all with $x_-(\Gamma_i) = 0$. We call $X_n(\ell, \Gamma_1, .., \Gamma_n)$ the set of all $(x_1, .., x_n)$ such that the collection $\{S_{x_i}(\Gamma_i)\}$, S_x denoting translation by x, fulfills (4.3) with $a = 0$, $b = \ell$ and $k_i = n$. We then have

$$G'_M \le \sum_{\ell>0} e^{-b\ell^\alpha} \sum_{n\ge0} \sum_{\substack{m_1,\dots,m_n \\ m_1+m_2\cdots+m_n+\ell=M}} \sum_{\substack{\Gamma_1..\Gamma_n \\ |\Gamma_i|=m_i,\, x_-(\Gamma_i)=0}} |X_n(\ell, \Gamma_1, .., \Gamma_n)| \prod_{i=1}^n w_b(\Gamma_i)$$

Writing $[a \wedge b] := \min\{a, b\}$, we have

$$|X_n(\ell, \Gamma_1, .., \Gamma_n)| \le (n+1) \prod_{i=1}^n [cm_i^3 \wedge \ell]$$

where $n+1$ counts the number of values that p_i can take when $k_i = n$ in (4.3). Using the induction assumption we then get

$$G'_M \leq \sum_{\ell>0} \sum_{n\geq 0} \sum_{\substack{m_1,\dots,m_n \\ m_1+m_2\cdots+m_n+\ell=M}} 2^n e^{-b(\ell^\alpha+m_1^\alpha+\cdots+m_n^\alpha)}(n+1)\prod_{i=1}^n [cm_i^3 \wedge \ell]$$

To select the maximal among all masses, we rewrite the above as

$$G'_M \leq \sum_{\ell>0} \sum_{n\geq 0} 2^n(n+1) \sum_{\substack{m_1,\dots,m_n \\ m_1+m_2\cdots+m_n+\ell=M}} e^{-b(\ell^\alpha+m_1^\alpha+\cdots+m_n^\alpha)} \prod_{i=1}^n [cm_i^3 \wedge \ell] \cdot \Big[\mathbf{1}_{\{\ell\geq m_i\ \forall i\}} + \mathbf{1}_{\left\{\substack{m_1\geq m_i\ \forall i\neq 1 \\ m_1\geq \ell}\right\}} + \cdots + \mathbf{1}_{\left\{\substack{m_n\geq m_i\ \forall i\neq n \\ m_n\geq \ell}\right\}}\Big] \tag{4.7}$$

In the term where the maximum is m_i, we bound $[cm_i^3 \wedge \ell] = \ell \leq c\ell^3$ and get

$$\begin{aligned} G'_M &\leq \sum_{n\geq 0} 2^n(n+1)^2 \sum_{\substack{x_1+\cdots+x_n+y=M \\ 0\leq x_i\leq y}} e^{-by^\alpha} \prod_{i=1}^n [e^{-bx_i^\alpha}(cx_i^3)] \\ &\leq \sum_{n\geq 0} (n+1)^2(2c)^n \sum_{\substack{x_1+\cdots+x_n+y=M \\ 0\leq x_i\leq y}} e^{-by^\alpha+\sum_i^n(b-a)x_i^\alpha} \prod_{i=1}^n [e^{-ax_i^\alpha} x_i^3] \end{aligned} \tag{4.8}$$

By Lemma E.1, if b/a is large enough,

$$\exp\{-by^\alpha + \sum_i^n (b-a)x_i^\alpha\} \leq \exp\{-b[y+x_1+\cdots+x_n]^\alpha\} \tag{4.9}$$

so that

$$G'_M \leq e^{-bM^\alpha} \sum_{n\geq 0} (n+1)^2(2c)^n \Big(\sum_x e^{-ax^\alpha} x^3\Big)^n$$

Calling $\delta(a)$ the sum in the last bracket and noticing that $\delta(a) \to 0$ as $a \to \infty$, for a large enough

$$G'_M \leq e^{-bM^\alpha}\Big(1 + \sum_{n\geq 1} (n+1)^2(2c)^n[\delta(a)]^n\Big) \leq \frac{3}{2}\, e^{-bM^\alpha}$$

Bound on G''_M. We now call $n \geq 2$ the number of [black and white] triangles generated by the root, m_i their masses. We fix all the contours with such specifications and sum over the spheres between two consecutive triangles. Denote by $k_i \geq 0$ the number of spheres between the i-th and $(i+1)$-th triangles; m^i_j their masses. The space interval where such spheres can be located is determined by the position of the triangles i and $i+1$, by (4.2) its length is bounded by $cm^3_{i,i+1}$, where $m_{i,i+1} := [m_i \wedge m_{i+1}]$. Then the sum over the spheres, once their number and masses are fixed, is bounded as in (4.7). We can also sum over all possible realizations of the n black and white triangles, given their number and masses using (4.2) and the induction assumption. We then get (with an extra 2^n factor counting the number of ways to color, either black or white, the n triangles)

$$\begin{aligned}
G''_M \quad \le \quad & \sum_{n\ge 2} 2^n \sum_{\substack{m_1,\dots,m_n\\ m_i>0}} \left\{\prod_{i=1}^{n-1} c m_{i,i+1}^3\right\}\left\{\prod_{i=1}^{n}[2e^{-bm_i^\alpha}]\right\} \\
& \times \sum_{k_1\ge 0}\sum_{\substack{m_1^1\dots m_{k_1}^1\\ m_j^1>0}} \left\{(k_1+1)\prod_{j=1}^{k_1} c[(m_j^1)^3\wedge m_{1,2}^3]\right\}\left\{\prod_{j=1}^{k_1}[2e^{-b(m_j^1)^\alpha}]\right\}\cdots \\
& \cdots \sum_{k_{n-1}\ge 0}\sum_{\substack{m_1^{n-1}\dots m_{k_1}^{n-1}\\ m_j^{n-1}>0}} \left\{(k_{n-1}+1)\prod_{j=1}^{k_{n-1}} c[(m_j^{n-1})^3\wedge m_{n-1,n}^3]\right\}\left\{\prod_{j=1}^{k_{n-1}}[2e^{-b(m_j^{k_{n-1}})^\alpha}]\right\} \\
& \times \mathbf{1}_{\{\sum_i m_i+\sum_{k,l} m_k^l = M\}}
\end{aligned}$$

We fix $n, k_1, \dots, k_{n-1}$ and sum over all masses. As in (4.8) we split the sum by fixing which one of the masses is larger. This will give a factor $(n + k_1 + \cdots + k_{n-1})$ equal to the number of masses which are present. Except for the largest mass we write the generic factor $e^{-bm^\alpha} = e^{-(b-a)m^\alpha}e^{-am^\alpha}$. In order to apply Lemma E.1 and get the analogue of (4.9), we must check that there is not a term with the maximal mass to the cube. If the maximal is one of the masses m_i, then it does not appear because we have products of $m_{i,i+1}^3$ which automatically select the smaller and avoid the larger. If the maximal mass is one of those relative to spheres, say m_j^i, we use the same trick as for G'_M and bound $[(m_j^i)^3 \wedge m_{i,i+1}^3] \le m_{i,i+1}^3$, so that the term $(m_j^i)^3$ does not appear. Notice that in this way there could be factors $m_{i,i+1}^6$. We then get

$$\begin{aligned}
\le \quad & e^{-bM^\alpha}\sum_{n\ge 2} 2^n \sum_{k_1\ge 0,\dots,k_{n-1}\ge 0} (n+k_1+\cdots+k_{n-1})(k_1+1)\cdots(k_{n-1}+1) \\
& \times \left(\sum_{x\ge 1} e^{-ax^\alpha}(2c)x^6\right)^{(n-1+k_1+\cdots+k_{n-1})}
\end{aligned}$$

calling $\delta = \sum_{x\ge 1} e^{-ax^\alpha}(2c)x^6$:

$$\le \quad e^{-bM^\alpha}\sum_{n\ge 2} 2^n\delta^{n-1}\sum_{k_1\ge 0}\delta^{k_1}(k_1+1)\cdots\sum_{k_{n-1}\ge 0}\delta^{k_{n-1}}(k_{n-1}+1)(n+k_1+\cdots+k_{n-1})$$

Since $(n+k_1+\cdots+k_{n-1}) = [(k_1+1)+\cdots+(k_{n-1}+1)+1] \le 2[(k_1+1)+\cdots+(k_{n-1}+1)]$,

$$\begin{aligned}
\le \quad & e^{-bM^\alpha}\sum_{n\ge 2} 2^n\delta^{n-1}2n\Big(\sum_{k\ge 0}\delta^k(k+1)^2\Big)^n \\
\le \quad & e^{-bM^\alpha}\sum_{n\ge 2}\delta^{n-1}2^{2n+1}n \le \frac{e^{-bM^\alpha}}{2}
\end{aligned}$$

because for a large enough $\sum_{k\ge 0} \delta^k (k+1)^2 \le 2$ and $\sum_{n\ge 2} \delta^{n-1} 2^{2n+1} n \le 1/2$. We have thus proved that

$$G_M = G'_M + G''_M \le (\frac{3}{2} + \frac{1}{2})\, e^{-bM^\alpha}$$

hence (4.4) and Theorem 4.1 are proved.

Acknowledgments

We are indebted to Vladas Sidoravicius, Herbert Spohn and Milos Zahradnik for many helpful comments.

PAF and MC acknowledge kind hospitality resp. at the Universities of Roma Tor Vergata and São Paulo.

PAF acknowledges financial support from FAPESP, CNPq and PRONEX; MC, IM and EP from MURST; IM and EP from NATO Grant PST.CLG.976552 and GNFM.

Appendix A.

In this appendix we will prove (2.8) identifying the value of the parameter ζ. We set

$$\zeta_\alpha := 1 - 2(2^\alpha - 1) > 0, \qquad 0 < \alpha < \alpha_+ := \frac{\ln 3}{\ln 2} - 1 \tag{A.1}$$

observing that $\alpha_+ > 1/2$. We call $W_\alpha(L)$ the r.h.s. of (2.7), the subscript underlining the dependence on α.

Lemma A.1. *Given $\alpha \in [0, \alpha_+)$, for $J(1)$ large enough*

$$W_\alpha(L) \ge \begin{cases} \zeta_\alpha L^\alpha & \textit{if } \alpha > 0 \\ 2\ln L + 8 & \textit{if } \alpha = 0 \end{cases} \tag{A.2}$$

PROOF. We first consider the case $\alpha > 0$. Using (1.2), (2.7) reads

$$
\begin{aligned}
W_\alpha(L) &= \sum_{x=1}^{L}\left(\sum_{\substack{y\in[L+1,2L]\\ y\in[-L+1,0]}}\frac{1}{|x-y|^{2-\alpha}} - \sum_{\substack{y\in[2L+1,\infty]\\ y\in[-\infty,-L]}}\frac{1}{|x-y|^{2-\alpha}}\right) + 2(J(1)-1)\\
&= 2\sum_{x=1}^{L}\left(\sum_{y=L+1}^{2L}\frac{1}{|x-y|^{2-\alpha}} - \sum_{y=2L+1}^{\infty}\frac{1}{|x-y|^{2-\alpha}}\right) + 2(J(1)-1)\\
&= 2\sum_{x=1}^{L}\left(\sum_{y=L+1-x}^{2L-x}\frac{1}{y^{2-\alpha}} - \sum_{y=2L+1-x}^{\infty}\frac{1}{y^{2-\alpha}}\right) + 2(J(1)-1)
\end{aligned}
$$

and using monotonicity to replace sums by integrals:

$$
\begin{aligned}
&\geq 2\sum_{x=1}^{L}\left[\int_{L+1-x}^{2L+1-x}\frac{dz}{z^{2-\alpha}} - \int_{2L-x}^{\infty}\frac{dz}{z^{2-\alpha}}\right] + 2(J(1)-1)\\
&\geq 2\sum_{x=1}^{L}\frac{1}{\alpha-1}\left\{\left[(2L-x)^{\alpha-1} - (L+1-x)^{\alpha-1}\right] + (2L-x)^{\alpha-1}\right\} + 2(J(1)-1)\\
&\geq 2\sum_{x=1}^{L}\frac{1}{1-\alpha}\left\{\left[-2(2L-x)^{\alpha-1} + (L+1-x)^{\alpha-1}\right]\right\} + 2(J(1)-1)\\
&\geq \frac{2}{1-\alpha}\left[-2\sum_{y=0}^{L-1}(L+y)^{\alpha-1} + \sum_{y=0}^{L-1}(y+1)^{\alpha-1}\right] + 2(J(1)-1)\\
&\geq \frac{2}{1-\alpha}\left[-2\int_{y=-1}^{L-1}(L+y)^{\alpha-1} + \int_{y=0}^{L}(y+1)^{\alpha-1}\right] + 2(J(1)-1)\\
&\geq \frac{2}{\alpha(1-\alpha)}\left[-2[(2L-1)^{\alpha} - (L-1)^{\alpha}] + (L+1)^{\alpha} - 1\right] + 2(J(1)-1) \geq \zeta_\alpha L^\alpha
\end{aligned}
$$

The last inequality holds for L large enough if $\alpha \in (0, \alpha_+)$, and for all L if $J(1)$ is large enough.

In the case $\alpha = 0$ we can repeat the same computations obtaining:

$$W_0(L) \geq 2\ln(L+2) + [2J(1) - 4\ln(3) - 2] \geq 2\ln L + 8$$

for $J(1)$ large enough.The lemma is proved. □

Appendix B.

We start by a preliminary lemma.

Lemma B.1. *Let $\mathcal{R}(\underline{T})$ satisfy P.0, P.1 and P.2, $\Gamma \in \mathcal{R}(\underline{T})$ and $\underline{T}'$ the configuration of triangles in Γ. Then $\mathcal{R}(\underline{T}') = \{\Gamma\}$.*

Proof. Writing $\mathcal{R}(\underline{T}) = \{\Gamma_i, i = 1, .., n\}$, denote by $\underline{T}^{(i)}$ the triangles in Γ_i and write

$$\mathcal{R}(\underline{T}^{(i)}) = \left\{\Gamma_1^{(i)}, \dots \Gamma_{n_i}^{(i)}\right\}$$

Each pair (Γ_i, Γ_j), $i \neq j$, verifies P.1, we want to show that P.1 is also verified by each distinct pair $\Gamma_j^{(i)}, \Gamma_{j'}^{(i')}$. This is by definition if $i = i'$, let us then suppose $i \neq i'$. If $T(\Gamma_i) \sqcap T(\Gamma_{i'}) = \emptyset$, then the same holds for $T(\Gamma_j^{(i)})$ and $T(\Gamma_{j'}^{(i')})$ and (3.1) holds. If instead $T(\Gamma_i) \sqsubset T(\Gamma_{i'})$ (or viceversa)

If instead $T(\Gamma_i) \sqsubset T(\Gamma_{i'})$ (or viceversa) then $\mathrm{dist}(\Gamma_j^{(i)}, \Gamma_{j'}^{(i')}) \geq \mathrm{dist}(\Gamma_j^{(i)}, \Gamma_{i'}) \geq \mathrm{dist}(\Gamma_i, \Gamma_{i'}) \geq c|\Gamma_i|^3 \geq c|\Gamma_j^{(i)}|^3 \geq c \min\{|\Gamma_j^{(i)}|^3, |\Gamma_{j'}^{(i')}|^3\}$ so that $\Gamma_j^{(i)}, \Gamma_{j'}^{(i')}$ verifies P.1.

By applying P.2, $\mathcal{R}(\underline{T}) = \{\Gamma_j^{(i)}\}$ which must therefore coincide with $\{\Gamma_i\}$. Hence the decomposition of each Γ_i into $\Gamma_j^{(i)}$ is trivial, i.e. $n_i = 1$ and $\Gamma_1^{(i)} = \Gamma_i$. The lemma is proved. $\square$

Proof of Theorem 3.1. Uniqueness. Suppose there are two algorithms, $\mathcal{R}^{(i)}, i = 1, 2$ which both satisfy P.1 and P.2 and let $\mathcal{R}^{(i)}(\underline{T}) = \{\Gamma_j^{(i)}, j = 1, ..., n_i\}$. Let

$$A_h^{(1)} = \Gamma_1^{(1)} \sqcap \Gamma_h^{(2)} \tag{B.3}$$

be the collection of those triangles which are both in $\Gamma_1^{(1)}$ and $\Gamma_h^{(2)}$. Of course the union of $A_h^{(1)}$ over h is equal to $\Gamma_1^{(1)}$.

Call $\{K_{h,j}^{(1)}, j = 1, .., m_h\} = \mathcal{R}^{(1)}(A_h^{(1)})$. Each distinct pair $K_{h,j}^{(1)}, K_{h',j'}^{(1)}$ verifies P.1, by an argument similar to that used in the proof of Lemma B.1 and which is omitted. Then, by P.2, $\{K_{h,j}^{(1)}, h = 1, .., n_2, j = 1, .., m_h\} = \mathcal{R}^{(1)}(\underline{T}^{(1)})$, $\underline{T}^{(1)}$ the collection of all triangles in $\Gamma_1^{(1)}$. By Lemma B.1 the decomposition is then trivial, which means that $\Gamma_1^{(1)} = \Gamma_i^{(2)}$ for some i. By iteration we then conclude that the two systems of contours $\{\Gamma_j^{(1)}\}$ and $\{\Gamma_i^{(2)}\}$ are identical. Thus $\mathcal{R}^{(1)} = \mathcal{R}^{(2)}$.

Existence. Given a configuration $\underline{T}$ of triangles, we call $\mathcal{C}$ the collection of all partitions $\underline{C} = (C_1, .., C_n)$ of $\underline{T}$ so that each pair (C_i, C_j), $i \neq j$, verifies P.1 and P.0 (relative to $\underline{T}$. $\mathcal{C}$ is non empty as the trivial partition in a single atom verifies P.0 and P.1 (as in that case there is nothing to check). We order $\mathcal{C}$ by setting $\underline{C} \geq \underline{C}'$ if the partition $\underline{C}$ is finer than $\underline{C}'$. We claim that $\mathcal{C}$ has a unique maximal element, which will be called $\mathcal{M}(\underline{T})$; we will then prove that $\mathcal{M}(\cdot)$ satisfies P.1 and P.2 and conclude the proof of existence.

The claim will follow from showing that the partition

$$\underline{C} \vee \underline{C}' = \{C_i \sqcap C_j'\}, \qquad \underline{C} = (C_1, .., C_n),\ \underline{C}' = (C_1', .., C_m')$$

is in $\mathcal{C}$, if also $\underline{C}$ and $\underline{C}'$ are in $\mathcal{C}$.

Without loss of generality we must thus prove that any distinct pair $(C_i \sqcap C_j', C_{i'} \sqcap C_{j'}')$ verifies the alternatives in P.1. By symmetry between the two clusters, we may suppose $i \neq i'$. If $T(C_i) \sqcap T(C_{i'}) = \emptyset$, then also $T(C_i \sqcap C_j') \sqcap T(C_{i'} \sqcap C_{j'}') = \emptyset$ and (3.1) holds. Let us suppose (again without loss of generality) that $T(C_i) \sqsubset T(C_{i'})$, then for any $T_k \in C_{i'}$, either $T(C_i) \sqsubset T_k$ or $T(C_i) \sqcap T_k = \emptyset$. If $T_k \in C'_{j'}$, in correspondence with the previous alternative, either $T(C_i \sqcap C'_j) \sqsubset T_k$ or $T(C_i \sqcap C'_j) \sqcap T_k = \emptyset$. If instead $T_k \notin C'_{j'}$, then $T_k \notin C_{i'} \sqcap C'_{j'}$ and there is nothing to check. In conclusion the pair $(C_i \sqcap C'_j, C_{i'} \sqcap C'_{j'})$ verifies the alternatives (i)–(ii) and in case the latter is verified, (3.2) holds.

To complete the proof we must show that $\mathcal{M}(\underline{T})$ satisfies $P.2$. Let $\underline{T}$ and $\underline{T}^{(i)} : \sqcup \underline{T}^{(i)} = \underline{T}$ be as in P.2 and suppose that the elements of $\{\mathcal{M}(\underline{T}^{(i)}), i = 1,..,k\}$ satisfy (3.1)–(3.2). Suppose by contradiction that $\mathcal{M}(\underline{T})$ is not equal to $\{\mathcal{M}(\underline{T}^{(i)}), i = 1..,k\}$, since the latter is in $\mathcal{C}$ (relative to $\underline{T}$), $\mathcal{M}(\underline{T})$ must then be finer than $\{\mathcal{M}(\underline{T}^{(i)}), i = 1..,k\}$. But then we would have a finer partition of $\underline{T}^{(i)}$, for some i, which still verifies P.1. We have thus reached a contradiction.

The theorem is proved. $\square$

As a consequence of P.1 and P.2 we have the following obvious property, namely that by adding triangles it cannot happen that contours split; the new triangles can either form separate contours or join other pre-existent ones and possibly cause them to merge.

Lemma B.2. *Monotonicity*
Let $\underline{T}$, $\underline{T}'$ be two configurations of triangles, $\underline{T} \sqsubset \underline{T}'$, then for any $\Gamma \in \mathcal{R}(\underline{T})$, there is $\Gamma' \in \mathcal{R}(\underline{T}')$ so that $\Gamma \sqsubset \Gamma'$.

Proof. Let $\Gamma_0 \in \mathcal{R}(\underline{T})$, $\mathcal{R}(\underline{T}') = \{\Gamma'_j\}$ and (recalling the notation in (B.3)) $A_j := \Gamma_0 \sqcap \Gamma'_j$. We must prove that for any j, either $A_j = \emptyset$ or $A_j = \Gamma_0$. Suppose by contradiction that this is not the case. We then consider the new partition of $\underline{T}$: $\underline{C} := \big[\mathcal{R}(\underline{T}) \vee \big\{(\mathcal{R}(\underline{T}) \setminus \Gamma_0), A_1, \dots, A_m\big\}\big]$. $\underline{C}$ is then in $\mathcal{C}$ (relative to $\underline{T}$) and is finer than $\mathcal{R}(\underline{T})$, which contradicts the fact that $\mathcal{R}(\underline{T})$ is the unique, finest partition of $\underline{T}$ verifying P.0, P.1 and P.2. The lemma is proved. $\square$

Appendix C.

An interval $[a, b]$ is compatible with $\underline{T}$ if a is an endpoint of a triangle of $\underline{T}$, b is also an endpoint of a triangle of $\underline{T}$ and for all $T \in \underline{T}$, $T \sqcap (a, b)$ is either void or equal to (a, b).

Lemma C.1. *Let $\underline{T}'' \sqsubset \underline{T}$, with $\mathcal{R}(\underline{T}'')$ a singleton; $[a,b]$ a $\underline{T}''$-compatible interval; $\underline{T}'$ the collection of all triangles of $\underline{T}$ with basis in (a,b). Then if $\mathcal{R}(\underline{T}', \underline{T}'')$ is not a singleton, also $\mathcal{R}(\underline{T})$ is not a singleton.*

Proof. Since $\mathcal{R}(\underline{T}'')$ is a singleton, by Lemma B.2 there is a contour Γ_0 in $\mathcal{R}(\underline{T}', \underline{T}'')$ which contains $\underline{T}''$. In order to prove the lemma we must consider the case

$$\mathcal{R}(\underline{T}', \underline{T}'') = \{\Gamma_0, \dots \Gamma_n\}, \qquad n \geq 1$$

Since Γ_i, $i \geq 1$, is distinct from Γ_0, it is a subset of $\underline{T}'$ and therefore $T(\Gamma_i)$ is strictly contained in (a,b). Let

$$\mathcal{R}(\{\underline{T} \setminus \underline{T}'\} \sqcup \Gamma_0) = \{\Gamma'_1 \dots \Gamma'_k\}$$

We claim that

$$\mathcal{R}(\underline{T}) = \{\Gamma_1, \dots, \Gamma_n, \Gamma'_1 \dots \Gamma'_k\} \tag{C.1}$$

hence $\mathcal{R}(\underline{T})$ is not a singleton and the lemma is proved.

To prove (C.1), it is enough to show that $\{\Gamma_1, \dots, \Gamma_n, \Gamma'_1 \dots \Gamma'_k\}$ satisfy properties P.0 and P.1 in the definition of contours, see Section 3, because (C.1) would then follow from P.2.

P.0 is obviously satisfied. Pairs Γ_i, Γ_j and Γ'_i, Γ'_j satisfy P.1 by definition, it thus remain to check P.1 for pairs Γ_i, Γ'_j. By Lemma B.2, Γ_0 is contained in one of the contours Γ'_j, say Γ'_1, and let us start from this case. Then $\mathrm{dist}(\Gamma_i, \Gamma'_1) = \mathrm{dist}(\Gamma_i, \Gamma_0)$, (because the triangles in $\Gamma'_1 \setminus \Gamma_0$ are outside $[a,b]$, while Γ_0 contains triangle(s) whose endpoints are a, b and Γ_i has support inside (a,b)). Since the pair Γ_i, Γ_0 satisfies P.1 then also Γ_i, Γ'_1 satisfies P.1. For the same reason as before, also for $j > 1$, $\mathrm{dist}(\Gamma_i, \Gamma'_j) \geq \mathrm{dist}(T(\Gamma_i), \{a,b\}) = \mathrm{dist}(\Gamma_i, \Gamma_0) > c|\Gamma_i|^3$, (the latter inequality by (ii) of P.1). Hence $\mathrm{dist}(\Gamma_i, \Gamma'_j) > c \min\{|\Gamma_i|^3, |\Gamma'_j|^3\}$. We have thus completed the proof of P.1: (C.1) and Lemma C.1 are thus proved. □

Proof of (ii) of Lemma 4.2. We apply Lemma C.1 with $\underline{T}'' = T$ (the maximal triangle), a, b the endpoints of T. Since $\mathcal{R}(\underline{T}'') = T$, $\mathcal{R}(\underline{T}'')$ is a singleton; $\underline{T}' = \{\underline{T}\}_S \setminus T$. By assumption $\mathcal{R}(\underline{T})$ is a singleton, then, by Lemma C.1, also $\mathcal{R}(\underline{T}', \underline{T}'')$ is a singleton, hence (ii) of Lemma 4.2 because $\mathcal{R}(\underline{T}', \underline{T}'') = \mathcal{R}(\{\underline{T}\}_S)$. □

Proof of (ii) of Lemma 4.3. By definition of squares, $\mathcal{R}(\{\underline{T}\}_{S_i})$, $i = 1, 2$, are singletons; then $\mathcal{R}(\{\underline{T}\}_{S_1}, \{\underline{T}\}_{S_2})$ is a singleton as well, because S_1 and S_2 are a-connected. We then apply Lemma C.1, identifying $T'' = \{\{\underline{T}\}_{S_1}, \{\underline{T}\}_{S_2}\}$ and a, b as the endpoints of the squares S_1, S_2 which face each other. The argument is hereafter the same as in the proof of (ii) of Lemma 4.2. □

Lemma C.2. *Let $\underline{S}$ be the square configuration at time $t = n$. Call $\underline{S}'$ the collection of squares in a maximal a-connected component of $\underline{S}$, $\{\underline{T}\}_{\underline{S}'})$ the set of triangles represented by the squares in $\underline{S}'$. Then $\mathcal{R}(\{\underline{T}\}_{\underline{S}'})$ is a singleton and $\underline{S}'$ will be represented by a square in the square configuration at time $t = n + 1$.*

Proof. Suppose $\underline{S}'$ is not a singleton (otherwise the statement of the lemma would trivially hold). If $S_1 \in \underline{S}'$ there must be $S_2 \in \underline{S}'$ with S_1 and S_2 endpoints of an arrow. Then either S_1 is an endpoint of a primary arrow, or it is in a shadow of a primary arrow. By the assumed maximality of $\underline{S}'$ it then follows that $\underline{S}'$ is made of a sequence of squares each one connected by a primary arrow to the successive one and all other squares contained in the shadow of these primary arrows.

By (ii) of Lemma 4.3, the collection of all the triangles in the shadow of a primary arrow and those in the two squares connected by the primary arrow form a single contour. The statement of the lemma then follows by monotonicity, Lemma B.2.

Proof of (i) of Lemma 4.2. We will first prove that $\Gamma_1, \dots, \Gamma_n$ are sequential, where

$$\{\Gamma_1, \dots, \Gamma_n\} = \mathcal{R}(\{\underline{T}\}_S \setminus T) \tag{C.2}$$

Suppose by contradiction that $T(\Gamma_i) \sqsubset T(\Gamma_j)$, $i \neq j$. By property P.1 in the definition of contours, there is a minimal contour Γ_k distinct from Γ_i such that $T(\Gamma_i) \sqsubset T(\Gamma_k)$. Let $[a, b]$ be the smallest interval containing $T(\Gamma_i)$ and compatible with Γ_k. Let $\underline{T}'$ be the collection of triangles with basis in (a, b). Then $\mathcal{R}(\underline{T}', \Gamma_k)$ contains at least Γ_i and Γ_k (by Lemma B.2), so that, by Lemma C.1, $\mathcal{R}(\underline{T})$ is not a singleton, against the assumption. Therefore $\Gamma_1, \dots, \Gamma_n$ are sequential.

We will next prove the analogue of (4.3), calling a^*, b^* the endpoints of T, shorthanding $\{\Gamma_i\} := \{\Gamma_1, \dots, \Gamma_n\}$ and labelling the contours so that Γ_i is before Γ_j when $i < j$ (we have already proved that $\{\Gamma_i\}$ is sequential). There is Γ_j such that $\mathrm{dist}(\Gamma_j, \{a^*, b^*\}) \equiv \mathrm{dist}(T(\Gamma_j), \{a^*, b^*\}) \leq c|\Gamma_j|^3$, otherwise all Γ_i would be contours in $\mathcal{R}(\underline{T})$, by the argument already used several times above. Supposing for the sake of definiteness that $\mathrm{dist}(\Gamma_j, a^*) \leq c|\Gamma_j|^3$, we then claim that

$$\mathrm{dist}(\Gamma_1, a^*) \leq c|\Gamma_1|^3 \tag{C.3}$$

Suppose by contradiction that there is $1 < k \leq j$ so that for any $i < k$, $\mathrm{dist}(\Gamma_i, a^*) > c|\Gamma_i|^3$ while $\mathrm{dist}(\Gamma_k, a^*) \leq c|\Gamma_k|^3$. This would imply that, for any $i < k$, $c|\Gamma_i|^3 < \mathrm{dist}(\Gamma_i, a^*) \leq \mathrm{dist}(\Gamma_k, a^*) \leq c|\Gamma_k|^3$. Thus $|\Gamma_i| \leq |\Gamma_k|$ for any $i \leq k$. Since $\{\Gamma_1, .., \Gamma_n\}$ are distinct contours, $\mathrm{dist}(\Gamma_i, \Gamma_k) > c \min\{|\Gamma_i|^3, |\Gamma_k|^3\} = c|\Gamma_i|^3$, for any $i < k$.

Call $\underline{T}'$ the collection of all triangles in $\{\Gamma_1, .., \Gamma_{k-1}\}$, $[a, b] = [a^*, x_-(\Gamma_k)]$, $\underline{T}'' = \{T, \Gamma_k\}$. We can then apply Lemma C.1 because $\mathcal{R}(\underline{T}'')$ is a singleton, since $\mathrm{dist}(\Gamma_k, a^*) \leq c|\Gamma_k|^3$: then $\{\Gamma_1, .., \Gamma_{k-1}\}$ are contours for $\mathcal{R}(\underline{T})$ which contradicts the assumption that the latter is a singleton. (C.3) is proved.

By (C.3), $\mathcal{R}(\{T\}_{\Gamma_1}, T)$ is a singleton, so that the previous analysis applies again with T replaced by $\{\Gamma_1, T\}$ and a^* replaced by $a_1^* = x_+(\Gamma_1)$, showing that if $\mathrm{dist}(\Gamma_j, a^*) \leq c|\Gamma_j|^3$ with $j > 2$, then $\mathrm{dist}(\Gamma_2, a_1^*) \leq c|\Gamma_2|^3$. By iterating the argument we then conclude the proof of (i) of Lemma 4.2. $\square$

Proof of (i) of Lemma 4.3. Since $\mathcal{R}(\{S_1, S_2\})$ is a singleton (see the proof above of (ii) of Lemma 4.3) the previous applies unchanged with a^* and b^* the endpoints of S_1 and S_2 which face each other. □

Appendix D.

Lemma D.1. *A squares configuration $\underline{S}$ is (new a)-connected iff it is (old a)-connected.*

Proof. If $\underline{S}$ is (new a)-connected, then it is also (old a)-connected, as the new arrows are also old arrows. We thus only need to prove the reverse implication: we suppose $\underline{S}$ (old a)-connected but not (new a)-connected and want to show that this leads to a contradiction.

We call "odd" a pair S and S' of squares when there is an old arrow between S and S' (denoted by $(S, S')_{\text{old}}$) while S and S' are not (new a)-connected. We will first show that if $\underline{S}$ is (old a)-connected but not (new a)-connected then there exist odd pairs; we will then prove that odd pairs "can be shortened" in the sense that if S and S' is an odd pair, then there is another square S'' in between S and S' such that either S and S'' or S' and S'' is an odd pair. The endless iteration of the argument leads to a contradiction.

Existence of odd pairs. If $\underline{S}$ is not (new a)-connected, there are two squares S and S' which are not (new a)-connected; since S and S' are (old a)-connected, there is a sequence $\{S_{\ell_i}, i = 1, .., j\}$ such that each pair $S_{\ell_i}, S_{\ell_{i+1}}$ is connected by an old arrow, and $S_{\ell_1} = S$ and $S_{\ell_j} = S'$. Then one of the pairs $S_{\ell_i}, S_{\ell_{i+1}}$ must be odd, otherwise S and S' would be (new a)-connected.

Shortening odd pairs. Writing $S \prec S'$ if the square S is before S' (recall that a square configuration is sequential), we label $\underline{S}$ so that $S_1 \prec S_2 \prec \cdots \prec S_n$. Let S_k, S_m be an odd pair and suppose, without loss of generality, that $S_k \prec S_m$ and that the old arrow which connects them goes from S_k to S_m. The old arrow which connects S_k to S_m is not a new arrow, otherwise S_k and S_m would be (new a)-connected, therefore there exists $S_\ell : S_k \prec S_\ell \prec S_m$; $|S_\ell| \geq |S_k|$ such that there is a new arrow from S_k to S_ℓ. Consider separately the two possible cases 1) $|S_\ell| \leq |S_m|$ and 2) $|S_\ell| > |S_m|$.

1). Since $c|S_\ell|^3 \geq c|S_k|^3 \geq \text{dist}(S_k, S_m) > \text{dist}(S_m, S_\ell)$, there is an old arrow connecting S_ℓ and S_m; on the other hand, by definition, S_ℓ and S_k are (new a)-connected, hence S_ℓ and S_m cannot be (new a)-connected, hence S_ℓ, S_m is an odd pair.

2). As in 1), $c|S_m|^3 > \text{dist}(S_m, S_\ell)$, which implies that there is an old arrow from S_m to S_ℓ, as well as an old arrow from S_k to S_ℓ. There are two subcases, *a*) S_m and S_ℓ are also connected by a new arrow or else *b*) they are not. In subcase *a*), S_ℓ is (new a)-connected to S_m, hence it cannot be (new a)-connected to S_k, thus S_k, S_ℓ is an odd pair. In subcase *b*), there is $S_h : S_\ell \prec S_h \prec S_m$; $|S_h| \geq |S_m|$, such that there is a new arrow from S_m to S_h.

Again, as S_h is (new a)-connected to S_m, it is not (new a)-connected to S_k. On the other hand $|S_h| \geq |S_m| \geq |S_k|$, hence there is an old arrow from S_k to S_h, thus S_k, S_h is an odd pair.

This concludes the analysis of case 2), and the proof of the shortening property of odd pairs. Thus the lemma is proved. $\square$

Proposition D.2. *The shadows of two new arrows have either empty intersection or else, one is contained in the other.*

PROOF. Suppose by contradiction that there are four squares $S_a \prec S_u \prec S_b \prec S_z$, with the crossing arrows $\vec{\xi}_{ab}$ (denoting a new arrow from a to b) and $\vec{\xi}_{uz}$. By definition of new arrow, this implies that there is no arrow $\vec{\xi}_{au}$ (there could be however an arrow in the opposite direction $\vec{\xi}_{ua}$) and that $\text{dist}(S_a, S_b) \leq c|S_a|^3$. Recalling that $S_u \prec S_b$ the only compatible sizes with the new arrow $\vec{\xi}_{a,b}$ are : $|S_u| < |S_a| < |S_b|$. On the other hand, $\text{dist}(S_u, S_b) < \text{dist}(S_u, S_z) \leq c|S_u|^3$ then, being $|S_b| > |S_u|$ there should be an arrow $\vec{\xi}_{u,b} : S_u \to S_b$ contradicting the fact $\vec{\xi}_{uz}$ is a new arrow (i.e that S_z is the first square connected with S_u).

Consider now the case in which the crossing arrows are $\vec{\xi}_{b,a}$ and $\vec{\xi}_{u,z}$ (that implies that $|S_b| > |S_a|$ and $|S_z| > |S_u|$). The existence of these arrows implies that there are no the arrows $\vec{\xi}_{b,u}$ and $\vec{\xi}_{u,b}$, and , since $\text{dist}(S_b, S_u) < \text{dist}(S_a, S_b) \leq c|S_b|^3$, this implies that $|S_u| \leq c|S_b|$. We get a contradiction by observing that, since $\text{dist}(S_u, S_b) \leq \text{dist}(S_u, S_z) \leq |S_u|^3$, there should be an arrow $\vec{\xi}_{u,b}$ that is incompatible with $\vec{\xi}_{u,z}$.

The other possible crossing cases are reduced to those above by reflection and the proposition is proved.

$\square$

Appendix E.

Lemma E.1. *Let $\alpha \in [0, 1/2]$, a and b positive and b/a large enough. Then for any $n \geq 2$, any $x_1, .., x_{n-1}, y$ such that $1 \leq x_i \leq y$,*

$$b\ h_\alpha(y) + (b-a) \sum_{i=1}^{n-1} h_\alpha(x) \ \geq\ b\ h_\alpha\big(\sum_{i=1}^{n-1} x_i + y\big) \tag{E.1}$$

where $h_\alpha(L)$ is defined in (2.8).

Proof. We will prove (E.1) by induction on $n \geq 2$ showing that

$$f_n(x_1, .., x_{n-1}, y) := \frac{b}{b-a} h_\alpha(y) + \sum_{i=1}^{n-1} h_\alpha(x) \ - \frac{b}{b-a} h_\alpha\big(y + \sum_{i=1}^{n-1} x_i\big)$$

is non negative in the set $1 \leq x_i \leq y$.

We start the induction by supposing that for $n > 2$, for any $2 \le m \le n$, $f_m \ge 0$ and want to prove that $f_{n+1}(x_1, .., x_n, y) \ge 0$. Since f_{n+1} is symmetric in the first n variables, we may suppose, without loss of generality, that $x_i \le x_n \le y \le x_1 + \cdots + x_{n-1} + y =: L$. Then

$$\begin{aligned} f_{n+1}(x_1, .., x_n, y) &= f_n(x_1, .., x_{n-1}, y) + h_\alpha(x_n) + \frac{b}{b-a} h_\alpha(L) - \frac{b}{b-a} h_\alpha(L + x_n) \\ &= f_n(x_1, .., x_{n-1}, y) + f_2(x_n, L) \ge 0 \end{aligned}$$

To complete the induction we need to prove that $f_2(x, y) \ge 0$.

The case $\alpha > 0$. We have

$$f_2(x, y) = y^\alpha g(x/y), \quad g(x) := x^\alpha + \frac{b}{b-a} - \frac{b}{b-a}(x+1)^\alpha, \;\; 0 \le x \le 1$$

If b/a is large enough, $g'(x) > 0$ and $g(x) \ge g(0) = 0$ and the induction is proved. Thus (E.1) is proved in the case $\alpha > 0$.

The case $\alpha = 0$:
Let

$$p := \frac{b}{b-a}, \quad \text{choose } a \text{ so that } 1 < p < 2$$

We have

$$\begin{aligned} f_2(x, y) &= p(\ln y + 4) + (\ln x + 4) - p(\ln[x + y] - 4) \\ &= -p \ln\big(1 + x/y\big) + \ln x + 4 \\ &\ge -2p + 4 + \ln x \ge 0 \end{aligned}$$

because $x \ge 1$ and $p < 2$.

$\square$

Appendix F.

In this appendix we sketch the proof of the analogue of (4.1) in the case $\alpha = 0$, namely that

$$\sum_{\Gamma : |\Gamma| = m, 0 \in \Gamma} w_b^0(\Gamma) \le 2m e^{-b(\ln m + 4)} \tag{F.1}$$

where

$$w_b^0(\Gamma) := \prod_{T \in \Gamma} e^{-b(\ln(|T|+4)} = \prod_{T \in \Gamma} \Big(|T|^{-b} e^{-4b}\Big)$$

(F.1) yields the analogue of (3.20), i.e.

$$\mu_\Lambda^+\Big(\{0 \in \Gamma\}\Big) \le 2 \sum_{m \ge 1} m e^{-\beta(\ln(m)+4)} = 2\, e^{-4\beta} \sum_{m \ge 1} m^{1-\beta} \tag{F.2}$$

The sum in (F.1) is bounded using the same iterative procedure as when $\alpha \in (0, 1/2]$, with the fundamental inequality (4.9) replaced by the "convexity" inequality

$$b\ h_0(y) + (b-a) \sum_{i=1}^{n-1} h_0(x) \ \geq\ b\ h_0\big(\sum_{i=1}^{n-1} x_i + y\big)$$

proved in Appendix D for $0 < a < b/2$. The proof of (F.1) then follows closely that of (4.1) for $\alpha > 0$, and it is omitted.

□

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