# Spectral distributions for long range perturbations

Jean-Marc Bouclet *
Centre for Mathematical Analysis and its Applications
University of Sussex,
Falmer, Brighton BN1 9QH
United Kingdom
e-mail : j.m.bouclet@sussex.ac.uk

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**Abstract**

We study distributions which generalize the concept of spectral shift function, for pseudo-differential operators on $\mathbb{R}^d$. These distributions are called spectral distributions. Relations between relative scattering determinants and spectral distributions are established; they lead to the definition of regularized scattering phases. These relations are analogous to the one valid for the usual spectral shift function. We give several asymptotic properties in the high energy and semiclassical limits where both non trapping and trapping cases are considered. In particular, we prove Breit-Wigner formulae for the regularized scattering phases, for semiclassical Schrödinger operators with long range potentials.

**Key words:** Scattering theory. Spectral shift function. Relative determinants. Microlocal analysis.

# 0 Introduction

## 0.1 Statement of the problem

The spectral shift function (SSF in what follows) of the pair of self-adjoint operators $(H_0, H_1)$ is defined, under some conditions on $V = H_1 - H_0$, which we discuss below, as the function $s_1$ (measurable and $L^1_{loc}$ in general) such that

$$\operatorname{Tr}\left(f(H_1) - f(H_0)\right) = -\int_{\mathbb{R}} f'(\lambda) s_1(\lambda) d\lambda, \qquad f \in C_0^\infty(\mathbb{R}). \tag{0.1}$$

When the operators are lower semi bounded, which we will always assume, $s_1$ is uniquely defined by the condition $s_1(\lambda) = 0$ for $\lambda \ll 0$. This function was first studied by Krein [27] from

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the mathematical point of view. It is a fundamental object in scattering theory with several physical meanings (time delay, scattering phase) and various mathematical applications (including geometry, see Guillopé-Zworski [21] and Froese-Hislop [15]); for a survey on these applications see Birman-Yafaev [3], Gesztesy-Makarov-Naboko [18] and Robert [35].

The SSF can be considered as a *scattering phase* for the following two reasons:

$$\mathrm{Det}_1\ S(\lambda) = e^{2i\pi s_1(\lambda)}, \qquad \text{a.e. on } \sigma_{ac}(H_0), \tag{0.2}$$

$$s_1(\lambda) = -\frac{1}{\pi}\lim_{\varepsilon\searrow+0} \arg \mathrm{Det}_1\left(1 + VR_0(\lambda+i\varepsilon)\right), \qquad \text{a.e. on } \mathbb{R}. \tag{0.3}$$

The first relation, which justify historically this terminology, is known as Krein's formula and was proved by Birman-Krein [2]. Here $S(\lambda)$ is the scattering matrix of $(H_0, H_1)$ defined almost everywhere on $\sigma_{ac}(H_0)$, the absolutely continuous spectrum of $H_0$; $R_0(z) = (H_0 - z)^{-1}$ is the free resolvent, and $\mathrm{Det}_1$ stands for the Fredholm determinant defined for *trace class perturbations* of identity (see section 1 for the determinants). All these formulae hold when $V \in \mathbf{S_1}$, the Schatten ideal of order 1 of trace class operators. Actually (0.1) and (0.2) are still valid when $(H_1 + E)^{-N} - (H_0 + E)^{-N} \in \mathbf{S_1}$ for some $E$ and $N$ large enough, since one can use the Birman-Kato invariance principle; this principle allows to replace $H_j$ by $\tilde{H}_j = (H_j + E)^{-N}$ in the sense that $s_1(\lambda) = -\tilde{s}_1\big((\lambda + E)^{-N}\big)$ is the SSF of $(H_0, H_1)$ if $\tilde{s}_1$ is the SSF of $(\tilde{H}_0, \tilde{H}_1)$.

If $H_0 = -\Delta$ and $H_1 = -\Delta + V$ are Schrödinger Hamiltonians on $\mathbb{R}^d$, the SSF can only be defined when $V$ decays fast enough at infinity. A natural condition on the decay of $V$, which ensure the existence of the SSF, is to ask that for all $\alpha$

$$\partial^\alpha V(x) = \mathcal{O}(\langle x\rangle^{-\rho}), \qquad \rho > d,$$

with the usual notation $\langle x\rangle = (1 + |x|^2)^{1/2}$. Weaker conditions on the smoothness of $V$ can be considered (see for example [32]), but the integrability of $V$ at infinity is always required.

The goal of this article is to relax this assumption on the decay of the perturbation, and thus to be able to consider *long range* potentials, *i.e.* potentials with $\rho > 0$. In that case, one can always find an integer $p$ such that $p\rho > d$ and for such a $p$, it is easy to prove that, for some $N$ and $E$

$$(H_1 + E)^{-N} - (H_0 + E)^{-N} \in \mathbf{S_p} \tag{0.4}$$

where $\mathbf{S_p}$ is the Schatten class of order $p$ (see section 1). Thus it is natural to deal with $\mathbf{S_p}$-perturbations or *relatively-$\mathbf{S_p}$-perturbations* (*i.e* such that (0.4) holds). In these cases, the SSF $s_1$ cannot be defined since $\mathbf{S_1} \subsetneq \mathbf{S_p}$ when $p > 1$, and some regularization has to be done.

The regularization that we shall use is based on Taylor's formula. When $H_0$ is a general self-adjoint operator, and $V \in \mathbf{S_p}$ (self-adjoint too), Koplienko [25] has proposed a way of regularizing the formula (0.1) by considering a functional $u_p$ defined as follows:

$$< u_p, f >= \mathrm{Tr}\left(f(H_0 + V) - \sum_{j=0}^{p-1}\frac{1}{j!}\frac{d^j}{d\epsilon^j}f(H_0 + \epsilon V)_{|_{\epsilon=0}}\right). \tag{0.5}$$

Here, the functions $f$ are rational fractions vanishing at infinity and without poles on the spectra of the operators. It is clear that the operator considered on the right hand side of (0.5) is trace class, since the ideal $\mathbf{S_p^{\mathit{p}}} = \mathbf{S_1}$ and, if $f_z(\lambda) = (\lambda - z)^{-1}$, we get

$$< u_p, f_z >= (-1)^p \mathrm{Tr}\left((H_0 + V - z)^{-1}\big(V(H_0 - z)^{-1}\big)^p\right). \tag{0.6}$$

However, as mentioned in [25], one cannot use the invariance principle any longer in (0.5), when $p \geq 2$: this means that, if we only assume that (0.4) holds, we cannot defined a distribution $u_p$

simply by replacing the operators by some power of their resolvents. In particular, Koplienko's work cannot be applied to differential operators (except for the special case $p=2$, see [25, 26], see also Rybkin [36]). This problem has been considered by the author in [4, 7], where the following result is proved:

**Theorem 0.1.** *Let $H_0 = p(x,D)$ and $H_0+V = H_0+v(x,D)$ be self-adjoint differential operators on $\mathbb{R}^d$ with smooth bounded coefficients, together with their derivatives, and uniformly elliptic, (i.e. their principal symbols are lower bounded on $\mathbb{R}^d \times \mathbb{S}^{d-1}$ by some $c>0$). Moreover, we suppose*

$$|\partial^\beta v_\alpha(x)| \leq C\langle x\rangle^{-\rho-\kappa|\alpha|}, \qquad \rho > d/p, \tag{0.7}$$

*with $\kappa \in [0,1]$, and $v(x,D) = \sum_{|\alpha|\leq 2m} v_\alpha(x) D^\alpha$.*

*Then for all $f \in \mathcal{S}(\mathbb{R})$, formula (0.5) makes sense and it defines a temperate distribution $u_p$, called spectral distribution of order $p$.*

Remark that $u_p \equiv 0$ below the spectra of the operators, which is obvious by formula (0.5).

We emphasize that $H_0$ might have variable coefficients and that $V$ might be of the same order as $H_0$. This theorem is actually proved for a wider class of pseudo-differential operators in [4, 7], but we only quote it in this less technical setting, for stating our main results in the next subsection.

Several natural questions arise when we consider the distributions $u_p$ for pseudo-differential operators. When one looks at formula (0.3), it is natural to want to establish a relation between $u_p$ and some scattering determinants. Moreover, for general $\rho > 0$, it is known that the scattering matrix $S(\lambda)$ may not be a compact perturbation of identity, and its spectrum can be the whole unit circle (see Yafaev [41]). Thus, one cannot hope to compute any determinant of $S(\lambda)$, and we may ask what should play the role of a *scattering phase* in this case.

To address these questions, it seems rather natural to consider the *relative scattering determinants* defined for $\mathbf{S_p}$-perturbations of identity (see section 1):

$$D_p(z) = \mathrm{Det}_p(1 + VR_0(z)). \tag{0.8}$$

One important advantage of (0.8) is that there is no distinction between the short-range, $\rho > 1$, and long-range conditions, $\rho > 0$. But recall that the perturbations $V$, that we consider here, are in general $H_0$-bounded and not $H_0$-compact, thus the meaning of $D_p(z)$ is not clear. To define it, we will have to use the meromorphic continuation of a so called *regularized zeta function*, as it is can be done for the SSF $s_1$. This is a rather usual process and we refer to the nice paper of Müller [29] on this subject (see also Bruneau [8] and Carron [12]).

Finally we mention that another kind of regularization, inspired by Taylor's formula too, has been considered by Melin in [28] and recent results associated to a related regularized scattering phase, in connection with some heat invariants, have been announced by Hitrik-Polterovich [23].

## 0.2 Main results of this article

We are going to prove results partially announced in [6]. In all what follows, $\rho > 0$ is a fixed positive real number and $p \geq 1$ an integer such that

$$\rho > d/p.$$

Unless something else is specified, the dimension $d$ satisfies $d \geq 1$.

In the following theorem, we define the determinant $D_p(z)$ and the associated scattering phase $s_p$, when the perturbation $V$ is only $H_0$-bounded. We consider the same differential operators $H_0$

and $H_0+V$ as in the theorem 0.1 with $\kappa=0$. We shall use the so-called *regularized Zeta function* defined by

$$\zeta_s(z):=< u_p, (.-z)^{-s} >$$

where $(.-z)^{-s}$ is the function $\lambda \mapsto (\lambda-z)^{-s}$.

**Theorem 0.2.** *i) $\zeta_s(z)$ is well defined for* Re $s \gg 1$ *and $z \in \mathbb{C} \setminus [\inf \operatorname{supp} u_p, \infty)$. It has a meromorphic continuation, with respect to $s$, to the complex plane and this continuation is regular at* $s=0$.
*ii) The function $D_p(z) := \exp(-\partial_s \zeta_s(z)|_{s=0})$ is holomorphic on $\mathbb{C} \setminus [\inf \operatorname{supp} u_p, \infty)$ and the following limit holds,*

$$\lim_{\delta \searrow 0} \frac{d}{d\lambda} \arg D_p(\lambda + i\delta) = -\pi u_p(\lambda), \qquad \textit{in } \mathcal{D}'(\mathbb{R}). \tag{0.9}$$

Notice that $H_0$ can have variable coefficients.
**Remark.** Our definition of $D_p(z)$ coincide with $\mathrm{Det}_p(1+V(H_0-z)^{-1})$, the usual Fredholm determinant of order $p$ when $V(H_0-z)^{-1} \in \mathbf{S_p}$ (see section for more details 1).

We add that the convergence in (0.9) cannot be pointwise in general, since it was proved in [4, 7] that $u_2 = \eta''$ (in the distributions sense) for some locally integrable function $\eta$, called Koplienko's function (see also [25]). Formula (0.9) can be viewed as a generalization of (0.3), and it suggests the following definition:

**Definition 0.3.** *The regularized scattering phase $s_p \in \mathcal{S}'(\mathbb{R})$ is the unique primitive of $u_p$ which vanishes near $-\infty$.*

Recall that $u_p$ vanishes below the spectra of the operators, so this definition makes sense. Moreover, if $p=1$, we get the usual scattering phase *i.e.* the Birman-Krein's SSF.

Now we give results in Euclidean scattering, in the high energy and semiclassical limits. For all these results we now assume that $d \geq 2$.

**Theorem 0.4.** *Assume that $H_0 = p_0(D)$, with $p_0(\xi) = \sum_{|\alpha|=2m} p_\alpha \xi^\alpha$ constant coefficients elliptic positive polynomial, and consider $V$ as in theorem* 0.1 *with $\kappa=1$. Then*
*i) $s_p$ is $C^\infty$ on $(0,\infty) \setminus \sigma_{pp}(H_0+V)$.*
*ii) If the classical flow is non trapping, then the following complete high energy expansion holds:*

$$s_p'(\lambda) \sim \lambda^{\frac{d}{2m}-1} \sum_{k \geq 0} a_k \lambda^{-k/m}, \qquad \lambda \nearrow +\infty.$$

*Moreover this expansion can be differentiated at any order.*

Recall that the classical flow is non trapping if $|\Pi(\phi^t(x,\xi))| \to \infty$, as $|t| \to \infty$, uniformly on each compact subset of $T^\star\mathbb{R}^d \setminus 0$, where $\phi^t$ is the Hamiltonian flow of the principal symbol of $H_0+V$, and $\Pi : T^\star\mathbb{R}^d \to \mathbb{R}^d$ is the projection. Here the principal symbol is the classical one, that is the leading homogeneous part of order $2m$ of the full symbol of $H_0+V$.

This theorem is well known for $p=1$ [11, 13, 20, 30, 33, 34] and it was proved by the author for $p=2$ [4, 5, 7], but the result is completely new for $p \geq 3$. A similar result has been announced by Hitrik-Polterovich in [23], where Melin's regularization [28] is considered, but only for a perturbation by a potential, for which the non trapping assumption is always valid.

Now we quote our results for the semiclassical Schrödinger operator with a long-range potential. Assume that $H_0 = -h^2\Delta$ with $h \in (0,1]$, and that $V=V(x)$ is a potential such that

$$\partial^\alpha V(x) = \mathcal{O}(\langle x \rangle^{-\rho-|\alpha|}), \qquad \alpha \in \mathbb{N}^d, \tag{0.10}$$

and let $I \subset (0,\infty)$ be an interval. This interval is said to be non trapping if $|\Pi(\phi^t(x,\xi))| \to \infty$, as $|t| \to \infty$, uniformly on each compact subset of $p^{-1}(I)$ where $p(x,\xi) = \xi^2 + V(x)$ is the semiclassical principal symbol of $H_0 + V$.

**Theorem 0.5.** *Assume that $I$ is non critical for $V$.*
*i) If $I$ is non trapping, we have the following complete asymptotic expansion in $C^\infty(I)$:*

$$s_p'(\lambda,h) \sim h^{-d}\sum_{k\geq 0} h^k a_{k,p}^V(\lambda), \qquad h \searrow 0, \qquad a_{k,p}^V \in C^\infty(I), \quad k \in \mathbb{N},$$

*where, if we set $c_d = (2\pi)^{-d}\mathrm{vol}(\mathbb{S}^{d-1})/2$, the leading coefficient is*

$$a_{0,p}^V(\lambda) = c_d \int (\lambda - V(x))_+^{\frac{d-2}{2}} - \sum_{j=0}^{p-1}(-1)^j (d/2-1)\cdots(d/2-j)\lambda^{\frac{d}{2}-1-j} V(x)^j/j!\; dx, \tag{0.11}$$

*with the convention that $(d/2-1)\cdots(d/2-j) = 1$ if $j=0$. Moreover $(t)_+ = \max(t,0)$ and, when $d=2$, $(t)_+^{d/2-1} = 1$ if $t>0$, and $0$ otherwise (Heaviside function).*
*ii) If the following estimates holds for some $n \geq 1$ and $s > 0$, locally uniformly on $I$,*

$$\langle x\rangle^{-s}(H_0 + V - \lambda \pm i0)^{-1}\langle x\rangle^{-s} = \mathcal{O}(\exp(h^{-n})) \tag{0.12}$$

*then, for all $\nu \geq 0$, the Riesz mean of order $\nu$ has the following expansion, locally uniform on $I$*

$$\int_{-\infty}^{\lambda} s_p'(\mu,h)(\lambda-\mu)^\nu d\mu = h^{-d}\sum_{k=0}^{[\nu]_+} C_{k,V}^\nu(\lambda) h^k + \mathcal{O}(h^{1+\nu-d}), \tag{0.13}$$

*where $[\nu]_+$ is the smallest integer $\geq \nu$. In particular, if $\nu = 0$, we get the Weyl formula*

$$s_p(\lambda,h) = h^{-d}C_{0,V}^0(\lambda) + \mathcal{O}(h^{1-d}).$$

We precise some details about the definition of the Riesz mean, defined by the left hand side of (0.13). It must be understood as

$$< fu_p, (\lambda - .)^\nu > + \int_{-\infty}^{\lambda} (1-f)(\mu) u_p(\mu,h)(\lambda-\mu)^\nu d\mu$$

where $f \in C_0^\infty(\mathbb{R})$ equals 1 near $[\inf\sigma(H_1), 0]$. The above integral makes sense since one know that $s_p'(.,h) = u_p(.,h)$ is smooth on $(0,\infty)\setminus\sigma_{pp}(H_1)$ by theorem 0.4. Moreover it is known that, in this case, $\sigma_{pp}(H_1)\cap(0,\infty)$ is empty (see [1, 34]).

We add two others remarks on this theorem. First, when $d$ is even, all the terms corresponding to $j \geq d/2$ vanish in (0.11) (if ever $d/2 \leq p-1$). In particular, if $d=2$, the leading term is the same as for the usual SSF, and more generally, when $d$ is even, $a_{0,p}^V = a_{0,d/2}^V$ for all $p \geq d/2$.

The second point is that sufficient conditions for the validity of the estimates (0.12) can be found in the work of Burq [10], combined with the one of Bruneau-Petkov [9].

In the next theorem, we prove that the phase shift $\pm 1/2$ which occurs for the SSF $s_1$ close to a resonance, holds for all $s_p$. This phenomenon is known as Breit-Wigner formula (see the book [39] for its physical interpretation) and has already been considered by several authors. Let us quote the papers of Combes-Duclos-Klein-Seiler [14], Petkov-Zworski [31], and especially the ones of Gérard-Martinez-Robert [17], for $s_1$ and the scattering cross section, and Robert [34], which we have followed.

We consider an energy level $\lambda_0 \in I$ (which can be critical for $V$) and the set of resonances $\Gamma(h)$ close to $\lambda_0$ defined by Helffer-Sjöstrand in [22]. We suppose, as in [17], that:
• there exists a connected open subset $\hat{O} \subset \mathbb{R}^d$ and a compact connected set $U \Subset \hat{O}$ such that $V > \lambda_0$ in $\hat{O} \setminus U$, and $V < \lambda_0$ in $U \cup \mathbb{R}^d \setminus \hat{O}$.
• $V$ is holomorphic in $\{\tilde{x} \in \mathbb{C}^d \mid |\text{Im } \tilde{x}| \leq \varepsilon_0 |\text{Re } \tilde{x}|, \text{ Re } \tilde{x} \text{ in a neighborhood of } \mathbb{R}^d \setminus \hat{O}\}$,
• $\lim_{|t| \nearrow \infty} |\Pi(\phi^t(x,\xi))| = \infty$, for all $(x,\xi) \in \mathbb{R}^d \setminus \hat{O} \times \mathbb{R}^d$ such that $p(x,\xi) = \xi^2 + V(x) = \lambda_0$. $\phi^t$ is the Hamiltonian flow of $p$ and $\Pi : T^\star\mathbb{R}^d \to \mathbb{R}^d$ the projection.
• There exists a family of open sets $\Omega(h) \in \mathbb{C}$, such that $\cap_{h>0}\Omega(h) = \{\lambda_0\}$, $I(h) := \Omega(h) \cap \mathbb{R} \neq \emptyset$, and such that $\Omega(h) \cap \Gamma(h)$ contains only one resonance that we note $r(h)$. Moreover we assume that there exist $\varepsilon > 0$ and $c > 0$ such that $\text{dist}(\Gamma(h), \partial\Omega(h)) \geq ce^{-\varepsilon/h}$.

Under these assumptions and (0.10), we have the following result:

**Theorem 0.6 (Breit-Wigner type formula).** *Let $\delta(h)$ be a family of positive real numbers such that, $\lim_{h\searrow 0} h^{-d}\delta(h) = 0$, $\lim_{h\searrow 0} |\text{Im } r(h)|^{-1}\delta(h) = +\infty$, and satisfying $\text{Re } r(h) \pm \delta(h) \in I(h)$, for all $h$ small enough. Then*

$$\lim_{h\searrow 0} s_p(\text{Re } r(h) \pm \delta(h), h) - s_p(\text{Re } r(h), h) = \pm\frac{1}{2}.$$

# 1 Fredholm determinants and spectral distributions

We are going to use the Schatten ideals $\mathbf{S_p}$ (on $L^2(\mathbb{R}^d)$) thus we recall some basic definitions and properties (for more details see Gohberg-Krein [19], Simon [38] or Yafaev [40]). An operator $A$ is said to be in $\mathbf{S_p}$ if $|A| = (A^\star A)^{1/2}$ is compact with a spectrum in $l^p(\mathbb{N})$. If this is true, then $A$ is compact as well and its spectrum $(\lambda_j)_{j\in\mathbb{N}} \in l^p(\mathbb{N})$, thus we can define

$$\text{Det}_p(1+A) = \prod_{j=0}^{\infty}(1+\lambda_j)\exp\left(\sum_{k=1}^{p-1}\frac{(-1)^k}{k}\lambda_j^k\right),$$

where the product converges, since the factors are $1 + \mathcal{O}(\lambda_j^p)$. If $p = 1$ we get trace-class operators, and when $A_1, \cdots, A_p \in \mathbf{S_p}$, then $A_1 A_2 \cdots A_p \in \mathbf{S_1}$. Let us also quote an important example of operators in the Schatten classes: the pseudo-differential operator $< x >^{-\mu} < D >^{-\mu}$ is in $\mathbf{S_p}$ if $\mu > d/p$ (it is sufficient to consider $p \in \mathbb{N}$ for our purpose). This justifies the relation between $\rho$ and the Schatten class $\mathbf{S_p}$ in the introduction and in our results.

## 1.1 The Stieltjes transform

For all $u \in \mathcal{E}'(\mathbb{R})$, the space of compactly supported distributions, one can define the so called *Stieltjes transform* of $u$ as the following holomorphic function on $\mathbb{C} \setminus \text{supp}(u)$

$$U(z) = < u, (. - z)^{-1} >,$$

and one can recover $u$ from $U$ thanks to the inversion formula:

$$\lim_{\varepsilon \searrow +0} U(\lambda + i\varepsilon) - U(\lambda - i\varepsilon) = 2i\pi u(\lambda), \qquad \text{in } \mathcal{D}'(\mathbb{R}).$$

This is essentially equivalent to the fact that $2i\pi\delta_0(\lambda) = (\lambda - i0)^{-1} - (\lambda + i0)^{-1}$ and very elementary. When considering temperate distributions in $\mathcal{S}'(\mathbb{R})$, one cannot define the Stieltjes transform in general. Nevertheless, under some conditions at infinity on $u \in \mathcal{S}'(\mathbb{R})$, described below, we'll show that it is possible.

Assuming that $u$ is temperate, the following formula makes sense

$$U_s(z) = < u, (. - z)^{-s} > \tag{1.1}$$

if Re $s \gg 1$ and $z \notin \mathbb{R}$ (or $z \in \mathbb{C} \setminus [\inf \operatorname{supp}(u), \infty)$ if the inf is finite). Since we have

$$(\lambda - z)^{-1} = \partial_z \left(\partial_s (\lambda - z)^{-s}\right)_{|s=0}, \tag{1.2}$$

where $z \in \mathbb{C} \setminus [\lambda, \infty)$, it is natural to consider the existence of a meromorphic continuation of $U_s(z)$, with respect to $s$, at 0. Note that the complex powers are defined on $\mathbb{C} \setminus (-\infty, 0]$ with the principal determination of the log.

We give now sufficient conditions to get a meromorphic continuation. Assume that $u \in \mathcal{S}'(\mathbb{R})$, with $\inf(\operatorname{supp}(u)) > -\infty$, and that its Laplace transform, defined as $Lu(t) = < u, f_t >$ (with $f_t(\lambda) = e^{-t\lambda}$) has the following finite expansion

$$Lu(t) = \sum_{j=0}^{n} \sum_{k=0}^{k(j)} a_{jk} t^{\alpha_j} \log^k t + \mathcal{O}(t^{\beta_n}), \qquad t \searrow 0^+, \tag{1.3}$$

where the $a_{jk}$ are complex numbers, $\alpha_0 < \alpha_1 < \cdots < \alpha_n$ are real numbers and $\beta_n > 0$. Moreover we assume, as in [29], that

$$-\alpha_j \in \mathbb{N} \Rightarrow a_{jk} = 0 \quad \forall\ k > 0. \tag{1.4}$$

Notice that these conditions hold if $u \in \mathcal{E}'(\mathbb{R})$ since, in that case, $Lu$ is an entire function of $t$. Under the assumptions (1.3) and (1.4) we have

**Proposition 1.1.** *i) The map $s \mapsto U_s(z)$ has a meromorphic continuation to $\{\text{Re } s > -\beta_n\}$. This continuation is regular at 0 and is holomorphic with respect to $z \in \mathbb{C} \setminus [\inf \operatorname{supp}(u), \infty)$.*
*ii) If we set $U(z) := \partial_z \partial_s U_s(z)_{|s=0}$, the following inversion formula holds:*

$$\lim_{\varepsilon \searrow 0} U(\lambda + i\varepsilon) - U(\lambda - i\varepsilon) = 2i\pi u(\lambda), \qquad \text{in } \mathcal{D}'(\mathbb{R}).$$

We shall call the function $U$, defined in this proposition, the Stieltjes transform of $u$; it extends obviously the definition valid on $\mathcal{E}'(\mathbb{R})$, because of formula (1.2).

**Proof.** $i)$ The result is local with respect to $z$ so we fix $r > 0$ and assume that Re $z < r$. If $\chi \in C_0^\infty$ is identically 1 near $(-r-1, r+1)$, the proposition is true for $\chi u$, therefore we may assume that $u$ is supported in $[r+1, \infty)$, since replacing $u$ by $(1-\chi)u$ doesn't change the assumptions. Then we use the fact that $(\lambda - z)^{-s} = \Gamma(s)^{-1} \int_0^\infty e^{-t(\lambda - z)} t^{s-1} dt$ when Re $s > 0$ and $\lambda > r$ to write

$$U_s(z) = \frac{1}{\Gamma(s)} \int_0^\infty Lu(t) e^{tz} t^{s-1} dt, \qquad \text{Re } s > x_0$$

since $Lu(t) = \mathcal{O}(e^{-(r'+1)t})$ when $t \to \infty$, for any $r' < r$. Using $\int_0^1 t^\nu \log^k t \, dt = (-1)^k k!/(\nu+1)^{k+1}$ combined with (1.3), we get

$$\int_0^1 Lu(t) e^{tz} t^{s-1} dt = \sum_{j=0}^{n} \sum_{k=0}^{k(j)} \sum_{l=0}^{\infty} a_{jk} \frac{z^l}{l!} \frac{(-1)^k k!}{(s + \alpha_j + l)^{k+1}} + R_n(z, s)$$

with $R_n(z,s)$ holomorphic with respect to $s$ when Re $s > -\beta_n$, and holomorphic with respect to $z$. This proves the existence of the meromorphic continuation. The other integral, $\int_1^\infty$ doesn't cause any problem since it is entire in $s$. Then, using condition (1.4) and the fact that $\Gamma(s)^{-1}$ vanishes at 0 we get the regularity at $s=0$.

*ii*) By the same reduction as for *i*), we may assume that Re $z<r$ and that $u$ is supported in $[r+1,+\infty)$; in that case we know, thanks to the proof of *i*), that $U_s(z)$ is holomorphic on $V(0)\times\{\text{Re } z<r\}$, where $s\in V(0)$ a neighborhood of 0. Hence $\partial_z\partial_s U(0,z)-\partial_z\partial_s U(0,\bar z)\to 0$ when Im $(z)\searrow 0$, locally uniformly with respect to Re $z$, and this completes the proof. □

Remark that, if we replace the assumption (1.3) by the following

$$Lu(t)\sim\sum_{j=0}^{\infty}\sum_{k=0}^{k(j)}a_{jk}t^{\alpha_j}\log^k t,\qquad t\searrow 0^+,\tag{1.5}$$

with $\alpha_j<\alpha_{j+1}$ and $\alpha_j\to\infty$, we get a meromorphic continuation on $\mathbb{C}$ and the poles are the points

$$-\alpha_j-l,\qquad \text{such that } \alpha_j+l\notin\mathbb{N}$$

where $j,l\in\mathbb{N}$. Moreover, if for all $j$ and $l$, $\alpha_j+l\notin\mathbb{N}$, then the meromorphic continuation of $U_s(z)$ vanishes when $-s\in\mathbb{N}$ (see [4, 7] for applications).

Now we turn to an example of computation of Stieltjes transform. Consider a family of abstract self-adjoint operators $H_\epsilon=H_0+\epsilon V$, depending on $\epsilon\in[0,1]$. Assume moreover that $H_0$ is semi bounded from below and that $V$ is $H_0$ compact; more precisely, we assume that, for one $z\notin\sigma(H_0)$ (and hence for all) we have

$$V(H_0-z)^{-\nu}\in\mathbf{S_p},\qquad \text{for some }\ \nu\in[0,1).$$

Then the domain of $H_\epsilon$ is independent of $\epsilon$, $H_\epsilon\geq\min(\inf\sigma(H_0),\inf\sigma(H_0+V))$ and one can consider the functional

$$<u,f>=\operatorname{Tr}\left(f(H_0+V)-\sum_{j=0}^{p-1}\frac{1}{j!}\frac{d^j}{d\epsilon^j}f(H_\epsilon)_{|_{\epsilon=0}}\right).\tag{1.6}$$

To prove that (1.6) makes sense for all test functions we proceed as follows. The right hand side of (1.6) vanishes when $f$ is supported in $(-\infty,\min(\inf\sigma(H_0),\inf\sigma(H_0+V)))$ and is well defined when $f=f_z$ with $f_z(\lambda)=(\lambda-z)^{-1}$. In that case, we have

$$<u,f_z>=(-1)^p\operatorname{Tr}\left((H_0+V-z)^{-1}\big(V(H_0-z)^{-1}\big)^p\right),$$

since $\partial_\epsilon^j R_\epsilon(z)=j!(-1)^jR_\epsilon(z)(VR_\epsilon(z))^j$. Using the Cauchy formula, we can define $<u,f>$ for $f(\lambda)=(\lambda-E)^{-s}$, Re $s>1$ and $E$ real, below the spectra. Then, by the Mellin transform, we can extend $u$ to the class of Schwartz functions supported in $(E,\infty)$, and thus $u$ defines a temperate distribution.

Using these considerations, we can define $U_s(z)$ as in (1.1) and we get the following result.

**Proposition 1.2.** *For all complex number* $z\notin[\min\big(\inf\sigma(H_0),\inf\sigma(H_0+V)\big),\infty)$, $s\mapsto U_s(z)$ *is holomorphic in the region* Re $s>-p(1-\nu)$ *and we have*

$$\partial_s U_s(z)_{|_{s=0}}=-\log\operatorname{Det}_p\big(1+VR_0(z)\big),$$

*where the branch of* $\log Det_p\big(1+VR_0(z)\big)$ *is determined by the condition that it goes to* 0 *when* Re $z-|\text{Im } z|\to-\infty$.

**Proof.** Firstly we assume that $z \in K$, with $K$ compact. Thus we can consider a contour $\mathcal{C}_K$, surrounding $[\min\big(\inf\sigma(H_0), \inf\sigma(H_0+V)\big), \infty)$, such that $k - z \notin (-\infty, 0]$, for all $k \in \mathcal{C}_K$ and $z \in K$ (roughly speaking, the spectrum lies inside the contour and $K$ lies outside). By the Cauchy formula, if we note $R_\epsilon(k) = (H_\epsilon - k)^{-1}$, we have

$$(H_\epsilon - z)^{-s} = \frac{i}{2\pi}\int_{\mathcal{C}_K}(k-z)^{-s}R_\epsilon(k)dk \tag{1.7}$$

Notice that we can choose $\mathcal{C}_K$ such that, outside a compact set, it coincides with the union of the two half lines $e^{\pm i\pi/4}[1,\infty)$ so that $R_\epsilon(k) = \mathcal{O}(|k|^{-1})$ at infinity. Then we get easily the following formula

$$U_s(z) = (-1)^p \mathrm{Tr}\left(\frac{i}{2\pi}\int_{\mathcal{C}_K}(k-z)^{-s}R_1(k)\big(VR_0(k)\big)^p dk\right). \tag{1.8}$$

Since the trace norm of $R_1(k)\big(VR_0(k)\big)^p$ is $\mathcal{O}(|k|^{-1-p(1-\nu)})$, the right hand side is holomorphic with respect to $s$ when Re $s > -p(1-\nu)$, thus it is regular at $s = 0$. Then if we differentiate with respect to $z$, we get, thanks to the formula (1.2)

$$\partial_z\partial_s U_s(z)_{|_{s=0}} = (-1)^p \mathrm{Tr}\left(R_1(z)\big(VR_0(z)\big)^p\right)$$

whose right hand side is precisely the opposite of the logarithmic derivative of $\mathrm{Det}_p\big(1+VR_0(z)\big)$ (see for example Yafaev [40]).

We still have to show that $\partial_s U_s(z)_{|_{s=0}} \to 0$ when $z \to -\infty$. If $z$ is real and $z \to -\infty$, we can choose a contour $\mathcal{C}$ as above and independant of $z$, such that Re $(k-z) > 0$ for all $k \in \mathcal{C}$. We have

$$\partial_s U_s(z)_{|_{s=0}} = (-1)^p \mathrm{Tr}\left(\frac{i}{2\pi}\int_{\mathcal{C}} -\log(k-z)R_1(k)\big(VR_0(k)\big)^p dk\right) \tag{1.9}$$

where we shall do an integration by part. To justify this integration by part, we consider the function $\varphi(k) = Tr\big(R_1(k)\big(VR_0(k)\big)^p\big)$ and its primitive vanshing when $k \to -\infty$ defined by

$$\Phi(k) = \int_{-\infty}^{c}\varphi(u)du + \int_{[c,k]}\varphi(u)du,$$

where $c \in \mathbb{R}$ is such that $c < \min\big(\inf\sigma(H_0), \inf\sigma(H_0+V)\big)$, and $[c,k]$ is the line segment joining $c$ to $k$. We can assume that $\mathcal{C} = c + \big(e^{i\pi/4}[0,\infty) \cup e^{-i\pi/4}[0,\infty)\big)$ and then it is easy to see that $|\Phi(k)| = \mathcal{O}(|k|^{-p(1-\nu)})$, when $k \in \mathcal{C}$, by writing $\int_{\gamma_k}\varphi(u)du = 0$ combined with the fact that $|\varphi(k)| = \mathcal{O}(|k|^{-1-p(1-\nu)})$. Here $\gamma_k$ is the union of the segments $[-|k|, c]$ and $c + e^{i\pi/4}[0,|k|]$, and of the portion of circle centered at $c$ with radius $|k|$ where the argument ranges between $\pi/4$ and $\pi$ (or the same with $-\pi/4$ everywhere and $-\pi$). Thus we can integrate $\varphi$ and differentiate the log in (1.9), and then we see easily that $\partial_s U_s(z) \to 0$ when $z \to -\infty$ which completes the proof. □

Proposition 1.2 was proved by Koplienko in [25] under the condition $V \in \mathbf{S_p}$, and is essentially well known. We have given the proof when $V(H_0 - z)^{-\nu} \in \mathbf{S_p}$ for the sake of completeness, and to justify the remark after theorem 0.2 (see the next subsection).

## 1.2 Proof of theorem 0.2

Consider $H_0 = p(x,D)$ and $H_0 + V = p(x,D) + v(x,D)$ two uniformly elliptic differential operators, where $p(x,\xi) = \sum_{|\alpha|\le 2m} p_\alpha(x)\xi^\alpha$, and $v(x,\xi) = \sum_{|\alpha|\le 2m} v_\alpha(x)\xi^\alpha$, with $p_\alpha$ and $v_\alpha$ smooth functions such that for all $|\alpha| \le 2m$ and $\beta \in \mathbb{N}^d$

$$|\partial^\beta p_\alpha(x)| \le C_\beta, \qquad |\partial^\beta v_\alpha(x)| \le C_\beta\langle x\rangle^{-\rho}, \qquad x \in \mathbb{R}^d.$$

Their order is the integer $2m$ which is even since we do the following ellipticity assumption

$$\sum_{|\alpha|=2m} p_\alpha(x)\xi^\alpha \geq C|\xi|^{2m}, \qquad \sum_{|\alpha|=2m} (p_\alpha + v_\alpha)(x)\xi^\alpha \geq C|\xi|^{2m}, \qquad x, \xi \in \mathbb{R}^d$$

for some constant $C > 0$. We also assume that these operators are symmetric on $L^2$, which allows us to consider their self-adjoint realization with domain $H^{2m}(\mathbb{R}^d)$, the usual Sobolev space; because of the ellipticity condition, these operators are lower semi-bounded.

Now, if $\rho > d/p$ and $u_p$ is the spectral distribution associated to $(H_0, H_0 + V)$, we know by [4, 7] that

$$< u_p, e^{-t(.)} > \sim t^{-d/2m} \sum_{k \geq 0} a_k t^{k/m}, \qquad t \searrow 0,$$

therefore we can apply proposition 1.1, which proves the theorem.

Moreover, if $V$ is of order $m' < 2m$, it is $H_0$-compact and if $2m - m' > d/p$, $V(H_0 + i)^{-1} \in \mathbf{S_p}$. Furthermore, for some $\nu \in [0, 1)$ we still have $\nu 2m - m' > d/p$ and thus $V(H_0 + i)^{-\nu} \in \mathbf{S_p}$ which allows us to use proposition 1.2; this proves that $D_p(z)$ coincides, in this case, with the usual determinant $\mathrm{Det}_p(1 + V(H_0 - z)^{-1})$.

# 2 The asymptotic representation formula

In this section we prove an asymptotic formula for $u_p(\lambda, h)$ for positive and non critical energies of the free Hamiltonian $H_0$. For this technical result, $u_p$ is associated to the same kind of pseudo-differential operators, $H_0, H_0 + V$, as the one considered in [7, 34]. We assume that

$$H_0 = \omega(hD),$$

with $\omega(\xi) \to +\infty$ when $\xi \to \infty$, and $\omega \geq 0$, with $\omega$ is smooth and temperate, in the sense that, for some $C$ and $M$ positive numbers, $\omega(\xi) \leq C(1 + \omega(\eta))\langle \xi - \eta \rangle^M$, for all $\xi, \eta$. Then we consider a perturbation given by Weyl quantization of real symbols

$$\begin{aligned} V &= q_0^w(x, hD) + \cdots + h^d q_d^w(x, hD) + h^{d+1}\tilde{q}^w(x, hD, h), \\ &= q^w(x, hD, h) + h^{d+1}\tilde{q}^w(x, hD, h). \end{aligned} \tag{2.1}$$

(We shall explain a little further why we split $V$ into these two parts.) Recall that, when $a$ is a symbol, $a^w(x, hD)$ is the operator defined by the oscillatory integral

$$a^w(x, hD)u(x) = (2\pi h)^{-d} \int\int e^{i\langle x-y, \xi\rangle/h} a\left(\frac{x+y}{2}, \xi\right) u(y) dy d\xi, \qquad u \in \mathcal{S}(\mathbb{R}^d).$$

We assume that $h^{d+1}\tilde{q}_{d+1} \sim \sum_{k \geq d+1} h^k q_k$, which means, as usual, that for all $N > d$ the difference $h^{d+1}\tilde{q}_{d+1} - \sum_{d+1 \leq k \leq N} q_k$ is in $S_1(\omega, -N)$. We also assume that there exists $\rho > 0$ such that

$$q_k \in S_1(\omega, -k) \cap S_1(\omega, -\rho), \qquad \forall\ k \in \mathbb{N}.$$

This is the *long-range* condition on the perturbation $V$. The symbol class $S_1(\omega, \mu)$, with $\mu$ real, is defined by asking that, for all $a$ in this class, the semi-norms

$$\sup_{\mathbb{R}^{2d}} \max_{|\alpha+\beta| \leq N} |\langle x \rangle^{-\mu+k+|\alpha|} (1 + \omega(\xi))^{-1} \partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \tag{2.2}$$

are finite for all $N$. Sometimes we shall refer to $S_1(w^n,\mu)$ which is the same class with $\omega^n$ instead of $\omega$; we shall also need $S_1(-\infty,\mu)$ which is $\cap_{n>0}S_1(\Omega^{-n},\mu)$ with $\Omega(\xi)=\langle\xi\rangle$. Notice that for all these symbols, differentiation with respect to $x$ increases the decay with respect to this variable.

Finally we assume that $q$ and $\tilde q$ are real valued and that, for some constant $C>0$

$$C+w(\xi)+q_0(x,\xi)>C^{-1}(1+\omega(\xi)),\qquad \forall\, x,\xi\in\mathbb{R}^d. \tag{2.3}$$

These conditions entail the essential self-adjointness of $H_0+\epsilon V$ for all $\epsilon\in[0,1]$ and $h\in(0,h_0]$ ($h_0$ small enough); the domain of the self-adjoint realization is independent of $\epsilon$ (see [4, 7]).

Let $I\subset(0,\infty)$ be a non critical interval for $\omega$ (see section 3). Our main result is the following:

**Theorem 2.1.** *Let $p\in\mathbb{N}$ be such that $\rho>d/p$. There exists $\chi\in C_0^\infty(\mathbb{R}^d)$, and a family of function $u_{reg,p}(.,h)\in C^\infty(I)$, such that for all $J\Subset I$ one can find sequences of operators $R_N^\pm(\lambda,h)$, such that for all $N>0$*

$$u_p(\lambda,h)=Tr\left(\chi\frac{\partial E}{\partial\lambda}(\lambda,h)\chi\right)+u_{reg,p}(\lambda,h)+h^{k(N)}\mathcal{R}_N(\lambda,h),\qquad h\searrow 0 \tag{2.4}$$

$$\mathcal{R}_N(\lambda,h)=Tr\left(\Theta_N^+(\lambda,h)R(\lambda+i0,h)\right)+Tr\left(\Theta_N^-(\lambda,h)R(\lambda-i0,h)\right), \tag{2.5}$$

*where $k(N)\nearrow+\infty$ when $N\nearrow\infty$. Here $E(\lambda,h)$ is the spectral resolution of $H_0+V$ and $R(z,h)$ its resolvent. The function $u_{reg,p}(.,h)$ has a full asymptotic expansion in $C^\infty(J)$ which means that there exists a sequence $u^j_{reg,p}\in C^\infty(J)$ such that, for all $N$*

$$u_{reg,p}(.,h)-h^{-d}\sum_{j=0}^N h^j u^j_{reg,p}=\mathcal{O}(h^{N-d}),\qquad \textit{in the } C^\infty(J) \textit{ topology.}$$

*The operators $\Theta_N^\pm(\lambda,h)$ are such that*

$$\langle x\rangle^{k(N)}\Theta_N^\pm(.,h)\langle x\rangle^{k(N)}\ \textit{is bounded in } C^{k(N)}(J,\mathbf{S_1}),\ \textit{when } h\searrow 0.$$

By the same arguments as in [34, 4, 7], theorems 0.4, 0.5 and 0.6 are simple corollaries of theorem 2.1 and the rest of this section is devoted to its proof.

Remark that our assumptions on the operators are satisfied by the long range Schrödinger operator ($\omega(\xi)=\xi^2$, $\tilde q(x,\xi,h)=0$, $q(x,\xi,h)=q_0(x,\xi)=V(x)$)

$$H_0=-h^2\Delta,\qquad V=V(x)$$

and, if $\kappa=1$ (see (0.7)), by the (rescaled) elliptic operators used in subsection 1.2:

$$H_0=h^{2m}p_0(D),\qquad V=h^{2m}\sum_{|\alpha|\le 2m}v_\alpha(x)D^\alpha.$$

This last case is useful to prove high energy asymptotics by considering these operators near the rescaled energy level $\lambda'=1$, and by choosing $h^{2m}=\lambda^{-1}$, with $\lambda\nearrow\infty$ since one sees easily that

$$h^{2m}u_p(\lambda',h)=u_p(\lambda'/h^{2m})$$

if $u_p(.,h)$ is associated to the pair $H_0,H_0+V$, and $u_p(.)=u_p(.,1)$.

The main tool of the proof of the theorem is the Isozaki-Kitada parametrix (see [24, 16, 34]) for the operators $H_\epsilon=H_0+\epsilon V$, $\epsilon\in[0,1]$ and especially its differentiability with respect to $\epsilon$. For the sake of completeness, but also to be able to state precise results, we have chosen to add the section 3 on this parametrix, and its control with respect to $\epsilon$. Moreover, as we shall see later, we are going to approximate our perturbation by a fast enough decaying one, and thus section 3 includes the control of the construction with respect to the principal symbol of $V$.

We have to precise one point on what we shall use as a semiclassical principal symbol. In our constructions, $q = q(x, \xi, h)$ will play the role of the principal symbol (see (2.1) for the notation). It is more convenient, for technical reasons, to do so since $\tilde{q}(h)$ is integrable with respect to $x$ which is useful when one needs to write traces.

We are going to approximate the principal symbol of $V$ by a fast decaying one so that we can define the usual SSF for the pair $H_0, H_0 + \epsilon V$. Thus we choose a subset

$$\mathcal{B} \subset S_1(\omega, -\mu), \qquad \mu > d \tag{2.6}$$

in which all symbols are real and satisfie (2.3) (they also can depend on $h$). We can choose $\mathcal{B}$ to be bounded in $S_1(\omega, -\rho)$. We also assume, which is the most important, that this subset contains a sequence which converges to $q$ in the topology of $S_1(\omega, -\rho_1)$, for all $\rho_1 < \rho$. The existence of such a subset is clear by standard properties of these classes of symbols. Moreover one can choose $\rho_1$ such that $\rho > \rho_1 > d/p$, thus there is no loss of generality if we choose $\rho$ slightly smaller, since the only property that we use is $\rho > d/p$. Then we can assume that the sequence converges to $q$ in $S_1(\omega, -\rho)$. Finally, we can also assume that $\rho^{-1} \notin \mathbb{N}$ which will be useful for technical reasons which appear in the appendix.

## 2.1 Two technical lemmas

Let $\chi \in C_0^\infty(\mathbb{R}^d)$ be a cutoff function such that $\chi \equiv 1$ on the ball of radius $R > 0$. (2.3).

**Lemma 2.2.** *One can choose $R > 0$ large enough such that, for all functions $f_1, f_0 \in C_0^\infty(\mathbb{R})$ satisfying $f_1 \equiv 1$ near* supp $f_0$, *we have the following estimates in* $\mathbf{S_1}$*:*

$$\langle x \rangle^N (1 - \chi)\big(1 - f_1(H_0)\big) f_0(H_0 + \epsilon V) \langle x \rangle^N = \mathcal{O}(h^N), \qquad \forall\ N.$$

*These estimates are uniform with respect to $\epsilon \in [0, 1]$ and to $q$ in $\mathcal{B}$. Moreover, for all $j \geq 0$, the following operator is pseudo-differential*

$$\frac{d^j}{d\epsilon^j} \left((1 - \chi)(1 - f_1(H_0)) f_0(H_0 + \epsilon V)\right)_{|\epsilon=0}. \tag{2.7}$$

*Its symbol is $\mathcal{O}(h^\infty)$ in $S_1(1, -\infty)$ and depends continuously on $q$ in $\mathcal{B}$.*

**Proof.** The first statement is a simple consequence of the fact that, for all $N$ large enough

$$f_0(H_0 + \epsilon V) = a_{N,\epsilon}(x, hD, h) + h^N R_{N,\epsilon}(h)$$

with $|||\langle x \rangle^{N/4} R_{N,\epsilon} \langle x \rangle^{N/4}|||_{\mathbf{1}} = \mathcal{O}(h^{N/4})$ uniformly with respect to $\epsilon \in [0, 1]$ and $q \in \mathcal{B}$ (see [4, 7]). The symbol $a_{N,\epsilon}(x, \xi, h)$ is a linear combination of $a_n(x, \xi) f_0^{(n)}(\omega(\xi) + \epsilon q(x, \xi, h))$ with $a_n \in S_1(\omega^{k(n)}, -n)$, for some $k(n) \geq 0$. If $x$ is large enough, and $\omega(\xi) + \epsilon q(x, \xi, h) \in$ supp $f_0$, then $\omega(\xi) \in f_1^{-1}(1)$, and then the composition formula of pseudo-differential operators yields

$$(1 - \chi)\big(1 - f_1(H_0)\big) a_{N,\epsilon}(x, hD, h) \in h^\infty Op(\mathcal{S}),$$

where $\mathcal{S}$ is the Schwartz class. The form of (2.7) is simply due to the fact that $\partial_\epsilon^j f(H_0 + \epsilon V)_{|\epsilon=0}$ is purely pseudo-differential, and to the functional calculus of [7]. □

In all what follows, we use the notation $U_\epsilon(t) = \exp(-itH_\epsilon/h)$, and $I$ is an open interval. Consider a family of bounded operators $B_\epsilon$ which is $C^p$ with respect to $\epsilon \in [0,1]$, and such that $B_\epsilon \langle x \rangle^{\bar{d}}$ is still $C^p$ in the class of bounded operators, for some $\bar{d} > d$. In the applications $B_\epsilon$ will be an operator with a rather fast decaying kernel. Then we have

**Lemma 2.3.** *The family of operators $B_\epsilon f(H_\epsilon)U_\epsilon(t)$ is $C^p$ in the trace class (with respect to $\epsilon \in [0,1]$) for all $f \in C_0^\infty(I)$, and all $t \in \mathbb{R}$. Moreover, for all $j \geq 1$*

$$\frac{d^j}{d\epsilon^j} Tr(B_\epsilon f(H_\epsilon)U_\epsilon(t))_{|_{\epsilon=0}}$$

*can be written as a linear combination of the traces of the following operators with $0 \leq n \leq j$:*

$$\frac{\partial_\epsilon^{j-n} B_\epsilon}{h^n}|_{\epsilon=0} \int_0^t \int_0^{t-t_1} \cdots \int_0^{t-t_1-\cdots t_{n-1}} a_0(x,hD,h) \prod_{k=0}^{n} U_0(t_k - t_{k+1}) a_{k+1}(x,hD,h) dt_n \cdots dt_2 dt_1 .$$

*Here the product must be understood as the composition from the left to the right (with the term $k=0$ on the left). We have used the following conventions:*

$$t_0 = t, \qquad t_{n+1} = 0$$

*and if $n=0$ there is no integral sign. The symbols $a_l$ are in $S_1(-\infty,-\rho)$ if $1 \leq l \leq n$, whereas $a_0$ and $a_{n+1}$ are in $S_1(-\infty,0)$. They all have an expansion with respect to $h$ and are compactly supported with respect to $\xi$ in a small neighborhood of $\omega^{-1}(\mathrm{supp}\ f)$. As in the previous lemma, everything depends continuously on $q \in \mathcal{B}$.*

**Proof.** This lemma is a simple consequence of

$$\frac{d}{d\epsilon} U_\epsilon(t) = -\frac{i}{h} \int_0^t U_\epsilon(t-t_1) V U_\epsilon(t_1) dt_1, \tag{2.8}$$

together with the fact that for all $n$, $(d/d\epsilon)^n f(H_\epsilon)|_{\epsilon=0}$ is pseudo-differential operator with symbol in $S_1(1,-n\rho)$ (see [7]). Moreover, using the fact that $(d/d\epsilon)(f(H_\epsilon)U_\epsilon(t))$ can always be written as

$$\left(\frac{d}{d\epsilon} f(H_\epsilon)\right) U_\epsilon(t) f_1(H_\epsilon) + f(H_\epsilon) \left(\frac{d}{d\epsilon} U_\epsilon(t)\right) f_1(H_\epsilon) + f(H_\epsilon) U_\epsilon(t) \left(\frac{d}{d\epsilon} f_1(H_\epsilon)\right),$$

with $f_1 f = f$, we see easily that the symbols $a_n$ are supported, with respect to $\xi$, in any arbitrary small neighbourhood of $\omega^{-1}(\mathrm{supp}\ f)$. $\square$

## 2.2 A non stationary phase lemma

To prove theorem 2.1, we shall need to consider inverse Fourier transforms of traces similar to the one of lemma 2.3. Since we want to prove that the traces are smooth, we need to prove fast decay in time, which will be obtained by a non stationary phase argument. If one looks at the critical points of the phases naturally involved in the traces of lemma 2.3, we find that $t$ must be zero. However it is not clear how to do the integrations by parts, to get the fast decay away from 0, that's why we need to establish the next lemma.

Let's consider the set of all complex valued functions $f$ such that, for all

$$\alpha_2, \cdots, \alpha_k \in \mathbb{N}^d,\ \beta_1, \cdots, \beta_{k-1} \in \mathbb{N}^d,\ \beta_k \in \mathbb{N}^d,\ \ |\beta_k| \leq N$$

one can find a constant $C$ such that

$$\begin{aligned} |\partial^{\alpha_2\cdots\alpha_k}_{x_2\cdots x_k}\partial^{\beta_1\cdots\beta_{k-1}}_{\xi_1\cdots\xi_{k-1}}\partial^{\beta_k}_{\xi_k} f(x_0,x_1,\cdots,x_k,\xi_1,\cdots,\xi_k,h)| &\leq C\langle x_0,x_1\rangle^{|\beta_k|}\times \\ \langle x_0\rangle^{-N}\langle x_{0,1}\rangle^{-\delta_1}\cdots\langle x_{0,n}\rangle^{-\delta_1}\langle x_1\rangle^{-N-\delta}\langle x_2\rangle^{-\delta-|\alpha_2|}\cdots\langle x_k\rangle^{-\delta-|\alpha_k|} \end{aligned} \tag{2.9}$$

with $\delta_1 > 1$, $\delta = d\delta_1$, and where the variables are such that

$$x_0 = (x_{0,1},\cdots,x_{0,n}) \in \mathbb{R}^n,\ x_1,\cdots,x_k,\ \xi_1,\cdots,\xi_k \in \mathbb{R}^d,\ h \in (0,1].$$

We also assume that these functions $f$ are supported in $\left(\omega^{-1}(I)\right)^k$ with respect to $\xi_1,\cdots,\xi_k$. The best constants $C$ in (2.9) are semi-norms and define the topology on the set.

**Remark.** This type of functions is what we get as amplitudes if we expand the kernel in lemma 2.3, except that $\delta$ must be replaced by $\rho$. This is not a strong restriction since, up to a fixed number of integrations by part, the problem can be reduced to the "$\delta$-case" that we consider in this subsection (see the proof of proposition 2.5).

Our non stationary phase lemma is the following:

**Lemma 2.4.** *There exists a constant $C > 0$ which depends on a finite number of semi-norms of $f$ such that, for all $t \neq 0$ and $h \in (0,1]$, we have*

$$|\int_{(\mathbb{R}^d)^k\times(\mathbb{R}^d)^k} e^{\frac{i}{h}\phi_k} f dx_1\cdots dx_k d\xi_1\cdots d\xi_k| \leq C\langle x_{0,1}\rangle^{-\delta_1}\cdots\langle x_{0,n}\rangle^{-\delta_1}\left(\frac{h}{t}\right)^N. \tag{2.10}$$

*Here the phase function is given by*

$$\phi_k = \sum_{j=1}^k \langle x_j - x_{j+1},\xi_j\rangle - t_j\omega(\xi_j), \qquad t_1+\cdots+t_k = t$$

*where all the $t_j$'s are non negative. Moreover, we have $x_{k+1} = x_1$.*

**Proof.** We proceed by induction on $k \geq 2$. We shall do integrations by parts using the following differential operators, when $1 \leq l \leq k$, which all leave $\exp(i\phi_k/h)$ invariant

$$P_l = -\frac{h}{i}\frac{\sum_{j=1}^l t_j\nabla\omega(\xi_j)}{|\sum_{j=1}^l t_j\nabla\omega(\xi_j)|^2}.\Big(\sum_{j=1}^l \nabla_{\xi_j}\Big) - (x_{l+1}-x_1).\frac{\sum_{j=1}^l t_j\nabla\omega(\xi_j)}{|\sum_{j=1}^l t_j\nabla\omega(\xi_j)|^2}$$

where one should notice that, if $l = k$, $x_1 - x_{l+1} = 0$. We shall use as well

$$Q_l = \frac{h}{i}\frac{\xi_l-\xi_{l-1}}{|\xi_l-\xi_{l-1}|^2}.\nabla_{x_l}, \qquad l \geq 2.$$

Let's start with $k = 2$. We choose $\chi \in C_0^\infty(\mathbb{R}^d)$, $\chi \equiv 1$ near 0 , and we write $f = f_1 + f_2$, with $f_1 = \chi(\xi_1-\xi_2)f$. By choosing the support of $\chi$ small enough, and using the fact that $|\nabla\omega(\xi)| \geq \varepsilon > 0$ on $\omega^{-1}(I)$, we get

$$|t_1\nabla\omega(\xi_1) + t_2\nabla\omega(\xi_2)| \geq \frac{\varepsilon}{2}t$$

on the support of $f_1$, and we do $N$ integrations by parts with $P_2$ (which is independent of $x_1,x_2$ since $l = k$). This proves that the estimate (2.10) holds for $f_1$.

To estimate the same integral with $f_2$, we first do $N$ integrations by parts with $Q_2$ which gives a gain of $h^N\langle x_2\rangle^{-N}$. Then:
• either $t_1 \geq t/2$, and we do $N$ integrations by parts with $P_1$, which gives a gain of $\mathcal{O}(t_1^{-N}) = \mathcal{O}(t^{-N})$. We get also $\mathcal{O}(\langle x_1 - x_2\rangle^N)$ but this doesn't cause any trouble thanks to (2.9) and to to the gain of $\langle x_2\rangle^{-N}$,
• or $t_2 \geq t/2$, and we do $N$ integrations by part with $\tilde{P}_1$, which is the same operator as $P_1$ but where we swap $x_1$ and $x_2$, and $\xi_1$ and $\xi_2$. As above, we get a gain of $\mathcal{O}(t_2^{-N}) = \mathcal{O}(t^{-N})$.
Thus the result is proved when $k = 2$.

Now assume that $k \geq 3$ and that the result holds to all orders $\leq k-1$. We define, for $1 \leq l \leq k-1$, the function $\chi_l = \chi(\xi_l - \xi_{l+1})$ and then write

$$f = \chi_1 \cdots \chi_{k-1} f + (1-\chi_1) f + \sum_{l=2}^{k-1} \chi_1 \cdots \chi_{l-1}(1-\chi_l) f.$$

By choosing the support of $\chi$ small enough (and $\chi \equiv 1$ near 0), we get $|\sum_{j=1}^k t_j \nabla\omega(\xi_j)| \geq ct$ on the support of $\chi_1 \cdots \chi_{k-1}$ for some $c > 0$; then doing $N$ integrations by parts with $P_k$ shows that

$$\Big|\int e^{\frac{i}{h}\phi_k} \chi_1 \cdots \chi_{k-1} f dx_1 \cdots dx_k d\xi_1 \cdots d\xi_k\Big| \leq C\langle x_{0,1}\rangle^{-\delta_1} \cdots \langle x_{0,n}\rangle^{-\delta_1} \left(\frac{h}{t}\right)^N.$$

Now we consider the integral with $(1-\chi_1)f$ in which we can do $N$ integrations by parts with $Q_2$, from which we get a gain of $h^N\langle x_2\rangle^{-N}$. There are again two cases:
• either $t_1 \geq t/2$, and we do exactly as for $k = 2$,
• or $t_2 + \cdots + t_k \geq t/2$, and we use the induction assumption. Precisely, the function that we integrate has the following form

$$e^{\frac{i}{h}\tilde{\phi}_{k-1}} \tilde{f}(\tilde{x}_0, \tilde{x}_2, \cdots, \tilde{x}_k, \tilde{\xi}_2, \tilde{\xi}_k, h)$$

where the new variables are $\tilde{x}_0 = (x_0, x_1, \xi_1)$, $(\tilde{x}_{j-1}, \tilde{\xi}_{j-1}) = (x_j, \xi_j)$, when $2 \leq j \leq k$ and the new phase function is

$$\tilde{\phi}_{k-1} = \sum_{j=1}^{k-1} \langle \tilde{x}_j - \tilde{x}_{j+1}, \tilde{\xi}_j\rangle - t_{j+1}\omega(\tilde{\xi}_j), \qquad \tilde{x}_k = \tilde{x}_1.$$

Note that $\phi_k - \tilde{\phi}_{k-1} = \langle x_1 - x_2, \xi_1 - \xi_k\rangle - t_1\omega(\xi_1)$ so (2.9) holds for $\tilde{f}$. We must note as well that $N$ differentiations of $\tilde{f}$ with respect to $\tilde{\xi}_k$ causes a lost $h^{-N}$ but this is absorbed by the $h^N$ term that we got thanks to $Q_2$ before. Thus we get

$$\Big|\int_{(\mathbb{R}^d)^{k-1}\times(\mathbb{R}^d)^{k-1}} e^{\frac{i}{h}\tilde{\phi}_{k-1}} \tilde{f} dx_2 \cdots dx_k d\xi_2 \cdots d\xi_k\Big| \leq C\langle \tilde{x}_{0,1}\rangle^{-\delta_1} \cdots \langle \tilde{x}_{0,n+2d}\rangle^{-\delta_1} \left(\frac{h}{t}\right)^N$$

from which we get the result after integration with respect to $x_1, \xi_1$.

Finally we have to consider the integrals involving $\chi_1 \cdots \chi_{l-1}(1-\chi_l)f$ with $2 \leq l \leq k-1$. Thanks to $(1-\chi_l)$, we can integrate by parts $N$ times with $Q_{l+1}$ from which we have a gain of $h^N\langle x_l\rangle^{-N}$ and there are again two possibilities:
• either $t_1 + \cdots + t_l \geq t/2$, and then, by choosing the support of $\chi$ small enough we can integrate by parts $N$ times with $P_{l-1}$. Then we get

$$\Big|\int e^{\frac{i}{h}\phi_k} \chi_1 \cdots \chi_{l-1}(1-\chi_l) f dx_1 \cdots dx_l d\xi_1 \cdots d\xi_l\Big| \leq C\big(\frac{h}{t}\big)^N \prod_{j=1}^n \langle x_{0,j}\rangle^{-\delta_1} \prod_{j=l+1}^k \langle x_j\rangle^{-\delta}$$

from which we deduce the result easily,
• or $t_{l+1}+\cdots+t_k\geq t/2$. We use again the induction assumption, considering $(x_0,x_1,\xi_1)$ as a new variable $\tilde{x}_0$ and $x_2,\xi_2,\cdots,x_{l-1},\xi_{l-1}$ as parameters. We introduce the phase function

$$\tilde{\phi}_{k-l+1}=\sum_{j=l}^{k-1}\langle x_j-x_{j+1},\xi_j\rangle-t_j\omega(\xi_j)+\langle x_k-x_l,\xi_k\rangle-t_k\omega(\xi_k),$$

and we remark that $\phi_k-\tilde{\phi}_{k-l+1}-\langle x_l-x_1,\xi_k\rangle+\langle x_l,\xi_l\rangle$ depends only on $\tilde{x}_0$ and the parameters, and we can get the result as before. The proof is now complete. □

## 2.3 The regular terms

In that subsection we will study the distributions defined, when $\tilde{\chi}\in S_1(1,-\infty)$ and $j\geq 0$, by

$$<u(h),f>=\mathrm{Tr}\left(\tilde{\chi}(x,hD)\frac{d^j}{d\epsilon^j}f(H_\epsilon)_{|\epsilon=0}\right),\qquad f\in C_0^\infty(I).$$

It is already known that these distributions have a full asymptotic expansion in $h$ in the weak sense (see [4, 7]); this means that there exists a sequence $(u_k)_{k\in\mathbb{N}}$ of elements of $\mathcal{D}'(I)$, independent of $h$, such that for all $N\geq 0$

$$u(h)=h^{-d}\sum_{k=0}^{N}h^ku_k+h^{N+1-d}\tilde{u}_N(h),\qquad h\searrow 0, \tag{2.11}$$

where $\tilde{u}_N(h)$ is a bounded family of $\mathcal{D}'(I)$ (*i.e.* $<\tilde{u}_N(h),f>$ is bounded for all $f\in C_0^\infty(I)$). Our goal is to prove that the asymptotic (2.11) holds in the strong sense, that is in the $C^\infty$ sense. This is the purpose of the following proposition. Recall that $I\subset(0,\infty)$ is an open an non critical interval for $\omega$.

**Proposition 2.5.** *The distributions $u_k$ are $C^\infty$ on $I$ and $\tilde{u}_N(h)$ is a bounded family in $C^\infty(I)$.*

**Proof.** Because of the uniqueness of the asymptotic expansion, it is sufficient to show that $u(h)$ has a full asymptotic expansion in $C^\infty(J)$ for all $J\Subset I$. By choosing a cutoff function $g\in C_0^\infty(I)$, $g\equiv 1$ near $J$, we only have to study $gu(h)$, and we can consider its semiclassical Fourier transform which is

$$w(h,t)=\mathrm{Tr}\left(\tilde{\chi}(x,hD)\frac{d^j}{d\epsilon^j}\big(g(H_\epsilon)U_\epsilon(t)\big)\right)_{|\epsilon=0}.$$

By lemma 2.3, $w(h,t)$ is a linear combination (with coefficients independent on $h$ and $t$) of the following type of functions, with $0\leq k\leq j$:

$$h^{-k}\int_0^t\int_0^{t_1}\cdots\int_0^{t_{k-1}}\mathrm{Tr}\big(\mathcal{K}_k(t-t_1,t_1-t_2,\cdots,t_{k-1}-t_k,t_k,h)\big)dt_k\cdots dt_2dt_1, \tag{2.12}$$

where, by convention, there is no $\int$ sign if $k=0$, and just $\int_0^t\cdots dt_1$ when $k=1$. Moreover we have

$$\mathcal{K}_k(t_1,t_2,\cdots,t_{k+1},h)=a_1(x,hD,h)U_0(t_1)a_2(x,hD,h)U_0(t_2)\cdots a_{k+1}(x,hD,h)U_0(t_{k+1}), \tag{2.13}$$

with symbols $a_1,\cdots,a_{k+1}$ all supported in a small neighbourhood of $\omega^{-1}(\mathrm{supp}(g))$ with respect to the $\xi$ variable, having an asymptotic expansion with respect to $h$, and in the following classes

$$a_1\in\mathcal{S}(\mathbb{R}^{2d}),\qquad a_j\in S_1(-\infty,-\rho),\qquad 2\leq j\leq k+1.$$

The Schwartz kernel $K_k(x_1, x_{k+2})$ of $\mathcal{K}_k(t_1, \cdots, t_{k+1})$ is given by the oscillatory integral

$$\int e^{\frac{i}{h}\phi_{k+1}} a_1(x_1,\xi_1)a_2(x_2,\xi_2)\cdots a_{k+1}(x_{k+1},\xi_{k+1})dx_2\cdots dx_{k+1}d\xi_1\cdots d\xi_{k+1}, \tag{2.14}$$

where the phase function $\phi_{k+1} = \phi_{k+1}(x_1, \cdots, x_{k+2}, \xi_1, \cdots, \xi_{k+1})$ is given by

$$\phi_{k+1} = \sum_{j=1}^{k+1} \langle x_j - x_{j+1}, \xi_j\rangle - t_j\omega(\xi_j), \qquad x_{k+2} = x_1.$$

First we rewrite (2.14) as a sum of convergent integrals: using the decay of $K_k(x_1, x_{k+1})$ with respect to $x_1$, one can do $kd$ integrations by parts in (2.14) with the operator $h|x_1 - x_2|^{-2}(x_1 - x_2).\nabla_{\xi_1}$, when $x_2 \neq x_1$ (when $x_1 - x_2$ is small, the decay in $x_1$ entails the decay in $x_2$). Thus, by loosing $\mathcal{O}(t_1^{kd})$, we get a decay of $\langle x_2\rangle^{-kd-\rho}$. By repeating this operation with the other $x_j$ variables, we can assume that the symbol is integrable with respect to each $x_j$ since it decays like $\langle x_j\rangle^{-(k+2-j)d-\rho}$ with respect to each $x_j$, with $k+2-j \geq 1$. Notice that this generates a factor $\mathcal{O}(t^{k(k+1)d/2})$ but anyway, we are now in position to use the lemma 2.4: it proves that $w(t,h) = \mathcal{O}((h/t)^\infty)$, when $|t| \geq c$ and $h \in (0, h_0]$, thus, if $\varphi \in C_0^\infty(-2c, 2c)$ is such that $\varphi \equiv 1$ on $(-c, c)$, we have

$$\lambda \mapsto (2\pi h)^{-1} \int e^{\frac{i}{h}t\lambda}(1-\varphi)(t)w(t,h)dt = \mathcal{O}(h^\infty) \ \text{ in } C^\infty.$$

Now we consider the region $|t| \leq 2c$. The operator $\mathcal{K}_k$ appears in (2.12) as the product of $a_0(x, hD, h)U_0(t)$ and of the operators $U_0(-t_j)a_j(x, hD, h)U_0(t_j)$, $1 \leq j \leq k$, with $|t_j| \leq |t|$, which are, by the Egorov's theorem, pseudo-differential operators. Their product can be written $\tilde{a}_k(x, hD, t, k)$, with $\tilde{a}_k$ depending smoothly on $t$. Hence $\int \exp(it\lambda/h)w(t,h)\varphi(t)dt$ can be written as a linear combination of the following functions

$$h^{-k}\int e^{\frac{i}{h}t\lambda}\mathrm{Tr}(a_0(x,hD,h)U_0(t)\tilde{a}_k(x,hD,t,h))\varphi(t)dt$$

which all have a full expansion in $C^\infty(J)$ (in increasing integer powers of $h$) thanks to the stationary phase theorem, and this completes the proof. $\square$

## 2.4 Proof of theorem 2.1

We have to investigate, for $\lambda \in J \Subset I$, the following distribution

$$u_p(\lambda, h) = u_{1,1}(\lambda, h) - \sum_{j=0}^{p-1}\frac{1}{j!}\partial_\epsilon^j u_{1,\epsilon}(\lambda,h)|_{\epsilon=0}$$

where, in the weak sense, we have

$$u_{1,\epsilon}(\lambda, h) = \mathrm{Tr}\left(f_0(H_\epsilon)\frac{\partial E_\epsilon}{\partial\lambda} - f_0(H_0)\frac{\partial E_0}{\partial\lambda}\right),$$

with $f_0 \in C_0^\infty(I)$ and $f_0 \equiv 1$ near $J$. Actually, $u_{1,\epsilon}$ is well defined, if the coefficients of the perturbation $V$ decay as $\langle x\rangle^{-\mu}$ for some $\mu > d$. That is why we approximate the principal symbol of $V$ by a elements of $\mathcal{B}$. So we first assume that $q \in \mathcal{B}$.

Let $\chi \in C_0^\infty$ be a cutoff function such that $\chi \equiv 1$ on the ball of radius $R$. Thanks to the result of the previous subsection, we only have to consider

$$\operatorname{Tr}\left(f_0(H_\epsilon)\frac{\partial E_\epsilon}{\partial \lambda}(1-\chi^2) - f_0(H_0)\frac{\partial E_0}{\partial \lambda}(1-\chi^2)\right), \tag{2.15}$$

since the others terms are $\operatorname{Tr}\big(\chi f(H_1)E_1'(\lambda, h)\chi\big)$ plus $p$ smooth terms with asymptotic expansion in $h$ (proposition 2.5). We split (2.15) into two terms that we are going to study separately:

$$\operatorname{Tr}\left(\left(\frac{\partial E_\epsilon}{\partial \lambda} - \frac{\partial E_0}{\partial \lambda}\right)(1-\chi^2)f(H_0)\right), \tag{2.16}$$

$$\operatorname{Tr}\left(\frac{\partial E_\epsilon}{\partial \lambda}(1-\chi^2)\big(f_0(H_\epsilon) - f_0(H_0)\big)\right). \tag{2.17}$$

Notice that these two formulas make sense when tested against a Schwartz function (which is actually the rigorous way to define it). The traces are well defined since $q \in \mathcal{B}$.

### 2.4.1 Study of (2.16)

We use a microlocal partition of unit

$$(1-\chi^2)f(H_0) = \chi^+(x, hD) + \chi^-(x, hD) \tag{2.18}$$

with supp $\chi^\pm \subset \Gamma^\pm(R, J, \sigma^\pm)$ (see the next section for the notations) and then by a usual trick (see Robert [33, 34]) we are left to the study, after a semiclassical Fourier transform in $\lambda$, of

$$\operatorname{Tr}\left(U_\epsilon(t)\chi_\pm(x, hD) - U_0(t)\chi_\pm(x, hD)\right), \qquad \pm t \geq 0. \tag{2.19}$$

From now on we only consider the case $+$ (the case $-$ is completely analogous). Using theorem 3.5 and the notation of (3.2), we can write (2.19) as the sum

$$\operatorname{Tr}\left(A_N^+(h,\epsilon)U_0(t)B_N^+(h,\epsilon)^\star - A_N^+(h,0)U_0(t)B_N^+(h,0)^\star\right) + h^N \operatorname{Tr}\left(R_N^+(h,\epsilon,t)\right), \qquad t \geq 0. \tag{2.20}$$

Using the form of $R_N^+(h,\epsilon,t)$ given by theorem 3.5 and the lemma 2.4, we see easily that the inverse Fourier transform of $\partial_\epsilon^j \operatorname{Tr}\left(R_N^+(h,\epsilon,t)\right)_{|\epsilon=0}$ is $\mathcal{O}(h^{n(N)})$ in $C^{n(N)}(J)$, with $n(N) \to \infty$, when $j \geq 1$. The term for $j = 0$ can be written as $Tr(\Theta_{N,1}^+(\lambda, h)R(\lambda + i0, h))$ with a notation similar to the one of theorem 2.1 and is the same as for the usual SSF (see Robert [34]).

We still have to consider the first term of (2.20) that we can write, using the cyclicity trick of Robert [34] (see [7] for its justification)

$$\operatorname{Tr}\left(\left(B_N^+(h,\epsilon)^\star A_N^+(h,\epsilon) - B_N^+(h,0)^\star A_N^+(h,0)\right)U_0(t)\right)$$

which, by (3.14), can be easily studied. The symbol $r_N(x,\xi,\epsilon,h)$ is smooth with respect to $\epsilon$ and decays as $\langle x\rangle^{-N}$ (see estimate (3.19)), thus after Taylor formula and inverse Fourier transform we get a term which is $\mathcal{O}(h^{n(N)})$ in $C^{n(N)}(J)$.

### 2.4.2 Study of (2.17)

First we remark that, up to terms that can be studied as above, we can insert a factor $f_1(H_0)$ between $(1-\chi^2)$ and $f_0(H_\epsilon) - f(H_0)$ in (2.17), where $f_1 \in C_0^\infty(I)$ and $f_1 \equiv 1$ near supp$f_0$. This is an easy consequence of lemma 2.2. Thus we get $(1-\chi^2)f_1(H_0)$ that we split into two terms

$\tilde{\chi}^{\pm}(x, hD)$, using the same kind of partition as for (2.18). Then we obtain the following kind of terms (after Fourier transform)

$$\operatorname{Tr}\left(U_0(t)\tilde{B}_N^{\pm}(h,\epsilon)^{\star}\big(f_0(H_\epsilon)-f_0(H_0)\big)\tilde{A}_N^{\pm}(h,\epsilon)\right), \tag{2.21}$$

where $\tilde{A}_N^{\pm}(h,\epsilon)$ and $\tilde{B}_N^{\pm}(h,\epsilon)$ are associated to $\tilde{\chi}^{\pm}(x,hD)$. There are also remainder terms due to $\tilde{R}_N^{\pm}(h,\epsilon,t)$ (see formula (3.2)), but they can be treated as in (2.16). Then we use the expansion of $f(H_\epsilon)$ into pseudodifferential operators $f(H_\epsilon) \sim \sum_j h^k c_k(x, hD, \epsilon)$, where $\partial_{x,\xi}^{\gamma}\partial_\epsilon^j c_k(x,\xi,\epsilon) = \mathcal{O}(\langle x\rangle^{-j\rho})$ (see [4, 7] for details). We get terms like $Tr\left(U_0(t)\mathcal{A}_N^{\pm}(\epsilon,h)\right)$ where

$$\mathcal{A}_N^{\pm}(\epsilon,h) = \tilde{B}_N^{\pm}(h,\epsilon)^{\star}c_k(x,hD,\epsilon)\tilde{A}_N^{\pm}(h,\epsilon).$$

We perform the Taylor expansion with respect to $\epsilon$; the estimates on the symbols of $A_N^+(\epsilon,h)$ and $B_N^+(\epsilon,h)$ given by (3.12), (3.13) and proposition (3.4) show that

$$\mathcal{A}_N^{\pm}(1,h) - \sum_{j=0}^{p-1}\frac{1}{j!}\partial_\epsilon^j\mathcal{A}_N^{\pm}(\epsilon,h)_{|\epsilon=0} = \tilde{a}_N(x,hD,h)$$

with a symbol $\tilde{a}_N \in S_1(-\infty,-p\rho)$ (compactly supported in $\omega^{-1}(I)$ with respect to $\xi$) with a full asymptotic expansion in $h$, and after inverse Fourier transform, we get a smooth function with complete asymptotic expansion in $h$, by the same method as in [34].

### 2.4.3 conclusion

By the previous steps, the theorem is proved when $q \in S_1(\omega,-\mu)$. All our constructions depend continuously on $q \in \mathcal{B}$ for the $S_1(\omega,-\rho)$ topology, therefore by an easy density argument we get the result in the general case, and the proof is complete. □

# 3 Isozaki-Kitada's parametrix with parameters

## 3.1 A review of the construction

Let $I$ be an open interval in $(0,\infty)$, non critical for $\omega$, *i.e.* $\omega(\xi) \in I \Rightarrow \nabla\omega(\xi) \neq 0$. For all $R > 0$, $J$ open interval such that $J \Subset I$ and $-1 < \sigma < 1$ one defines the outgoing area $\Gamma^+(R,J,\sigma)$ (resp. incoming $\Gamma^-$) by

$$\Gamma^{\pm}(R,J,\sigma) = \{(x,\xi) \in \mathbb{R}^{2d} \mid |x| > R,\ \omega(\xi) \in J,\ \pm\cos\big(x, v(\xi)\big) > \sigma\} \tag{3.1}$$

where $v(\xi) = \nabla\omega(\xi)$ and $\cos(x,y) = \langle\hat{x},\hat{y}\rangle$ with $\hat{z} = z/|z|$.

The Isozaki-Kitada parametrix (see [24]) is a microlocal approximation of $U_\epsilon(t) = \exp(-itH_\epsilon/h)$ of the following form

$$U_\epsilon(t)\chi_{\pm}(x,hD) = A_N^{\pm}(h,\epsilon)U_0(t)B_N^{\pm}(h,\epsilon)^{\star} + h^N R_N^{\pm}(h,\epsilon,t), \qquad \pm t \geq 0. \tag{3.2}$$

Here $\chi_+$ (resp. $\chi_-$) is supported in the outgoing (resp. incoming) area $\Gamma^+(R,J,\sigma_+)$ (resp. $\Gamma^-(R,J,\sigma^-)$), $R_N^{\pm}(h,\epsilon,t)$ are remainders that we'll describe later, and $A_N^{\pm}(h,\epsilon), B_N^{\pm}(h,\epsilon)$ are FIO with phase functions *independent* of $t$, given by $A_N^{\pm}(h,\epsilon) = \sum_{k\leq N} h^k J_{\varphi^{\pm}}(a_k^{\pm})$ where

$$J_{\varphi^{\pm}}(a^{\pm})u(x) = (2\pi h)^{-d}\iint e^{i(\varphi^{\pm}(x,\xi,\epsilon,h)-\langle y,\xi\rangle)/h}a^{\pm}(x,\xi,\epsilon,h)u(y)dyd\xi, \qquad u \in \mathcal{S}(\mathbb{R}^d),$$

and the same with $b$ instead of $a$ for $B$.

Usually, the phase functions $\varphi^{\pm}$ and the symbols $a_k^{\pm}, b_k^{\pm}$ are independent of $h$; here we allow a $h$-dependance, for some minor technical reasons, and the reader may forget it in the first reading. Moreover, when $H_\epsilon = -h^2\Delta + \epsilon V$, the phases doesn't depend on $h$. This dependance is due to the fact that the principal symbol of $V$ can depend on $h$.

The main point of this section is to study the behavior of Isozaki-Kitada's parametrix with respect to $\epsilon$, especially when one differentiates with respect to this parameter. The usual estimates on $\varphi^{\pm}$ are $|\partial_x^\alpha(\varphi^{\pm}(x,\xi) - \langle x, \xi\rangle)| = \mathcal{O}(\langle x\rangle^{1-\rho-|\alpha|})$ and, here, our goal is to show that $\varphi^{\pm}$ depends smoothly on $\epsilon$ and that we get a power $\langle x\rangle^{-\rho}$ at each differentiation with respect to $\epsilon$. We also need to control the constructions with respect to the principal symbol of the perturbation, $q \in \mathcal{B}$ (see the previous section).

We only treat the outgoing case, since the incoming is similar and we drop the supscrit + on the phases and the symbols.

Let $\sigma_0 \in (-1,1)$, and $J_0$ an open interval such that $J_0 \Subset I$. We shall use the following notation

$$H_\epsilon^{cl}(x,\xi,\epsilon,h) = \omega(\xi) + \epsilon q(x,\xi,h).$$

This first proposition solves the Hamilton-Jacobi equation (3.3) and lists the properties of the phase.

**Proposition 3.1.** *There exists $R_0 > 0$ such that for all $q \in \mathcal{B}$, one can find a function $\varphi \in C^\infty(\mathbb{R}^{2d} \times [0,1] \times [0,h_0], \mathbb{R})$ satisfying*

$$H_\epsilon^{cl}(x, \partial_x\varphi(x,\xi,\epsilon,h), h) = \omega(\xi), \tag{3.3}$$

*for all $(x,\xi) \in \Gamma^+(R_0, J_0, \sigma_0)$, $\epsilon \in [0,1]$ and $h \in [0,h_0]$. $\varphi$ satisfies the additional conditions:*

$$\varphi(x,\xi,0,h) = \langle x,\xi\rangle, \tag{3.4}$$

$$|\nabla_x^t \nabla_\xi \varphi(x,\xi,\epsilon,h) - 1_d| \leq 1/2, \tag{3.5}$$

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\epsilon^n \partial_h^l \big(\varphi(x,\xi,\epsilon,h) - \langle x,\xi\rangle\big)| \leq C(\alpha,\beta,n,l)\langle x\rangle^{1-|\alpha|-\rho n_+}, \tag{3.6}$$

*for all $(x,\xi,\epsilon,h) \in \mathbb{R}^{2d} \times [0,1] \times [0,h_0]$ and $\alpha,\beta,n,l$, with the notation $n_+ = \max(1,n)$. Moreover we can choose the function $\varphi$ depending continuously on $q$, that is*

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\epsilon^n \partial_h^l \big(\varphi_1(x,\xi,\epsilon,h) - \varphi_2(x,\xi,\epsilon,h)\big)| \leq C'(\alpha,\beta,n,l)\mathcal{N}(q^1 - q^2)\langle x\rangle^{1-|\alpha|-\rho n_+} \tag{3.7}$$

*for some semi-norm $\mathcal{N}$ on $S_1(\omega, -\rho)$, when $\varphi_j$ is associated to $q^j$, $j = 1,2$.*

The proof of this proposition will be established in the next subsection. In the present subsection, we are only interested in the construction of the parametrix and we quote the results that we are going to use.

Once the phase function is defined, we have to consider $X = X(t,x,\xi,\epsilon,h)$, the solution of the following differential equation

$$\begin{cases} \dot{X} = \Theta(X,\xi,\epsilon,h), \\ X_{|_{t=0}} = x, \end{cases}$$

where $\dot{X} = \partial_t X$ and $\Theta$ is the vector field defined by

$$\Theta(x,\xi,\epsilon,h).\nabla_x = (\nabla_\xi H_\epsilon^{cl})\big(x, \partial_x\varphi(x,\xi,\epsilon,h), h\big).\nabla_x.$$

Let $J_1$ be an open interval such that $J_1 \Subset J_0$, and let $\sigma_1$ be such that $-1 < \sigma_1 < \sigma_0$.

**Proposition 3.2.** *There exists $R_1 > 0$ and $e_1 > 0$ such that, for all $(x,\xi) \in \Gamma^+(R_1, J_1, \sigma_1)$, $(\epsilon, h) \in [0,1] \times [0, h_0]$, and $q \in \mathcal{B}$, the solution $X(t,x,\xi,\epsilon,h)$ is defined for $t \geq 0$, and satisfies*

$$|X(t,x,\xi,\epsilon,h)| \geq e_1(t + |x|), \tag{3.8}$$

$$\big(X(t,x,\xi,\epsilon,h), \xi\big) \in \Gamma^+(R_0, J_0, \sigma_0). \tag{3.9}$$

*Moreover, for all $\alpha, \beta, n, l$ there exists $C > 0$ such that*

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\epsilon^n \partial_h^l \big(X(t,x,\xi,\epsilon,h) - x - tv(\xi)\big)| \leq C\langle t\rangle \langle x\rangle^{-|\alpha| - \rho n_+}, \tag{3.10}$$

*for all $(x,\xi) \in \Gamma^+(R_1, J_1, \sigma_1)$, $(\epsilon, h) \in [0,1] \times [0, h_0]$, $t \geq 0$ and $q \in \mathcal{B}$.*
*More generally, if the supscripts $j$ refer to functions associated to $q^j$, $j = 1, 2$, then, for all $\alpha, \beta, n, l$, there exists a constant $C$ and a semi-norm $\mathcal{N}$ in $S_1(\omega, -\rho)$ such that*

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\epsilon^n \partial_h^l \big(X_1 - X_2\big)(t,x,\xi,\epsilon,h)| \leq C\mathcal{N}(q^1 - q^2)\langle t\rangle \langle x\rangle^{-|\alpha| - \rho n_+} \tag{3.11}$$

*with respect to the same variables as in* (3.10).

**Proof.** We use the proposition $A.5$ to prove (3.8) and (3.9), and proposition $A.1$, applied to $y = X - x - tv(\xi)$ and $\eta = 0$ to prove (3.10). The details are essentially the same as for the previous proposition and we don't go into details. □

Now we are in position to define the symbols $a_0, a_1, \cdots, a_N$ in $\Gamma^+(R_1, J_1, \sigma)$. For $j = 0$

$$a_0(x,\xi,\epsilon,h) \quad = \quad \exp\left(\int_0^{+\infty} p_0\big(X(t,x,\xi,\epsilon,h), \xi, \epsilon, h\big) dt\right),$$

and for $j \geq 1$ we have

$$a_j(x,\xi,\epsilon,h) \quad = \quad \int_0^{+\infty} p_j\big(X(t,x,\xi,\epsilon,h), \xi, \epsilon, h\big) \exp\left(\int_0^t p_0\big(X(s,x,\xi,\epsilon,h), \xi, \epsilon, h\big) ds\right) dt.$$

The symbols $p_0, p_1, \cdots$ are defined inductively by

$$\begin{aligned} p_0(x,\xi,\epsilon,h) &= \frac{1}{2} tr\left(\big(\partial^2_{\xi,\xi} H^{cl}_\epsilon\big)(x, \partial_x\varphi(x,\xi,\epsilon,h), h).\big(\partial^2_{x,x}\varphi\big)(x,\xi,\epsilon,h)\right), \\ p_j &= \sum_{j_0 + j_1 + j_2 = j+1, \ j_1 < j} (\tilde{H}^{cl}_{j_0} \# a_{j_1})_{j_2}, \qquad j \geq 1, \end{aligned}$$

where $\tilde{H}^{cl}_{j_0} = q_{j_0}$ if $j_0 > d$, $\tilde{H}^{cl}_0 = H^{cl}_\epsilon$, $\tilde{H}^{cl}_{j_0} = 0$ for $1 \leq j_0 < d$, and $(H \# a)_j$ is the $j^{th}$ term in the expansion of the symbol of $H(x, hD) J_\varphi(a)$. Then we have the following estimates, if $(x,\xi) \in \Gamma^+(R_1, J_1, \sigma)$, $\epsilon \in [0,1]$, $h \in [0, h_0]$ and $q \in \mathcal{B}$:

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\epsilon^n \partial_h^l (a_0(x,\xi,\epsilon,h) - 1)| \quad \leq \quad C\langle x\rangle^{-|\alpha| - \rho n}, \tag{3.12}$$

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\epsilon^n \partial_h^l a_k(x,\xi,\epsilon,h)| \quad \leq \quad C\langle x\rangle^{-k - |\alpha| - \rho n}, \qquad k \geq 1. \tag{3.13}$$

Note that all these symbols depend continuously on $q \in \mathcal{B}$, because of (3.7) and (3.11). The estimate (3.12) proves that, by choosing $R_1$ large enough, one can assume that $a_0 \geq 1/2$ in the outgoing area that we consider ($A_N(\epsilon, h)$ is then said to be elliptic in this area). Then, after multiplication by a cutoff function in $S_1(-\infty, 0)$ supported in $\Gamma^+(R_1, J_1, \sigma)$ and which is $\equiv 1$ near $\Gamma^+(R_2, J_2, \sigma_2)$, for arbitrary $R_2 > R_1$, $J_2 \Subset J_1$ and $-1 < \sigma_2 < \sigma_1$, the symbols $a_0, \cdots, a_N$ are globally defined.

Now we can construct the symbols $b_0, b_1, \cdots, b_N$; they are defined inductively by requiring that

$$A_N(h,\epsilon)B_N(h,\epsilon)^\star = \chi_+(x,hD) + h^{N+1}r_N(x,hD,\epsilon,h), \tag{3.14}$$

where $r_N$ decays faster and faster in the $x$ variable (see the estimate (3.19)). This is, of course, a well known consequence of the Egorov's theorem. However, since our result is based on a careful control of the constructions with respect to $\epsilon$, we recall a few details. The kernel of $A_N(h,\epsilon)B_N(h,\epsilon)^\star$ is the sum, when $k_1, k_2 \leq N$ of

$$\frac{h^{k_1+k_2}}{(2\pi h)^d}\int e^{i(\varphi(x,\xi,\epsilon,h)-\varphi(y,\xi,\epsilon,h))/h} a_{k_1}(x,\xi,\epsilon,h)\overline{b_{k_2}(y,\xi,\epsilon,h)}d\xi. \tag{3.15}$$

Then, by Taylor's formula, we write

$$\varphi(x,\xi,\epsilon,h) - \varphi(y,\xi,\epsilon,h) = \langle x-y, \tilde{\eta}(x,y,\xi,\epsilon,h)\rangle$$

where $\tilde{\eta}(x,y,\xi,\epsilon,h) = \int_0^1 \nabla_x\varphi(y+u(x-y),\xi,\epsilon,h)du$ satisfies the properties which we sum up in the following lemma.

**Lemma 3.3.** *i) The map $\xi \mapsto \tilde{\eta}(x,y,\xi,\epsilon,h)$ is a global diffeomorphism of $\mathbb{R}^d$.*
*ii) If we call $\tilde{\xi}(x,y,.,\epsilon,h)$ its inverse, we have the following estimates*

$$C_0^{-1}\langle\eta\rangle \leq \langle\tilde{\xi}(x,y,\eta,\epsilon,h)\rangle \leq C_0\langle\eta\rangle, \qquad \textit{for some } C_0 > 0, \tag{3.16}$$

$$|\partial_x^\alpha\partial_y^{\alpha'}\partial_\eta^\beta\partial_\epsilon^n\partial_h^n(\tilde{\xi}(x,y,\eta,\epsilon,h)-\eta)| \leq C_{\alpha,\alpha',\beta,n,l}\langle x\rangle^{-m}\langle y\rangle^{-m'-\rho n_+}\langle x-y\rangle^{m+m'+\rho n_+} \tag{3.17}$$

*for all $x,y,\eta \in \mathbb{R}^d$, $\epsilon \in [0,1]$, $h \in [0,h_0]$, and $m, m'$ such that $m \leq |\alpha|$, $m' \leq |\alpha'|$. Moreover all these estimates are uniform with respect to $q \in \mathcal{B}$ and we have continuity estimates for $\tilde{\xi}$ analogous to* (3.7).

**Proof.** The fact that we get a global diffeomorphism is just due to (3.5) and to a global inversion theorem of [37]. Then the estimates are simple consequences of proposition 3.1. □

After the change of variable $\xi = \tilde{\xi}(x,y,\eta,\epsilon,h)$, (3.15) can be written as the sum, for $k_1, k_2 \leq N$, of the kernels of $(2\pi h)^{-d}h^{k_1+k_2}c_{k_1,k_2}(x,hD,\epsilon,h)$, where

$$c_{k_1,k_2}(x,\eta,\epsilon,h) \sim \sum_\alpha h^{|\alpha|}\frac{1}{\alpha!}\partial_y^\alpha D_\eta^\alpha\left(a_{k_1}(x,\tilde{\xi},\epsilon,h)\overline{b_{k_2}(y,\tilde{\xi},\epsilon,h)}|\frac{\partial\tilde{\xi}}{\partial\eta}|\right)_{|y=x} \tag{3.18}$$

in which one must keep in mind that $\tilde{\xi} = \tilde{\xi}(x,y,\eta,\epsilon,h)$. This allows to compute explicitly $b_0, \cdots, b_N$ by saying that the right hand side of (3.18) must be equal to $\chi_+(x,\eta)$. This yields

$$\overline{b_0(x,\xi,\epsilon,h)} = \chi_+(x,\tilde{\eta}(x,x,\xi,\epsilon,h))\left(a_0(x,\xi,\epsilon,h)|\frac{\partial\tilde{\xi}}{\partial\eta}(x,x,\tilde{\eta}(x,x,\xi,\epsilon,h),\epsilon,h)|\right)^{-1}$$

which is well defined since the term in $(\cdots)^{-1}$ cannot vanish on the support of $\chi_+(.,\tilde{\eta}(.,.,.,\epsilon,h))$. We don't go any further into details and summarize the results on the symbols $b_0,\cdots,b_N$ in the following proposition. Let $J_3 \Subset J_2$ and $-1 < \sigma_3 < \sigma_2$.

**Proposition 3.4.** *If $\chi_+$ is supported in $\Gamma^+(R_3,J_3,\sigma_3)$ with $R_3$ large enough, one can find symbols $b_0,\cdots,b_N$ supported in $\Gamma^+(R_2,J_2,\sigma_2)$, depending smoothly on $\epsilon, h$ such that:*

$$|\partial_x^\alpha\partial_\xi^\beta\partial_\epsilon^n\partial_h^l b_k(x,\xi,\epsilon,h)| \leq C\langle x\rangle^{-k-|\alpha|-\rho n}, \qquad 0 \leq k \leq N,$$

*and $b_k$ depends continuously, in $S_1(-\infty,-k)$, on $q \in \mathcal{B}$. Moreover* (3.14) *holds with $r_N(.,.,\epsilon,h)$ in a bounded subset of $S_1(-\infty,-N)$, depending continuously on $q$ and such that*

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\epsilon^n r_N(x,\xi,\epsilon,h)| \leq C\langle x\rangle^{-N-|\alpha|-\rho n}, \tag{3.19}$$

All what we have done leads to the following theorem.

**Theorem 3.5.** *For all $J \Subset I$, and $-1 < \sigma_+ < 1$, there exists $R > 0$, such that, if* supp $\chi_+ \subset \Gamma^+(R,J,\sigma_+)$, *then $U_\epsilon(t)\chi_+(x,hD) - A_N(\epsilon,h)U_0(t)B_N(\epsilon,h)^\star$ is equal to*

$$h^N \int_0^t U_\epsilon(t-s)K_N(h,\epsilon,s)ds + h^{N+1}U_\epsilon(t)r_N(x,hD,\epsilon,h), \tag{3.20}$$

*for all $t \geq 0$, $h \in [0,h_0]$, $\epsilon \in [0,1]$ and $q \in \mathcal{B}$. $r_N(.,.,\epsilon,h)$ is a symbol in $S_1(-\infty,-N)$ which depends smoothly on $\epsilon$, and is bounded with respect to $h$. The kernel $\mathcal{K}_N(x,y,h,\epsilon,s)$ of $K_N(h,\epsilon,s)$ satisfies the following estimates:*

$$\partial_x^\alpha \partial_y^\beta \partial_\epsilon^n \mathcal{K}_N(x,y,h,\epsilon,s) = \mathcal{O}(\langle x\rangle^{-[N/4]-|\alpha|}\langle y\rangle^{-[N/4]-|\beta|}\langle s\rangle^{-[N/4]}),$$

*for all $s \geq 0$, $x,y \in \mathbb{R}^d$, $h \in (0,h_0]$, $\epsilon \in [0,1]$ and uniformly with respect to $q \in \mathcal{B}$.*

## 3.2 Proof of proposition 3.1

Following [24, 16, 34], we first recall the formula which defines $\varphi$ in some suitable outgoing area

$$\varphi(x,\xi,\epsilon,h) = \langle x,\xi\rangle + \int_0^{+\infty} \frac{\partial F}{\partial t}(t,x,\xi,\epsilon,h)dt \tag{3.21}$$

where we have, for some $R > 0$ big enough (whose existence will be proved below)

$$\begin{aligned} \frac{\partial F}{\partial t}(t,x,\xi,\epsilon,h) &= H_\epsilon^{cl}(\tilde{z}(t,x,\xi,\epsilon,h),\xi,h) - H_\epsilon^{cl}(\tilde{z}(t,Rv(\xi),\xi,\epsilon,h),\xi,h), & (3.22)\\ \tilde{z}(t,x,\xi,\epsilon,h) &= \overline{x}(t,x,\overline{\eta}(t,x,\xi,\epsilon,h),\epsilon,h) & (3.23)\end{aligned}$$

with $\overline{\eta}(t,x,.,\epsilon,h)$ the inverse of the diffeomorphism

$$\Gamma_x^+(R',J',\sigma') \ni \eta \mapsto \overline{\xi}(t,x,\eta,\epsilon,h) \tag{3.24}$$

which is well defined on

$$\Gamma_x^+(R',J',\sigma') = \{\eta \in \mathbb{R}^d \mid (x,\eta) \in \Gamma^+(R',J',\sigma')\} \tag{3.25}$$

for a suitable outgoing area $\Gamma^+(R',J',\sigma')$ (see lemma 3.7 below). Everywhere, $(\overline{x}(t,.),\overline{\xi}(t,.))$ denotes the classical flow, that is the solution to

$$\begin{cases} \partial_t \overline{x} = \partial_\xi H_\epsilon^{cl}(\overline{x},\overline{\xi},h), & \overline{x}(0,x,\xi,\epsilon,h) = x, \\ \partial_t \overline{\xi} = -\partial_x H_\epsilon^{cl}(\overline{x},\overline{\xi},h), & \overline{\xi}(0,x,\xi,\epsilon,h) = \xi. \end{cases}$$

The proof just consists in a technical review of proofs of [16, 34] to show that all the objects can be constructed and estimated uniformly with respect to the parameters $\epsilon, h$ and $q$.

Let $J \Subset I$, and $\sigma \in (-1,1)$. The first step is to estimate the classical flow; this is the goal of the the next lemma where, if $q_1, q_2 \in \mathcal{B}$, we denote by $(\overline{x}_1,\overline{\xi}_1)$ and $(\overline{x}_2,\overline{\xi}_2)$ the respective Hamiltonian flows of $\omega + \epsilon q_1$ and $\omega + \epsilon q_2$.

**Lemma 3.6.** *There exists $R > 0$ such that, for all $\gamma$ defined by $\partial^\gamma = \partial_x^\alpha \partial_\xi^\beta \partial_\epsilon^n \partial_h^l$, one can find $C > 0$ and $\mathcal{N}$ semi-norm in $\mathcal{S}_1(\omega, -\rho)$ such that*

$$\begin{aligned} |\partial^\gamma(\overline{x}_1 - \overline{x}_2)(t,x,\xi,\epsilon,h)| &\leq C\mathcal{N}(q_1 - q_2)\langle t\rangle\langle x\rangle^{-|\alpha|-\rho n_+}, \\ |\partial^\gamma(\overline{\xi}_1 - \overline{\xi}_2)(t,x,\xi,\epsilon,h)| &\leq C\mathcal{N}(q_1 - q_2)\langle x\rangle^{-|\alpha|-\rho n_+}, \end{aligned}$$

*for all $(x,\xi) \in \Gamma^+(R,J,\sigma)$, $(\epsilon,h) \in [0,1] \times [0,h_0]$, $t \geq 0$ and $q_1, q_2 \in \mathcal{B}$.*

**Proof.** By reviewing the proof of lemma 2.2 of [16] (see also proposition $A.5$ of this paper), one get easily the existence of $R > 0$ and $c > 0$ such that

$$\begin{aligned} |\overline{x}(t,x,\xi,\epsilon,h) - x - tv(\xi)| &\leq c\langle t\rangle\langle x\rangle^{-\rho}, && (3.26) \\ |\overline{\xi}(t,x,\xi,\epsilon,h) - \xi| &\leq c\langle x\rangle^{-\rho}, && (3.27) \\ |\overline{x}(t,x,\xi,\epsilon,h)| &\geq c^{-1}(|x| + t) && (3.28) \end{aligned}$$

for all $t \geq 0$, $q \in \mathcal{B}$, $\epsilon \in [0,1]$, $h \in [0,h_0]$ and $(x,\xi) \in \Gamma^+(R,J,\sigma)$. From this we deduce that our lemma is true when $q_2 \equiv 0$; indeed, by considering $y = \partial_\epsilon \overline{x}$ and $\eta = \partial_\epsilon \overline{\xi}$, wee see that they are solutions of the system $(A.1)$ with $y_0 = \partial_\xi q(\overline{x},\overline{\xi})$, $\eta_0 = -\partial_x q(\overline{x},\overline{\xi})$ (verifying $(A.4)$ with $\mu = 0$) and

$$\begin{aligned} A &= \epsilon\partial^2_{x,\xi} q(\overline{x},\overline{\xi}), & B &= \partial^2_{\xi,\xi}\omega(\overline{\xi}) + \epsilon\partial^2_{\xi,\xi} q(\overline{x},\overline{\xi}), \\ C &= -\epsilon\partial^2_{x,x} q(\overline{x},\overline{\xi}), & D &= -\epsilon\partial^2_{x,\xi} q(\overline{x},\overline{\xi}), \end{aligned}$$

which satisfies $(A.2)$, $(A.3)$, because $q \in \mathcal{B}$ and thanks to the estimates (3.26), (3.27) and (3.28) . More generally, by induction on $|\gamma| \geq 0$, we get the estimates on $\partial^\gamma\partial_\epsilon\overline{x}$, $\partial^\gamma\partial_\epsilon\overline{\xi}$ by mean of the proposition $A.1$. This proves the result when $q_2 \equiv 0$. Then we get the general result by considering the system $(A.1)$ with $y = \overline{x}_2 - \overline{x}_1$, $\eta = \overline{\xi}_2 - \overline{\xi}_1$ and

$$\begin{aligned} A &= \int_0^1 \epsilon\partial^2_{x,\xi} q_2(\overline{x}_1 + u(\overline{x}_2 - \overline{x}_1), \overline{\xi}_1)du, \\ B &= \int_0^1 \partial^2_{\xi,\xi}\omega(\overline{\xi}_1 + u(\overline{\xi}_2 - \overline{\xi}_1)) + \epsilon\partial^2_{\xi,\xi} q_2(\overline{x}_2, \overline{\xi}_1 + u(\overline{\xi}_2 - \overline{\xi}_1))du, \\ C &= -\int_0^1 \epsilon\partial^2_{x,x} q_2(\overline{x}_1 + u(\overline{x}_2 - \overline{x}_1), \overline{\xi}_2)du, \\ D &= -\int_0^1 \epsilon\partial^2_{x,\xi} q_2(\overline{x}_1, \overline{\xi}_1 + u(\overline{\xi}_2 - \overline{\xi}_1))du, \end{aligned}$$

and with the second terms $y_0$ and $\eta_0$ given by

$$\begin{aligned} y_0 &= \epsilon\left(\partial_\xi q_2(\overline{x}_1,\overline{\xi}_1) - \partial_\xi q_1(\overline{x}_1,\overline{\xi}_1)\right), \\ \eta_0 &= \epsilon\left(\partial_x q_1(\overline{x}_1,\overline{\xi}_1) - \partial_x q_2(\overline{x}_1,\overline{\xi}_1)\right). \end{aligned}$$

By increasing $R$ if necessary (uniformly with respect to $\epsilon, h, q$) so that, for some $c' > 0$

$$|\overline{x}_1 + u(\overline{x}_2 - \overline{x}_1)| \geq c'(|x| + t), \qquad \forall\, u \in [0,1],$$

we see that $A, B, C, D, y_0$ and $\eta_0$ satisfy $(A.2)$, $(A.3)$ and $(A.4)$. This allows us to use again proposition $A.1$ from which we get the result by applying $\partial^\gamma$ to this new system, arguing by induction on $|\gamma| \geq 0$ (remark that because of the forms of $y_0$ and $\eta_0$ we'll get estimates with $\mathcal{N}(q_2 - q_1)$). □

**Lemma 3.7.** *There exists $\tilde{R} \geq R$ such that, for all $|x| > \tilde{R}$, $t \geq 0$, $(\epsilon, h) \in [0,1] \times [0, h_0]$ and $q \in \mathcal{B}$*

$$\Gamma_x^+(\tilde{R}, J, \sigma) \ni \eta \mapsto \overline{\xi}(t, x, \eta, \epsilon, h)$$

*is a diffeomorphism onto its range. We call $\overline{\eta}(t, x, ., \epsilon, h)$ its inverse.*

**Proof:** We use the same method as Gérard-Martinez in [16]: thanks to lemma 3.6, $\partial_\eta \overline{\xi}$ is invertible when $|x| \geq \tilde{R} >> 1$, so we have a local diffeomorphism, and then it is sufficient to show that $\overline{\xi}$ is injective. This is a consequence of the fact that, when $\eta, \eta' \in \Gamma_x^+(\tilde{R}, J, \sigma)$

$$\left|\overline{\xi}(t, x, \eta, \epsilon, h) - \overline{\xi}(t, x, \eta', \epsilon, h)\right| \geq \frac{1}{2}|\eta - \eta'|, \qquad \forall\ |x| \geq \tilde{R},\ t \geq 0, \epsilon \in [0,1],\ h \in [0, h_0],\ q \in \mathcal{B}$$

if $\tilde{R}$ is big enough, by application of the Taylor formula to the second order, and using again the previous lemma. The proof is complete. □

The following proposition, which "controls" the range of $(x, \eta) \mapsto (x, \overline{\xi}(t, x, \eta, \epsilon, h))$ will be useful to define (3.23) (and hence (3.21)) in an outgoing area independent on $\epsilon$, $h$ and $q$.

**Proposition 3.8.** *For all open interval $J'$, with $\overline{J}' \subset J$ and all $-1 < \sigma' < \sigma$, there exists $R' > 0$ such that*

$$\Gamma^+(R', J', \sigma') \subset \Phi_{t,\epsilon,h,q}\big(\Gamma^+(R, J, \sigma)\big) \tag{3.29}$$

*for all $t \geq 0$, $(\epsilon, h) \in [0,1] \times [0, h_0]$ and $q \in \mathcal{B}$, where $\Phi_{t,\epsilon,h,q}$ denotes the following map*

$$\Gamma^+(R, J, \sigma) \ni (x, \eta) \mapsto \big(x, \overline{\xi}(t, x, \eta, \epsilon, h)\big)$$

*which is a diffeomorphism onto its (open) range.*
*Moreover, we can choose $R'$ such that, for all $\gamma$ there exists $C \geq 0$ and $\mathcal{N}$ semi-norm in $\mathcal{S}_1(\omega, -\rho)$ such that, for all $(x, \xi) \in \Gamma^+(R', J', \sigma')$, $\epsilon \in [0,1]$, $h \in [0, h_0]$ and $q_j \in \mathcal{B}$ we have*

$$|\partial^\gamma \big(\overline{\eta}_1 - \overline{\eta}_2\big)(t, x, \xi, \epsilon, h)| \leq C\mathcal{N}(q_1 - q_2)\langle x\rangle^{-|\alpha| - \rho n_+}, \tag{3.30}$$

*where $\overline{\eta}_j$ is associated to $q_j$, $j = 1, 2$.*

**Proof.** (3.29) is based on the following basic topological fact (see [33]): if $A, B \subset \mathbb{R}^n$ with $A$ connected, and if $f : \overline{B} \to \mathbb{R}^n$ is an homeomorphism onto its range, then

$$A \cap f(B) \neq \emptyset \quad \text{and} \quad A \cap f(\partial B) = \emptyset\ \Rightarrow A \subset f(B).$$

To use this property, we write $\omega^{-1}(J') = \coprod_j \Omega_j$, where the $\Omega_j$'s are the (open) connected components of $\omega^{-1}(J')$. For all $R'$, $\Gamma_j^+(R') = \{(x, \xi)\ |\ |x| > R',\ \xi \in \Omega_j,\ \cos(x, v(\xi)) > -\sigma'\}$ is connected (simple exercise), thus it is sufficient to prove that:

$$\forall\ R' \geq R,\ \forall j,\ \forall t, \epsilon, h, q, \qquad \Gamma_j^+(R') \cap \Phi_{t,\epsilon,h,q}\big(\Gamma^+(R, J, \sigma)\big) \neq \emptyset, \tag{3.31}$$

$$\exists\ R' \geq R,\ \forall j,\ \forall t, \epsilon, h, q, \qquad \Gamma_j^+(R') \cap \Phi_{t,\epsilon,h,q}\big(\partial\Gamma^+(R, J, \sigma)\big) = \emptyset. \tag{3.32}$$

(3.31) is a simple consequence of the fact that $\overline{\xi}(t, x, \xi, \epsilon, h) \to \xi$ as $|x| \to \infty$, because it implies that $\Phi_{t,\epsilon,h,q}(x, \xi) \in \Gamma_j^+(R')$ if $(x, \xi) \in \Gamma_j^+(R')\big(\subset \Gamma^+(R, J, \sigma)\big)$ with $|x|$ big enough. In order to prove (3.32), we remark that, if $(x, \xi) \in \Gamma_j^+(R') \cap \Phi_{t,\epsilon,h,q}\big(\partial\Gamma^+(R, J, \sigma)\big)$, with $R' > R$ then there exists $\xi'$ such that

$$\xi = \overline{\xi}(t, x, \xi', \epsilon, h) \quad \text{and} \quad \big(\omega(\xi') \in \partial J \quad \text{or} \quad \cos(x, v(\xi')) = -\sigma\big). \tag{3.33}$$

By lemma 3.6, it is clear that, given any $\delta > 0$, we have $|\xi - \xi'| < \delta$ if $R' >> 1$ (uniformly with respect to $t, \epsilon, h, q$); as $J$ is a neighborhood of $\overline{J'}$ and $\sigma \neq \sigma'$, (3.33) is impossible if $R'$ is big enough. This completes the proof of the first point.
To prove the estimate (3.30), we use the trivial relation

$$0 = \left(\overline{\xi}_1(t,x,\overline{\eta}_1,\epsilon,h) - \overline{\xi}_1(t,x,\overline{\eta}_2,\epsilon,h)\right) + \left(\overline{\xi}_1(t,x,\overline{\eta}_2,\epsilon,h) - \overline{\xi}_2(t,x,\overline{\eta}_2,\epsilon,h)\right). \tag{3.34}$$

Since, for any $\delta > 0$, we can choose $R' > 0$ big enough, such that $|\overline{\eta}_j - \xi| \leq \delta$, $(j = 1,2)$ and $|\partial_\xi \overline{\xi}_1(t,x,u\overline{\eta}_1 + (1-u)\overline{\eta}_2,\epsilon,h) - 1_d| \leq \delta$, for all $u \in [0,1]$, $(x,\xi) \in \Gamma^+(R',J',\sigma'))$, $\epsilon \in [0,1]$, $h \in [0,h_0]$ and $q_j \in \mathcal{B}$, we get by Taylor formula:

$$\int_0^1 \partial_\xi \overline{\xi}_1(t,x,u\overline{\eta}_1 + (1-u)\overline{\eta}_2,\epsilon,h)du(\overline{\eta}_2 - \overline{\eta}_1) = \mathcal{O}\big(\mathcal{N}(q_1 - q_2)\langle x\rangle^{-\rho}\big)$$

which proves the result when $|\gamma| = 0$, by using the lemma 3.6. Then we get the general result by induction on $|\gamma|$ by differentiating the relation (3.34). We don't go further into details. $\square$

**Proof of the proposition** 3.1. First we mention that the fact that $\varphi$, given by (3.21), satisfies the Hamilton-Jacobi equation (3.3), is standard (see again [16, 34]) and we only insist on the proof of the estimates.

Let $J_0$ and $\sigma_0$ be fixed and let's choose open intervals $J, J'$ and real numbers $\sigma, \sigma'$ such that $J_0 \Subset J' \Subset J \Subset I$ and $-1 < \sigma_0 < \sigma' < \sigma < 1$.
According to the formula (3.21) and following [16], the estimate (3.6) will be a consequence of the fact that

$$|\partial^\gamma\big(\partial_x \tilde{z}(t,x,\xi,\epsilon,h) - 1_d\big)| \leq c_\gamma \langle x\rangle^{-|\alpha| - \rho n_+} \tag{3.35}$$

where $\partial^\gamma = \partial_x^\alpha \partial_\xi^\beta \partial_\epsilon^n \partial_h^l$ and $c_\gamma$ is a constant independent of $t \geq 0$, $(x,\xi) \in \Gamma^+(R',J',R')$, and $\epsilon, h, q$. To prove this estimate, we study, when $0 \leq s \leq t$,

$$\begin{aligned} \overline{x}(s,\overline{\eta}) = \overline{x}(s,\overline{\eta}(t)) &= \overline{x}(s,x,\overline{\eta}(t,x,\xi,\epsilon,h),\epsilon,h), \\ \overline{\xi}(s,\overline{\eta}) = \overline{\xi}(s,\overline{\eta}(t)) &= \overline{\xi}(s,x,\overline{\eta}(t,x,\xi,\epsilon,h),\epsilon,h) \end{aligned}$$

which are solutions of the following equations

$$\overline{x}(s,\overline{\eta}(t)) = x + \int_0^s \partial_\xi \omega(\overline{\xi}(u,\overline{\eta})) + \epsilon \partial_\xi q(\overline{x}(u,\overline{\eta}),\overline{\xi}(u,\overline{\eta}),h)du, \tag{3.36}$$

$$\overline{\xi}(s,\overline{\eta}(t)) = \xi + \int_s^t \epsilon \partial_x q(\overline{x}(u,\overline{\eta}),\overline{\xi}(u,\overline{\eta}),h)du. \tag{3.37}$$

We remark that the following three estimates hold, when $(x,\xi) \in \Gamma^+(R',J',\sigma')$, $\epsilon \in [0,1]$, $h \in [0,h_0]$, $q \in \mathcal{B}$ and $t \geq 0$, if we choose $R' > 0$ large enough

$$|\overline{x}(s,\overline{\eta}(t))| \geq e_0(s + |x|), \tag{3.38}$$

$$|\partial^\gamma\big(\overline{x}(s,\overline{\eta}(t)) - x - sv(\xi)\big)| \leq c_\gamma \langle s\rangle \langle x\rangle^{-\rho n_+ - |\alpha|}, \tag{3.39}$$

$$|\partial^\gamma\big(\overline{\xi}(s,\overline{\eta}(t)) - \xi\big)| \leq c_\gamma \langle x\rangle^{-\rho n_+ - |\alpha|}. \tag{3.40}$$

(3.38) is a consequence of (3.39) which, as (3.40), is a consequence of (3.30) and lemma 3.6. Now we can prove (3.35) when $|\gamma| = 0$ because, by applying $\partial_x$ to (3.36) and (3.37), we find that $(y,\eta)$ defined as

$$y = \partial_x\big(\overline{x}(s,\overline{\eta}(t)) - x\big), \tag{3.41}$$

$$\eta = \partial_x\big(\overline{\xi}(s,\overline{\eta}(t))\big), \tag{3.42}$$

is solution of a system as $(A.10)$ (notice that the fact that $y$ and $\eta$ are matrices and not vectors is irrelevant), so proposition $A.3$ proves that, uniformly with respect to the parameters, we have

$$|y| \leq c|x|^{-\rho}, \qquad |\eta| \leq c(s+|x|)^{-1-\rho'}|x|^{\rho'-\rho}$$

(see proposition $A.3$ for the definition of $\rho'$). Then we get, by induction on $|\gamma| \geq 0$, that

$$\begin{aligned} |\partial^\gamma y| &\leq c_\gamma |x|^{-|\alpha|-\rho n_+}, & (3.43)\\ |\partial^\gamma \eta| &\leq c_\gamma (s+|x|)^{-1-\rho'}|x|^{\rho'-|\alpha|-\rho n_+}, & (3.44)\end{aligned}$$

again by application of proposition $A.3$ and using the fact that, when $a \in S_1(\omega, -\mu)$ with $\mu \geq 0$,

$$\partial^\gamma\big(a(\overline{x}(s,\overline{\eta}(t)), \overline{\xi}(s,\overline{\eta}(t)))\big) = \mathcal{O}\big((s+|x|)^{-\mu}|x|^{-|\alpha|-\rho n}\big),$$

so the estimate (3.35) is proved. In the same way, (3.7) is essentially a consequence of the fact that

$$|\partial^\gamma\big(\partial_x \tilde{z}_1(t,x,\xi,\epsilon,h) - \partial_x \tilde{z}_2(t,x,\xi,\epsilon,h)\big)| \leq c_\gamma \mathcal{N}(q_1-q_2)\langle x\rangle^{-|\alpha|-\rho n_+}$$

with $\tilde{z}_j$ associated to $q_j$. The proof of this new estimate is done as before applying proposition $A.3$ (with $C_1 = \mathcal{O}(\mathcal{N}(q_1-q_2))$) to $y = y_1 - y_2$ and $\eta = \eta_1 - \eta_2$, if $y_j$ and $\eta_j$ are defined by (3.41) and (3.42) with $\overline{x}_j$ and $\overline{\eta}_j$, instead of $\overline{x}$ and $\overline{\eta}$. We can apply proposition $A.3$, once we have chosen $R'$ large enough so that

$$|\overline{x}_1 + u(\overline{x}_2 - \overline{x}_1)| \geq c'(|x|+t), \qquad \forall\, u \in [0,1]$$

with $c' > 0$, which allows to get the right estimates for the matrices involved in the system from which $(y,\eta)$ is solution.
Finally, choosing $R_0 > R'$, and multiplying $\partial F/\partial t$ by a cutoff function in $S_1(-\infty,0)$ supported in $\Gamma^+(R',J',\sigma')$ and identically 1 in a neighborhood of $\Gamma^+(R_0,J_0,\sigma_0)$, we get the result. $\square$

# A Estimates on solutions of some ODE

In this appendix, we are going to give estimates on $y = y(t,x)$ and $\eta = \eta(t,x)$ solutions of some ordinary differential equations, or integral equations.

The first proposition will prove estimates on the solutions of the following linear differential system (where $\dot{z} = \partial_t z$)

$$\begin{cases} \dot{y} = Ay + B\eta + y_0, \\ \dot{\eta} = Cy + D\eta + \eta_0, \end{cases} \tag{A.1}$$

where $A, B, C, D$ are $d \times d$ matrices and $y_0, \eta_0$ are vectors of $\mathbb{R}^d$ all depending continuously on $(t,x) \in [0,\infty) \times \mathcal{E}(R)$, with $\mathcal{E}(R) \subset B(0,R)^c$, and satisfying the following estimates for some $C_0 \geq 0$, $C_1 \geq 0$ and $\mu \geq 0$

$$\begin{aligned} |A(t,x)| \leq C_0(t+|x|)^{-\rho-1}, &\qquad |B(t,x)| \leq C_0, & (A.2)\\ |C(t,x)| \leq C_0(t+|x|)^{-\rho-2}, &\qquad |D(t,x)| \leq C_0(t+|x|)^{-\rho-1}, & (A.3)\\ |y_0(t,x)| \leq C_1|x|^{-\mu-\rho}, &\qquad |\eta_0(t,x)| \leq C_1|x|^{-\mu}(t+|x|)^{-1-\rho} & (A.4)\end{aligned}$$

for all $t \geq 0$ and $x \in \mathcal{E}(R)$ (the matrix norm, also denoted by $|.|$ is such that $|Ez| \leq |E|.|z|$).

**Proposition A.1.** *There exists $R \geq 1$ large enough, and $C_2$ which both depend only on $C_0$ and $\rho$, such that for all $x \in \mathcal{E}(R)$, the solution $(y, \eta)$ which vanishes at $t = 0$ satisfies*

$$\begin{aligned} |y(t,x)| &\leq C_2 C_1 \langle t \rangle |x|^{-\mu-\rho}, && \text{(A.5)} \\ |\eta(t,x)| &\leq C_2 C_1 |x|^{-\mu-\rho}. && \text{(A.6)} \end{aligned}$$

The proof is a consequence of the following lemma.

**Lemma A.2.** *Assume that $C_1 = 1$. Then for all $0 < \varepsilon \leq 1$ there exists $R = R(C_0, \rho, \varepsilon) \geq 1$, such that for all $|x| \geq R$ and $t \geq 0$*

$$|y(t,x)| \leq \varepsilon \langle t \rangle |x|^{\frac{\rho}{2}-\mu-\rho}, \qquad |\eta(t,x)| \leq \varepsilon |x|^{\frac{\rho}{2}-\mu-\rho}.$$

**Proof.** Let's choose $R$ such that

$$2e^{2C_0} R^{-\frac{\rho}{2}} \leq \varepsilon, \tag{A.7}$$

$$2C_0 \int_0^\infty (|x|+s)^{-\rho-1} ds + |x|^{\rho-\frac{\rho}{2}} \int_0^\infty (|x|+s)^{-1-\rho} ds \leq \tfrac{\varepsilon}{4} |x|^{-\frac{\rho}{4}}, \qquad \forall\, x \in \mathcal{E}(R), \tag{A.8}$$

$$C_0 \int_0^\infty (R+s)^{-\rho-1} ds + C_0 R^{-\frac{\rho}{4}} + R^{-\frac{\rho}{2}} \leq \tfrac{\varepsilon}{2}. \tag{A.9}$$

We shall prove that for all $x \in \mathcal{E}(R)$, we have $I(x) = [0, \infty)$ where

$$I(x) = \{T \geq 0 \mid |y(t,x)| \leq \varepsilon \langle t \rangle |x|^{\frac{\rho}{2}-\mu-\rho} \text{ and } |\eta(t,x)| \leq \varepsilon |x|^{\frac{\rho}{2}-\mu-\rho},\ \forall\, t \in [0,T]\}.$$

For $t \in [0,1]$ we have easily

$$|y(t,x)| + |\eta(t,x)| \leq 2C_0 \int_0^t |y(s,x)| + |\eta(s,x)| ds + 2|x|^{-\mu-\rho},$$

so by the Gronwall's lemma we get

$$|y(t,x)| + |\eta(t,x)| \leq 2|x|^{-\mu-\rho} e^{2C_0}, \qquad \forall\, t \in [0,1]$$

and the right-hand side is lower than $\varepsilon |x|^{\frac{\rho}{2}-\mu-\rho}$ if $(A.7)$ is satisfied, hence $I(x)$ is not empty. Moreover $I(x)$ is clearly an interval so we just have to prove that $T(x) := \sup I(x) = +\infty$. We argue by contradiction so we assume that $T(x) < \infty$. Then for all $t \in [0, T(x)]$ we get, by integrating the second equation of $(A.1)$ between $0$ and $t$

$$\begin{aligned} |y(t,x)| &\leq 2C_0 \int_0^t (|x|+s)^{-\rho-1} \varepsilon |x|^{\frac{\rho}{2}-\mu-\rho} ds + |x|^{-\mu} \int_0^t (s+|x|)^{-1-\rho} ds \\ &\leq \frac{\varepsilon}{4} |x|^{-\frac{\rho}{4}+\frac{\rho}{2}-\mu-\rho} \end{aligned}$$

if $(A.8)$ is satisfied; then using this new estimate and by integrating the first equation of $(A.1)$ between $0$ and $t$ we get

$$\begin{aligned} |y(t,x)| &\leq \langle t \rangle |x|^{\frac{\rho}{2}-\mu-\rho} \left( \varepsilon C_0 \int_0^t (s+|x|)^{-1-\rho} ds + \varepsilon C_0 |x|^{-\frac{\rho}{4}} + |x|^{-\frac{\rho}{2}} \right) \\ &\leq \frac{\varepsilon}{2} \langle t \rangle |x|^{\frac{\rho}{2}-\mu-\rho} \end{aligned}$$

if $(A.9)$ is satisfied. This shows that the estimates on $y$ and $\eta$ are valid on $[0, T']$ with $T' > T(x)$ which is a contradiction. $\square$

**Proof of the proposition.** If $C_1 = 0$, $y = \eta = 0$ and the result is trivial. If $C_1 > 0$, then $y/C_1, \eta/C_1$ is solution of the same system with $y_0, \eta_0$ replaced by $y_0/C_1, \eta_0/C_1$ which satisfies $|y_0/C_1| \le |x|^{-\mu-\rho}$ and $|\eta_0/C_1| \le |x|^{-\mu}(t+|x|)^{-1-\rho}$, so we may use the previous lemma. We choose $\varepsilon = 1$ so, for some $R \ge 1$ depending only on $C_0$ and $\rho$, we have

$$|y(t,x)/C_1| \le \langle t\rangle |x|^{\frac{\rho}{2}-\mu-\rho}, \quad \text{and} \quad |\eta(t,x)/C_1| \le |x|^{\frac{\rho}{2}-\mu-\rho}, \qquad \forall\, t \ge 0,\ |x| \ge R.$$

Then by integrating the second line of $(A.1)$ we get

$$\begin{aligned} |\eta(t,x)/C_1| &\le C_0 \int_0^{\infty} (s+|x|)^{-2-\rho}(1+s)|x|^{\frac{\rho}{2}-\mu-\rho} ds + \\ & \quad C_0 \int_0^{\infty} (s+|x|)^{-1-\rho}|x|^{\frac{\rho}{2}-\mu-\rho} ds + |x|^{-\mu} \int_0^{\infty} (s+|x|)^{-1-\rho} ds \\ &\le C_2 |x|^{-\mu-\rho} \end{aligned}$$

with $C_2$ which depends only on $C_0$ and $\rho$. In the same way we get the expected estimate on $y/C_1$ (by using the final estimate on $\eta/C_1$). The proof is complete. □

Our second proposition proves estimates on the solutions of the following system of integral equations

$$\begin{cases} y(s) = y_0(s) + \int_0^s A(u)y(u)du + \int_0^s B(u)\eta(u)du, \\ \eta(s) = \eta_0(s) + \int_s^t C(u)y(u)du + \int_s^t D(u)\eta(u)du, \end{cases} \tag{A.10}$$

where $0 \le s \le t$, and $y = y(s) = y(s,x)$ (as $\eta$, $y_0$, $\eta_0$, $A$, $B$, $C$ and $D$), with $A, B, C, D$ $d \times d$ matrices satisfying $(A.2), (A.3)$ and $y_0, \eta_0$ vectors satisfying $(A.4)$. We assume moreover that the solution $y, \eta$ of the system satisfies the following *a priori* estimates:

$$|y| \le C_1 \langle s\rangle |x|^{-\rho-\mu-1}, \tag{A.11}$$
$$|\eta| \le C_1 |x|^{-\rho-\mu-1}. \tag{A.12}$$

The fact that we integrate on $[s,t]$ on the second line of $(A.10)$ will provides estimates on $y$ and $\eta$ independent of $s$. Precisely we have

**Proposition A.3.** *Assume that $\rho^{-1} \notin \mathbb{N}$. There exists $C_2$ which depends only on $\rho$ and $C_0$, such that*

$$|y| \le C_2 C_1 |x|^{-\rho-\mu}, \tag{A.13}$$
$$|\eta| \le C_2 C_1 (s+|x|)^{-1-\rho'} |x|^{\rho'-\rho-\mu}, \tag{A.14}$$

*where $\rho' > 0$ is defined by $1+\rho' = (k+1)\rho$, if $k \ge 0$ is the lowest integer such that $(k+1)\rho > 1$.*

This proposition gives a $L^\infty$ estimate for $y$ with respect to $s$, and a $L^1$ one for $\eta$, which will be crucial for the proof of proposition 3.1.

**Proof.** By putting the *a priori* estimates $(A.11)$ and $(A.12)$ in the second equation of the system $(A.10)$, we get easily that

$$\eta(s) = \mathcal{O}\big(C_1 (s+|x|)^{-\rho} |x|^{-\mu-1}\big)$$

which means that $|\eta(s)| \le cC_1(s+|x|)^{-\rho}|x|^{-\mu-1}$ for some $c$ depending only on $\rho$ and $C_0$. Then using this new estimate together with $(A.11)$ in the first equation of $(A.10)$ we get

$$y(s) = \mathcal{O}\big(C_1|x|^{-\rho-\mu}\big) + \mathcal{O}\big(C_1 \int_0^s (u+|x|)^{-\rho}|x|^{-\mu-1} du\big)$$

and there are two cases; if $\rho > 1$: then we get the result and if $\rho < 1$, we have the estimate

$$y(s) = \mathcal{O}\big(C_1|x|^{-\rho-\mu}\big) + \mathcal{O}\big(C_1(s+|x|)^{1-\rho}|x|^{-\mu-1}\big)$$

so we are in position to apply the following lemma from which the proof of the proposition is a trivial consequence by a finite induction. $\square$

**Lemma A.4.** *Assume that for some $n \geq 1$, we have*

$$\begin{aligned} \eta(s) &= \mathcal{O}\big(C_1(s+|x|)^{-\rho-1}|x|^{-\mu}\big) + \mathcal{O}\big(C_1(s+|x|)^{-n\rho}|x|^{-1-\mu}\big), && \text{(A.15)} \\ y(s) &= \mathcal{O}\big(C_1|x|^{-\rho-\mu}\big) + \mathcal{O}\big(C_1(s+|x|)^{1-n\rho}|x|^{-1-\mu}\big), && \text{(A.16)} \end{aligned}$$

*then we have*

$$\eta(s) = \mathcal{O}\big(C_1(s+|x|)^{-\rho-1}|x|^{-\mu}\big) + \mathcal{O}\big(C_1(s+|x|)^{-(n+1)\rho}|x|^{-1-\mu}\big), \tag{A.17}$$

*and*

$$y(s) = \begin{cases} \mathcal{O}\big(C_1|x|^{-\rho-\mu}\big) + \mathcal{O}\big(C_1(s+|x|)^{1-(n+1)\rho}|x|^{-1-\mu}\big), & \text{if } 1-(n+1)\rho > 0 \\ \mathcal{O}\big(C_1|x|^{-\rho-\mu}\big), & \text{if } 1-(n+1)\rho < 0. \end{cases}$$

**Proof.** Putting the estimates $(A.16)$ and $(A.15)$ in the second equation of the system $(A.10)$ leads directly to the estimate $(A.17)$. Then by putting $(A.17)$ and $(A.16)$ in the first equation of the system, one proves that

$$y(s) = \mathcal{O}\big(C_1|x|^{-\rho-\mu}\big) + \mathcal{O}\big(C_1\int_0^s (u+|x|)^{-(n+1)\rho}|x|^{-\mu-1}du\big)$$

from which we deduce the estimates on $y(s)$ in each case, and this completes the proof. $\square$

Our final result gives estimates on $y = y(t) = y(t,x,\xi)$, the solution of

$$\begin{cases} \dot{y} = v(\xi) + w(y,\xi), \\ y_{|t=0} = x, \end{cases} \tag{A.18}$$

when $(x,\xi) \in \Gamma^+(R_0, J_0, \sigma_0)$. We assume that $w \in S_1(\omega, -\rho)$ (with values in $\mathbb{R}^d$) which means that, for all $\alpha, \beta$

$$|\partial_x^\alpha \partial_\xi^\beta w(x,\xi)| \leq C_{\alpha\beta}(1+\omega(\xi))\langle x\rangle^{-\rho-|\alpha|}.$$

Then we have

**Proposition A.5.** *There exists $\tilde{R}_0 \geq R_0$ and $c_0$ (both depending only on $J_0$, $\sigma_0$ and a finite number of semi-norms of $w$ in $S_1(\omega, -\rho)$) such that, for all $(x,\xi) \in \Gamma^+(\tilde{R}_0, J_0, \sigma_0)$, $y(t,x,\xi)$ is defined for all $t \geq 0$ and*

$$|y(t,x,\xi) - x - tv(\xi)| \leq c_0\langle t\rangle\langle x\rangle^{-\rho}, \qquad \forall\ t \geq 0.$$

**Proof.** By differentiating the expression $\langle \dot{y}, y\rangle$ with respect to $t$, we get easily

$$\frac{\partial^2}{\partial t^2}|y|^2 \geq 2|\dot{y}|^2 - C\langle y\rangle^{-\rho} \tag{A.19}$$

where $C$ depends only on $J_0$ and some semi-norms of $w$. Independently, let's remark that we can choose $\delta_0, e_0 > 0$ small enough and $\tilde{R}_0$ large enough (all depending only on $J_0, \sigma_0$ and some semi-norms of $w$) such that

$$(1-\delta_0)t^2|v(\xi)|^2 + 2t\langle x, v(\xi) + w(x,\xi)\rangle + |x|^2 \geq e_0(|x|+t)^2 \tag{A.20}$$

for all $t \geq 0$ and $(x,\xi) \in \Gamma^+(\tilde{R}_0, J_0, \sigma_0)$.
Now we introduce the set

$$I(x,\xi) = \{t \geq 0 \mid 2|\dot{y}(s)|^2 - C\langle y(s)\rangle^{-\rho} \geq 2(1-\delta_0)|v(\xi)|^2, \ \forall\ s \in [0,t]\}$$

and we shall prove that $T(x,\xi) := \sup I(x,\xi)$ is $+\infty$, when $(x,\xi) \in \Gamma^+(\tilde{R}_0, J_0, \sigma_0)$, possibly after increasing $\tilde{R}_0$. First, we remark that, when $t = 0$, $2|\dot{y}|^2 - C\langle y\rangle^{-\rho} \geq 2(1-\delta_0/2)|v(\xi)|^2$ by choosing $\tilde{R}_0$ large enough, which proves that $T(x,\xi) > 0$. Thus, for all $t \in [0, T(x,\xi))$ we get, by integrating two times $(A.19)$ on $[0,t]$,

$$|y(t,x,\xi)|^2 \geq e_0(|x|+t)^2 \tag{A.21}$$

using $(A.20)$ and the definition of $I(x,\xi)$. This proves that $|\dot{y}(t)|^2 = v(\xi) + \mathcal{O}(\langle x\rangle^{-\rho})$, and then by increasing $\tilde{R}_0$ (in a way depending only on $J_0, C$ and a finite number of semi-norms of $w$) we get for all $t \in [0, T(x,\xi))$

$$|\dot{y}(t)|^2 - C\langle y(t)\rangle^{-\rho} \geq 2(1-\delta_0/2)|v(\xi)|^2$$

which proves that $|\dot{y}|^2 - C\langle y\rangle^{-\rho} \geq (1-\delta_0)|v(\xi)|^2$ on $[0, T'(x,\xi))$ with $T'(x,\xi) > T(x,\xi)$. This proves that $T(x,\xi) = +\infty$ and that $(A.21)$ holds on $\mathbb{R}_+$. The result follows easily by integrating $(A.18)$ on $[0,t]$, taking $(A.21)$ into account. $\square$

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