Enhanced Binding in a General Class of Quantum Field Models

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Abstract

We consider, in an abstract form, a system of “quantum particles” coupled to a Bose field. It is shown that, under suitable hypotheses, the composed system can have a ground state even if the uncoupled particle system has no ground state.

Keywords: quantum field, Fock space, enhanced binding, ground state, non-relativistic QED

1 Introduction

In a quantum system with a coupling parameter $\lambda \in \mathbb{R}$, it may occur that the Hamiltonian of the system has a ground state for a non-zero λ even if it has no ground state at the zero-coupling $\lambda = 0$. If such a phenomenon occur, then we call it the *enhanced binding* in the quantum system under consideration. A typical example is a quantum mechanical system whose Hamiltonian is given by the Schrödinger operator $H_{\mathrm{S}}(\lambda) := -\Delta + \lambda V$ on $L^2(\mathbb{R}^d)$, where Δ is the d-dimensional generalized Laplacian and $V : \mathbb{R}^d \to \mathbb{R}$ is a potential. Indeed, it is well known that $H_{\mathrm{S}}(0)$ has no ground state, but, for a general class of V, $H_{\mathrm{S}}(\lambda)$ with $\lambda \neq 0$ has a ground state (e.g., [14]). As a next stage, it is interesting to investigate if enhanced binding occurs in a quantum system of particles coupled to a quantum field. Recently enhanced binding in non-relativisic quantum electrodynamics (QED) has been discussed [11, 8]. In [11] the Pauli-Fierz model in the dipole approximation is considered and it is shown that, under suitable conditions, the enhanced binding occurs for large coupling constants. On the other hand, in [8], the enhanced binding of the Pauli-Fierz model without the dipole approximation is shown to occur for small coupling constants and for a class of potentials. In this paper we consider enhanced binding in an abstract model of "quantum particles" coupled to a multi-component Bose field. We prove that, under suitable hypotheses, enhanced binding occurs in this model. This suggests that enhanced binding in quantum particle-field interaction systems is a general phenomenon, although it may depend on the type of interactions.

The present paper is organized as follows. In Section 2 we describe the model considered and state the main results. The model is essentially same as that discussed in the previous papers [4, 5, 6] except that the Bose field is a multi-component one. In Section 3 we prove the self-adjointness of the total Hamiltonian of the model, where we present a method different from the one used in [4, 5]. In considering the problem of enhanced binding in the model, we distinguish two cases: the case where the Bose field is massive and the one where the Bose field is massless, but, without infrared singularity. We first prove the existence of enhanced binding in the massive case. This is done in Section 4. Section 5 is devoted to proof of the existence of enhanced binding in the massless case. In the last section we apply these general results to the Pauli-Fierz type model without A^2-term in the dipole approximation. In particular, we show that, if the regime of momenta of photons interacting with the quantum particle becomes sufficiently large with an infrared cutoff fixed, then the model has a ground state at least for the coupling constant in some bounded open interval even if the unperturbed particle Hamiltonian has no ground state. The present paper has two appendices. In Appendix A, we formulate, in an abstract form, the weak differentiability of Heisenberg type operators. In Appendix B, we establish, in an abstract framework, a theorem on the existence of a ground state of a self-adjoint operator and a limit theorem of ground states. Each theorem clarifies a general structure underlying methods used in proofs of existence of ground states in non-relativistic QED [7, 9, 10]. These theorems may be interesting also in its own right in the spectral theory of self-adjoint operators.

2 Definition of the Model and the Main Results

We consider, in an abstract form, a model of a quantum system S coupled to a multi-component Bose field. We denote the Hilbert space of the system S by $\mathcal{H}$, which is taken to be an arbitrary separable complex Hilbert space. In concrete realizations, S may be a system of quantum particles or a quantum field system. In general we denote the inner product and the norm of a Hilbert space $\mathcal{X}$ by $\langle\cdot,\cdot\rangle_{\mathcal{X}}$ and $\|\cdot\|_{\mathcal{X}}$ respectively, where we use the convention that the inner product is antilinear (resp. linear) in the first (resp. second) variable. If there is no danger of confusion, then we omit the subscript $\mathcal{X}$ in $\langle\cdot,\cdot\rangle_{\mathcal{X}}$ and $\|\cdot\|_{\mathcal{X}}$. For a linear operator T on a Hilbert space, we denote its domain by $D(T)$. For a subspace $D \subset D(T)$, $T|D$ denotes the restriction of T to D. If T is densely defined, then the adjoint of T is denoted T^*. For linear operators S and T on a Hilbert space, $D(S+T) := D(S) \cap D(T)$ unless otherwise stated.

For a self-adjoint operator S on a Hilbert space, we denote its spectrum (resp. essential spectrum) by $\sigma(S)$ (resp. $\sigma_{\rm ess}(S)$) and its spectral measure by $E_S(\cdot)$. If S is bounded from below, then we set

$$E_0(S) := \inf \sigma(S) \tag{2.1}$$

the *ground state energy* of S. We say that S has a ground state if $E_0(S)$ is an eigenvalue of S; in this case, each non-zero vector in $\ker(S - E_0(S))$ is called a *ground state* of S. To describe the Bose field, one uses the Boson Fock space over a separable complex Hilbert space $\mathcal{X}$:

$$\begin{aligned} \mathcal{F}_{\rm b}(\mathcal{X}) &:= \oplus_{n=0}^{\infty} \otimes_{\rm s}^{n} \mathcal{X} \\ &= \left\{ \psi = \{\psi^{(n)}\}_{n=0}^{\infty} \middle| n \geq 0, \psi^{(n)} \in \otimes_{\rm s}^{n} \mathcal{X}, \sum_{n=0}^{\infty} \|\psi^{(n)}\|^2 < \infty \right\}, \end{aligned} \tag{2.2}$$

where $\otimes_{\rm s}^{n}\mathcal{X}$ denotes the n-fold symmetric tensor product of $\mathcal{X}$ with $\otimes_{\rm s}^{0}\mathcal{X} := \mathbb{C}$ (the set of complex numbers).

As is well known [13, §X.7], one of basic objects on $\mathcal{F}_{\rm b}(\mathcal{X})$ is the annihilation operator $a(f)$ ($f \in \mathcal{X}$) which is a densely defined closed linear operator on $\mathcal{F}_{\rm b}(\mathcal{X})$ such that, for all $\psi = \{\psi^{(n)}\}_{n=0}^{\infty} \in D(a(f)^*)$, $(a(f)^*\psi)^{(0)} = 0$ and

$$(a(f)^*\psi)^{(n)} = \sqrt{n} S_n \left(f \otimes \psi^{(n-1)} \right), \quad n \geq 1,$$

where S_n is the symmetrization operator on $\otimes^n \mathcal{X}$ ($S_n^* = S_n, S_n^2 = S_n, \otimes_{\rm s}^{n}\mathcal{X} = S_n(\otimes^n \mathcal{X})$). The adjoint $a(f)^*$, called the creation operator, and the annihilation operator $a(g)$ ($g \in \mathcal{X}$) obey the canonical commutation relations

$$[a(f), a(g)^*] = \langle f, g \rangle, \quad [a(f), a(g)] = 0, \quad [a(f)^*, a(g)^*] = 0 \tag{2.3}$$

for all $f, g \in \mathcal{X}$ on the dense subspace

$$\begin{aligned} \mathcal{F}_0(\mathcal{X}) := \{\psi \in \mathcal{F}_{\rm b}(\mathcal{X}) \ \mid \ & \text{there exists a number } n_0 \text{ such that} \\ & \psi^{(n)} = 0 \text{ for all } n \geq n_0\}, \end{aligned} \tag{2.4}$$

where $[X, Y] := XY - YX$. Let

$$\phi(f) := \frac{a(f) + a(f)^*}{\sqrt{2}}, \quad f \in \mathcal{X}, \tag{2.5}$$

which is called the Segal field operator. It is shown that $\phi(f)$ is essentially self-adjoint on $\mathcal{F}_0(\mathcal{X})$ [13, §X.7]. *We denote its closure by the same symbol* $\phi(f)$. It follows from (2.3) that, for all $f, g \in L^2(\mathbb{R}^d)$,

$$[\phi(f), \phi(g)] = i\Im \langle f, g \rangle \tag{2.6}$$

on $\mathcal{F}_0(\mathcal{X})$. Moreover we have

$$e^{i\phi(f)} e^{i\phi(g)} = e^{-i\Im\langle f,g \rangle} e^{i\phi(g)} e^{i\phi(f)}, \quad f, g \in \mathcal{X}, \tag{2.7}$$

which is called the Weyl relations of $\{\phi(f) | f \in \mathcal{X}\}$ [13, §X.7]. For every self-adjoint operator S on $\mathcal{X}$, one can define a self-adjoint operator $d\Gamma(S)$, called the second quantization of S ([12, p.302], [13, §X.7]), by

$$d\Gamma(S) := \oplus_{n=0}^{\infty} S^{(n)}, \tag{2.8}$$

with $S^{(0)} = 0$ and $S^{(n)}$ is the closure of

$$\left(\sum_{j=1}^{n} I \otimes \cdots \otimes \overset{j\text{th}}{\smile}{S} \otimes \cdots \otimes I\right) \Bigg| \otimes_{\mathrm{alg}}^{n} D(S),$$

where I denotes identity and $\otimes_{\mathrm{alg}}^{n}$ algebraic tensor product. If S is nonnegative, then so is $d\Gamma(S)$.

We assume that the Bose field is an N-component quantum field over $\mathbb{R}^d$ ($d, N \in \mathbb{N}$). Hence the one-boson Hilbert space is taken to be

$$\mathcal{W} := \oplus^N L^2(\mathbb{R}^d) \tag{2.9}$$

(the N direct sum of $L^2(\mathbb{R}^d)$) and the Hilbert space of the Bose field is taken to be $\mathcal{F}_{\mathrm{b}}(\mathcal{W})$. Let ω be a Borel measurable function on $\mathbb{R}^d$ which is injective and $0 < \omega(k) < \infty$ for a.e. $k \in \mathbb{R}^d$ with respect to (w.r.t.) the Lebesgue measure on $\mathbb{R}^d$. Then ω defines a multiplication operator on $\mathcal{W}$, which is nonnegative, injective and self-adjoint. We denote it by the same symbol ω ($\omega f := (\omega f_1, \cdots, \omega f_N)$, $f = (f_1, \cdots, f_N) \in \mathcal{W}$ with $f_i \in D(\omega), i = 1, \cdots, N$). The function ω represents a dispersion relation of one free boson associated with the Bose field under consideration.

The free Hamiltonian of the Bose filed is defined by

$$H_{\mathrm{b}} := d\Gamma(\omega) \tag{2.10}$$

acting on $\mathcal{F}_{\mathrm{b}}(\mathcal{W})$. The Hilbert space of the coupled system of S and the Bose field is given by the tensor product

$$\mathcal{F} := \mathcal{H} \otimes \mathcal{F}_{\mathrm{b}}(\mathcal{W}). \tag{2.11}$$

Let A be a self-adjoint operator on $\mathcal{H}$, which denotes physically the Hamiltonian of the quantum system S and B_j ($j = 1, \cdots, J$, $J \in \mathbb{N}$) be a self-adjoint operator on $\mathcal{H}$ such that $\cap_{j=1}^{J} D(B_j)$ is dense in $\mathcal{H}$. Let $g_j \in \mathcal{W}$, $j = 1, \cdots, J$. As a total Hamiltonian of the coupled system, we take the following operator:

$$H(\lambda) := A \otimes I + I \otimes H_{\mathrm{b}} + \lambda \sum_{j=1}^{J} B_j \otimes \phi(g_j), \tag{2.12}$$

where $\lambda \in \mathbb{R}$ is a constant parameter denoting the coupling constant of the system S and the Bose field system. The Hamiltonian $H(\lambda)$ gives a unification of Hamiltonians of some particle-field interaction models (cf. [4, 5]). In the previous papers [4, 5], the existence of a ground state of $H(\lambda)$ with $N = 1$ is discussed under the assumption that A has a ground state (hence $H(0)$ has a ground state). In the present paper, we consider the problem of enhanced binding on the model, i.e., the problem whether or not $H(\lambda)$ with $\lambda \neq 0$ has a ground state *even if A has no ground state*(hence $H(0)$ has no ground state). We show that, under suitable hypotheses, the problem is solved affirmatively. For results on the problem of enhanced binding on the Pauli-Fierz model in non-relativistic QED, see [11, 8]. The method taken in the present paper is similar to that used in [11], but we do not need such scalings as done in [11], at least on the level of a general theory.

We now formulate basic hypotheses. To do this, we first recall an important notion on commutativity of self-adjoint operators:

Definition 2.1 We say that two self-adjoint operators T and S on a Hilbert space strongly commute (or T strongly commutes with S) if their spectral meausres commute, i.e., for all Borel sets $I_1, I_2 \subset \mathbb{R}$, $E_T(I_1)E_S(I_2) = E_S(I_2)E_T(I_1)$. A family of self-adjoint operators $\{S_j\}_{j=1}^n$ on a Hilbert space is said to be strongly commuting if S_j strongly commutes with S_l for all $j, l = 1, \cdots, n$ with $j \neq l$.

In what follows, we assume that A is of the form

$$A = A_0 + A_1 \tag{2.13}$$

with A_0 a nonnegative self-adjoint operator and A_1 a symmetric operator.

Hypothesis I. g_j, $g_j/\omega^{3/2} \in \mathcal{W}$ $(j = 1, \cdots, J)$ and $\langle g_j(k), g_l(k)\rangle_{\mathbb{C}^N} \in \mathbb{R}$, a.e.$k$, $j, l = 1, \cdots, J$.

Hypothesis II. The operator A_1 is A_0-bounded, i.e., $D(A_0) \subset D(A_1)$ and there exist constants $a, b \geq 0$ such that, for all $u \in D(A_0)$,

$$\|A_1 u\| \leq a\|A_0 u\| + b\|u\|. \tag{2.14}$$

Hypothesis III. The operator A_0 strongly commutes with each B_j $(j = 1, \cdots, J)$ and

$$D(A_0) \subset \cap_{j,l=1}^J D(B_j B_l). \tag{2.15}$$

Moreover, there exist constants $c_j, d_j \geq 0$ such that, for all $u \in D(A_0^{1/2})$,

$$\|B_j u\| \leq c_j \|A_0^{1/2} u\| + d_j \|u\| \quad (j = 1, \cdots, J). \tag{2.16}$$

Hypothesis IV. The set $\{B_j\}_{j=1}^J$ is a family of strongly commuting self-adjoint operators.

Hypothesis V. $D(A_0) \subset \cap_{j=1}^J D(B_j A_1) \cap D(A_1 B_j)$ and $[B_j, A_1]|D(A_0)$ is bounded $(j = 1, \cdots, J)$. We denote the operator norm of $[B_j, A_1]$ by $\|[B_j, A_1]\|$.

We introduce an operator

$$R_B := \frac{1}{2}\sum_{j,l=1}^J \left\langle \frac{g_j}{\sqrt{\omega}}, \frac{g_l}{\sqrt{\omega}} \right\rangle_{\mathcal{W}} B_j B_l. \tag{2.17}$$

and define

$$A(\lambda) := A - \lambda^2 R_B. \tag{2.18}$$

Under Hypotheses I–III, we have $D(A(\lambda)) = D(A_0)$. Let

$$\Lambda := \{\lambda \in \mathbb{R} \setminus \{0\} | A(\lambda) \text{ is self-adjoint and bounded from below}\} \tag{2.19}$$

Hypothesis VI. $\Lambda \neq \emptyset$.

Remark 2.1 *Assume Hypotheses I–III and suppose that*

$$a + \frac{\lambda^2}{2} \sum_{j,l=1}^{J} \left| \left\langle \frac{g_j}{\sqrt{\omega}}, \frac{g_l}{\sqrt{\omega}} \right\rangle \right| c_j c_l < 1. \tag{2.20}$$

Then Hypothesis VI holds. This is proved as follows. By Hypothesis III, we can show that

$$\|B_j B_l u\| \leq c_j c_l \|A_0 u\| + (c_j d_l + c_l d_j)\|A_0^{1/2} u\| + d_j d_l \|u\|, \quad u \in D(A_0). \tag{2.21}$$

Hence

$$\begin{aligned} \|(A_1 - \lambda^2 R_B)u\| \leq & \left(a + \frac{\lambda^2}{2} \sum_{j,l=1}^{J} \left| \left\langle \frac{g_j}{\sqrt{\omega}}, \frac{g_l}{\sqrt{\omega}} \right\rangle \right| c_j c_l \right) \|A_0 u\| \\ & + \lambda^2 \left(\sum_{j,l=1}^{J} \left| \left\langle \frac{g_j}{\sqrt{\omega}}, \frac{g_l}{\sqrt{\omega}} \right\rangle \right| c_j d_l \right) \|A_0^{1/2} u\| \\ & + \left(b + \frac{\lambda^2}{2} \sum_{j,l=1}^{J} \left| \left\langle \frac{g_j}{\sqrt{\omega}}, \frac{g_l}{\sqrt{\omega}} \right\rangle \right| d_j d_l \right) \|u\|. \end{aligned}$$

Since $A_0^{1/2}$ is infinitesimally small w.r.t. A_0, it follows under condition (2.20) that $A_1 - \lambda^2 R_B$ is relatively bounded w.r.t. A_0 with relative bound small than 1. Hence, by the Kato-Rellich theorem, $A(\lambda) = A_0 + A_1 - \lambda^2 R_B$ is self-adjoint with $D(A(\lambda)) = D(A_0)$ and bounded from below.

A result on the self-adjointness of $H(\lambda)$ is given by the following theorem.

Theorem 2.2 *Assume Hypotheses I–VI. Then for all $\lambda \in \Lambda$, $H(\lambda)$ is self-adjoint with $D(H(\lambda)) = D(A_0 \otimes I) \cap D(I \otimes H_{\rm b})$ and bounded from below.*

This theorem is proved in Section 3. To establish an existence theorem of a ground state of $H(\lambda)$ without the assumption that A has a ground state, we need additional conditions.

Hypothesis VII. The function ω is continuous on $\mathbb{R}^d$ with

$$\lim_{|k| \to \infty} \omega(k) = \infty$$

and there exist constants $\gamma > 0$ and $C > 0$ such that

$$|\omega(k) - \omega(k')| \leq C|k - k'|^{\gamma}(1 + \omega(k) + \omega(k')), \quad k, k' \in \mathbf{R}^d.$$

In general the existence of a ground state of $H(\lambda)$ may depend on whether

$$m := \text{ess.} \inf_{k \in \mathbb{R}^d} \omega(k) \tag{2.22}$$

is positive or zero [6], where ess. inf means essential infimum. We say that the Bose field under the consideration is *massive* (resp. *massless*) if $m > 0$ (resp. $m = 0$). We first establish a theorem on the existence of a ground state of $H(\lambda)$ in the massive case.

For $s \geq 0$, we introduce a constant $c_s(g)$ by

$$c_s(g) := \sqrt{2} \sum_{j=1}^{J} \|[B_j, A_1]\| \left\| \frac{g_j}{\omega^s} \right\| \tag{2.23}$$

provided that $g_j/\omega^s \in \mathcal{W}$. We set

$$\Sigma_\lambda := \inf \sigma_{\mathrm{ess}}(A(\lambda)). \tag{2.24}$$

Remark 2.2 If $m > 0$, then the condition $g_j \in \mathcal{W}$ implies that $g_j/\omega^s \in \mathcal{W}$ for all $s > 0$. Hence, in this case, Hypothesis I is replaced by the condition that $g_j \in \mathcal{W}$ $(j = 1, \cdots, J)$ and $\langle g_j(k), g_l(k)\rangle_{\mathbb{C}^N} \in \mathbb{R}$, a.e.$k$, $j, l = 1, \cdots, J$.

Theorem 2.3 (Enhanced binding in the massive case). *Consider the case $m > 0$. Assume Hypotheses I–VII. Suppose that $\lambda \in \Lambda$ and*

$$\Sigma_\lambda - E_0(A(\lambda)) > m + \frac{1}{2}\lambda^2 c_{3/2}(g)^2 + |\lambda| c_1(g). \tag{2.25}$$

Then $H(\lambda)$ has purely discrete spectrum in the interval $[E_0(H(\lambda)), E_0(H(\lambda)) + m)$. In particular, $H(\lambda)$ has has a ground state.

Remark 2.3 Condition (2.25) implies that $E_0(A(\lambda))$ is a discrete eigenvalue of $A(\lambda)$ and hence $A(\lambda)$ has a finite number of ground states.

A new point of Theorem 2.3 is in that *the existence of a ground state of A is not assumed.*. Theorem 2.3 is proved in Section 4.

Theorem 2.4 (Enhanced binding in the massless case). *Consider the case $m = 0$. Assume Hypotheses I–VII with $g_j/\omega^2 \in \mathcal{W}$ $(j = 1, \cdots, J)$ in addition. Suppose that*

$$\Sigma_\lambda - E_0(A(\lambda)) > \frac{1}{2}\lambda^2 c_{3/2}(g)^2 + |\lambda| c_1(g). \tag{2.26}$$

and

$$\frac{\lambda^2 c_1(g)^2}{[\Sigma_\lambda - E_0(H(\lambda))]^2} + \left\{ \frac{2\lambda^2 c_1(g)^2}{[\Sigma_\lambda - E_0(H(\lambda))]^2} + 1 \right\} \frac{\lambda^2}{2} c_2(g)^2 < 1. \tag{2.27}$$

Then $H(\lambda)$ has a ground state.

We prove this theorem in Section 5.

3 Self-adjointness of the Total Hamiltonian

Generally speaking, in considering the enhanced binding problem of a quantum field model, it would be desirable to establish the self-adjointness of the Hamiltonian of the model for a wider range of the coupling constant. For this purpose the method used in [4], which employs the Kato-Rellich theorem, is not useful. Here we take another approach which is used in [1]: we prove Theorem 2.2 by making a suitable unitary transformation of $H(\lambda)$. We need some lemmas.

Lemma 3.1 *Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert spaces. Let $\{X_j\}_{j=1}^J$ (resp. $\{Y_j\}_{j=1}^J$) be a family of strongly commuting self-adjoint adjoint operators on $\mathcal{X}$ (resp. $\mathcal{Y}$). Then $\{X_j \otimes Y_j\}_{j=1}^J$ is a family of strongly commuting self-adjoint operators on $\mathcal{X} \otimes \mathcal{Y}$. Moreover, if W (resp. U) is a self-adjoint operator on $\mathcal{X}$ (resp. $\mathcal{Y}$) strongly commuting with each X_j (resp. Y_j), then $W \otimes I$ (resp. $I \otimes U$) strongly commutes with each $X_j \otimes Y_j$.*

Proof. It is a well known fact that each $X_j \otimes Y_j$ is self-adjoint on $\mathcal{X} \otimes \mathcal{Y}$ (e.g., [12, §VIII.10]). Moreover there exists a two-dimensional spectral measure E_j such that, for all Borel sets $I_1, I_2 \subset \mathbb{R}$, $E_j(I_1 \times I_2) = E_{X_j}(I_1) \otimes E_{Y_j}(I_2)$ and $X_j \otimes I = \int_{\mathbb{R}^2} x dE_j(x,y), Y_j = \int_{\mathbb{R}^2} y dE_j(x,y), f(X_j \otimes Y_j) = \int_{\mathbb{R}^2} f(xy) dE_j(x,y)$. It follows that, for all Borel sets $K \subset \mathbb{R}$, $E_{X_j \otimes Y_j}(K) = E_j(\{(x,y) \in \mathbb{R}^2 | xy \in K\})$. On the other hand, the strong commutativity of X_j's and that of Y_j's imply that $\{E_j(\cdot)\}_{j=1}^J$ is a family of commuting orthogonal projections. Hence, for all Borel sets $I_1, I_2 \subset \mathbb{R}$ and $j, l = 1, \cdots, J$, $E_{X_j \otimes Y_j}(I_1)$ commutes with $E_{X_l \otimes Y_l}(I_2)$. Let W and U be as above. Then $E_W(I_1)E_{X_j}(I_2) = E_{X_j}(I_2)E_W(I_1)$, which implies the commutativity of $E_{W \otimes I}$ and E_j. Hence $E_{W \otimes I}$ commutes with $E_{X_j \otimes Y_j}$. Thus $W \otimes I$ strongly commutes with $X_j \otimes Y_j$. Similarly one can show that $I \otimes U$ strongly commutes with $X_j \otimes Y_j$. ∎The following fact is well known or easy to prove (e.g., [3, Lemma 2.33]).

Lemma 3.2 *Let $\{S_j\}_{j=1}^J$ be a family of strongly commuting self-adjoint operators on a Hilbert space. Then $S := \sum_{j=1}^J S_j$ is essentially self-adjoint and, for all $t \in \mathbb{R}$,*

$$e^{it\bar{S}} = \prod_{j=1}^J e^{itS_j}, \tag{3.1}$$

where $\bar{S}$ denotes the closure of S and the order of the factors on the right hand side (r.h.s.) is arbitrary.

Lemma 3.3 *Let $\mathcal{X}$ be a separable complex Hilbert space and $h_j \in \mathcal{X}$ ($j = 1, \cdots, J$) such that $\langle h_j, h_l \rangle_{\mathcal{X}} \in \mathbb{R}$, $j, l = 1, \cdots, J$. Then $\{\phi(h_j)\}_{j=1}^J$ is a family of strongly ommuting self-adjoint operators on $\mathcal{F}_{\mathrm{b}}(\mathcal{X})$.*

Proof. By the present assumption, $\Im \langle h_j, h_l \rangle = 0$. Hence, by the Weyl relations (2.7), $e^{it\phi(h_j)}$ commutes with $e^{is\phi(h_l)}$ for all $s, t \in \mathbb{R}$ and $j, l = 1, \cdots, J$. Hence, by a general criterion [12, Theorem VIII.13], $\phi(h_j)$ strongly commutes with $\phi(h_l)$ ($j, l = 1, \cdots, J$). ∎

Let

$$T_j := B_j \otimes \phi\left(i\frac{g_j}{\omega}\right). \tag{3.2}$$

Then the operator

$$T := \sum_{j=1}^{J} T_j \tag{3.3}$$

is a symmetric operator with $D(T) \supset \cap_{j=1}^{J} D(B_j) \otimes_{\mathrm{alg}} \mathcal{F}_0(\mathcal{W})$. We denote the closure of T by the same symbol T.

Lemma 3.4 *Assume Hypotheses I and IV. Then:*

(i) $\{T_j\}_{j=1}^{J}$ *is a family of strongly commuting self-adjoint operators.*

(ii) T *is essentially self-adjoint on* $\cap_{j=1}^{J} D(T_j)$ *and, for all* $s \in \mathbb{R}$,

$$e^{isT} = \prod_{j=1}^{J} e^{isT_j}, \tag{3.4}$$

where the order of the factors on the r.h.s. is arbitrary.

Proof. (i) By Lemmas 3.1, 3.3 and Hypothesis IV, $\{T_j\}_{j=1}^{J}$ is a family of strongly commuting self-adjoint operators. (ii) By part (i) and Lemma 3.2, T is essentially self-adjoint and (3.4) holds. ∎

Lemma 3.5 *Assume Hypotheses I, III and IV. Then T strongly commutes with $A_0 \otimes I$.*

Proof. By Lemma 3.1 and Hypothesis III, $A_0 \otimes I$ strongly commutes with each T_j, which implies that, for all $s, t \in \mathbb{R}$, $e^{itA_0 \otimes I} e^{isT_j} = e^{isT_j} e^{itA_0 \otimes I}$. By this equation and (3.4), $e^{itA_0 \otimes I} e^{isT} = e^{isT} e^{itA_0 \otimes I}$. Hence, by a general criterion [12, Theorem VIII.13], T strongly commutes with $A_0 \otimes I$. ∎

The following fact is well known (e.g., [3, p.516, Lemma 12-5]).

Lemma 3.6 *Let $\mathcal{X}$ be a Hilbert space and S be a nonnegative, injective self-adjoint operator on $\mathcal{X}$. Let $g \in D(S)$. Then*

$$e^{i\phi(ig)} D(d\Gamma(S)) = D(d\Gamma(S)) \tag{3.5}$$

and

$$e^{i\phi(ig)} d\Gamma(S) e^{-i\phi(ig)} = d\Gamma(S) + \phi(Sg) + \frac{1}{2} \langle g, Sg \rangle . \tag{3.6}$$

Suppose that Hypotheses I and IV hold. Then, by Lemma 3.4, we can define a unitary operator

$$U(\lambda) := e^{-i\lambda T}. \tag{3.7}$$

We set

$$L := A_0 \otimes I + I \otimes H_{\mathrm{b}}, \tag{3.8}$$

which is nonnegative. We introduce an operator

$$\widetilde{H}(\lambda) := A(\lambda) \otimes I + I \otimes H_{\mathrm{b}} + \delta A_1(\lambda), \tag{3.9}$$

where

$$\delta A_1(\lambda) := U(\lambda)(A_1 \otimes I)U(\lambda)^{-1} - A_1 \otimes I. \tag{3.10}$$

Lemma 3.7 *Assume Hypotheses I–IV. Then, for all* $\lambda \in \mathbb{R}$,

$$U(\lambda)D(L) = D(L) \tag{3.11}$$

and, for all $\Psi \in D(L)$,

$$U(\lambda)H(\lambda)U(\lambda)^{-1} = \widetilde{H}(\lambda)\Psi. \tag{3.12}$$

Proof. By Hypothesis IV, there exists a J-dimensional spectral measure E such that, for all Borel sets $I_j \subset \mathbb{R}$ $(j = 1, \cdots, J)$, $E(I_1 \times \cdots \times I_J) = E_{B_1}(I_1) \cdots E_{B_J}(I_J)$ and $B_j = \int_{\mathbb{R}^J} \xi_j dE(\xi)$ $(\xi = (\xi_1, \cdots, \xi_J) \in \mathbb{R}^J)$. Let $u, v \in D(A_0)$ and $\psi, \varphi \in D(H_\mathrm{b})$. Set $\Psi = u \otimes \psi$ and $\Phi = v \otimes \varphi$. Then $\Psi, \Phi \in D(L)$ and

$$\langle I \otimes H_b \Psi, U(\lambda)\Phi\rangle = \int_{\mathbb{R}^J} \left\langle H_\mathrm{b}\psi, e^{-i\phi(iG_\xi)}\varphi\right\rangle d\,\langle u, E(\xi)v\rangle\,,$$

where $G_\xi = \lambda \sum_{j=1}^J \xi_j g_j/\omega \in \mathcal{W}$. By Lemma 3.6, we have

$$\left\langle H_\mathrm{b}\psi, e^{-i\phi(iG_\xi)}\varphi\right\rangle = \left\langle \psi, e^{-i\phi(iG_\xi)}(H_\mathrm{b} + \phi(\omega G_\xi) + \frac{1}{2}\langle G_\xi, \omega G_\xi\rangle)\varphi\right\rangle.$$

Hence

$$\langle I \otimes H_b \Psi, U(\lambda)\Phi\rangle = \left\langle \Psi, U(\lambda)\left(I \otimes H_\mathrm{b} + \lambda \sum_{j=1}^J B_j \otimes \phi(g_j) + \lambda^2 R_B \otimes I\right)\Phi\right\rangle. \tag{3.13}$$

This extends to all $\Psi, \Phi \in D(A_0) \otimes_\mathrm{alg} D(H_\mathrm{b})$. Using the well known estimates

$$\|a(f)\psi\| \le \left\|\frac{f}{\sqrt{\omega}}\right\| \|H_\mathrm{b}^{1/2}\psi\|, \tag{3.14}$$

$$\|a(f)^*\psi\| \le \left\|\frac{f}{\sqrt{\omega}}\right\| \|H_\mathrm{b}^{1/2}\psi\| + \|f\|\|\psi\|, \quad \psi \in D(H_\mathrm{b}^{1/2}), f \in D(\omega^{-1/2}), \tag{3.15}$$

and Hypothesis III, we can show that

$$\|B_j \otimes \phi(g_j)\Psi\| \le C\|(L+1)\Psi\|, \quad \Psi \in D(L), \tag{3.16}$$

where $C > 0$ is a constant. By (2.21), R_B is A_0-bounded. Hence

$$\left\|\left(I \otimes H_\mathrm{b} + \lambda \sum_{j=1}^J B_j \otimes \phi(g_j) + \lambda^2 R_B \otimes I\right)\Xi\right\| \le C'\|(L+1)\Xi\|, \quad \Xi \in D(L), \tag{3.17}$$

where $C' > 0$ is a constant. By this estimate and the fact that $D(A_0) \otimes_\mathrm{alg} D(H_\mathrm{b})$ is a core for L, (3.13) extends (via a limiting argument) to all $\Phi \in D(L)$. Moreover, $D(A_0) \otimes_\mathrm{alg} D(H_\mathrm{b})$ is a core for $I \otimes H_\mathrm{b}$, (3.13) extends also to all $\Psi \in D(I \otimes H_\mathrm{b})$. Thus, for all $\Phi \in D(L)$, the vector $U(\lambda)\Phi$ is in $D(I \otimes H_\mathrm{b})$ and

$$I \otimes H_\mathrm{b} U(\lambda)\Phi = U(\lambda)\left(I \otimes H_\mathrm{b} + \lambda \sum_{j=1}^J B_j \otimes \phi(g_j) + \lambda^2 R_B \otimes I\right)\Phi. \tag{3.18}$$

The strong commutativity of $A_0 \otimes I$ and T (Lemma 3.5) implies that $U(\lambda)D(A_0 \otimes I) \subset D(A_0 \otimes I)$. Thus $U(\lambda)D(L) \subset D(L)$. Since $U(\lambda)$ is unitary, it follows that $D(L) \subset U(\lambda)^{-1}D(L) = U(-\lambda)D(L)$. Since $\lambda \in \mathbb{R}$ is arbitrary, we obtain (3.11). Eq.(3.18) implies (3.12). ∎In view of Lemma 3.7, we first prove the self-adjointness of $\widetilde{H}(\lambda)$ (Theorem 3.11 below). We denote by $\overline{[B_j, A_1]}$ the closure of $[B_j, A_1]|D(A_0)$ which, by Hypothesis V, is bounded with $D\left(\overline{[B_j, A_1]}\right) = \mathcal{H}$.

Lemma 3.8 *Assume Hypotheses I and V. Then $D(I \otimes H_{\rm b}^{1/2})) \subset D\left(\overline{[B_j, A_1]} \otimes \phi(ig_j/\omega)\right)$ and, for all $\Psi \in D(I \otimes H_{\rm b}^{1/2})$,*

$$\|\overline{[B_j, A_1]} \otimes \phi(ig_j/\omega)\Psi\| \le \|[B_j, A_1]\| \left(\sqrt{2}\left\|\frac{g_j}{\omega^{3/2}}\right\| \|I \otimes H_{\rm b}^{1/2}\Psi\| + \frac{1}{\sqrt{2}}\left\|\frac{g_j}{\omega}\right\| \|\Psi\|\right). \quad (3.19)$$

Proof. By (3.14) and (3.15), we have

$$\|\phi(f)\psi\| \le \sqrt{2}\left\|\frac{f}{\sqrt{\omega}}\right\| \|H_{\rm b}^{1/2}\psi\| + \frac{1}{\sqrt{2}}\|f\|\|\psi\|, \quad f \in D(\omega^{-1/2}),\ \psi \in D(H_{\rm b}^{1/2}). \quad (3.20)$$

Hence, for all $\Psi \in D(A_0) \otimes_{\rm alg} D(H_{\rm b}^{1/2})$,

$$\begin{aligned} \|\overline{[B_j, A_1]} \otimes \phi(ig_j/\omega)\Psi\| &\le \|[B_j, A_1]\|\|I \otimes \phi(ig_j/\omega)\Psi\| \\ &\le \|[B_j, A_1]\|(\sqrt{2}\|g_j/\omega^{3/2}\|\|I \otimes H_{\rm b}^{1/2}\Psi\| + \|g_j/\omega\|\|\Psi\|/\sqrt{2}). \end{aligned}$$

Hence (3.19) holds for all $\Psi \in D(A_0) \otimes_{\rm alg} D(H_{\rm b}^{1/2})$. Since $D(A_0) \otimes_{\rm alg} D(H_{\rm b}^{1/2})$ is a core for $I \otimes H_{\rm b}^{1/2}$, (3.19) extends to all $\Psi \in D(I \otimes H_{\rm b}^{1/2})$ showing $D(I \otimes H_{\rm b}^{1/2})) \subset D(\overline{[B_j, A_1]} \otimes \phi(ig_j/\omega))$ as well. ∎We set

$$Y := \sum_{j=1}^{J} \overline{[B_j, A_1]} \otimes \phi\left(i\frac{g_j}{\omega}\right). \quad (3.21)$$

By Lemma 3.8, we have

$$D(I \otimes H_{\rm b}) \subset D(Y).$$

Lemma 3.9 *Assume Hypotheses I–III and Hypothesis V. Then, for all $\Psi, \Phi \in D(L)$*

$$\langle T\Psi, A_1 \otimes I\Phi\rangle - \langle A_1 \otimes I\Psi, T\Phi\rangle = \langle \Psi, Y\Phi\rangle. \quad (3.22)$$

Proof. It is easy to see that, for all $\Psi, \Phi \in D(A_0) \otimes_{\rm alg} D(H_{\rm b})$, (3.22) holds. By (3.16), T is L-bounded. By Hypothesis II, one can show that $A_1 \otimes I$ is L-boounded. By Lemma 3.8, Y also is L-bounded. Since $D(A_0) \otimes_{\rm alg} D(H_{\rm b})$ is a core for L, (3.22) extends to all $\Psi, \Phi \in D(L)$. ∎

Lemma 3.10 *Assume Hypotheses I–V. Then, $D(L) \subset D(\delta A_1(\lambda))$ and, for all $\Psi \in D(L)$,*

$$\|\delta A_1(\lambda)\Psi\| \le |\lambda| \left(c_{3/2}(g)\|I \otimes H_{\rm b}^{1/2}\Psi\| + \frac{1}{2}c_1(g)\|\Psi\|\right), \quad (3.23)$$

where $c_s(g)$ is defined by (2.23).

Proof. Let $A_1(\lambda) = U(\lambda)A_1 \otimes IU(\lambda)^{-1}$ and $\Psi, \Phi \in D(L)$. Then, applying Proposition A.1 in Appendix A with $H = -\lambda T$, $S = A_1 \otimes I$ and $K = A_0 \otimes I$, we see that the function:$t \mapsto \langle \Phi, A_1(t\lambda)\Psi \rangle$ ($t \in \mathbb{R}$) is differentiable and

$$\begin{aligned}\frac{d}{dt}\langle \Phi, A_1(t\lambda)\Psi \rangle &= -i\lambda\{\left\langle TU(t\lambda)^{-1}\Phi, A_1 \otimes IU(t\lambda)^{-1}\Psi \right\rangle \\ &\quad - \left\langle A_1 \otimes IU(t\lambda)^{-1}\Phi, TU(t\lambda)^{-1}\Psi \right\rangle\},\end{aligned}$$

which, together with Lemma 3.9, yields that

$$\frac{d}{dt}\langle \Phi, A_1(t\lambda)\Psi \rangle = -i\lambda \left\langle U(t\lambda)^{-1}\Phi, YU(t\lambda)^{-1}\Psi \right\rangle.$$

Integrating this equation from $t = 0$ to $t = 1$, we obtain

$$\langle \Phi, \delta A_1(\lambda)\Psi \rangle = -i\lambda \int_0^1 \left\langle U(t\lambda)^{-1}\Phi, YU(t\lambda)^{-1}\Psi \right\rangle dt.$$

Hence

$$|\langle \Phi, \delta A_1(\lambda)\Psi \rangle| \le |\lambda| \int_0^1 \|\Phi\| \|YU(t\lambda)^{-1}\Psi\| dt,$$

which implies that

$$\|\delta A_1(\lambda)\Psi\| \le |\lambda| \int_0^1 \|YU(t\lambda)^{-1}\Psi\| dt.$$

We have

$$\begin{aligned}\|YU(t\lambda)^{-1}\Psi\| &\le \sum_{j=1}^{J} \|[B_j, A_1]\| \|I \otimes \phi(ig_j/\omega)U(t\lambda)^{-1}\Psi\| \\ &= \sum_{j=1}^{J} \|[B_j, A_1]\| \|I \otimes \phi(ig_j/\omega)\Psi\|,\end{aligned}$$

where we have used the strong commutativity of $I \otimes \phi(ig_j/\omega)$ and $U(t\lambda)^{-1}$. Hence

$$\|\delta A_1(\lambda)\Psi\| \le |\lambda| \sum_{j=1}^{J} \|[B_j, A_1]\| \|I \otimes \phi(ig_j/\omega)\Psi\|. \tag{3.24}$$

Using this estimate and (3.20), we obtain (3.23). ∎Let

$$L(\lambda) := A(\lambda) \otimes I + I \otimes H_{\mathrm{b}}. \tag{3.25}$$

Theorem 3.11 *Assume Hypotheses I-VI. Then, for all $\lambda \in \Lambda$, $\widetilde{H}(\lambda)$ is self-adjoint with $D(\widetilde{H}(\lambda)) = D(L)$ and bounded from below. Moreover, every core of $L(\lambda)$ is a core of $\widetilde{H}(\lambda)$.*

Proof. We can write

$$\widetilde{H}(\lambda) = L(\lambda) + \delta A_1(\lambda). \tag{3.26}$$

Let $\lambda \in \Lambda$. Then, by the definition of Λ, $L(\lambda)$ is self-adjoint and bounded from below. It is easy to see that $I \otimes H_{\mathrm{b}}^{1/2}$ is infinitesimally small with respect to (w.r.t.) $I \otimes H_{\mathrm{b}}$.

Since $A(\lambda) \otimes I$ is bounded from below, it follows that $I \otimes H_{\rm b}^{1/2}$ is infinitesimally small with respect to (w.r.t.) $L(\lambda)$. Therefore, by Lemma 3.10, $\delta A_1(\lambda)$ is infinitesimally small w.r.t. $L(\lambda)$. Thus, by the Kato-Rellich theorem (e.g., [13, p.162, Theorem X.12]), $\widetilde{H}(\lambda)$ is self-adjoint on $D(L(\lambda)) = D(A_0 \otimes I) \cap D(I \otimes H_{\rm b})$, bounded from below and every core of $L(\lambda)$ is a core of $\widetilde{H}(\lambda)$. ∎

Proof of Theorem 2.2 We have $D(L(\lambda)) = D(L)$. By Lemma 3.7 and Theorem 3.11, $H(\lambda)$ is self-adjoint on $D(L)$ and bounded from below. ∎

Corollary 3.12 *Assume Hypotheses I-VI. Then, for all $\lambda \in \Lambda$,*

$$E_0(A(\lambda)) - \frac{\lambda^2}{4} c_{3/2}(g)^2 - \frac{|\lambda|}{2} c_1(g) \le E_0(H(\lambda)) \le E_0(A(\lambda)) + \frac{1}{2}|\lambda| c_1(g). \tag{3.27}$$

Proof. By Lemma 3.7 and Theorem 2.2, we have

$$E_0(H(\lambda)) = E_0(\widetilde{H}(\lambda)). \tag{3.28}$$

Hence we need only to prove (3.27) with $E_0(H(\lambda))$ replaced by $E_0(\widetilde{H}(\lambda))$. Let $\Psi \in D(L)$ with $\|\Psi\| = 1$. Then, using (3.23), we have

$$\begin{aligned} \left\langle \Psi, \widetilde{H}(\lambda)\Psi \right\rangle &\ge \langle \Psi, A(\lambda) \otimes I\Psi\rangle + \langle \Psi, I \otimes H_{\rm b}\Psi\rangle - \|\delta A_1(\lambda)\Psi\| \\ &\ge E_0(A(\lambda)) + \|I \otimes H_{\rm b}^{1/2}\Psi\|^2 - |\lambda| c_{3/2}(g) \|I \otimes H_{\rm b}^{1/2}\Psi\| - \frac{|\lambda|}{2} c_1(g) \\ &\ge E_0(A(\lambda)) - \frac{\lambda^2 c_{3/2}(g)^2}{4} - \frac{|\lambda|}{2} c_1(g), \end{aligned}$$

which, combined with the variational principle, yields the first inequality in (3.27). Let $\Omega \in \mathcal{F}_{\rm b}(\mathcal{W})$ be the Fock vacuum: $\Omega^{(0)} = 1,\ \Omega^{(n)} = 0,\ n \ge 1$. Then we have

$$H_{\rm b}\Omega = 0. \tag{3.29}$$

Hence, for all $u \in D(A_0)$ with $\|u\| = 1$, we have by the variational principle

$$E_0(\widetilde{H}(\lambda)) \le \langle u, A(\lambda)u\rangle + \langle u \otimes \Omega, \delta A_1(\lambda) u \otimes \Omega\rangle .$$

By (3.23) and the Schwarz inequality, we have

$$\langle u \otimes \Omega, \delta A_1(\lambda) u \otimes \Omega\rangle \le \frac{1}{2}|\lambda| c_1(g).$$

Hence

$$E_0(\widetilde{H}(\lambda)) \le \langle u, A(\lambda)u\rangle + \frac{1}{2}|\lambda| c_1(g).$$

Applying the variational principle again, we obtain (3.27) with $E_0(H(\lambda))$ replaced by $E_0(\widetilde{H}(\lambda))$. ∎

4 Existence of a Ground State in the Massive Case

In the present case, methods used in [4, 5] is not applied directly to proving Theorem 2.3, because the existence of a ground state of A is not assumed. Thus we need a new idea. We note Lemma 3.7, which tells us that $H(\lambda)$ has a ground state if and only if $\widetilde{H}(\lambda)$ does. Hence one may prove the existence of a ground state of $H(\lambda)$ by proving that of $\widetilde{H}(\lambda)$. We use this structure.

Throughout this section we assume Hypothesis I–VII. For a parameter $V > 0$, we define a lattice

$$\Gamma_V := \frac{2\pi \mathbb{Z}^d}{V} = ft\{k = (k_1, \cdots, k_d) \,\Big|\, k_j = \frac{2\pi n_j}{V},\ n_j \in \mathbb{Z},\ j = 1, \cdots, d.$$

We denote by $\ell^2(\Gamma_V)$ the Hilbert space of square sumable sequences indexed by Γ_V:

$$\ell^2(\Gamma_V) := \left\{ f : \Gamma_V \to \mathbb{C} \,\Big|\, \sum_{k\in\Gamma_V} |f(k)|^2 < \infty \right\}.$$

Each element f in $\ell^2(\Gamma_V)$ can be identified with a piecewise constant function in $L^2(\mathbb{R}^d)$ which is a constant on each cube

$$C(k, V) := \left[k_1 - \frac{\pi}{V}, k_1 + \frac{\pi}{V}\right) \times \cdots \times \left[k_d - \frac{\pi}{V}, k_d + \frac{\pi}{V}\right) \subset \mathbb{R}^d$$

centered about a lattice point $k \in \Gamma_V$. In this identification, $\ell^2(\Gamma_V)$ is a closed subspace of $L^2(\mathbb{R}^d)$. Then, putting

$$\mathcal{W}_V := \oplus^N \ell^2(\Gamma_V). \tag{4.1}$$

we have a natural orthogonal decomposition

$$\mathcal{W} = \mathcal{W}_V \oplus \mathcal{W}_V^{\perp}.$$

Hence

$$\mathcal{F}_{\mathrm{b}}(\mathcal{W}) = \mathcal{F}_{\mathrm{b}}(\mathcal{W}_V) \otimes \mathcal{F}_{\mathrm{b}}(\mathcal{W}_V^{\perp}).$$

(4.2)

We define

$$\omega_V(k) = \omega(k_V), \quad k \in \mathbb{R}^d$$

with k_V a lattice point closed to k:

$$k_V \in \Gamma_V, \quad |k_j - (k_V)_j| \le \frac{\pi}{V}, \quad j = 1, \cdots, d, \quad k \in C(k_V, V).$$

Let

$$H_{\mathrm{b},V} := d\Gamma(\omega_V),$$

the second quantization of ω_V. For technical reasons, we assume the following as a preliminary hypothesis:

Hypothesis VIII. Each $g_j : \mathbb{R}^d \to \mathbb{C}^N$ is continuous with $g_j, g_j/\omega^{3/2} \in \mathcal{W}$ and $\langle g_j(k), g_l(k)\rangle_{\mathbb{C}^N} \in \mathbb{R}$, $j, l = 1, \cdots, J$, $k \in \mathbb{R}^d$.

Let C, γ be the constants in Hypothesis VII. In what follows we assume that $m > 0$ (m is defined by (2.22)) and

$$C_V := C d^{\gamma/2} \left(\frac{\pi}{V}\right)^{\gamma} \left(1 + \frac{1}{2m}\right) < 1. \tag{4.3}$$

Condition (4.3) is equivalent to $V > V_0$, where V_0 is the constant defined by

$$C_{V_0} = 1. \tag{4.4}$$

For a constant $K > 0$, we define a function $g_{j,K,V} : \mathbb{R}^d \to \mathbb{C}^N$ ($j = 1, \cdots, J$) by

$$g_{j,K,V}(k) \;\; := \sum_{\ell \in \Gamma_V, |\ell_i| \le K, i=1,\cdots,d} g_j(\ell) \chi_{C(\ell,V)}(k), \;\; k \in \mathbb{R}^d, \tag{4.5}$$

where χ_S denotes the characteristic fucntion of the set S. We introduce a lattice approximation version for $H(\lambda)$:

$$H_{K,V}(\lambda) := A \otimes I + I \otimes H_{\mathrm{b},V} + \lambda \sum_{j=1}^{J} B_j \otimes \phi(g_{j,K,V}). \tag{4.6}$$

As in the case of T_j, one can show that $\{B_j \otimes \phi(i g_{j,K,V}/\omega_V)\}_{j=1}^{J}$ is a family of strongly commuting self-adjoint operators. Hence

$$T_{K,V} := \overline{\sum_{j=1}^{J} B_j \otimes \phi\left(i \frac{g_{j,K,V}}{\omega_V}\right)} \tag{4.7}$$

is self-adjoint. We set

$$U_{K,V}(\lambda) := e^{-i\lambda T_{K,V}}. \tag{4.8}$$

Let

$$A_{K,V}(\lambda) := A - \frac{\lambda^2}{2} \sum_{j,l=1}^{J} \left\langle \frac{g_{j,K,V}}{\sqrt{\omega_V}}, \frac{g_{l,K,V}}{\sqrt{\omega_V}} \right\rangle B_j B_l, \tag{4.9}$$

$$L_{K,V}(\lambda) := A_{K,V}(\lambda) \otimes I + I \otimes H_{\mathrm{b},V}, \tag{4.10}$$

$$\delta A_{1,K,V}(\lambda) := U_{K,V}(\lambda) A_1 \otimes I U_{K,V}(\lambda)^{-1} - A_1 \otimes I. \tag{4.11}$$

and

$$\widetilde{H}_{K,V}(\lambda) := L_{K,V}(\lambda) + \delta A_{1,K,V}(\lambda). \tag{4.12}$$

Lemma 4.1 *For all* $\lambda \in \mathbb{R}$,

$$U_{K,V}(\lambda) D(L) = D(L) \tag{4.13}$$

and, for all $\Psi \in D(L)$,

$$U_{K,V}(\lambda) H_{K,V}(\lambda) U_{K,V}(\lambda)^{-1} = \widetilde{H}_{K,V}(\lambda)\Psi. \tag{4.14}$$

Proof. By [4, Lemma 3.1], we have

$$D(H_{\mathrm{b},V}) = D(H_{\mathrm{b}}), \tag{4.15}$$

which implies that

$$D(A \otimes I + I \otimes H_{\mathrm{b},V}) = D(L). \tag{4.16}$$

Then, in a way similar to the proof of Lemma 3.7, one can prove (4.13) and (4.14). ∎

Lemma 4.2 *For all sufficiently large V, $D(L) \subset D(\delta A_{1,K,V}(\lambda))$ and, for all $\Psi \in D(L)$,*

$$\|\delta A_{1,K,V}(\lambda)\Psi\| \le |\lambda| \left(c_{3/2,K,V}(g)\|I \otimes H_{\mathrm{b},V}^{1/2}\Psi\| + \frac{1}{2}c_{1,K,V}(g)\|\Psi\| \right), \tag{4.17}$$

where $c_{s,K,V}(g)$ is the constant $c_s(g)$ with g_j (resp. ω) replaced by $g_{j,K,V}$ (res. ω_V).

Proof. Similar to the proof of Lemma 3.10. ∎

Let

$$g_{j,K}(k) := \chi_{[-K,K]}(k_1)\cdots\chi_{[-K,K]}(k_d)g_j(k), \quad k \in \mathbb{R}^d,\ j = 1, \cdots, J. \tag{4.18}$$

Then the following hold [4, Lemma 3.2]:

$$\lim_{V\to\infty} \|g_{j,K,V} - g_{j,K}\| = 0, \tag{4.19}$$

$$\lim_{K\to\infty} \|g_{j,K} - g_j\| = 0, \tag{4.20}$$

$$\lim_{V\to\infty} \left\| \frac{g_{j,K,V}}{\sqrt{\omega_V}} - \frac{g_{j,K}}{\sqrt{\omega}} \right\| = 0, \tag{4.21}$$

$$\lim_{K\to\infty} \left\| \frac{g_{j,K}}{\sqrt{\omega}} - \frac{g_j}{\sqrt{\omega}} \right\| = 0. \tag{4.22}$$

Let

$$A_K(\lambda) := A - \frac{\lambda^2}{2}\sum_{j,l=1}^{J} \left\langle \frac{g_{j,K}}{\sqrt{\omega}}, \frac{g_{l,K}}{\sqrt{\omega}} \right\rangle B_j B_l. \tag{4.23}$$

Lemma 4.3 *Let $\lambda \in \Lambda$. Then, there exists a constant $K_0(\lambda) > 0$ such that, for all $K > K_0(\lambda)$, $A_K(\lambda)$ is self-adjoint with $D(A_K(\lambda)) = D(A_0)$ and bounded from below.*

Proof. We write

$$A_K(\lambda) = A(\lambda) + \lambda^2 D_K, \tag{4.24}$$

where

$$D_K := \frac{1}{2}\sum_{j,l=1}^{J} c_{j,l}(K,V) B_j B_l \tag{4.25}$$

with

$$c_{j,l}(K) := \left\langle \frac{g_j}{\sqrt{\omega}}, \frac{g_l}{\sqrt{\omega}} \right\rangle - \left\langle \frac{g_{j,K}}{\sqrt{\omega}}, \frac{g_{l,K}}{\sqrt{\omega}} \right\rangle.$$

By (2.21),

$$\|B_jB_lu\| \le (c_jc_l+\varepsilon)\|A_0u\| + \left(\frac{(c_jd_l+c_ld_j)^2}{4\varepsilon}+d_jd_l\right)\|u\|, \quad u \in D(A_0),$$

where $\varepsilon > 0$ is arbitrary. Since $A(\lambda)$ is self-adjoint with $D(A(\lambda)) = D(A_0)$, it follows from the closed graph theorem that there exists constant $\nu(\lambda) > 0$ and $\mu(\lambda) > 0$ such that, for all $u \in D(A_0)$,

$$\|A_0u\| \le \nu(\lambda)\|A(\lambda)u\| + \mu(\lambda)\|u\| \tag{4.26}$$

Hence, for all $u \in D(A_0)$,

$$\|D_Ku\| \le \alpha_K(\lambda)\|A(\lambda)u\| + \beta_K(\lambda)\|u\|, \tag{4.27}$$

where

$$\alpha_K(\lambda) := \frac{\nu(\lambda)}{2}\sum_{j,l=1}^{J}|c_{j,l}(K)|(c_jc_l+\varepsilon),$$
$$\beta_K(\lambda) := \frac{\mu(\lambda)}{2}\sum_{j,l=1}^{J}|c_{j,l}(K)|\left(\frac{(c_jd_l+c_ld_j)^2}{4\varepsilon}+d_jd_l\right).$$

By (4.22), $\lim_{K\to\infty}\alpha_K(\lambda) = 0$. Hence there exists a constant $K_0(\lambda) > 0$ such that, for all $K > K_0(\lambda)$, $\lambda^2\alpha_K(\lambda) < 1$. Then, by the Kato-Rellich theorem, $A_K(\lambda)$ is self-adjoint with $D(A_K(\lambda)) = D(A(\lambda)) = D(A_0)$ and bounded from below. ∎In what follows we assume that $K > K_0(\lambda)$ $(\lambda \in \Lambda)$.

Lemma 4.4 *Let $\lambda \in \Lambda$. Then, for all sufficiently large V, $A_{K,V}(\lambda)$ is self-adjoint with $D(A_{K,V}(\lambda)) = D(A_0)$ and bounded from below.*

Proof. We write

$$A_{K,V}(\lambda) = A_K(\lambda) + \lambda^2 D_{K,V}, \tag{4.28}$$

where

$$D_{K,V} := \frac{1}{2}\sum_{j,l=1}^{J} c_{j,l}(K,V)B_jB_l$$

with

$$c_{j,l}(K,V) := \left\langle \frac{g_{j,K}}{\sqrt{\omega}}, \frac{g_{l,K}}{\sqrt{\omega}}\right\rangle - \left\langle \frac{g_{j,K,V}}{\sqrt{\omega_V}}, \frac{g_{l,K,V}}{\sqrt{\omega_V}}\right\rangle.$$

In the same way as in the proof of Lemma 4.3 [use (4.21)], we can show that $\lambda^2 D_{K,V}$ is relatively bounded w.r.t. $A_K(\lambda)$ with relative bound small than 1 for all sufficiently large V. Thus, by the Kato-Rellich theorem, the assertion holds. ∎

Lemma 4.5 *Let $\lambda \in \Lambda$. Then, for all sufficiently large $V > 0$, $\widetilde{H}_{K,V}(\lambda)$ and $H_{K,V}(\lambda)$ are self-adjoint with $D(\widetilde{H}_{K,V}(\lambda)) = D(H_{K,V}(\lambda)) = D(A_0 \otimes I) \cap D(I \otimes H_{\mathrm{b}})$ and bounded from below.*

Proof. Similar to the proof of Theorem 2.2. ∎The following fact is well-known:

Lemma 4.6 *The operator $H_{\mathrm{b},V}$ is reduced by $\mathcal{F}_{\mathrm{b}}(\mathcal{W}_V)$ and its reduced part is equal to the second quantization of $\omega|\mathcal{W}_V$ in $\mathcal{F}_{\mathrm{b}}(\mathcal{W}_V)$.*

Let

$$\mathcal{F}_V := \mathcal{H} \otimes \mathcal{F}_{\mathrm{b}}(\mathcal{W}_V). \tag{4.29}$$

Then we have the orthogonal decomposition

$$\mathcal{F} = \mathcal{F}_V \oplus \mathcal{F}_V^{\perp}, \tag{4.30}$$

where

$$\mathcal{F}_V^{\perp} = \oplus_{n=1}^{\infty} \mathcal{F}_V \otimes \left[\otimes_{\mathrm{s}}^n \mathcal{W}_V^{\perp})\right]. \tag{4.31}$$

Lemma 4.7 *The operator $\widetilde{H}_{K,V}(\lambda)$ is reduced by $\mathcal{F}_V$ and*

$$\widetilde{H}_{K,V}(\lambda)|\mathcal{F}_V^{\perp} \geq E_0(H_{K,V}(\lambda)) + m. \tag{4.32}$$

Proof. It is easy to see that $g_{j,K,V}/\omega_V \in \mathcal{W}_V$. Hence, under the identifications (4.29) and (4.30), we have $B_j \otimes \phi(ig_{j,K,V}/\omega_V) = [B \otimes \phi(ig_{j,K,V}/\omega_V)] \oplus 0$. It follows that $T_{K,V}$ is reduced by $\mathcal{F}_V$ and so is $U_{K,V}(\lambda)$, which implies that $\delta A_{1,K,V}(\lambda)$ is reduced by $\mathcal{F}_V$. By this fact and Lemma 4.6, $\widetilde{H}_{K,V}(\lambda)$ is reduced by $\mathcal{F}_V$. Then a method similar to the proof of [4, Lemma 3.10] yields (4.32). ∎

Let

$$\Sigma_{\lambda,K} := \inf \sigma_{\mathrm{ess}}(A_K(\lambda)).$$

Lemma 4.8 *There exists a constant $\varepsilon_K > 0$ such that $\lim_{K\to\infty} \varepsilon_K = 0$ and*

$$\Sigma_\lambda \leq \Sigma_{\lambda,K} + \varepsilon_K \tag{4.33}$$

Proof. By (4.27) and a general theorem [13, p.168, Theorem X.18], one can show that

$$(1 - a_K)A(\lambda) - [\mu - E_0(A(\lambda))]a_K \leq A_K(\lambda) \leq (1 + a_K)A(\lambda) + [\mu - E_0(A(\lambda))]a_K \tag{4.34}$$

with

$$a_K := \lambda^2 \left\{\alpha_K(\lambda)\left(1 + \frac{|E_0(A(\lambda))|}{\mu}\right) + \frac{\beta_K(\lambda)}{\mu}\right\}$$

where $\mu > 0$ is arbitrary. For all sufficiently large K, we have $1 - a_K > 0$. We fix such a K. Then it follows from the first inequality in (4.34) and the min-max principle [14, p.76, Theorem XIII.1] that

$$(1 - a_K)\Sigma_\lambda \leq \Sigma_{\lambda,K} + [\mu - E_0(A(\lambda))]a_K,$$

which implies that

$$\Sigma_\lambda \leq \Sigma_{\lambda,K} + \varepsilon_K$$

with $\varepsilon_K := a_K[\Sigma_\lambda + \mu - E_0(A(\lambda))]$. We have $\lim_{K\to\infty} \varepsilon_K = 0$. ∎

Let

$$\Sigma_{\lambda,K,V} := \inf \sigma_{\mathrm{ess}}(A_{K,V}(\lambda)).$$

Lemma 4.9 *There exists a constant $\eta_{K,V} > 0$ such that $\lim_{V\to\infty} \eta_{K,V} = 0$ and*

$$\Sigma_{\lambda,K} \le \Sigma_{\lambda,K,V} + \eta_{K,V} \tag{4.35}$$

Proof. Similar to the proof of Lemma 4.8. ∎

Lemma 4.10

$$\lim_{K\to\infty} E_0(A_K(\lambda)) = E_0(A(\lambda)), \tag{4.36}$$
$$\lim_{V\to\infty} E_0(A_{K,V}(\lambda)) = E_0(A_K(\lambda)). \tag{4.37}$$

Proof. By (4.34) and the variational principle,

$$(1 - a_K)E_0(A(\lambda)) - \mu a_K \le E_0(A_K(\lambda)) \le (1 + a_K)E_0(A(\lambda)) + \mu a_K,$$

which implies (4.36). Similarly one can prove (4.37). ∎

Lemma 4.11 *Suppose that the same hypothesis as in Theorem 2.3 and Hypothesis VIII hold. Then, for all sufficiently large K and V, $H_{K,V}(\lambda)$ has purely discrete spectrum in $[E_0(H_{K,V}(\lambda)), E_0(H_{K,V}(\lambda)) + m)$.*

Proof. By Lemma 4.1, we need only to show that $\widetilde{H}_{K,V}(\lambda)$ has purely discrete spectrum in $[E_0(H_{K,V}(\lambda)), E_0(H_{K,V}(\lambda)) + m)$ [note that $E_0(h_{K,V}) = E_0(\widetilde{H}_{K,V}(\lambda)) = E_0(H_{K,V}(\lambda))$]. By Lemma 4.7, it is sufficient to show that the reduced part $h_{K,V} := \widetilde{H}_{K,V}(\lambda)|\mathcal{F}_V$ has such a property. By Lemma 4.2, we have

$$|\langle\Psi, \delta A_{1,K,V}(\lambda)\Psi\rangle| \le \varepsilon\langle\Psi, I \otimes H_{\mathrm{b},V}\Psi\rangle + b_\varepsilon\|\Psi\|^2,\ \Psi \in D(I \otimes H_{\mathrm{b},V}), \tag{4.38}$$

where $\varepsilon > 0$ is arbitrary and

$$b_\varepsilon := \left(\frac{\lambda^2 c_{3/2,K,V}(g)^2}{4\varepsilon} + \frac{1}{2}|\lambda|c_{1,K,V}(g)\right). \tag{4.39}$$

By condition (2.25) and Lemmas 4.8–4.10, we have

$$\Sigma_{\lambda,K,V} - E_0(A_{K,V}(\lambda)) > m + 2b_\varepsilon \tag{4.40}$$

if $\varepsilon < 1$ is sufficiently close to 1 and K and V are sufficiently large. Note that the spectrum of $h_0 := H_{\mathrm{b},V}|\mathcal{F}_{\mathrm{b}}(\mathcal{W}_V)$ is purely discrete with $\mathrm{Ran}(E_{h_0}([0, s])$ being finite dimensional for all $s > 0$. Hence we can apply Theorem B.3 with Remark B.1 in Appendix B to conclude that $h_{K,V}$ has purely discrete spectrum in $[E_0(h_{K,V}), E_0(h_{K,V}) + m)$. ∎

Proof of Theorem 2.3 Let

$$H_K(\lambda) := A \otimes I + I \otimes H_{\mathrm{b}} + \lambda \sum_{j=1}^{J} B_j \otimes \phi(g_{j,K}).$$

Then, in the same way as in [4, Lemma 3.5], one can show that $H_{K,V}(\lambda)$ converges to $H_K(\lambda)$ in the norm resolovent sense as $V \to \infty$. Hence, by Lemma 4.11 and an application of [4, Lemma 3.12], we conclude that $H_K(\lambda)$ has purely discrete spectrum in $[E_0(H_K(\lambda)), E_0(H_K(\lambda)) + m)$. In the same way as in [4, Lemma 3.11], one can show that $H_K(\lambda)$ converges to $H(\lambda)$ in the norm resolovent sense as $K \to \infty$. Hence, by the preceding result and [4, Lemma 3.12] again, we see that $H(\lambda)$ has purely discrete spectrum in $[E_0(H(\lambda)), E_0(H(\lambda))+m)$. Finally we consider the case where each g_j is not necessarily continuous. In this case we can take a sequence of continuous functions $\{g_j^{(n)}\}_{n=1}^{\infty}$ such that $g_j^{(n)} \in \mathcal{W}$ such that $\|g_j^{(n)} - g_j\| \to 0$ $(n \to \infty)$. Let H_n be the operator $H(\lambda)$ with g_j replaced $g_j^{(n)}$ $(j = 1, \cdots, J)$. Then one can show that H_n converges to $H(\lambda)$ in the norm resolovent sense as $n \to \infty$. By the result of the last paragraph, H_n has purely discrete spectrum in $[E_0(H_n), E_0(H_n) + m)$. Hence, by [4, Lemma 3.12] once again, we see that $H(\lambda)$ has purely discrete spectrum in $[E_0(H(\lambda)), E_0(H(\lambda)) + m)$.

5 Existence of a Ground State in the Massless Case

This section is devoted to proof of Theorem 2.4. Throughout the section, all the hypotheses of Theorem 2.4 are assumed. For each constant $M > 0$, we define $\omega_M : \mathbb{R}^d \to [M, \infty)$ by

$$\omega_M(k) := \omega(k) + M, \quad k \in \mathbb{R}^d,$$

so that $\inf_{k\in\mathbb{R}^d} \omega_M(k) = M > 0$. We set

$$g_j^{(M)} := \frac{\omega_M}{\omega} g_j,$$

which is in $\mathcal{W}$. We introduce a "regularized" version of the Hamiltonian $H(\lambda)$:

$$H_M := A \otimes I + I \otimes H_{\mathrm{b},M} + \lambda \sum_{j=1}^{J} B_j \otimes \phi(g_j^{(M)}),$$

where

$$H_{\mathrm{b},M} := d\Gamma(\omega_M).$$

Let

$$A_M := A - \lambda^2 R_B^{(M)}$$

with

$$R_B^{(M)} := \frac{1}{2} \sum_{j,l=1}^{J} \left\langle \frac{g_j^{(M)}}{\sqrt{\omega_M}}, \frac{g_l^{(M)}}{\sqrt{\omega_M}} \right\rangle B_j B_l$$

and

$$\widetilde{H}_M := A_M \otimes I + I \otimes H_{\mathrm{b},M} + \delta A_1(\lambda).$$

Then, by Lemma 3.7,

$$U(\lambda) H_M U(\lambda)^{-1} = \widetilde{H}_M.$$

By applying the Lebesgue dominated convergence theorem, one can show that

$$\lim_{M\to 0}\left\|\frac{g_j^{(M)}}{\omega_M^s}-\frac{g_j}{\omega^s}\right\|=0 \tag{5.1}$$

for all $s\geq 0$ such that $g_j/\omega^{s+1}\in\mathcal{W}$. We write

$$A_M=A(\lambda)+W_M$$

with

$$W_M:=\lambda^2\left(R_B-R_B^{(M)}\right).$$

We put

$$c_M:=\lambda^2\sum_{j,l=1}^{J}\left|\left\langle\frac{g_j}{\sqrt{\omega}},\frac{g_l}{\sqrt{\omega}}\right\rangle-\left\langle\frac{g_j^{(M)}}{\sqrt{\omega_M}},\frac{g_l^{(M)}}{\sqrt{\omega_M}}\right\rangle\right|.$$

Then we can show that

$$\|W_Mu\|\leq c_M(a\|A(\lambda)u\|+b\|u\|),\quad u\in D(A_0),$$

where a and b are constants independent of M (cf. the proof of Lemma 4.3). In the same way as in Lemma 4.10, one can show that

$$\lim_{M\to 0}E_0(A_M)=E_0(A(\lambda)). \tag{5.2}$$

By this fact, (5.1) and (2.26), we can take $M>0$ (sufficiently small) satisfying

$$\Sigma_\lambda-E_0(A_M)>M+\frac{1}{2}\lambda^2c_{3/2}^{(M)}(g)^2+|\lambda|c_1^{(M)}(g), \tag{5.3}$$

where $c_s^{(M)}(g)$ is the $c_s(g)$ with ω and g_j replaced by ω_M and $g_j^{(M)}$ respectively. It follows from Theorem 2.3 that H_M has a ground state and so does $\widetilde{H}_M$. We denote a normalized ground state of $\widetilde{H}_M$ by Ψ_M.

Lemma 5.1 *For all $f\in\mathcal{W}$ with $\omega f\in\mathcal{W}$, $I\otimes a(f)\Psi_M\in D(\widetilde{H}_M)$ and*

$$\begin{aligned}&(\widetilde{H}_M-E_0(\widetilde{H}_M))I\otimes a(f)\Psi_M\\&=\left\{-a(\omega_Mf)-\sum_{j=1}^{J}\frac{\lambda}{\sqrt{2}}\left\langle f,\frac{g_j}{\omega}\right\rangle U(\lambda)[B_j,A_1]U(\lambda)^{-1}\right\}\Psi_M.\end{aligned}$$

Proof. Similar to the proof of [4, Lemma 4.1] except that, in the present case, one uses an easily proven formula

$$U(\lambda)I\otimes a(f)U(\lambda)^{-1}=I\otimes a(f)-\frac{\lambda}{\sqrt{2}}\left\langle f,\frac{g_j}{\omega}\right\rangle B_j\otimes I$$

on $D(A_0\otimes I)\cap D(I\otimes H_{\mathrm{b},M})$. ∎

We set

$$N_{\rm b} := d\Gamma(I),$$

the number operator on $\mathcal{F}_{\rm b}(\mathcal{W})$. Then we have for all $f \in \mathcal{W}$ and $\psi \in D(N_{\rm b}^{1/2})$

$$\|a(f)\psi\| \le \|f\| \|N_{\rm b}^{1/2}\psi\|, \quad \|a(f)^*\psi\| \le \|f\| \|N_{\rm b}^{1/2}\psi\| + \|f\|,$$

which implies that

$$\|\phi(f)\psi\| \le \sqrt{2}\|f\|\|N_{\rm b}^{1/2}\psi\| + \frac{1}{\sqrt{2}}\|f\|\|\psi\|. \tag{5.4}$$

Now we can apply Theorem B.4 in Appendix B with $(K, B, C) = (A(\lambda), H_{\rm b}, \delta A_1(\lambda))$ so that $H = \widetilde{H}(\lambda)$. By (2.25), Hypothesis (B.1) holds. Hypothesis (B.2) is satisfied with $m = M$, $K_m = A_M$, $B_m = H_{{\rm b},M}$, $\psi_0 = \Omega$ (the Fock vacuum) and $D = \mathcal{F}_0(\mathcal{W}) \cap D(H_{\rm b})$. We denote by P_Ω the orthogonal projection onto the one-dimensional subspace $\{\alpha\Omega | \alpha \in \mathbb{C}\}$ and set $P_\Omega^\perp. := I - P_\Omega$. By (3.23) and the fact that $H_{{\rm b},M}\Omega = 0$ and $H_{\rm b}\Omega = 0$, we have for all $u \in D(A_0)$ with $\|u\| = 1$

$$\left\langle u \otimes \Omega, \widetilde{H}_M u \otimes \Omega \right\rangle \le \langle u, A_M u\rangle + \frac{|\lambda|}{2} c_1(g),$$

which implies that $E_0(\widetilde{H}_M) \le E_0(A_M) + |\lambda| c_1(g)/2$. By (5.2), for every $\eta > 0$, there exists a constant $M_0 > 0$ such that $|E_0(A_M) - E_0(A(\lambda))| < \eta$ for all $M < M_0$. By this fact and (2.26), we have

$$E_0(\widetilde{H}_M) \le \eta + E_0(A(\lambda)) + \frac{|\lambda|}{2} c_1(g) < \Sigma_\lambda.$$

for all $M < M_0$, where we take η sufficiently small. Thus, if we show that

$$\frac{1}{(\Sigma_\lambda - E_0(\widetilde{H}_M))^2} \|\delta A_1(\lambda)\Psi_M\|^2 + \|I \otimes P_\Omega^\perp \Psi_M\|^2 < \delta \tag{5.5}$$

for some $\delta < 1$, then $\widetilde{H}(\lambda)$ has a ground state and so does $H(\lambda)$. Let us prove (5.5). Using estimate (5.4) and (3.24), we have

$$\|\delta A_1(\lambda)\Psi_M\| \le |\lambda| c_1(g) \|N_{\rm b}^{1/2}\Psi_M\| + \frac{|\lambda| c_1(g)}{2}.$$

It is well known or easy to see that $N_{\rm b} \ge P_\Omega^\perp$. Hence

$$\|I \otimes P_\Omega^\perp \Psi_M\|^2 \le \|N_{\rm b}^{1/2}\Psi_M\|^2.$$

Therefore, if

$$\left(\frac{2|\lambda|^2 c_1(g)^2}{[\Sigma_\lambda - E_0(\widetilde{H}_M)]^2} + 1 \right) \|N_{\rm b}^{1/2}\Psi_M\|^2 + \frac{|\lambda|^2 c_1(g)^2}{[\Sigma_\lambda - E_0(\widetilde{H}_M)]^2} < \delta, \tag{5.6}$$

then (5.5) follows. To estimate $\|N_{\rm b}^{1/2}\Psi_M\|$,we follow the method given in the proof of [4, Lemma 4.3]. Indeed, by Lemma 5.1, we can show that

$$\|N_{\rm b}^{1/2}\Psi_M\| \le \frac{|\lambda|}{\sqrt{2}} \left(\sum_{j=1}^J \left\| \frac{g_j}{\omega\omega_M} \right\| \|[B_j, A_1]\| \right).$$

Hence, if

$$\left(\frac{2|\lambda|^2 c_1(g)^2}{[\Sigma_\lambda - E_0(\widetilde{H}_M)]^2} + 1\right)\frac{\lambda^2}{2}\left(\sum_{j=1}^{J}\left\|\frac{g_j}{\omega\omega_M}\right\|\|[B_j, A_1]\|\right)^2 + \frac{|\lambda|^2 c_1(g)^2}{[\Sigma_\lambda - E_0(\widetilde{H}_M)]^2} < \delta, \tag{5.7}$$

then (5.6) follows. In the same way as in [4, Lemma 4.11], we can show that

$$\lim_{M\to 0} E_0(H_M) = E_0(H(\lambda)).$$

We have $E_0(H_M) = E_0(\widetilde{H}_M)$. Hence condition (2.27) implies (5.7) with $M > 0$ sufficiently small. This completes the proof of Theorem 2.4.

6 The Pauli-Fierz Type Model

In this section, we apply Theorem 2.4 to a model of the Pauli-Fierz type in non-relativistic QED . Namely we consider the case where the system S is a system of n non-relativistic quantum particles moving in $\mathbb{R}^d$ under the influence of a potential $V : \mathbb{R}^{dn} \to \mathbb{R}$ ($d, n \in \mathbb{N}$). We set $\nu := nd$. We assume for simplicity that

$$\nu \geq 3, \quad V \in C_0^\infty(\mathbb{R}^\nu), \quad V_- := \min\{V, 0\} \neq 0.$$

The Hilbert space of the particle system is taken to be

$$\mathcal{H} = L^2(\mathbb{R}^\nu).$$

Hence the Hamiltonian of the particle system is

$$H_{\rm p} := -\Delta + \alpha V$$

acting in $L^2(\mathbb{R}^\nu)$, where Δ is the generalized Laplacian on $L^2(\mathbb{R}^\nu)$ and $\alpha > 0$ is a parameter. We write $x = (x_1, \cdots, x_\nu) \in \mathbb{R}^\nu$ and define

$$p_j := -iD_j$$

with D_j being the generalized partial differential operator in the variable x_j. By the Cwikel-Lieb-Rosenbljum bound [14, Theorem XIII.12], $H_{\rm p}$ has no ground state for all sufficiently small α.

Let $g_j : \mathbb{R}^d \to \mathbb{R}^N$ ($j = 1, \cdots, \nu$) be such that

$$g_j, \quad \frac{g_j}{\omega^2} \in \mathcal{W}. \tag{6.1}$$

We take as the total Hamiltonian of the composed system

$$H_{\rm PF}(\lambda) := H_{\rm p} \otimes I + I \otimes H_{\rm b} + \lambda \sum_{j=1}^{\nu} p_j \otimes \phi(g_j).$$

This model is a concrete realization of the abstract model $H(\lambda)$ with the following choice:

$$A_0 = -\Delta, \quad A_1 = \alpha V, \quad J = \nu, \quad B_j = p_j.$$

It is straightforward to see that Hypotheses I–V hold with

$$[B_j, A_1]|D(A_0) = -i\alpha D_j V.$$

Suppose that, for all $\xi = (\xi_1, \cdots, \xi_\nu) \in \mathbb{R}^\nu$,

$$\frac{1}{2}\sum_{j,l=1}^{\nu}\left\langle \frac{g_j}{\sqrt{\omega}}, \frac{g_l}{\sqrt{\omega}}\right\rangle \xi_j\xi_l = G(g)\xi^2 \tag{6.2}$$

with $G(g) > 0$ a constant independent of ξ. This condition is satisfied in the original Pauli-Fierz model without A^2-term in the dipole approximation [2] (see Example 6.2 below). Condition (6.2) implies that

$$\frac{1}{2}\sum_{j,l=1}^{\nu}\left\langle \frac{g_j}{\sqrt{\omega}}, \frac{g_l}{\sqrt{\omega}}\right\rangle p_jp_l = -G(g)\Delta.$$

Hence, in the present case, $A(\lambda)$ takes the following form:

$$A(\lambda) = -(1 - \lambda^2 G(g))\Delta + \alpha V$$

Therefore, in the present case,

$$\Lambda = (-\lambda(g), 0) \cup (0, \lambda(g)) \neq \emptyset,$$

where

$$\lambda(g) := \frac{1}{\sqrt{G(g)}}.$$

Thus Hypothesis VI holds. Also we have

$$c_s(g) = \sqrt{2}|\alpha|\sum_{j=1}^{\nu}\|D_jV\|_\infty \left\|\frac{g_j}{\omega^s}\right\|, \tag{6.3}$$

where $\|F\|_\infty := \sup_{x\in\mathbb{R}^\nu}|F(x)|$ ($F : \mathbb{R}^\nu \to \mathbb{C}$). We set

$$V_0 := \inf_{x\in\mathbb{R}^\nu} V(x) < 0.$$

We first consider the massive case.

Theorem 6.1 *Consider the case $m > 0$. Let $\nu \geq 3$ and Hypothesis VII be satified. Suppose that*

$$\alpha|V_0| > \frac{1}{2}\lambda(g)^2 c_{3/2}(g)^2 + \lambda(g)c_1(g) \tag{6.4}$$

and the constant m satisfies

$$\alpha|V_0| > m + \frac{1}{2}\lambda(g)^2 c_{3/2}(g)^2 + \lambda(g)c_1(g). \tag{6.5}$$

Then there exists a constant δ such that, for all $|\lambda| \in (\lambda(g) - \delta, \lambda(g))$, $H_{\rm PF}(\lambda)$ has purely discrete spectrum in the interval $[E_0(H_{\rm PF}(\lambda)), E_0(H_{\rm PF}(\lambda)) + m)$. In particular, $H_{\rm PF}(\lambda)$ has a ground state.

Proof. Let $0 < |\lambda| < \lambda(g)$. By [14, Theorem XIII.15],

$$\Sigma_\lambda = \inf \sigma_{\rm ess}(A(\lambda)) = 0.$$

Therefore, by Theorem 2.3, we need only to show

$$-E_0(A(\lambda)) > m + \frac{1}{2}\lambda^2 c_{3/2}(g)^2 + |\lambda| c_1(g) \tag{6.6}$$

for all $|\lambda|$ sufficiently close to $\lambda(g)$. We can take a constant $\varepsilon > 0$ such that $E := V_0 + \varepsilon < 0$. Then $D_E := \{x \in \mathbb{R}^\nu | V(x) < E\}$ is a non-empty bounded open set. Let $d_E := \inf_{u \in C_0^\infty(D_E), \|u\|=1} \langle u, -\Delta u\rangle$. Then, by the strict positivity of the Dirichlet Laplacian in a bounded open set, $d_E > 0$. By the variational principle and the fact that $\langle u, Vu\rangle \le E\|u\|^2$, $u \in C_0^\infty(D_E)$,

$$E_0(A(\lambda)) \le (1 - \lambda^2 G(g)) d_E + \alpha E. \tag{6.7}$$

Note that $-\alpha E = \alpha |V_0| - \alpha\varepsilon$. Hence, by (6.5), for all sufficiently small $\varepsilon > 0$ and $|\lambda|$ sufficiently close to $\lambda(g)$,

$$-\alpha E - (1 - \lambda^2 G(g)) d_E > m + \frac{1}{2}\lambda(g)^2 c_{3/2}(g)^2 + \lambda(g) c_1(g). \tag{6.8}$$

Hence

$$\begin{aligned} -E_0(A(\lambda)) &> m + \frac{1}{2}\lambda(g)^2 c_{3/2}(g)^2 + \lambda(g) c_1(g) \\ &> m + \frac{1}{2}\lambda^2 c_{3/2}(g)^2 + |\lambda| c_1(g). \end{aligned}$$

Thus (6.6) holds. ∎

We next consider the massless case.

Theorem 6.2 *Consider the case $m = 0$. Let $\nu \ge 3$. Assume Hypothesis VII and (6.4). Suppose in addition that*

$$\frac{4\lambda(g)^2 c_1(g)^2}{\alpha^2 V_0^2} + \frac{1}{2}\left(\frac{8\lambda(g)^2 c_1(g)^2}{\alpha^2 V_0^2} + 1\right)\lambda(g)^2 c_2(g)^2 < 1. \tag{6.9}$$

Then there exists a constant δ such that, for all $|\lambda| \in (\lambda(g) - \delta, \lambda(g))$, $H_{\rm PF}(\lambda)$ has a ground state.

Proof. Let $0 < |\lambda| < \lambda(g)$. By Theorem 2.4 and by the proof of Theorem 6.1, we need only to check that

$$\frac{\lambda^2 c_1(g)^2}{E_0(H_{\rm PF}(\lambda))^2} + \frac{1}{2}\left(\frac{2\lambda^2 c_1(g)^2}{E_0(H_{\rm PF}(\lambda))^2} + 1\right)\lambda^2 c_2(g)^2 < 1 \tag{6.10}$$

for all $|\lambda|$ sufficiently close to $\lambda(g)$. By (6.7) and Corollary 3.12, we have

$$\begin{aligned} E_0(H_{\rm PF}(\lambda)) &\le (1 - \lambda^2 G(g)) d_E + \alpha E + \frac{\lambda(g)}{2} c_1(g) \\ &\le (1 - \lambda^2 G(g)) d_E - \frac{\alpha}{2}|V_0| + \alpha\varepsilon, \end{aligned} \tag{6.11}$$

where, in the last step, we have used (6.4). For all $|\lambda|$ sufficiently close to $\lambda(g)$ and sufficiently small ε, the right hand side of (6.11) is negative. Hence, if we show that

$$\frac{\lambda^2 c_1(g)^2}{[\frac{\alpha|V_0|}{2} - \alpha\varepsilon - (1-\lambda^2 G(g))d_E]^2} + \frac{1}{2}\left(\frac{2\lambda^2 c_1(g)^2}{[\frac{\alpha|V_0|}{2} - \alpha\varepsilon - (1-\lambda^2 G(g))d_E]^2} + 1\right)\lambda^2 c_2(g)^2 < 1, \tag{6.12}$$

for all $|\lambda|$ sufficiently close to $\lambda(g)$ and sufficiently small ε, then (6.10) follows. It is easy to see that (6.9) implies (6.12) for all $|\lambda|$ sufficiently close to $\lambda(g)$ and sufficiently small $\varepsilon > 0$. ∎

Example 6.1 Assume Hypothesis VII. Let $\kappa > 0$ and $H_{\mathrm{PF}}^{\kappa}(\lambda)$ be the $H_{\mathrm{PF}}(\lambda)$ with ω replaced by $\kappa\omega$. Then conditions (6.4) and (6.9) take the following forms respectively:

$$\alpha|V_0| > \frac{1}{2\kappa^2}\lambda(g)^2 c_{3/2}(g)^2 + \frac{1}{\sqrt{\kappa}}\lambda(g)c_1(g), \tag{6.13}$$

$$\frac{4\lambda(g)^2 c_1(g)^2}{\kappa\alpha^2 V_0^2} + \frac{1}{2\kappa^3}\left(\frac{8\lambda(g)^2 c_1(g)^2}{\kappa\alpha^2 V_0^2} + 1\right)\lambda(g)^2 c_2(g)^2 < 1. \tag{6.14}$$

For a given $\alpha|V_0|$, these inequalities are satisfied if κ is sufficiently large. Thus $H_{\mathrm{PF}}^{\kappa}(\lambda)$ has a ground state for all sufficiently large κ and $|\lambda| < \sqrt{\kappa}\lambda(g)$ sufficiently close to $\sqrt{\kappa}\lambda(g)$. This result is somewhat analogous to the results by Hiroshima and Spohn [11, Lemma 3.3, Theorem 3.4], except that the regime of the coupling constant is different.

Example 6.2 Consider the original Pauli-Fierz model with one non-relativistic particle in $\mathbb{R}^3$ so that $n = 1, d = 3$ and $N = 2$. We take $\omega(k) = |k|, k \in \mathbb{R}^3$, and the momentum cutoff function $g_j : \mathbb{R}^3 \to \mathbb{R}^2$ $(j = 1,2,3)$ as

$$g_j(k) = \left(\frac{\chi_{[\sigma,L]}(|k|)}{\sqrt{(2\pi)^3|k|}}e_j^{(1)}(k), \frac{\chi_{[\sigma,L]}(|k|)}{\sqrt{(2\pi)^3|k|}}e_j^{(2)}(k)\right),$$

where $\chi_{[\sigma,L]}$ is the characteristic function of the interval $[\sigma, L]$ ($\sigma > 0$ is an infrared cutoff and $L > \sigma$ is an ultraviolet cutoff) and $e^{(r)} = (e_1^{(r)}, e_2^{(r)}, e_3^{(r)}) : \mathbb{R}^3 \to \mathbb{R}^3$ $(r = 1,2)$ is Borel measurable such that

$$\left\langle e^{(s)}(k), e^{(r)}(k)\right\rangle = \delta_{sr}, \quad \left\langle e^{(r)}(k), k\right\rangle = 0, \quad r, s = 1, 2, \text{ a.e.} k \in \mathbb{R}^3.$$

By the identity

$$\sum_{r=1}^{2} e_j^{(r)}(k)e_l^{(r)}(k) = \delta_{jl} - \frac{k_j k_l}{|k|^2}, \quad \text{a.e.} k$$

and the easily proven fact that

$$\int_{\mathbb{R}^3} f(|k|)k_j k_l dk = \frac{1}{3}\delta_{jl}\int_{\mathbb{R}^3} f(|k|)k^2 dk$$

for all $f : [0,\infty) \to \mathbb{C}$ such that $\int_{\mathbb{R}^3} f(|k|)k^2 dk < \infty$, we can show that

$$\begin{aligned}\left\langle \frac{g_j}{\omega^s}, \frac{g_l}{\omega^s} \right\rangle &= \frac{2}{3(2\pi)^3}\delta_{jl}\int_{\mathbb{R}^3} \frac{\chi_{[\sigma,L]}(|k|)}{|k|^{2s+1}}dk \\ &= \begin{cases} \delta_{jl}\frac{8\pi}{3(2\pi)^3}\log\frac{L}{\sigma} & ;\ s=1 \\ \delta_{jl}\frac{8\pi}{3(2\pi)^3}\frac{1}{2(1-s)}\left(L^{2(1-s)} - \sigma^{2(1-s)}\right) & ;\ s \neq 1 \end{cases}.\end{aligned}$$

Hence, in the present example, we have

$$\begin{aligned} G(g) &= \frac{4\pi}{(2\pi)^3}(L-\sigma), \quad \lambda(g) = \sqrt{\frac{(2\pi)^3}{4\pi}}\frac{1}{\sqrt{L-\sigma}}, \\ c_1(g) &= \sqrt{2}|\alpha|\left(\sum_{j=1}^3 \|D_jV\|_\infty\right)\sqrt{\frac{8\pi}{3(2\pi)^3}}\sqrt{\log\frac{L}{\sigma}}, \\ c_{3/2}(g) &= \sqrt{2}|\alpha|\left(\sum_{j=1}^3 \|D_jV\|_\infty\right)\sqrt{\frac{8\pi}{3(2\pi)^3}}\sqrt{\frac{1}{\sigma}-\frac{1}{L}}, \\ c_2(g) &= \sqrt{2}|\alpha|\left(\sum_{j=1}^3 \|D_jV\|_\infty\right)\sqrt{\frac{8\pi}{3(2\pi)^3}}\frac{1}{\sqrt{2}}\sqrt{\frac{1}{\sigma^2}-\frac{1}{L^2}}. \end{aligned}$$

¿From these formulas, we see that

$$\begin{aligned} \lambda(g)c_1(g) &\sim \text{const.}\sqrt{\frac{\log L}{L}}, \\ \lambda(g)c_{3/2}(g) &\sim \text{const.}\frac{1}{\sqrt{L}}, \\ \lambda(g)c_2(g) &\sim \text{const.}\frac{1}{\sqrt{L}} \end{aligned}$$

as $L \to \infty$, where "const." denotes a constant independent of L sufficiently large. Hence, for all sufficiently large L, all the assumptions of Theorem 6.2 are satisfied. Thus, in the present example, $H_{\mathrm{PF}}(\lambda)$ has a ground state for all sufficiently large L and $|\lambda| < \lambda(g)$ sufficiently close to $\lambda(g)$. A possible physical picture of this result is that the coupling of non-relativistic quantum particles to photons with *larger* momenta makes higher the possibility for $H_{\mathrm{PF}}(\lambda)$ to have a ground state.

Appendix

A Weak Differentiablity of a Heisenberg Operator

Let $\mathcal{X}$ be a Hilbert space. Let H be a self-adjoint operator and S a symmetric operator on $\mathcal{X}$. Then the Heisenberg operator ("time evolution") of S with respect to H is defined by

$$S(t) := e^{itH} S e^{-itH}, \quad t \in \mathbb{R}. \tag{A.1}$$

Proposition A.1 *Suppose that there exists a self-adjoint operator K on $\mathcal{X}$ such that the following (K.1) and (K.2) hold:*

(K.1) K strongly commutes with H.

(K.2) $D(K) \subset D(S)$ and there exist constants $a, b \geq 0$ such that

$$\|S\psi\| \leq a\|K\psi\| + b\|\psi\|, \quad \psi \in D(K).$$

Then, for all $\psi, \phi \in D(K) \cap D(H)$, the function: $t \mapsto \langle \psi, S(t)\phi \rangle$ ($t \in \mathbb{R}$) is differentiable and

$$\frac{d}{dt} \langle \psi, S(t)\phi \rangle = i\{\left\langle He^{-itH}\psi, Se^{-itH}\phi \right\rangle - \left\langle Se^{-itH}\psi, He^{-itH}\phi \right\rangle\}. \tag{A.2}$$

Proof. It follows from the strong commutativity of K with H and the two-variable functional calculus that $e^{itH} D(K) \cap D(H) = D(K) \cap D(H)$ for all $t \in \mathbb{R}$. Let $f(t) := \langle \psi, S(t)\phi \rangle$, $F_\varepsilon := (e^{-i\varepsilon H} - 1)/\varepsilon$ and $G_\varepsilon := e^{-i\varepsilon H} - 1$ with $\varepsilon \in \mathbb{R} \setminus \{0\}$. Then

$$\begin{aligned} \frac{f(t+\varepsilon) - f(t)}{\varepsilon} &= \left\langle F_\varepsilon e^{-itH}\psi, Se^{-itH} G_\varepsilon \phi \right\rangle + \left\langle F_\varepsilon e^{-itH}\psi, Se^{-itH}\phi \right\rangle \\ &\quad + \left\langle Se^{-itH}\psi, e^{-itH} F_\varepsilon \phi \right\rangle. \end{aligned}$$

The first term on the right hand side is estimated as follows:

$$\begin{aligned} |\left\langle F_\varepsilon e^{-itH}\psi, Se^{-itH} G_\varepsilon \phi \right\rangle| &\leq \|F_\varepsilon \psi\|(a\|Ke^{-itH} G_\varepsilon \phi\| + b\|G_\varepsilon \phi\|) \\ &= \|F_\varepsilon \psi\|(a\|G_\varepsilon K\phi\| + b\|G_\varepsilon \phi\|), \end{aligned}$$

where, in the last step, we have used the strong commutativity of K and H. Note that $F_\varepsilon \psi \to -iH\psi$ and $G_\varepsilon \phi \to 0$ strongly as $\varepsilon \to 0$. Hence

$$\lim_{\varepsilon \to 0} \left\langle F_\varepsilon e^{-itH}\psi, Se^{-itH} G_\varepsilon \phi \right\rangle = 0$$

and (A.2) follows. ∎

B Ground States of Self-adjoint Operators

In this section we establish two abstract theorems on existence of ground states of a self-adjoint operator on an abstract Hilbert space. They reveal general structures of methods used in previous papers [7, 9, 10] to prove the existence of ground states of particle-field interaction models.

B.1 Existence of a Ground State of a Self-adjoint Operator

For a self-adjoint operator S on a Hilbert space $\mathcal{X}$, we denote the form domain of S by $Q(S)$:

$$Q(S) := \left\{ \psi \in \mathcal{X} \middle| \int_{\mathbb{R}} |\lambda| d\|E_S(\lambda)\psi\|^2 < \infty \right\} = D(|S|^{1/2}), \tag{B.1}$$

where $E_S(\cdot)$ denotes the spectral measure of S (see Section 2 for notations). For $\psi, \phi \in Q(S)$, we define $\langle \psi, S\phi \rangle$ by

$$\langle \psi, S\phi \rangle := \int_{\mathbb{R}} \lambda d \langle \psi, E_S(\lambda)\phi \rangle .$$

For symmetric operators A, B and a subspace $D \subset D(A) \cap D(B)$, we mean by "$A \le B$ on D" that $\langle \psi, A\psi \rangle \le \langle \psi, B\psi \rangle$ for all $\psi \in D$.

Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces. Let A and B be self-adjoint operators on $\mathcal{H}$ and $\mathcal{K}$ respectively. We assume the following:

Hypothesis A. The operator A is bounded from below and B is nonnegative with $E_0(B) = 0$.

We set

$$T_0 := A \otimes I + I \otimes B, \tag{B.2}$$

which is self-adjoint and bounded from below by $E_0(A)$.

For a sesquilinear form Z on a Hilbert space, we denote its form domain by $Q(Z)$.

Let Z be a symmemtric sesquilinear form on $\mathcal{H} \otimes \mathcal{K}$ obeying the following conditions:

(i) $Q(Z) \supset Q(I \otimes B)$;

(ii) There exist constants $a \in [0, 1)$ and $b \ge 0$ such that, for all $\psi \in Q(I \otimes B)$,

$$|Z(\psi, \psi)| \le a \langle \psi, I \otimes B\psi \rangle + b\|\psi\|^2.$$

Lemma B.1 *Assume Hypothesis A and and let Z be as above. Then there exists a unique self-adjoint operator T on $\mathcal{H} \otimes \mathcal{K}$ such that $Q(T) = Q(T_0)$ and*

$$\langle \psi, T\phi \rangle = \langle \psi, T_0\phi \rangle + Z(\psi, \phi), \quad \psi, \phi \in Q(T_0).$$

T is bounded from below by $E_0(A) - b$ and every domain of essential self-adjointness for T_0 is a form core for T.

Proof. Let $\hat{A} := A - E_0(A)$, which is nonnegative. By the present assumption, we have for all $\psi \in Q(T_0)$,

$$|Z(\psi, \psi)| \le a \left\langle \psi, (\hat{A} \otimes I + I \otimes B)\psi \right\rangle + b\|\psi\|^2.$$

Note that $\hat{A} \otimes I + I \otimes B \ge 0$. Hence we can apply the KLMN theorem [13, Theorem X.17] to conclude that there exists a unique self-adjoint operator T' on $\mathcal{H} \otimes \mathcal{K}$ such that $Q(T') = Q(\hat{A} \otimes I + I \otimes B) = Q(T_0)$ and $T' = \hat{A} \otimes I + I \otimes B + Z$ in the sense of sesquilinear form on $Q(T_0)$ with $T' \ge -b$. Then the operator T defined by $T := T' + E_0(A)$ is the desired one. ∎

Lemma B.2 *Under the same hypothesis as in Lemma B.1,*

$$|E_0(T) - E_0(A)| \leq b. \tag{B.3}$$

Proof. By the variational principle and Lemma B.1, we have

$$E_0(T) \geq E_0(A) - b. \tag{B.4}$$

On the other hand, for all $f \in D(A)$ and $g \in D(B)$ with $\|f\| = 1$ and $\|g\| = 1$, we have

$$E_0(T) \leq (f, Af) + (1+a)(g, Bg) + b,$$

which, together with the variational principle, implies that $E_0(T) \leq E_0(A) + b$, where we have used the condition $E_0(B) = 0$. Hence (B.3) follows. ∎

We set

$$\Sigma := \inf \sigma_{\mathrm{ess}}(A) \tag{B.5}$$

and, for $s > 0$,

$$\beta(s) := \frac{E_0(T) - E_0(A) + b + s}{1-a}. \tag{B.6}$$

By (B.3), we have

$$\beta(s) \geq \frac{s}{1-a} > 0. \tag{B.7}$$

Theorem B.3 *Assume Hypothesis A and let Z be as above. Suppose that*

$$\Sigma - E_0(T) > b \tag{B.8}$$

and, for some $s_0 > 0$, $\mathrm{Ran}(E_B([0, \beta(s_0)])$ is finite dimensional. Let m be a constant such that

$$\Sigma - E_0(T) > m + b, \tag{B.9}$$
$$0 < m < s_0. \tag{B.10}$$

Then T has purely discrete spectrum in the interval $[E_0(T), E_0(T) + m)$. In particular, T has a ground state.

Remark B.1 By Lemma B.2, condition (B.8) is satisfied if

$$\Sigma - E_0(A) > 2b. \tag{B.11}$$

Proof. Let

$$\hat{A} := A - E_0(A), \quad T_m := T - E_0(T) - m.$$

Then we have on $D(T_0)$

$$\begin{aligned} T_m &= \hat{A} \otimes I + I \otimes B + Z + E_0(A) - E_0(T) - m \\ &\geq \hat{A} \otimes I + I \otimes (1-a)B - \alpha_0, \end{aligned}$$

where

$$\alpha_0 := E_0(T) - E_0(A) + b + m \geq m.$$

Since we have (B.9)

$$\alpha_0 < \Sigma - E_0(A),$$

one can take a constant $\delta > 0$ such that $\alpha_0 \leq \delta < \Sigma - E_0(A)$. Let $P_\delta := E_{\hat{A}}([0, \delta])$ and $P_\delta^\perp := I - P_\delta = E_{\hat{A}}((\delta, \infty))$, so that $P_\delta + P_\delta^\perp = I$. Then we have

$$\begin{aligned} & \hat{A} \otimes I + I \otimes (1-a)B - \alpha_0 \\ = \; & P_\delta \hat{A} \otimes I + P_\delta^\perp \hat{A} \otimes I + P_\delta \otimes (1-a)B + P_\delta^\perp \otimes (1-a)B - \alpha_0 P_\delta \otimes I - \alpha_0 P_\delta^\perp \otimes I. \end{aligned}$$

Note that

$$P_\delta \hat{A} \otimes I \geq 0, \quad P_\delta^\perp \hat{A} \otimes I \geq \delta P_\delta^\perp \otimes I, \quad P_\delta^\perp \otimes (1-a)B \geq 0.$$

Hence

$$\begin{aligned} \hat{A} \otimes I + I \otimes (1-a)B - \alpha_0 & \geq (\delta - \alpha_0) P_\delta^\perp \otimes I + P_\delta \otimes [(1-a)B - \alpha_0] \\ & \geq (P_\delta \otimes [(1-a)B - \alpha_0], \end{aligned}$$

where we have used the condition $\delta \geq \alpha_0$. Hence we have on $D(T_0)$

$$T_m \geq P_\delta \otimes [(1-a)B - \alpha_0] \geq P_\delta \otimes [(1-a)B - \alpha_0]_-,$$

where $[(1-a)B - \alpha_0]_-$ means the negative part of $(1-a)B - \alpha_0$. Let $J_m := E_{T_m}([-m, 0))$.

(i) The case where $\mathrm{Ran}(J_m)$ is finite dimensional. In this case T_m has a purely discrete spectrum in $[-m, 0)$. This means that the spectrum of T in $[E_0(T), E_0(T) + m)$ is purely discrete. In particular T has a ground state.

(ii) The case where $\mathrm{Ran}(J_m)$ is infinite dimensional. Note that

$$[(1-a)B - \alpha_0]_- = E_B([0, \beta(m))[(1-a)B - \alpha_0] E_B([0, \beta(m)).$$

By condition (B.10) and the present assumption, $\mathrm{Ran}(E_B([0, \beta(m))))$ is finite dimensional. Hence $[(1-a)B - \alpha_0]_-$ is trace class. Therefore $P_\delta \otimes [(1-a)B - \alpha_0]_-$ is trace class. Let $\{\psi_n\}_{n=1}^\infty$ be a complete orthonormal system of $\mathrm{Ran}(J_m)$. Then, for all $N \in \mathbb{N}$,

$$\begin{aligned} 0 & \geq \sum_{n=1}^N \langle \psi_n, T_m \psi_n \rangle \geq \sum_{n=1}^N \langle \psi_n, P_\delta \otimes [(1-a)B - \alpha_0]_- \psi_n \rangle \\ & \geq \mathrm{Tr}\{P_\delta \otimes [(1-a)B - \alpha_0]_-\}. \end{aligned}$$

Hence $\sum_{n=1}^N \langle \psi_n, T_m \psi_n \rangle$ is convergent as $N \to \infty$, which implies $J_m T_m J_m$ is trace class and hence it is compact. Thus T_m has purely discrete spectrum in $[-m, 0)$, which implies that T has purely discrete spectrum in $[E_0(T), E_0(T) + m)$. In particular T has a ground state. ∎

B.2 A Limit Theorem on Ground States

Let K be a self-adjoint operator on $\mathcal{H}$ bounded from below and B be a nonnegative self-adjoint operator on $\mathcal{K}$ with $E_0(B) = 0$. Let C be a symmetric operator on $\mathcal{H} \otimes \mathcal{K}$ with $D(K \otimes I) \cap D(I \otimes B) \subset D(C)$ such that

$$H := K \otimes I + I \otimes B + C \tag{B.12}$$

is self-adjoint and bounded from below. Let

$$\Sigma := \inf \sigma_{\rm ess}(K). \tag{B.13}$$

Hypothesis (B.1). $\Sigma > E_0(K)$.

Hypothesis (B.2). There are a family $\{K_m\}_{m\in(0,m_0]}$ of symmetric operators on $\mathcal{H}$ with $D(K) \subset \cap_{m\in(0,m_0]} D(K_m)$ and a family $\{B_m\}_{m\in(0,m_0]}$ of nonnegative self-adjoint operators on $\mathcal{K}$ with $E_0(B_m) = 0$ such that the following hold:

(i) There exists a constant $c_m > 0$ such that, for all $u \in D(K)$,

$$\|(K - K_m)u\| \le c_m(\|Ku\| + \|u\|)$$

and $\lim_{m\to 0} c_m = 0$.

(ii) There exists a non-zero vector ψ_0 such that, for all $m \in (0, m_0]$, $B_m\psi_0 = 0$. We denote the orthogonal projection onto the one-dimensional subspace $\{\alpha\psi_0 | \alpha \in \mathbf{C}\}$ by P_0.

(iii) For each $m \in (0, m_0]$, $D(K_m \otimes I) \cap D(I \otimes B_m) \subset D(C)$ and the operator

$$H_m := K_m \otimes I + I \otimes B_m + C \tag{B.14}$$

is self-adjoint and bounded from below.

(iv) There exists a dense subspace $D \subset [\cap_{m\in(0,m_0]} D(B_m)] \cap D(B)$ such that, for all $\psi \in D$, $\lim_{m\to 0} B_m\psi = B\psi$ and $D(K) \otimes_{\rm alg} D$ is a core of H, where $\otimes_{\rm alg}$ means algebraic tensor product.

For an orthogonal projection P on a Hilbert space, we set

$$P^\perp := I - P.$$

Theorem B.4 *Assume Hypotheses (B.1) and (B.2). Suppose that*

$$\inf_{m\in(0,m_0]} E_0(H_m) > -\infty, \qquad \sup_{m\in(0,m_0]} E_0(H_m) < \Sigma. \tag{B.15}$$

Suppose that, for all $m \in (0, m_0]$, H_m has a ground state Ψ_m with $\|\Psi_m\| = 1$ and there exists a constant $\delta < 1$ independent of $m \in (0, m_0]$ such that

$$\frac{1}{(\Sigma - E_0(H_m))^2}\|C\Psi_m\|^2 + \|I \otimes P_0^\perp \Psi_m\|^2 < \delta. \tag{B.16}$$

Then there exists a subsequence $\{\Psi_{m_j}\}_{j=1}^{\infty}$ *with*

$$m_1 > m_2 > \cdots > m_j > m_{j+1} > \cdots, \quad \lim_{j\to\infty} m_j = 0$$

such that the weak limit $\Psi_0 := \text{w-}\lim_{j\to\infty} \Psi_{m_j}$ *is a ground state of* H.

Proof. We divide the proof of this theorem into two steps. (1) By Hypothesis (B.2)-(i), there exists a positive constant $\varepsilon_0 < m_0$ such that, for all $0 < m < \varepsilon_0$, $c_m < 1$. Then, by the Kato-Rellich theorem, $K_m = K + (K_m - K)$ is self-adjoint with $D(K_m) = D(K)$ and bounded from below. We can take a constant ξ such that $\max\{\sup_{m\in(0,m_0]} E_0(H_m), E_0(K)\} < \xi < \Sigma$ and

$$\frac{1}{(\xi - E_0(H_m))^2}\|C\Psi_m\|^2 + \|I \otimes P_0^{\perp}\Psi_m\|^2 \le \delta. \tag{B.17}$$

Let

$$P_K := E_K([E_0(K), \xi]).$$

Then, by Hypothesis (B.1), $\dim \mathrm{Ran} P_K < \infty$. Let $K_m(\beta) := K + \beta L_m$ with $L_m := (K_m - K)/c_m$ and $\beta \in \mathbb{C}$. Since L_m is relatively bounded with respect to K, it follows from [14, p.16, Lemma] that $K_m(\beta)$ is an analytic family of type (A) near $\beta = 0$. Hence it is an analytic family in the sense of Kato [14, p.17, Theorem XII.9] and $K_m(\beta)$ is self-adjoint for real β with $|\beta|$ sufficiently small. We define

$$Q_m(\beta) := E_{K_m(\beta)}([E_0(K), \xi])$$

and

$$Q_m := E_{K_m}([E_0(K), \xi]).$$

Then $Q_m(\beta)$ is analytic near $\beta = 0$. In particular,

$$\lim_{\beta\to 0} \|Q_m(\beta) - P_K\| = 0$$

and hence $\dim \mathrm{Ran} Q_m(\beta) = \dim \mathrm{Ran} P_K < \infty$ for all sufficiently small $|\beta|$. Note that $K_m(c_m) = K_m$. Therefore, for every $\varepsilon > 0$, there exists a constant $\eta_0 > 0$ such that, for all $m \in (0, \eta_0)$,

$$\|Q_m - P_K\| < \varepsilon \tag{B.18}$$

and $\dim \mathrm{Ran} Q_m = \dim \mathrm{Ran} P_K$.

(2) By the weak compactness of the unit ball of a Hilbert space and condition (B.15), there exists a subsequence $\{\Psi_{m_j}\}_{j=1}^{\infty}$ ($m_1 > m_2 > \cdots > m_j > m_{j+1} > \cdots, \lim_{j\to\infty} m_j = 0$) such that the weak limit $\Psi_0 := \text{w-}\lim_{j\to\infty} \Psi_{m_j}$ and $E_0 := \lim_{j\to\infty} E_0(H_{m_j})$ exist. By Hypothesis (B.2)-(iii), we have $\lim_{m\to 0} H_m\Psi = H\Psi$ for all $\Psi \in D(K) \otimes_{\mathrm{alg}} D$. Hence, by an applicaiton of [4, Lemma 4.9], if we show that $\Psi_0 \neq 0$, then we can conclude that Ψ_0 is a ground state with $E_0 = E_0(H)$. We have $\dim \mathrm{Ran} P_0 = 1$. Hence, to show that $\Psi_0 \neq 0$, we need only to prove

$$\langle \Psi_m, P_K \otimes P_0 \Psi_m \rangle \ge 1 - \delta' \tag{B.19}$$

with a constant $\delta' < 1$ independent of m. Then, passing to the subsequence $\{\Psi_{m_j}\}_j$ and taking the limit $j \to \infty$, we obtain $\langle \Psi_0, P_K \otimes P_0 \Psi_0 \rangle \ge 1 - \delta' > 0$, which implies that $\Psi_0 \neq 0$.

To prove (B.19), we first prove

$$\langle \Psi_m, Q_m \otimes P_0 \Psi_m \rangle \geq 1 - \delta. \tag{B.20}$$

Then, by (B.18), we obtain (B.19) for all $m < \eta_0$ with $\delta' = \delta + \varepsilon < 1$ and hence the proof is completed. Now we note that (B.20) is equivalent to

$$\left\langle \Psi_m, (Q_m^\perp \otimes P_0 + I \otimes P_0^\perp) \Psi_m \right\rangle \leq \delta. \tag{B.21}$$

This is seen by using the identity $1 = \left\langle \Psi_m, (Q_m + Q_m^\perp) \otimes (P_0 + P_0^\perp) \Psi_m \right\rangle$. We prove (B.21). We have

$$(Q_m^\perp \otimes P_0) H_m = Q_m^\perp K_m \otimes P_0 + Q_m^\perp \otimes P_0 C.$$

Hence

$$\begin{aligned} 0 &= (Q_m^\perp \otimes P_0)(H_m - E_0(H_m))\Psi_m \\ &= (Q_m^\perp (K_m - E_0(H_m)) \otimes P_0)\Psi_m + (Q_m^\perp \otimes P_0) C \Psi_m, \end{aligned}$$

which implies that

$$\begin{aligned} \left\langle \Psi_m, (Q_m^\perp \otimes P_0) C \Psi_m \right\rangle &= -\left\langle \Psi_m, Q_m^\perp (K_m - E_0(H_m)) \otimes P_0 \Psi_m \right\rangle \\ &\leq -(\xi - E_0(H_m)) \left\langle \Psi_m, Q_m^\perp \otimes P_0 \Psi_m \right\rangle. \end{aligned}$$

Hence

$$\begin{aligned} \left\langle \Psi_m, Q_m^\perp \otimes P_0 \Psi_m \right\rangle &\leq -\frac{1}{\xi - E_0(H_m)} \left\langle \Psi_m, (Q_m^\perp \otimes P_0) C \Psi_m \right\rangle \\ &\leq \frac{1}{\xi - E_0(H_m)} \|Q_m^\perp \otimes P_0 \Psi_m\| \|C \Psi_m\|. \end{aligned}$$

Hecne

$$\left\langle \Psi_m, Q_m^\perp \otimes P_0 \Psi_m \right\rangle \leq \frac{1}{(\xi - E_0(H_m))^2} \|C \Psi_m\|^2,$$

which, together with (B.17), yields (B.21). ∎

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