Ornstein-Zernike behavior for the Bernoulli bond percolation on $\mathbb{Z}^d$ in the supercritical regime

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Abstract

We derive an Ornstein-Zernike asymptotic formula for the decay of the two point finite connectivity function $\tau^{\mathrm{f}}_{x,y}(p)$ of the Bernoulli bond percolation process on $\mathbb{Z}^d$, along the principal directions, for $d \geq 3$, and for supercritical values of p sufficiently near to $p = 1$.

Key Words: Supercritical bond percolation, polymer expansion, Ornstein-Zernike behavior.
Running title: Ornstein-Zernike behavior for supercritical bond percolation on $\mathbb{Z}^d$.

§1. *Introduction and results*

Consider the d-dimensional cubic lattice, i.e., the infinite graph $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$ with vertex set $\mathbb{Z}^d = \{x = (x_1, \ldots, x_d) :\ x_i \in \mathbb{Z}\}$ and edge set $\mathbb{E}^d = \{\{x, y\} \subset \mathbb{Z}^d :\ |x-y| = 1\}$ (nearest neighbors) where $\mathbb{Z}$ is the set of all integers and $|x - y|$ is the usual graph distance, i.e. $|x - y| = \sum_{i=1}^{d} |x_i - y_i|$.
In the translation invariant bond percolation process on $\mathbb{L}^d$ an edge $e \in \mathbb{E}^d$ is open with probability p and closed with probability $1-p$ independently of all other edges. The product measure on the configurations of edges, denoted by P_p, is the Bernoulli probability on $\mathbb{E}^d$. Let C_x denote the open cluster at x (i.e., the set of endpoints of the open edges that are connected to x) and let $\{x \leftrightarrow y\}$ denote the event x connected to y through open edges. The finite connectivity function is defined as

$$\tau^{\mathrm{f}}_{x,y}(p) = P_p(x \leftrightarrow y, |C_x| < \infty). \tag{1.1}$$

In this paper we study the behavior of $\tau^{\mathrm{f}}_{xy}(p)$ in the highly supercritical phase, i.e. for p sufficiently near 1. Our main result can be stated in terms of the following theorem

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Theorem 1.1 *Let $d \geq 3$ and let $x, y \in \mathbb{Z}^d$ such that $x - y = (0, \cdots, x_\nu - y_\nu, \cdots, 0)$ for some $\nu = 1, 2, \cdots, d$. Then there exists a positive number r such that the two point finite connectivity function can be written as*

$$\tau^{\mathrm{f}}_{x,y}(p) = C(p, x-y)\frac{e^{-m(p)|x-y|}}{|x-y|^{\frac{d-1}{2}}} \qquad \textit{whenever} \qquad \tfrac{1}{1+r} < p < 1 \tag{1.2}$$

where $e^{-m(p)} = (1-p)^{2d-2}[1 + f(p)]$ and $C(p, x-y)$ is such that

$$\lim_{x_\nu - y_\nu \to \infty} C(p, x-y) = \frac{(1-p)^{4d-d^2-1}}{(2\pi)^{\frac{d-1}{2}}}[1 + g(p)] \tag{1.3}$$

and $f(p)$, $g(p)$ are analytic functions of p, $\frac{1}{1+r} < p < 1$, such that $\lim_{p\to 1} f(p) = 0 = \lim_{p\to 1} g(p)$.

Asymptotic behavior of the form (1.2) is known as the *Ornstein-Zernike behavior*, after the work of L. Ornstein and F. Zernike [18]. In statistical mechanics of spin systems, Paes Leme [19] obtained the OZ behavior for the spin-spin correlation function of the d-dimensional Ising model at very high temperature, $d \geq 2$, while Schor [21] obtained the same result for the truncated spin-spin correlation function at very low temperature and $d \geq 3$.
Since then, the OZ behavior has been rigorously established for a variety of models in statistical mechanics and field theory: classical fluids at low density [2], strongly coupled lattice gauge theories [22, 23, 17, 4], random surfaces [1, 4, 11], $O(N)$ model at high temperature [4], self-avoiding walks at all noncritical temperatures [9, 14, 13], Ising type models at high temperatures [16, 15], finite range Ising type models for all temperatures above T_c [8], quantum Ising model in a transverse field [12] and surface tension for the low temperature 2D Blume-Capel model [10].
Concerning in particular the Bernoulli bond percolation process, the Ornstein-Zernike behavior of the two point connectivity function has been proved in the whole subcritical regime [6] and for all directions [7]. On the other hand, a rigorous proof of the the Ornstein-Zernike behavior for the *finite* connectivity function in the supercritical phase was still lacking. Such a behavior was conjectured to occur at least for $d \geq 3$ and for p near 1, on the basis of the analogy between bond percolation in the highly supercritical regime and the Ising model in the very low temperature regime.
In this paper we proceed with the analysis started in [3, 20] and we rewrite the finite connectivity in terms of an absolutely convergent polymer type expansion, a key ingredient in order to prove theorem 1.1. We then follow the strategy illustrated in [21, 22, 23, 17], using analyticity of the finite connectivity and Spencer's hyperplane decoupling technique [25, 26] to prove that the convolutive inverse of $\tau^{\mathrm{f}}_{x,y}(p)$ decay faster than $\tau^{\mathrm{f}}_{x,y}(p)$. Then, proceeding as in [24], we use Rouché's theorem and careful bounds on the first terms of the Newman series for the inverse of $\tau^{\mathrm{f}}_{x,y}(p)$ to check that such faster decay implies that the Fourier transform of the connectivity is meromorphic in a strip containing the real axis. Moreover, there are two simple poles in this strip localized on the imaginary axis

in symmetric positions with respect to the real axis. Finally, the OZ decay follows by a straightforward application of residue theorem.
We would like to comment that, once a polymer expansion is available, the strategy described above is not the only possibility, see e.g. [4, 12, 1] for possible alternative methods.

§2. *Analyticity of the finite connectivity.*

In this section we obtain an explicit polymer expansion formula for the finite connectivity following essentially the construction given in [3, 20]. To make this section self-contained, we make a brief review of some definitions and results we have obtained earlier.
If R denotes a finite set we denote by $|R|$ the number of its elements. Consider a set $\Lambda \subset \mathbb{Z}^d$ and let E_Λ be the set of all edges $e \in \mathbb{E}^d$ such that $e \subset \Lambda$. Let $\partial\Lambda = \{x \in \Lambda : \exists y \notin \Lambda \text{ such that } |y - x| = 1\}$. Let Ω be the set of all functions from $\mathbb{E}^d$ to $\{0,1\}$. An element $\omega \in \Omega$, i.e. a *configuration* of the system, is a function that assigns to each edge $e \in \mathbb{E}^d$ either the value 1 (open bond) or 0 (closed bond). Ω_Λ will denote the set of all functions from E_Λ to $\{0,1\}$. From now on we will assume that Λ is finite, e.g., the cube $[-N,N]^d$ in $\mathbb{Z}^d$. So, $\Lambda \to \infty$ means simply $N \to \infty$. For fixed $p \in (0,1)$, in Ω_Λ is naturally defined a probability measure P_p^Λ, which is simply the restriction to E_Λ of the Bernoulli probability P_p on $\mathbb{E}^d$. Given a configuration ω in Ω_{E_Λ} we denote by O_ω the set of all open bonds of ω and with C_ω the set of all closed bonds in ω. Hence the probability assigned to ω is simply $P_p^\Lambda(\omega) = p^{|O_\omega|}(1-p)^{|C_\omega|}$.
This probability is normalized, thus

$$\sum_{\omega\in\Omega_\Lambda} P_p^\Lambda(\omega) = \sum_{\omega\in\Omega_\Lambda} p^{|O_\omega|}(1-p)^{|C_\omega|} = p^{|E_\Lambda|} \sum_{\omega\in\Omega_\Lambda} \left(\frac{1-p}{p}\right)^{|C_\omega|} = 1 \tag{2.1}$$

Let us now define, for $\lambda \in \mathbb{C}$, the "partition function"

$$Z_\Lambda(\lambda) = \sum_{\omega\in\,\Omega_\Lambda} \lambda^{|C_\omega|} \tag{2.2}$$

then, by (2.1)

$$Z_\Lambda(\lambda)\Big|_{\lambda=\frac{1-p}{p}} = p^{-|E_\Lambda|} \tag{2.3}$$

Draw now for each *closed* edge e of the configuration ω a $(d-1)$-dimensional unit hyper-square Δ_e perpendicular to e, with edges parallel to the axis of $\mathbb{Z}^d$, and in such way that $\Delta_e \cap e$ is a point which coincides with the center of hyper-square Δ and the middle point of the segment e. We call such hyper-square the *dual plaquette to the edge* e. The vertices of a plaquette lie in the so called dual lattice shifted by the vector $(1/2,\ldots,1/2)$ with respect to $\mathbb{Z}^d$. We denote by $\mathbb{E}^{d^*}$ the set of all plaquettes dual of edges in $\mathbb{E}^d$. In general, given a subset $S \subset \mathbb{E}^d$, we denote by S^* its dual, i.e. $S^* = \{\Delta_e \in \mathbb{E}^{d^*} :\ e \in S\}$. Given two sets $\gamma \subset \mathbb{E}^{d^*}$ and $\gamma' \subset \mathbb{E}^{d^*}$ we denote $\gamma \cap \gamma' = \{\Delta \in \mathbb{E}^{d^*} : \Delta \in \gamma \text{ and } \Delta \in \gamma'\}$.

Two plaquettes in $\mathbb{E}^{d*}$ are *connected* if they share a $(d-2)$-dimensional edge. A set $\gamma \subset \mathbb{E}^{d*}$ is *connected* if it is formed by pairwise connected plaquettes. A *finite* connected set $\gamma \subset \mathbb{E}^{d*}$ will be called a *dual animal*. We will denote by Γ the set of all dual animals in $\mathbb{E}^{d*}$ and by Γ_Λ in the set of all dual animals in E^*_Λ. We recall that a connected subgraph of $\mathbb{L}^d$ is called an *animal* $\mathcal{A}$ and we denote by $V_{\mathcal{A}}$ ($E_{\mathcal{A}}$) the set of vertices (edges) of $\mathcal{A}$. A dual animal γ with the property that each $d-2$ dimensional edge is shared between two and only two plaquettes is called a *Peierls contour*. Note that a Peierls contour is always a closed polyedron, i.e., a $d-1$ dimensional closed surface which has the property that $V_{\mathbb{Z}^d}$ is uniquely partitioned in two disjoint components, one finite and the other infinite; the finite component is the *interior of* γ and is denoted by $I(\gamma)$. Hence $I(\gamma) \subset V_{\mathbb{Z}^d}$ and $|I(\gamma)| < \infty$. A *contour* is a dual animal γ such that there exists at least one $\gamma' \subset \gamma$ which is a Peierls contour.
Let $\gamma \in \Gamma$ be a contour and let $\gamma_1, \dots, \gamma_n$ be the set of all the Peierls contours contained in γ. Then we denote by $I(\gamma) = \cup_{i=1}^n I(\gamma_i)$. The set $I(\gamma)$, which is a finite subset of $\mathbb{Z}^d$, will be called *the interior of the contour* γ.
Given a contour $\gamma \in \Gamma$ and k vertices $x_1, \dots, x_k$ we say that γ *surrounds* $x_1, \dots, x_k$ and write $\gamma \bigodot \{x_1, \dots, x_k\}$ if there exists a Peierls contour $\gamma' \subset \gamma$ and an animal $\mathcal{A} \subset \mathbb{L}^d$ such that $\{x_1, \dots, x_k\} \subset V_{\mathcal{A}} \subset I(\gamma')$, and $E^*_{\mathcal{A}} \cap \gamma = \emptyset$.
We will say that two dual animals $\gamma, \gamma' \in \Gamma$ are *compatible*, and we write $\gamma \sim \gamma'$, if $\gamma \cup \gamma' \notin \Gamma$. We will denote by $|\gamma|$ the cardinality of γ, i.e. the number of plaquettes which form the dual animal γ. We associate to any dual animal γ an activity $\lambda^{|\gamma|}$.
With these notations, the partition function (2.2) can be rewritten as

$$Z_\Lambda(\lambda) = 1 + \sum_{n \geq 1} \sum_{\substack{\{\gamma\}_n:\ \gamma_i \in \Gamma_\Lambda \\ \gamma_i \sim \gamma_j}} \lambda^{|\{\gamma\}_n|} \tag{2.4}$$

where we denoted shortly $\{\gamma\}_n = \{\gamma_1, \dots, \gamma_n\}$, $|\{\gamma\}_n| = \sum_{i=1}^n |\gamma_i|$. Note that the term 1 is the contribution of the configuration in which all bonds are open.
We define *the finite volume finite connectivity function* as

$$\tau^\Lambda_{x,y}(p) = \mathrm{Prob}(x \leftrightarrow y : C_x \cap \partial\Lambda = \emptyset) = p^{|E_\Lambda|} \sum_{\substack{\omega \in \Omega_\Lambda \\ \exists C_x \subset O_\omega:\ x \leftrightarrow y \\ C_x \cap \partial\Lambda = \emptyset}} \left(\frac{1-p}{p}\right)^{|C_\omega|} \tag{2.5}$$

i.e. $\tau^\Lambda_{x,y}(p)$ is the probability that x and y are connected through a path of open bonds which does not intersect the boundary of Λ. Then,

$$\lim_{\Lambda \to \infty} \tau^\Lambda_{x,y}(p) = \tau^{\mathrm{f}}_{x,y}(p) \tag{2.6}$$

where $\tau^{\mathrm{f}}_{x,y}(p)$ is the *finite connectivity function* which is the probability that x and y belong to the same *finite* open cluster.
In term of dual animals the condition that ω contain a path C_x of open bonds which connects x to y and does not intersect the boundary, corresponds to a configuration of

dual animals $\{\gamma\}_n$ such that there is at least one *contour*, say γ_i, which surrounds the set $\{x,y\}$, i.e. $\gamma_i \odot \{x,y\}$. This means that in allowed configurations $\{\gamma\}_n$ no dual animal can be a contour surrounding x but not y and viceversa. It is important to stress that this is a condition *on the whole configuration* of dual animals. If an unordered n-uple $\{\gamma\}_n$ of contours satisfies such a condition we also denote it $\{\gamma\}_n \odot \{x,y\}$.
Thus, in the contour notation, recalling (2.3), (2.4) and (2.5), we can rewrite the finite connectivity as

$$\tau^\Lambda_{x,y}(p) = \tau^\Lambda_{x,y}(\lambda)\Big|_{\lambda=\frac{1-p}{p}} \tag{2.7}$$

where $\tau^\Lambda_{x,y}(\lambda)$ is a function of the complex variable λ (where defined) given by

$$\tau^\Lambda_{x,y}(\lambda) = \frac{\sum_{n\geq 1}\sum_{\substack{\{\gamma\}_n:\ \gamma_i\in\Gamma_\Lambda,\ \gamma_i\sim\gamma_j \\ \{\gamma\}_n \odot\{x,y\}}} \lambda^{|\{\gamma\}_n|}}{1+\sum_{n\geq 1}\sum_{\substack{\{\gamma\}_n:\ \gamma_i\in\Gamma_\Lambda \\ \gamma_i\sim\gamma_j}} \lambda^{|\{\gamma\}_n|}} \tag{2.8}$$

Our first aim is to show that in some small disk around $\lambda = 0$ the function $\tau^\Lambda_{x,y}(\lambda)$ and its $\Lambda \to \infty$ limit are analytic. To this goal we rewrite the ratio (2.8) as a power series in λ. In order to do this we will follow the ideas in [3] defining a new set of "connected" objects called *polymers*.

Definition 2.1 *A set $\gamma \in \mathbb{E}^{d^*}$ is said to be $\{x,y\}$-connected if $\gamma = \cup_{i=1}^k \gamma_i$ with $k \geq 1$ such that, for all $i = 1,2,\ldots,k$: $\gamma_i \in \Gamma$; γ_i is a contour; either $\gamma_i \odot x$, or $\gamma_i \odot y$; and $\gamma_i \sim \gamma_j$ for all $j \neq i$.*

We will denote by $\Gamma^{x,y}$ the set of all $\{x,y\}$-connected sets in $\mathbb{E}^{d^*}$ and by $\Gamma^{x,y}_\Lambda$ in the set of all $\{x,y\}$-connected sets in E^*_Λ. We will also put $\mathcal{G}^{x,y} = \Gamma \cup \Gamma^{x,y}$ and $\mathcal{G}^{x,y}_\Lambda = \Gamma_\Lambda \cup \Gamma^{x,y}_\Lambda$.

Definition 2.2 *A set $\gamma \in \mathcal{G}^{x,y}$ will be called a xy-polymer (or simply polymer when it is clear from the contest). We will say that two polymers $\gamma_i \in \mathcal{G}^{x,y}$ and $\gamma_j \in \mathcal{G}^{x,y}$ are compatible, and we write $\gamma_i \approx \gamma_j$, if $\gamma_i \cup \gamma_j \notin \mathcal{G}^{x,y}$; viceversa, $\gamma_i \in \mathcal{G}^{x,y}$ and $\gamma_j \in \mathcal{G}^{x,y}$ are incompatible, and we write $\gamma_i \not\approx \gamma_j$, if $\gamma_i \cup \gamma_j \in \mathcal{G}^{x,y}$.*

Note that if $\gamma \in \Gamma^{x,y}$ and $\gamma' \in \Gamma^{x,y}$ then necessarily $\gamma \not\approx \gamma'$. To any $\gamma \in \mathcal{G}^{x,y}$ we associate an activity $\lambda^{|\gamma|}$ where $|\gamma|$ is the cardinality of γ, i.e. the number of plaquettes which form the polymer γ. We say that a polymer $\gamma \in \mathcal{G}^{x,y}$ *surrounds* $\{x,y\}$ if either $\gamma \in \Gamma$ is a contour such that $\gamma \odot \{x,y\}$, or if $\gamma = \cup_{i=1}^k \gamma_i \in \Gamma^{x,y}$ and $\{\gamma\}_k \odot \{x,y\}$. Note that if two polymers γ and γ' are such that $\gamma \odot \{x,y\}$ and $\gamma' \odot \{x,y\}$, then necessarily $\gamma \not\approx \gamma'$.
With this definitions, the r.h.s. of (2.8) can be rewritten as

$$\tau^\Lambda_{x,y}(\lambda) = \frac{\sum_{n\geq 1}\sum_{\substack{\{\gamma\}_n:\ \gamma_i\in\mathcal{G}^{x,y}_\Lambda,\ \gamma_i\approx\gamma_j \\ \exists!\gamma_i:\ \gamma_i \odot\{x,y\}}} \lambda^{|\{\gamma\}_n|}}{1+\sum_{n\geq 1}\sum_{\substack{\{\gamma\}_n:\ \gamma_i\in\mathcal{G}^{x,y}_\Lambda \\ \gamma_i\approx\gamma_j}} \lambda^{|\{\gamma\}_n|}}$$

Define now, for $\alpha \in \mathbb{R}$ and $\gamma \in \mathcal{G}^{x,y}$

$$\rho_\alpha(\gamma) = \begin{cases} (1+\alpha)\lambda^{|\gamma|} & \text{if } \gamma \bigodot \{x,y\} \\ \\ \lambda^{|\gamma|} & \text{otherwise} \end{cases}$$

then

$$\tau^\Lambda_{x,y}(p) = \frac{d}{d\alpha} \ln \Xi_\Lambda(\alpha)\Big|_{\alpha=0} \tag{2.9}$$

where

$$\Xi_\Lambda(\alpha) = 1 + \sum_{n\geq 1} \sum_{\substack{\{\gamma\}_n:\ \gamma_i \in \mathcal{G}^{x,y}_\Lambda \\ \gamma_i \approx \gamma_j}} \prod_{i=1}^{n} \rho_\alpha(\gamma_i) \tag{2.10}$$

The function $\Xi_\Lambda(\alpha)$ is the grand-canonical partition function of a standard hard-core polymer gas in which the polymers γ are elements of $\mathcal{G}^{x,y}_\Lambda$, they have activity $\rho_\alpha(\gamma)$ and the hard core condition is expressed by the fact that any two pairs γ_i, γ_j in $\{\gamma\}_n$ must be compatible with the notion of compatibility being $\gamma_i \approx \gamma_j$.

As it is well known, its logarithm can be expressed in term of a formal power series (see e.g. [5], [24]). To simplify the expression of this formal power series let us introduce some further notations. Let's thus denote $\mathcal{B}_\Lambda = \cup_{n\in\mathbb{N}}(\mathcal{G}^{x,y}_\Lambda)^n$ ($\mathcal{B} = \cup_{n\in\mathbb{N}}(\mathcal{G}^{x,y})^n$), where $(\mathcal{G}^{x,y}_\Lambda)^n$ ($(\mathcal{G}^{x,y})^n$)) is the n times cartesian product of $\mathcal{G}^{x,y}_\Lambda$ ($\mathcal{G}^{x,y}$), and let's denote by η a generic element of $\mathcal{B}$. Thus an element $\eta \in \mathcal{B}$ is an the ordered n-uple $(\gamma_1, \ldots, \gamma_n)$ in $(\mathcal{G}^{x,y})^n$ for some $n \in \mathbb{N}$. If $\eta = (\gamma_1, \ldots, \gamma_n)$, we denote $|\eta| = \sum_{i=1}^n |\gamma_i|$ and $\|\eta\| = n$. The formal power series for $\ln \Xi_\Lambda(\alpha)$ can thus be written as

$$\ln \Xi_\Lambda(\alpha) = \sum_{\eta \in \mathcal{B}_\Lambda} \frac{1}{\|\eta\|!} \Phi^T(\eta) \rho_a(\eta) \tag{2.11}$$

where, if $\eta = (\gamma_1, \ldots, \gamma_n)$, then $\rho_a(\eta) = \prod_{i=1}^n \rho_a(\gamma_i)$ and where *the Ursell factor* $\Phi^T(\eta)$ is defined as follows:

$$\Phi^T(\eta) = \begin{cases} 1 & \text{if } \|\eta\| = 1 \\ \\ 0 & \text{if } \|\eta\| \geq 2 \ \text{ and } \ g_\eta \notin G_{\|\eta\|} \\ \\ \sum\limits_{\substack{f \in G_{\|\eta\|} \\ f \subset g_\eta}} (-1)^{|f|} & \text{if } \|\eta\| \geq 2 \ \text{ and } \ g_\eta \in G_{\|\eta\|} \end{cases} \tag{2.12}$$

with $G_{\|\eta\|}$ being the set of all connected graphs with vertex set $\{1, 2, \ldots, \|\eta\|\}$ and g_η is the graph with vertex set $V = \{1, \ldots, \|\eta\|\}$ and with edge set $E = \{\{i,j\} :\ \gamma_i \not\approx \gamma_j\}$.

Thus, inserting (2.11) in (2.9) we have an explicit (formal) expansion for $\tau^\Lambda_{x,y}(\lambda)$ given by

$$\tau^\Lambda_{x,y}(\lambda) = \sum_{\substack{\eta \in \mathcal{B}_\Lambda \\ \eta \bigodot \{x,y\}}} \frac{k_\eta}{\|\eta\|!} \Phi^T(\eta) \lambda^{|\eta|} \tag{2.13}$$

where, for $\eta = (\gamma_1, \dots, \gamma_n)$, the notation $\eta \bigodot \{x, y\}$ means that at least for one $i = 1, \dots, n$ it occurs that $\gamma_i : \gamma_i \bigodot \{x, y\}$. Moreover

$$k_\eta = \# \text{ of } \gamma_i \text{ such that } \gamma_i \bigodot \{x, y\} \tag{2.14}$$

By taking the limit $\Lambda \to \infty$ in (2.13) we can also define

$$\tau^{\mathrm{f}}_{x,y}(\lambda) = \sum_{\substack{\eta \in \mathcal{B} \\ \eta \bigodot \{x,y\}}} \frac{k_\eta}{\|\eta\|!} \Phi^T(\eta) \lambda^{|\eta|} \tag{2.15}$$

which represents an expansion for the finite connectivity function. It is now a rather standard application of cluster expansion methods to show that (2.15) (and hence *a fortiori* (2.13)) is actually an absolutely convergent power series in λ if $|\lambda|$ is sufficiently small. Therefore the r.h.s. of (2.15) provides an explicit polymer expansion representation of the finite connectivity function for p is sufficiently near to 1, via the following theorem.

Theorem 2.3 *Let $\lambda \in \mathbb{C}$ and let $\tau^{\mathrm{f}}_{x,y}(\lambda)$ as in (2.15). Then there exists a positive number r_0 such that the function $\tau^{\mathrm{f}}_{x,y}(\lambda)$ is an absolutely convergent power series of λ and hence an analytic function for all complex λ in the disk $|\lambda| < r_0$ and for all pairs $x, y \in \mathbb{Z}^d$. Moreover*

$$\tau^{f}_{x,y}(\lambda)\Big|_{\lambda=(1-p)/p} = \tau^{\mathrm{f}}_{x,y}(p) \qquad \textit{for all } p \textit{ such that } \quad \tfrac{1}{1+r_0} < p < 1 \tag{2.16}$$

where $\tau^{\mathrm{f}}_{x,y}(p)$ is defined via (2.6) and (2.5).

Proof. The proof is very similar to that presented in [3] for the analyticity of the percolation probability. For self consistency reasons we give here a (sketchy) description of this proof. Define, for $\epsilon \geq 0$

$$|\tau^{\mathrm{f}}|_{x,y}(\epsilon) = \sum_{\substack{\eta \in \mathcal{B} \\ \eta \bigodot \{x,y\}}} \frac{k_\eta}{\|\eta\|!} |\Phi^T(\eta)| \epsilon^{|\eta|} \tag{2.17}$$

Then we get the bound

$$|\tau^{\mathrm{f}}_{x,y}(\lambda)| \leq |\tau^{\mathrm{f}}|_{x,y}(|\lambda|) \leq \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \sum_{\tau \in T_n} \sum_{\substack{\eta \in \mathcal{B}:\ \|\eta\| = n \\ \eta \bigodot \{x,y\} \\ g_\eta \supset \tau}} |\lambda|^{|\eta|} \tag{2.18}$$

where the factor $|\Phi^T(\eta)|$ has been bounded, when non zero, via the well known tree graph domination formula by the number of tree sub-graphs in g_η (see e.g. [5]) and where τ is a tree graph in $\{1, \dots, n\}$, T_n is the set of all connected trees in $\{1, \dots, n\}$. Observe now that, for a fixed tree τ, if $\{i, j\} \in \tau$, then we have to sum over $\eta = (\gamma_1, \dots \gamma_n)$ in such way that $\gamma_i \not\approx \gamma_j$. Thus we have to bound quantities of the form $\sum_{\gamma \in \mathcal{G}^{x,y}:\ \gamma \not\approx \gamma_0} \lambda^{|\gamma|}$. Using the definitions of the previous section we write

$$\sum_{\substack{\gamma \in \mathcal{G}^{x,y} \\ \gamma \not\approx \gamma_0}} \lambda^{|\gamma|} \leq |\gamma_0| \Big[\sum_{\substack{\gamma \in \Gamma \\ \gamma \not\sim \Delta_0}} \lambda^{|\gamma|} + \sum_{\gamma \in \Gamma^{x,y}} \lambda^{|\gamma|} \Big] \doteq |\gamma_0| \sum_{\gamma \in \mathcal{G}^{x,y}}{}^{*} \lambda^{|\gamma|} \tag{2.19}$$

where, by symmetry, Δ_0 is any fixed plaquette; moreover we have used, due to definitions, 2.1 and 2.2, that $\mathcal{G}^{x,y}\backslash\Gamma^{x,y} \subset \Gamma$. Using (2.19) it is now a standard calculus in the cluster expansion framework (see [3]) to obtain

$$|\tau^{\rm f}|_{x,y}(|\lambda|) \leq \sum_{\substack{\gamma_1 \in \mathcal{G}^{x,y} \\ \gamma_1 \odot \{x,y\}}} (e|\lambda|)^{|\gamma_1|} | \sum_{n=1}^{\infty} n \left[\sum_{\gamma \in \mathcal{G}^{x,y}}{}^{*} (e|\lambda|)^{|\gamma|} \right]^{n-1} \tag{2.20}$$

The last sum converges if

$$\sum_{\gamma \in \mathcal{G}^{x,y}}{}^{*} (e|\lambda|)^{|\gamma|} < 1 \tag{2.21}$$

By definition (2.19)

$$\sum_{\gamma \in \mathcal{G}^{x,y}}{}^{*} (e|\lambda|)^{|\gamma|} = \sum_{n=1}^{\infty} (|\lambda| e)^n B_n + \sum_{n=2d}^{\infty} (|\lambda| e)^n C_n$$

where $B_n = \sum_{\gamma \in \Gamma:\, |\gamma|=n,\, \gamma \not\sim \Delta_0} 1$ and $C_n = \sum_{\gamma \in \Gamma^{x,y}:\, |\gamma|=n} 1$. It is easy to check that $B_n \leq B^n$ for some positive B depending only on d (see [3]). Concerning C_n, proceeding as in lemmas 4.2 and 4.3 of [3], we obtain

$$C_n \leq \sum_{k=1}^{n} \sum_{n_1 + \ldots + n_k = n} \prod_{i=1}^{k} \left[2 \sum_{\substack{\gamma \in \Gamma:\ |\gamma| = n_i \\ \text{either } \gamma \odot x, \text{ or } \gamma \odot y}} 1 \right] \leq \sum_{k=1}^{n} 2^k \sum_{n_1 + \ldots + n_k = n} \prod_{i=1}^{k} [2^d B]^{2n_i} =$$

$$= [2^d B]^{2n} \sum_{k=1}^{n} 2^k \sum_{n_1 + \ldots + n_k = n} 1 \leq [2^d B]^{2n} \sum_{k=1}^{n} 2^k \binom{n-1}{k-1} \leq [3 \cdot (2^d B)^2]^n \doteq C^n \tag{2.22}$$

We stress that bounds above are not at all optimal. Thus, if $|\lambda| < r_0$, where r_0 can be chosen so small to satisfy e.g. the inequality

$$\sum_{n=1}^{\infty} (Ber_0)^n + \sum_{n=1}^{\infty} (Cer_0)^n < \frac{1}{2} \tag{2.23}$$

then (2.20) converges and $|\tau^{\rm f}|_{x,y}(|\lambda|) \leq 2$. We stress that the size of the convergence radius r_0 in (2.23) is not at all optimal. *A fortiori* we can conclude that also $\tau^{\Lambda}_{x,y}(\lambda)$ is analytic as a function of λ for $|\lambda| < r_0$ and for all Λ. It is now a simple cluster expansion exercise to show that $\lim_{\Lambda \to \infty} |\tau^{\Lambda}_{x,y}(\lambda) - \tau^{\rm f}_{x,y}(\lambda)| = 0$ whenever $|\lambda| < r_0$ (see e.g corollary 3.1 [3]). This ends the proof the theorem 2.3. $\square$

Remark. It is immediate to obtain an upper bound for the (analytic continuation of the) finite connectivity function. Just observe that the power series (2.18) starts at least with a term proportional to $|\lambda|^{n_{x,y}}$ where $n_{x,y} = (2d-2)(|x-y|+1)+2$ is the the minimal number of plaquettes of a polymer γ such that $\gamma \odot \{x,y\}$, and hence, for all $|\lambda| < r_0$ we have

$$|\tau^{\rm f}_{x,y}(\lambda)| \le |\tau^{\rm f}|_{x,y}(|\lambda|) \le \left(\frac{\lambda}{r_0}\right)^{n_{x,y}} |\tau^{\rm f}|_{x,y}(r_0) \le 2\left(\frac{\lambda}{r_0}\right)^{n_{x,y}} \tag{2.24}$$

§3. *Hyperplane decoupling and the convolutive inverse of* $\tau^{\rm f}_{x,y}(\lambda)$

Let $k \in \mathbb{Z}$ and let $H_k = \{(x_1, x_2, \dots, x_d) \in \mathbb{Z}^d : x_1 = k\}$ be the hyperplanes perpendicular to the x_1 axis in $\mathbb{Z}^d$. We denote by $E_k \subset \mathbb{E}^d$ the set of edges in H_k. The set $E^*_k \subset \mathbb{E}^{d^*}$, formed by the plaquettes which are dual to edges in E_k, will be called the *dual hyperplane*. Moreover we put $E^*_c = \mathbb{E}^{d^*} \setminus (\cup_{k\in\mathbb{Z}} E^*_k)$, i.e. E^*_c is the set of plaquettes which belong to no E^*_k. Note that the dual hyperplane E^*_k is a set of plaquettes with a "waffle mold" shape containing all those plaquettes that cut edges in the hyperplane H_k.

Definition 3.1 *We associate to each hyperplane H_k a complex variable $w_k \in \mathbb{C}$. We will indicate shortly with w the set of all these variables $\{w_k\}_{k\in\mathbb{Z}}$. Then define, for any pair $u, v \in \mathbb{Z}^d$, the function $\tau^{\rm f}_{u,v}(\lambda, w)$ of variables λ, w as follows:*

$$\tau^{\rm f}_{u,v}(\lambda, w) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{(\gamma)_n \in (\mathcal{G}^{u,v})^n \\ \exists \gamma_i:\ \gamma_i \odot \{u,v\}}} k[(\gamma)_n] \Phi^T[(\gamma)_n] \prod_{j=1}^{n} \Big[\prod_{\Delta\in\gamma_j} \lambda_\Delta \Big] \tag{3.1}$$

where $\lambda_\Delta = \lambda$ if $\Delta \in E^*_c$, and $\lambda_\Delta = w_i$ if $\Delta \in E^*_i$.

By construction, $\tau^{\rm f}_{u,v}(\lambda, w)$ is analytic in the polydisk $|\lambda|$, $|w_i| < r_0$ and it is equal to $\tau^{\rm f}_{u,v}(\lambda)$ when $w_i = \lambda$. If $x \in \mathbb{Z}^d$ we will use the notation $x = (x_1, \vec{x})$ with $x_1 \in \mathbb{Z}$ and $\vec{x} \in \mathbb{Z}^{d-1}$. We now prove the following theorem

Theorem 3.2 *In the polydisk $|\lambda|$, $|w_i| < r_0$ the function defined in (3.1) satisfies*

$$\left.\frac{\partial^s \tau^{\rm f}_{x,y}(\lambda, w)}{\partial \lambda^s}\right|_{\lambda=0} = 0, \qquad \text{for} \quad s \le 2(|\vec{x} - \vec{y}| + 1) \tag{3.2}$$

Moreover, for all k such that $x_1 \le k \le y_1$

$$\left.\frac{\partial^s \tau^{\rm f}_{x,y}(\lambda, w)}{\partial w_k^s}\right|_{w_k=0} = 0, \qquad \text{for} \quad s \le 2d - 3 \tag{3.3}$$

Finally, for k such that $x_1 < k < y_1$ we have

$$\left.\frac{\partial^{2d-2} \tau^{\rm f}_{x,y}(\lambda, w)}{\partial w_k^{2d-2}}\right|_{\lambda_k=0} = \frac{(2d-2)!}{\lambda^2} \sum_{z\in H_k} \tau^{\rm f}_{x,z-e_1}(\lambda, w)\Big|_{w_k=0} \tau^{\rm f}_{z+e_1,y}(\lambda, w)\Big|_{w_k=0} \tag{3.4}$$

while for k such that $x_1 < k = y_1$ we have

$$\left.\frac{\partial^{2d-2} \tau^{\rm f}_{x,y}(\lambda, w)}{\partial w_k^{2d-2}}\right|_{\lambda_k=0} = \frac{(2d-2)!}{\lambda^2} \tau^{\rm f}_{x,y-e_1}(\lambda, w)\Big|_{w_k=0} \sigma_{y+e_1}(\lambda, w)\Big|_{w_k=0} \tag{3.5}$$

where $e_1 = (1, 0, \dots, 0) \in \mathbb{Z}^d$ *and* $\sigma^{\mathrm{f}}_{z+e_1,y}(\lambda, w)$ *is analytic in the polydisk* $|\lambda|$, $|w_i| < r_0$ *(see (3.12)).*

Proof. Concerning (3.3) and (3.2) we just observe that the lowest power monomial in the expansion (3.1) comes from the contribution in which only one Peierls contour, say γ, surrounding $\{x, y\}$ with minimal cardinality, i.e. $|\gamma| = (2d-2)(|x-y|+1)+2$. If $x_1 \le k \le y_1$, then at least $2d-2$ plaquettes of γ belong to the dual hyperplane E^*_k. Moreover, consider the projection $\gamma^\perp$ of γ onto the hyperplane H_k. Such a projection is the union of $|\vec{x} - \vec{y}| + 1$ unit hypersquares and in order to close γ we need two plaquettes belonging to E^*_c, (top and bottom) for each of the hypersquares forming $\gamma^\perp$. Thus γ has at least least $2(|\vec{x} - \vec{y}| + 1)$ plaquettes in E^*_c. So the lowest power of w_k in the expansion (3.1) will be w_k^{2d-2} and the lowest power in λ will be $\lambda^{2(|\vec{x}-\vec{y}|+1)}$.
Concerning (3.4), denote $E^{*+}_k = \{\Delta_{\{x,y\}} \in \mathbb{E}^{d^*} : y_1 > k\}$ and $E^{*-}_k = \{\Delta_{\{x,y\}} \in \mathbb{E}^{d^*} : y_1 < k\}$, i.e. E^{*+}_k is the set of all plaquettes which are above H_k while E^{*+}_k is the set of all plaquettes which are below H_k. Note that $E^{*\pm}_k \cap E^*_k = \emptyset$ and also $E^{*+}_k \cap E^{*-}_k = \emptyset$. Let now $z \in H_k$. Since the plaquette dual to the edge $\{x, z - e_1\}$ belongs to E^{*-}_k, in the explicit expansion for $\tau^{\mathrm{f}}_{x,z-e_1}|_{w_k=0}$ given by r.h.s. of (3.1) with $u = x$, $v = z - e_1$, the n-uples $(\gamma)_n$ will contribute to $\tau^{\mathrm{f}}_{x,z-e_1}|_{w_k=0}$ have to fulfill the following conditions: $(\gamma)_n \in \Gamma^n$ where Γ^n is the n-times Cartesian product of Γ (i.e. all γ_j are dual animals); for all γ_j, $\gamma_j \cap B^*_k = \emptyset$; and there exists a unique contour, say γ_i, such that $\gamma_i \bigodot \{x, z - e_1\}$. Hence

$$\tau^{\mathrm{f}}_{x,z-e_1}(\lambda, w)\Big|_{w_k=0} = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{(\gamma)_n \in \Gamma^n:\ \gamma_j \cap E^*_k = \emptyset\ \forall j \\ \exists! \gamma_i:\ \gamma_i \bigodot \{x, z-e_1\}}} \Phi^T[(\gamma)_n] \prod_{j=1}^{n} \Big[\prod_{\Delta \in \gamma_j} \lambda_\Delta \Big] \tag{3.6}$$

Now recall that, by definition (2.12), $\Phi^T[(\gamma)_n] = 0$ whenever $G[(\gamma)_n]$ is a disconnected graph (remind that $G[(\gamma)_n]$ is is the graph with vertex set $\{1, \dots, n\}$ which has the edge $\{i, j\}$ iff $\gamma_i \not\sim \gamma_j$). This means that the sum over ordered n-uples $(\gamma)_n$ in r.h.s. of (3.6) is submitted to the further restriction that $G[(\gamma)_n]$ has to be a connected graph i.e. the polymers $\gamma_1, \dots \gamma_n$ have to be pairwise incompatible and the set $\cup_{i=1}^n \gamma_i$, viewed as a subset of $E^*_{\mathbb{Z}^d}$, has to be connected. This condition and can happen simultaneously with the condition $\gamma_j \cap E^*_k$ only if all $\gamma_i \in E^{*-}_k$. Hence

$$\tau^{\mathrm{f}}_{x,z-e_1}(\lambda, w)\Big|_{w_k=0} = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{(\gamma)_n \in \Gamma^n:\ \gamma_j \subset E^{*-}_k\ \forall j \\ \exists! \gamma_i:\ \gamma_i \bigodot \{x, z-e_1\}}} \Phi^T[(\gamma)_n] \prod_{j=1}^{n} \Big[\prod_{\Delta \in \gamma_j} \lambda_\Delta \Big] \tag{3.7}$$

analogously for $\tau^{\mathrm{f}}_{z+e_1,y}|_{w_k=0}$ one obtains

$$\tau^{\mathrm{f}}_{z+e_1,y}(\lambda, w)\Big|_{w_k=0} = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{(\gamma)_n \in \Gamma^n:\, \gamma_j \subset E^{*+}_k\ \forall j \\ \exists \gamma_i:\, \gamma_i \bigodot \{z+e_1, y\}}} \Phi^T[(\gamma)_n] \prod_{j=1}^{n} \Big[\prod_{\Delta \in \gamma_j} \lambda_\Delta \Big] \tag{3.8}$$

We now derive an explicit expansion for $(\partial^{2d-2}\tau^{\rm f}_{x,y}/\partial w_k^{2d-2})|_{w_k=0}$ for all k such that $x_1 < k < y_1$ assuming that $x_1+1<y_1$. The derivative of order $2d-2$ with respect to w_k is the coefficient of w_k^{2d-2}, i.e. the lowest power of w_k in the expansion (3.1), times the factor $(2d-2)!$. The n-uples $(\gamma)_n$ for which the power w_k^{2d-2} appears in the r.h.s. of (3.1) are those for which all $\gamma_j\in\Gamma$ (i.e. all $\gamma_j\in\Gamma$ are dual animals) and there exists among the $(\gamma)_n$ a unique *contour*, say γ_i, which surrounds $\{x,z,y\}$, where $z\in H_k$, and has exactly $2d-2$ plaquettes in common with E_k^*. These n-uples corresponds to configurations in Ω for which one and only one site of H_k, namely z, belongs to the finite cluster containing x and y. Hence

$$\left.\frac{\partial^{2d-2}\tau^{\rm f}_{x,y}}{\partial w_k^{2d-2}}\right|_{w_k=0} = (2d-2)!\sum_{z\in H_k}\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{\substack{(\gamma)_n\in\Gamma^n:\ \exists!\gamma_i:\ \gamma_i\bigodot\{x,z,y\}\\ |\gamma_i\cap E_k^*|=2d-2\\ \gamma_j\cap E_k^*=\emptyset,\ \forall j\neq i}}\Phi^T[(\gamma)_n]\prod_{j=1}^{n}\Big[\prod_{\substack{\Delta\in\gamma_j\\ \Delta\notin E_k^*}}\lambda_\Delta\Big]=$$

$$=(2d-2)!\sum_{z\in H_k}\sum_{\substack{\gamma\in\Gamma:\ \gamma\bigodot\{x,z,y\}\\ |\gamma\cap E_k^*|=2d-2}}\prod_{\substack{\Delta\in\gamma\\ \Delta\notin E_k^*}}\lambda_\Delta\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\substack{(\gamma)_n\in\Gamma^n\\ \gamma_j\cap E_k^*=\emptyset,\ \forall j}}\Phi^T[\gamma,(\gamma)_n]\prod_{j=1}^{n}\Big[\prod_{\Delta\in\gamma_j}\lambda_\Delta\Big] \tag{3.9}$$

Consider now the plaquettes $\Delta_z^+\in E_k^{*+}$ and $\Delta_z^-\in E_k^{*-}$ dual to the edges $\{z,z+e_1\}$ and $\{z,z-e_1\}$ respectively and observe that $\lambda_{\Delta_z^+}=\lambda_{\Delta_z^-}=\lambda$ since they both are plaquettes perpendicular to the x_1 axis. Then, for γ contour such that $\gamma\bigodot\{x,z,y\}$ and $|\gamma\cap E_k^*|=2d-2$ we can associate uniquely to γ the pair $\{\gamma^+,\gamma^-\}$ where $\gamma^+=(\gamma\cap E_k^{*+})\cup\Delta_z^+$ and $\gamma^-=(\gamma\cap E_k^{*-})\cup\Delta_z^-$. Observe that $\gamma^-\bigodot\{x,z-e_1\}$ and $\gamma^+\bigodot\{y,z+e_1\}$. Hence the sum over γ factorizes as

$$\sum_{\substack{\gamma\in\Gamma:\ \gamma\bigodot\{x,z,y\}\\ |\gamma\cap E_k^*|=2d-2}}\prod_{\substack{\Delta\in\gamma\\ \Delta\notin E_k^*}}\lambda_\Delta=\frac{1}{\lambda^2}\sum_{\substack{\gamma^-\in\Gamma:\ \gamma^-\subset E^*(H^-_{k\nu})\\ \gamma^-\bigodot\{x,z-e_1\}}}\prod_{\Delta\in\gamma^-}\lambda_\Delta\sum_{\substack{\gamma^+\in\Gamma:\ \gamma^+\subset E^*(H^+_{k\nu})\\ \gamma\bigodot\{z+e_1,y\}}}\prod_{\Delta\in\gamma^+}\lambda_\Delta \tag{3.10}$$

Now observe that the n-uple $(\gamma)_n\in(\mathcal{G}^{x,y})^n$ such that $\gamma_j\cap E_k^*=\emptyset,\ \forall j$ can always be organized as $(\gamma)_n=((\gamma^+)_{n_1},(\gamma^-)_{n_2})$ such that $n_1+n_2=n$ and $\gamma_i^+\subset E_k^{*+}$ for all $i=1,\ldots,n_1$ and $\gamma_i^-\subset E_k^{*-}$ for all $j=1,\ldots,n_2$.
By definition (2.12) it is not difficult to check that

$$\Phi^T[\gamma,(\gamma)_n]=\Phi^T[\gamma^-,(\gamma^-)_{n_2}]\Phi^T[\gamma^+,(\gamma^+)_{n_1}] \tag{3.11}$$

Formulas (3.10) and (3.11) then imply that

$$\left.\frac{\partial^{2d-2}\tau^{\rm f}_{x,y}}{\partial w_k^{2d-2}}(\lambda,w)\right|_{w_k=0}=\frac{(2d-d)!}{\lambda^2}\ \times$$

$$\times\ \sum_{z\in H_k}\Bigg[\sum_{\substack{\gamma^-\in\Gamma:\ \gamma^-\subset E_k^{*-}\\ \gamma^-\bigodot\{x,z-e_\nu\}}}\prod_{\Delta\in\gamma^-}\lambda_\Delta\sum_{n_2=0}^{\infty}\frac{1}{n_2!}\sum_{\substack{(\gamma^-)_n\in\Gamma^n\\ \gamma_j^-\subset E_k^{*-},\ \forall j}}\lambda^{|(\gamma^-)_n|}\Phi^T[\gamma^-,(\gamma^-)_n]\Bigg]\ \times$$

$$\times \left[\sum_{\substack{\gamma^+ \in \Gamma:\ \gamma^+ \subset E_k^{*+} \\ \gamma^+ \odot \{y, z+e_\nu\}}} \prod_{\Delta \in \gamma^+} \lambda_\Delta \sum_{n_1=0}^{\infty} \frac{1}{n_1!} \sum_{\substack{(\gamma^+)_n \in \Gamma^n \\ \gamma_j^+ \subset E_k,\ \forall j}} \lambda^{|(\gamma^+)_n|} \Phi^T[\gamma^+, (\gamma^+)_n] \right] =$$

$$= \frac{(2d-d)!}{\lambda^2} \sum_{z \in H_k} \tau^{\mathrm{f}}_{x,z-e_1}\Big|_{w_k=0} \tau^{\mathrm{f}}_{z+e_1,y}\Big|_{w_k=0}$$

where in last line we used (3.7) and (3.8).
The case k such that $x_1 < k = y_1$, i.e. the proof of identity (3.5), can be treated analogously. We left to the reader to check that in this case formula (3.5) holds with

$$\sigma^{\mathrm{f}}_{z+e_1,y}(\lambda, w) = \tau^{\mathrm{f}}_{y+e_1,y+e_1}(\lambda, w) + \lambda \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{(\gamma)_n \in \Gamma^n:\ \gamma_i \oslash \{y+e_1\} \\ \gamma_i \oslash \{y+e_1\},\ \exists \gamma_j \ni \Delta_y^+}} \Phi^T[(\gamma)_n] \prod_{j=1}^{n} \Big[\prod_{\Delta \in \gamma_j} \lambda_\Delta \Big] \tag{3.12}$$

where Δ_y^+ denotes the plaquette dual to the edge $\{y, y+e_1\}$ and the sum runs over ordered n-uples of contours in Γ such that no contour surrounds $y + e_1$ and there exists a contour which contains the plaquette $\Delta_{\{y,y+e_1\}}$.

$\square$

As a direct consequence of the convergence of the cluster expansion and Theorem 3.2 we have the following corollary

Corollary 3.3 *Let $r_1 < r_0$. Then for all $|w_i|, |\lambda| \le r_1$ the following bound holds*

$$|\tau^{\mathrm{f}}_{x,y}(\lambda, w)| \le 2 \left(\frac{|\lambda|}{r_1} \right)^{2(|\vec{x}-\vec{y}|+1)} \prod_{i=x_1}^{y_1} \left(\frac{|w_i|}{r_1} \right)^{2d-2} \tag{3.13}$$

Proof. By theorem 3.2 and recalling (2.17) and comment below (2.23), we have for all $|\lambda|, |w| \le r_1$

$$|\tau^{\mathrm{f}}_{x,y}(\lambda, w)| \le \sum_{m \ge 0} r_1^m \sum_{\substack{\eta \in \mathcal{B} \\ |\eta| = m}} \frac{k_\eta}{\|\eta\|!} |\Phi^T[\eta]| = |\tau^{\mathrm{f}}|_{x,y}(r_1) \le 2$$

The analyticity of $\tau^{\mathrm{f}}_{x,y}(\lambda, w)$ in the polydisk $|\lambda|, |w| < r_0$ also implies that we can differentiate $\tau^{\mathrm{f}}_{x,y}(\lambda, w)$ by differentiating the series term by term. Thus, (3.2) and (3.3) imply that for any $\eta \in \mathcal{B}$ the monomial $\prod_{\Delta \in \eta} \lambda_\Delta$ is of the form $\lambda^{2(|\vec{x}-\vec{y}|+1)} \prod_{i=x_1}^{y_1} w_i^{2d-2} M_\eta(\lambda, w)$ where $M_\eta(\lambda, w)$ is again a monomial in the variables λ and w which depends on η whose total degree is $|\eta| - [2(|\vec{x} - \vec{y}| + 1) + (2d-2)(|y_1 - x_1| + 1)]$. Thus for $|\lambda|, |w| \le r_1$ and for all η contributing to $\tau^{\mathrm{f}}_{x,y}(\lambda, w)$ we get

$$|\prod_{\Delta\in\eta}\lambda_\Delta| \le |\lambda|^{2(|\vec{x}-\vec{y}|+1)}\prod_{i=x_1}^{y_1}|w_i|^{2d-2}|M_\eta(\lambda,w)| \le \left[\frac{|\lambda|}{r_1}\right]^{2(|\vec{x}-\vec{y}|+1)}\prod_{i=x_1}^{y_1}\left[\frac{|w_i|}{r_1}\right]^{2d-2} r_1^{|\eta|}$$

Whence, for all λ, w in the polydisk $|\lambda|, |w| \le r_1$

$$|\tau^{\mathrm{f}}_{u,v}(\lambda,w)| \le |\tau^{\mathrm{f}}|_{x,y}(r_1)\left[\frac{|\lambda|}{r_1}\right]^{2(|\vec{x}-\vec{y}|+1)}\prod_{i=x_1}^{y_1}\left[\frac{|w_i|}{r_1}\right]^{2d-2}$$

□

Let $l^2(\mathbb{Z}^d)$ the Hilbert space of the square summable functions $\phi : \mathbb{Z}^d \to \mathbb{C}$ (i.e functions $x \mapsto \phi_x$ such that $\|\phi\| = \sum_{x\in\mathbb{Z}^d}|\phi_x|^2 < \infty$). Define the operator $\tau^{\mathrm{f}} : l^2(\mathbb{Z}^d) \to l^2(\mathbb{Z}^d)$ as follows

$$(\tau^{\mathrm{f}}\phi)_x = \sum_{y\in\mathbb{Z}^d}\tau^{\mathrm{f}}_{x,y}(\lambda,w)\phi_y, \tag{3.14}$$

Note that, by (2.24), τ^{f} is a bounded operator in $l^2(\mathbb{Z}^d)$ whenever $|\lambda|, |w_i| < r_0$. We now consider the question of invertibility of the operator τ^{f} when $|\lambda|$ and $|w_i|$ are small. We have to circumvent the problem that $\tau^{\mathrm{f}} = 0$ when $w_i = \lambda = 0$. We follow the strategy explained in [21]. Namely, we write $\tau^{\mathrm{f}} = P + S$ where P and S are operators with entries $P_{x,y}(\lambda,w) = \tau^{\mathrm{f}}_{x,y}(\lambda,w)\delta_{x,y}$ and $S_{x,y}(\lambda,w) = \tau^{\mathrm{f}}_{x,y}(\lambda,w)(1-\delta_{x,y})$. Then one shows that P is invertible as long as $w_k, \lambda \ne 0$ and thus we can rewrite $\tau^{\mathrm{f}} = P(I + P^{-1}S)$ where I denotes the identity operator in $l^2(\mathbb{Z}^d)$. Finally one checks that $P^{-1}S$ has small norm when $|w_i|, |\lambda|$ are sufficiently small so that also $I + P^{-1}S$ is invertible. This will imply that τ^{f} is invertible when $|w_i|, |\lambda|$ are sufficiently small and different from 0. In order to develop this strategy we will need a lower bound for $\tau^{\mathrm{f}}_{x,x}(\lambda,w)$ for $\lambda, w_i \ne 0$. We thus show the following proposition.

Proposition 3.4 *There exists $r_2 < r_1$ such that, for all $|\lambda|, |w_i| \le r_2$,*

$$|\tau^{\mathrm{f}}_{x,x}(\lambda,w)| \ge \frac{1}{2}|\lambda|^2|w_{x_1}|^{2d-2} \tag{3.15}$$

and thus P is invertible in the punctued polydisk $0 < |\lambda|, |w| \le r_1$.

Proof. This is a direct consequence of the definition (3.1). Just observe that the function $f_{x,x}(\lambda,w) = \tau^{\mathrm{f}}_{x,x}(\lambda,w)/\lambda^2 w_{x_1}^{2d-2}$ is analytic and uniformly bounded by C_{r_1} for all $|\lambda|, |w_i| \le r_1$. Moreover via (3.1) it is easy to check that $f_{x,x}(\lambda = 0, w = 0) = 1$ and therefore we get $|f_{x,x}(\lambda,w) - 1| < 1/2$ if $|w_i|, |\lambda| \le r_2$ for a certain $r_2 < r_1$ and the proposition is proved. □

Let $Q = P^{-1}S$ and $R = I + Q$. We now prove the following theorem.

Theorem 3.5 *One can choose r_2 so small such that $\|Q\| \le 1/2$ in the polydisk $|\lambda|, |w| \le r_2$ and hence the inverse $M = R^{-1} = (I+Q)^{-1}$ exists, it is analytic and its norm $\|M\| \le 2$.*

Proof. The entries $Q_{x,y}(\lambda,w) = (1-\delta_{x,y})\tau_{xy}(\lambda,w)/\tau_{xx}(\lambda,w)$ are analytic for any $|w_i|,|\lambda| \le r_1$ and, by (3.13) and (3.15), admit the following upper bound for all $|w_i|,|\lambda| \le r_2 < r_1$

$$|Q_{x,y}(\lambda,w)| \le 4\left(\frac{|\lambda|}{r_1}\right)^{2|\vec{x}-\vec{y}|}\prod_{i=x_1+1}^{y_1}\left(\frac{|w_i|}{r_1}\right)^{2d-2} \tag{3.16}$$

To prove that $I+Q$ is an invertible operator in a certain polydisk $|\lambda|,|w_i| < r_3$ around the origin, we just have to show that in this polydisk the norm of the operator Q is small, which is an easy task using the upper bound (3.16). Put $\epsilon = r_2/r_1$. By hypothesis $\epsilon < 1$. Hence using (3.16), for all $|\lambda|,|w| \le r_2 < r_1$ we get:

$$\|Q\|^2 = \sup_{y\in\mathbb{Z}^d}\sum_{x\in\mathbb{Z}^d,\,x\neq y}|Q_{x,y}|^2 \le C(\epsilon)\epsilon^2$$

where $\lim_{\epsilon\to 0} C(\epsilon)\epsilon^2 = 0$. Thus choosing ϵ so small, say, such that $C(\epsilon)\epsilon^2 < 1/2$, we have that for all $|\lambda|,|w| \le r_2 = \epsilon\, r_1$ the operator norm $\|Q\| < 1/2$ and $I+Q$ is invertible. Note that the inverse

$$M = R^{-1} = (I+Q)^{-1} = I + \sum_{n=1}^{\infty}(-1)^n Q^n \tag{3.17}$$

is analytic in the polydisk $|\lambda|,|w| \le r_2$ and is a bounded operator with norm $\|M\| \le 2$. □

Remark. Clearly (3.16) is not such a good bound for $|Q_{x,y}(\lambda)|$ (i.e. when we make $w_i = \lambda$). But since we know that $Q_{x,y}(\lambda) = \tau^{\mathrm{f}}_{x,y}(\lambda)/\tau^{\mathrm{f}}_{x,x}(\lambda)$ is analytic in the disc $|\lambda| < r_2$ with $|Q_{x,y}(\lambda)| \le 1/2$, and its expansion as a power series of λ starts at least with a the power $\lambda^{(2d-2)|x-y|}$, this is enough to conclude that

$$|Q_{x,y}(\lambda)| \le \frac{1}{2}\left(\frac{|\lambda|}{r_2}\right)^{(2d-2)|x-y|} \tag{3.18}$$

Our aim for the rest of this section is to prove that $M_{x,y}$ decays faster than $\tau^{\mathrm{f}}_{x,y}$ along the principal direction x_1. With this in mind we prove the following theorem.

Theorem 3.6 *Let λ, w such that $|\lambda|,|w| \le r_2$. Take $x,y \in \mathbb{Z}^d$ such that $|x-y| \ge 2$. Then it holds*

$$\left.\frac{\partial^j M_{x,y}(\lambda,w)}{\partial w_k^j}\right|_{w_k=0} = 0 \qquad \textit{for}\ \ 0 \le j \le 2d-3 \ \ \textit{and} \ \ x_1 < k \le y_1 \tag{3.19}$$

and

$$\left.\frac{\partial^{2d-2} M_{x,y}(\lambda,w)}{\partial w_k^{2d-2}}\right|_{w_k=0} = 0 \qquad \textit{for} \quad x_1 < k+1 \le y_1 \tag{3.20}$$

Proof: Let $u, v \in \mathbb{Z}^d$. We have, by definition,

$$R_{u,v}(\lambda, w) = \frac{\tau^{\mathrm{f}}_{u,v}(\lambda, w)}{\tau^{\mathrm{f}}_{u,u}(\lambda, w)}$$

Hence, by theorem 3.2 and proposition 3.4, we also obtain

$$\left.\frac{\partial^\ell R_{u,v}(\lambda, w)}{\partial w_k^\ell}\right|_{w_k=0} = 0 \qquad \text{for} \quad 0 \le \ell \le 2d-3 \quad \text{and} \quad u_1 < k \le v_1 \tag{3.21}$$

and

$$\left.\frac{\partial^{2d-2} R_{u,v}(\lambda, w)}{\partial w_k^{2d-2}}\right|_{w_k=0} = \left.\left[\frac{1}{\tau^{\mathrm{f}}_{u,u}(\lambda, w)} \frac{\partial^{2d-2}\tau^{\mathrm{f}}_{u,v}(\lambda, w)}{\partial w_k^{2d-2}}\right]\right|_{w_k=0} \qquad \text{for} \quad k \neq u_1$$

Observe that (3.21) does not hold when $u_1 = k$ (in this case even $R_{u,v}(\lambda, w)|_{w_k=0} \neq 0$). Hence, for $u_1 < k < v_1$ we can use identity (3.4) and get

$$\left.\frac{\partial^{2d-2} R_{u,v}}{\partial w_k^{2d-2}}\right|_{w_k=0} = \frac{(2d-2)!}{\lambda^2} \sum_{z \in H_k} R_{u,z-e_1}(\lambda, w)\Big|_{w_k=0} \tau^{\mathrm{f}}_{z+e_1,v}(\lambda, w)\Big|_{w_k=0} \tag{3.22}$$

while, for $u_1 < k = v_1$, we can use (3.5) to get

$$\left.\frac{\partial^{2d-2} R_{u,v}}{\partial w_k^{2d-2}}\right|_{w_k=0} = \frac{(2d-2)!}{\lambda^2} R_{u,v-e_1}(\lambda, w)\Big|_{w_k=0} \sigma_{v+e_1}(\lambda, w)\Big|_{w_k=0} \tag{3.23}$$

¿From (3.21) (in the case $\ell = 0$) and representation (3.17) for $M_{x,y}$ in the case $x \neq y$ one can easily check that

$$M_{x,y}(\lambda, w)\Big|_{w_k=0} = 0 \qquad \text{for} \quad x_1 < k \le y_1 \tag{3.24}$$

Now we note that since $I = MR$, by the product rule we have,

$$\left.\frac{\partial^m M}{\partial w_k^m}\right|_{w_k=0} = -\left.\left[\left(\sum_{i=0}^{m-1} \binom{m}{i} \frac{\partial^i M}{\partial w_k^i} \cdot \frac{\partial^{m-i} R}{\partial w_k^{m-i}}\right) M\right]\right|_{w_k=0} \tag{3.25}$$

whence, by (3.21) and (3.24) above, we get immediately the identity (3.19).
Concerning the proof of identity (3.20) we have

$$\left.\frac{\partial^{2d-2} M_{x,y}(\lambda, w)}{\partial w_k^{2d-2}}\right|_{w_k=0} = -\Big[\sum_{\substack{u,v \in \mathbb{Z}^d \\ u_1 < k \le v_1}} M_{xu}(\lambda, w) \frac{\partial^{2d-2} R_{u,v}(\lambda, w)}{\partial w_k^{2d-2}} M_{vy}(\lambda, w)\Big]\Big|_{w_k=0} =$$

$$= -\Big[\sum_{\substack{u,v\in\mathbb{Z}^d\\ u_1<k<v_1}} M_{xu}\frac{\partial^{2d-2}R_{u,v}}{\partial w_k^{2d-2}}M_{vy}\Big]\Big|_{w_k=0} - \Big[\sum_{\substack{u,v\in\mathbb{Z}^d\\ u_1<k=v_1}} M_{xu}\frac{\partial^{2d-2}R_{u,v}}{\partial w_k^{2d-2}}M_{vy}\Big]\Big|_{w_k=0} \tag{3.26}$$

We now can use (3.22) and (3.23) to get

$$\frac{\partial^{2d-2}M_{x,y}}{\partial w_k^{2d-2}}\Big|_{w_k=0} = -\frac{(2d-2)!}{\lambda^2}\Big[\sum_{z\in H_k}\sum_{\substack{u\in\mathbb{Z}^d\\ u_1<k}} M_{xu}R_{u,z-e_1}\sum_{\substack{v\in\mathbb{Z}^d\\ k<v_1}}\tau^{\mathrm{f}}_{z+e_1,v}M_{vy}\Big]\Big|_{w_k=0} -$$

$$-\frac{(2d-2)!}{\lambda^2}\Big[\sum_{\substack{v\in\mathbb{Z}^d\\ v_1=k}}\sum_{\substack{u\in\mathbb{Z}^d\\ u_1<k}} M_{xu}R_{u,v-e_1}\sigma_{v+e_1}M_{vy}\Big]\Big|_{w_k=0} \tag{3.27}$$

Now, since $M_{x,u}|_{w_k=0} = 0$ if $u_1 \geq k$ we have

$$\sum_{\substack{u\in\mathbb{Z}^d\\ u_1<k}} M_{xu}R_{u,z-e_1}\Big|_{w_k=0} = \sum_{u\in\mathbb{Z}^d} M_{xu}R_{u,z-e_1}\Big|_{w_k=0} = \delta_{x,z-e_1} = 0 \tag{3.28}$$

and

$$\sum_{\substack{u\in\mathbb{Z}^d\\ u_1<k}} M_{xu}R_{u,v-e_1} = \sum_{u\in\mathbb{Z}^d} M_{xu}R_{u,v-e_1} = \delta_{x,v-e_1} = 0 \tag{3.29}$$

where $\delta_{x,z-e_1} = 0$ and $\delta_{x,v-e_1} = 0$ follow from the hypothesis that $x_1 < k+1$. Inserting now (3.28) and (3.29) in (3.27) the theorem is proved. □

As an immediate corollary of theorem 3.6 above we have the following

Corollary 3.7 *Let λ, w such that $|\lambda|, |w| \leq r_2$. Take $x, y \in \mathbb{Z}^d$ such that $|x-y| \geq 2$. Then*

$$|M_{x,y}(\lambda, w)| \leq 2\left(\frac{|w_{x_1+1}|}{r_2}\right)^{(2d-2)}\prod_{k=x_1+2}^{y_1}\left(\frac{|w_k|}{r_2}\right)^{(2d-1)} \tag{3.30}$$

Proof. Since by theorem 3.5 $M_{x,y}(\lambda, w)$ is analytic in the polydisk $|\lambda|, |w| \leq r_2$ and $\|M\| \leq 2$, one can use identities (3.19) and (3.20), and proceed as in the proof of corollary 3.3. □

The auxiliary variables have done their job and we can now let all variables $\{\lambda, w\}$ be equal to λ to get the following immediate corollary.

Corollary 3.8 *Let λ such that $|\lambda| \leq r_2$. Take $x, y \in \mathbb{Z}^d$ such that $|x-y| \geq 2$. Then*

$$|M_{x,y}(\lambda)| \leq 2\left(\frac{|\lambda|}{r_2}\right)^{(2d-1)\|x-y\|-1} \qquad \textit{where } \|x-y\| = \max_{i=1}^d |x_i - y_i| \tag{3.31}$$

Proof. The bound (3.31) follows immediately from the fact that (3.30) can be obtained for any principal direction and by making all w equal to λ. $\square$

Remark. The bound (3.31) is optimal when x is a vector along some principal direction, but clearly is not a good bound in all the other cases. A more efficient bound for general $x-y$ can be obtained by noting that, if $\lambda^{m_{x,y}}$ is the lowest power in the expansion of $M_{x,y}(\lambda)$, then, since we know that $M_{x,y}(\lambda)$ is analytic in the disc $|\lambda|<r_2$ and that $|M_{x,y}(\lambda)|\leq 2$ in in this disc, this is enough to conclude that $|M_{x,y}|\leq 2(|\lambda|/r_2)^{m_{x,y}}$.
By (3.17) the lowest power of λ in the expansion of $M_{x,y}(\lambda)$ is at least equal to the the lowest power in the expansion of $Q_{x,y(\lambda)}=\tau^{\mathrm{f}}_{x,y}(\lambda)/\tau^{\mathrm{f}}_{x,y}(\lambda)$, i.e. $\lambda^{(2d-2)|x-y|}$. On the other hand, bound (3.31) clearly shows that some cancellations are occurring in the expansion for $M_{x,y}$. So one should reasonably expect that the lowest power in the expansion $M_{x,y}(\lambda)$ is actually bigger that $\lambda^{(2d-2)|x-y|}$. As a matter of fact, looking at (3.17), the lowest power $\lambda^{(2d-2)|x-y|}$ is contained in the first $|x-y|$ terms of the Neuman expansion of $M_{x,y}(\lambda)$, namely $Q_{x,y}, [Q^2]_{x,y}, [Q^3]_{x,y}, \ldots, [Q^{|x-y|}]_{x,y}$. We now show that these terms cancel in the alternate sum. Consider first for simplicity the case in which $x=(0,\vec{0})$ and $y=(n,\vec{0})$. In this case we have that the power $\lambda^{(2d-2)n}$ appears only one time in the expansion of Q since there is only one minimal contour (i.e. with $(2d-2)(n+1)+2$ plaquettes) surrounding x and y. It appears n times in the expansion of Q^2, corresponding to the $n-1$ configurations in which there is a minimal tube from x to z and a minimal tube from z to y (where of course $z=(m,\vec{0})$ with $1\leq m\leq n-1$. It appears a number of times, say $C_{m,n}$, in the expansion of Q^m, each time corresponding to a configuration in which there is a minimal contour surrounding x, z_1, then a minimal contour surrounding z_1, z_2 and so one finishing with a minimal contour surrounding z_{m-1}, y where of course $1\leq z_1<z_2\ldots<z_{m-1}\leq n-1$. Thus $C_{m,n}$ is exactly the number of ways in which we can choose $m-1$ objects among $n-1$ objects. Since in the term Q^m appears the factor $(-1)^m$ we have that the coefficient in front of $\lambda^{(2d-2)n}$ in the expansion of $M_{x,y}(\lambda)$, at least in the case in which $x=(0,\vec{0})$ and $y=(n,\vec{0})$, is

$$\sum_{m=1}^{n}\binom{n-1}{m-1}(-1)^m=0$$

When x and y are not in the same principal direction then there are $|x-y|!/(|x_1-y_1|!\cdots|x_d-y_d|!)$ minimal contours surrounding $\{x,y\}$. But we can repeat for each of these minimal contours the same reasoning seen before (which does not depend on the shape of the minimal contour but only on its length). Hence we can conclude that the coefficient in front of $\lambda^{(2d-2)|x-y|}$ in the expansion of $M_{x,y}(\lambda)$ is actually zero and thus we have actually a bound as follows

$$|M_{x,y}|\leq 2\left[\frac{|\lambda|}{r_2}\right]^{(2d-2)|x-y|+1}\qquad \text{whenever } |x-y|\geq 2 \tag{3.32}$$

We will use both estimates (3.32) and (3.31) in the sequel. We will need also a lower bound for the terms $M_{x,y}(\lambda)|_{|x-y|=0}$ and $M_{x,y}(\lambda)|_{|x-y|=1}$. Again, by symmetry these terms are constant and independent of x, y. We will put

$$M_0(\lambda) = M_{x,y}(\lambda)|_{|x-y|=0} \qquad M_1(\lambda) = M_{x,y}(\lambda)|_{|x-y|=1} \tag{3.33}$$

¿From (3.16) and (3.17) and the polymer expansion (2.15) it follows easily that, for all $|\lambda| < r_2$,

$$M_0(\lambda) = 1 + f(\lambda), \qquad M_1(\lambda) = -\lambda^{2d-2}(1 + f'(\lambda)), \tag{3.34}$$

where $f(\lambda)$ and $f'(\lambda)$ are analytic for $|\lambda| < r_2$ and $f(\lambda)|_{\lambda=0} = f'(\lambda)|_{\lambda=0} = 0$. For a shorter notation we will denote $O(1)$ any analytic function of the form $1 + f(\lambda)$ and, for any $a > 0$ we denote $O(\lambda^a)$ any analytic function of the form $C\lambda^a[1 + f(\lambda)]$ with $f(\lambda)|_{\lambda=0} = 0$ and C constant. In what follow $O(1)$ and $O(\lambda^a)$ may refer to different functions. Moreover, since $\tau^{\rm f}_{x,x}(\lambda, w = \lambda)$ does not depend on x, we have that

$$[\tau^{\rm f}(\lambda)]^{-1} = \frac{1}{\tau^{\rm f}_0(\lambda)} M(\lambda) \tag{3.35}$$

where $\tau^{\rm f}_0(\lambda) = \tau^{\rm f}_{0,0}(\lambda)$.

§4. *Location of zeroes of* $\hat{M}(k_1, \vec{k})$

Put $m_2(\lambda) = (2d - 1)(|\ln \lambda| - |\ln r_2|)$. By (3.31) the Fourier transform $\hat{M}(k_1, \vec{k})$ of $M_{x,y}$ is analytic for fixed $\vec{k} \in \mathbb{R}^{d-1}$ and $|\lambda| < r_2$, as a function of $k_1 \in \mathbb{C}$, at least in the strip $\{|\mathrm{Im} k_1| < m_2(\lambda)\}$. We will further show that when λ is real positive the Fourier transform $\hat{M}(k_1, \vec{k})$ has two simples zeroes inside this strip on the imaginary axis at the points $k_1 = \pm i\omega(\vec{k})$, where $\omega(\vec{k})$ is real positive. Thus we can state the following theorem.

Theorem 4.1 *There exists $r_3 < r_2$ such that for any fixed $\vec{k} \in \mathbb{Z}^{d-1}$ and $0 < \lambda < r_3$, the function $\hat{\tau^{\rm f}}(k_1, \vec{k})$ is meromorphic in the strip $|\mathrm{Im} k_1| < \tilde{m}_2(\lambda)$, where $\tilde{m}_2(\lambda) = (2d - 7/4)(|\ln \lambda| - |\ln r_2|)$, and has two simple poles in this strip at the points $k_1 = \pm i\omega(\vec{k})$ with $\omega(\vec{k})$ being a real positive even analytic function of $\vec{k} \in \mathbb{R}^{d-1}$ such that $e^{-\omega(\vec{k})} = \lambda^{(2d-2)} O(1)$.*

Proof. Equation (3.35) implies that $\hat{\tau^{\rm f}}(k_1, \vec{k}) = \tau^{\rm f}_0 \times [1/\hat{M}(k_1, \vec{k})]$. Hence we just have to show that $\hat{M}(k_1, \vec{k})$ has two simple zeroes inside the strip above on the imaginary axis at the points $k_1 = \pm i\omega(\vec{k})$.

Let us study in detail the Fourier transform $\hat{M}(k_1, \vec{k})$. First, recalling that $M_{x,y} = M_{x-y,0}$, denote $M_x = M_{x,0}$. Recall also the definition of $M_0 = M_{x=0}$ and $M_1 = M_x|_{|x|=1}$. We also use the notation $x = (x_1, \vec{x})$. With these notations the Fourier transform of $M_x \equiv M_{x_1,\vec{x}}$ is defined as

$$\hat{M}(k_1, \vec{k}) = \sum_{x \in \mathbb{Z}^d} e^{-ik_1 x_1} e^{-i\vec{k}\cdot\vec{x}} M_{x_1,\vec{x}}$$

Using that $M_{x_1,-\vec{x}} = M_{x_1,\vec{x}} = M_{-x_1,\vec{x}}$ one can rewrite $\hat{M}(k_1,\vec{k})$ as follows.

$$\hat{M}(k_1,\vec{k}) = g_0(\vec{k}) + \left[e^{ik_1} + e^{-ik_1}\right] g_1(\vec{k}) + \sum_{n=2}^{\infty} \left[e^{-ik_1 n} + e^{+ik_1 n}\right] g_n(\vec{k}) \tag{4.1}$$

where

$$g_n(\vec{k}) = \sum_{\vec{x}\in\mathbb{Z}^{d-1}} \cos(\vec{k}\cdot\vec{x}) M_{n,\vec{x}} \qquad \text{for } n = 0, 1, \dots \tag{4.2}$$

Observe that the functions $g_n(\vec{k})$ are analytic in $\vec{k}$, real for $\vec{k}$ real and even, i.e., $g(\vec{k}) = g(-\vec{k})$. Moreover by (3.34) and (3.31) it is easy to check that

$$g_0(\vec{k}) = O(1), \qquad g_1(\vec{k}) = -\lambda^{2d-2} O(1) \tag{4.3}$$

Let now $z = e^{ik_1}$ and define the function of the complex variable z

$$C(z,\vec{k}) = C_0(z,\vec{k}) + C_s(z,\vec{k})$$

where

$$C_0(z,\vec{k}) = g_0(\vec{k}) + \left[z + \frac{1}{z}\right] g_1(\vec{k}) \qquad C_s(z,\vec{k}) = \sum_{n=2}^{\infty} \left[z^n + \frac{1}{z^n}\right] g_n(\vec{k}) \tag{4.4}$$

Since $C(e^{ik_1},\vec{k}) = \hat{M}(k_1,\vec{k})$, to find the zeroes of the function $\hat{M}(k_1,\vec{k})$ in the rectangle $\{|\mathrm{Im}\, k_1| < m_2(\lambda), -\pi < |\mathrm{Re}\, k_1| \leq \pi\}$ is equivalent to find the zeroes of $C(z,\vec{k})$ in the annulus $A \equiv \{e^{-m_2(\lambda)} < |z| < e^{m_2(\lambda)}\}$. Note that the zeroes of $\hat{M}(k_1,\vec{k})$ are on the imaginary axis if the zeroes of $C(z,\vec{k})$ are real and positive. Moreover, since the functions $g_n(\vec{k})$ are even in the variable $\vec{k}$ then, if $z_0(\vec{k})$ is a zero of $C(z,\vec{k})$, $z_0(-\vec{k}) = z_0(\vec{k})$, or equivalently, if $k_1^0(\vec{k})$ is a zero of $\hat{M}(k_1,\vec{k})$, then $k_1^0(-\vec{k}) = k_1^0(\vec{k})$. Finally, due to the fact that $C(z,\vec{k}) = C(1/z,\vec{k})$, if $z_0(\vec{k})$ is a zero of $C(z,\vec{k})$ then $1/z_0(\vec{k})$ is also a zero, or, equivalently, if $k_1^0(\vec{k})$ is a zero of $\hat{M}(k_1,\vec{k})$, then $-k_1^0(\vec{k})$ is also a zero.
We will now show that for any $\vec{k} \in \mathbb{Z}^{d-1}$ and $\lambda \in \mathbb{R}^+$, $\lambda < r_3$, one can find an annulus $\tilde{A} \subset A$ such that the function $C(z,\vec{k})$ has only two real zeroes, say $\tilde{z}_1(\lambda,\vec{k})$ and $\tilde{z}_2(\lambda,\vec{k}) = 1/\tilde{z}_1(\lambda,\vec{k})$, in $\tilde{A}$, and they are real positive.
First observe that, by (4.3), for fixed $\vec{k} \in \mathbb{Z}^{d-1}$ and $\lambda \in \mathbb{R}^+$, $\lambda < r_3$, the function $C_0(z,\vec{k})$ has two real positive zeroes at $z_1(\lambda,\vec{k})$ and $z_2(\lambda,\vec{k})$, given by

$$z_1(\lambda,\vec{k}) = \frac{-2g_1(\vec{k})/g_0(\vec{k})}{1 + \sqrt{1 - 4\left[g_1(\vec{k})/g_0(\vec{k})\right]^2}} = \lambda^{2d-2} O(1) \quad ; \quad z_2(\lambda,\vec{k}) = \frac{1}{z_1(\lambda,\vec{k})} = \lambda^{-(2d-2)} O(1)$$

Note that the minus in the second function of (4.3), which is a consequence of the expansion (3.17), is crucial to conclude that $z_1(\lambda,\vec{k})$ and $z_2(\lambda,\vec{k})$ are real positive.
On the other hand, using bound (3.30), we get easily that

$$|g_n(\vec{k})| \leq C_d\, n^d\, \lambda^{-1} e^{-m_2(\lambda) n} \quad \text{for} \quad n \geq 2 \tag{4.5}$$

hence

$$|C_s(z,\vec{k})| \le \frac{C_d}{|\lambda|}\sum_{n=2}^{\infty}\left[|z|^n + \frac{1}{|z|^n}\right] n^d\, e^{-m_2(\lambda)n}$$

Note that, as expected, the series in r.h.s. above converges in the annulus A given by $\frac{1}{\epsilon}e^{-m_2(\lambda)} < |z| < \epsilon e^{+m_2(\lambda)}$ for any $\epsilon < 1$; moreover, for $\epsilon < 1/2$, we also have

$$|C_s(z,\vec{k})| \le \tilde{C}_d \lambda^{-1}\epsilon^2 \tag{4.6}$$

where $\tilde{C}_d = 2C_d \sum_{n=0}^{\infty} n^d 2^{-n}$. Let us choose $\epsilon = (\lambda/r_2)^{3/4}$ such that $\epsilon e^{+m_2(\lambda)} = e^{+\tilde{m}_2(\lambda)}$ where $\tilde{m}_2(\lambda) = (2d - 7/4)\,(|\ln\lambda| - |\ln r_2|)$ and define the annulus $\tilde{A} \subset A$ as $\tilde{A} = \{e^{-\tilde{m}_2(\lambda)} < |z| < e^{+\tilde{m}_2(\lambda)}\}$. We will denote γ_1 the circle $|z| = e^{-\tilde{m}_2(\lambda)}$ and γ_2 the circle $|z| = e^{+\tilde{m}_2(\lambda)}$. Thus $\partial\tilde{A} = \gamma_1 \cup \gamma_2$ is the boundary of $\tilde{A}$. Since the root $z_1(\lambda,\vec{k})$ is of order λ^{2d-2} while $e^{-\tilde{m}_2(\lambda)}$ is of the order $\lambda^{2d-7/4}$, one can choose $r_3 < r_2$ such small that $\{z_1(\lambda,\vec{k}), 1/z_1(\lambda,\vec{k})\} \subset \tilde{A}$ and $\epsilon = (\lambda/r_2)^{3/4} < 1/2$ whenever $\lambda < r_3$. Thus, by (4.6) we get

$$|C_s(z,\vec{k})| \le \frac{C_d}{r_2^{3/2}}\lambda^{1/2} \qquad \text{for } z \in \tilde{A}\cup\partial A \text{ and } \lambda < r_3 \tag{4.7}$$

i.e. $C_s(z,\vec{k})$ is of order $\lambda^{1/2}$ in $\tilde{A}\cup\partial\tilde{A}$. Finally it is easy to show that $|C_0(z,\vec{k})|$ is of order $\lambda^{-1/4}$ for $z \in \partial A$ and is of order 1 for z in the real negative interval $I^- = (-e^{+\tilde{m}_2(\lambda)}, -e^{-\tilde{m}_2(\lambda)})$.

We can now apply Rouché's theorem to find the zeroes of the function $C(z,\vec{k}) = C_0(z,\vec{k}) + C_s(z,\vec{k})$ in the interior $I(\gamma)$ of the closed curve $\gamma = \gamma_1 \cup \gamma_2 \cup I$ (with I^- traversed twice in opposite directions) and conclude that $C(z,\vec{k})$ has still two zeroes in the interior $I(\gamma)$ of γ (hence these zeroes are the only two zeroes in the annulus). Note that $I(\gamma) = \tilde{A}\backslash I^-$ and it is easy to show that $C(z,\vec{k})$ is, as $C_0(z,\vec{k})$, of order $O(1)$ in the whole interval I^-. Hence we conclude that $C(z,\vec{k})$ has two zeroes inside the annulus $\tilde{A}$.

Finally it is easy to check that the two zeroes $\tilde{z}_1(\lambda,\vec{k})$ and $\tilde{z}_2(\lambda,\vec{k}) = 1/\tilde{z}_1(\lambda,\vec{k})$ in the annulus $\tilde{A}$ lay actually in the real positive interval $I^+ = (e^{-\tilde{m}_2(\lambda)}, e^{+\tilde{m}_2(\lambda)})$ of $C(z,\vec{k})$ and

$$\tilde{z}_1(\lambda,\vec{k}) = e^{-\omega(\vec{k})} = \lambda^{2d-2}O(1) \tag{4.8}$$

As a matter of fact this follows from the fact that the function $C_s(z,\vec{k})$ (see (4.4)) is real (for λ real) and by (4.7) $|C_s(z,\vec{k})| \le C\lambda^{1/2}$ for some constant C in the interval I^+. This immediately implies that the function $C_0(z,\vec{k}) + C_s(z,\vec{k})$ has at least two zeroes in the interval I^+. More precisely, it has at least one zero laying in the interval $I_1 = (z_1^-, z_1^+)$ around z_1 and at least one zero laying in the interval $I_2 = (z_2^+, z_2^-)$ around z_1, where z_1^+ and $z_2^+ = 1/z_1^+$ are the solution of the equation $C_0(z,\vec{k}) = +C\lambda^{1/2}$ and z_1^- and $z_2^- = 1/z_1^-$ are the solution of the equation $C_0(z,\vec{k}) = -C\lambda^{1/2}$. Since $z_1^{\pm} = \lambda^{2d-2}O(1)$, (4.8) follows.

Concerning the analyticity of $\omega(\vec{k})$, just observe that $\hat{M}(k_1,\vec{k})$ is analytic as a function of k_1 and $\vec{k}$. Since $i\omega(\vec{k})$ is a simple zero of $\hat{M}$, the conclusion is a straightforward application of the implicit function theorem. Finally we recall that $\omega(\vec{k}) = \omega(-\vec{k})$ follows from the fact that the functions $g_n(\vec{k})$ defined in (4.2) are even. $\square$

In order to proof our main theorem we just need to show that the function $\omega(\vec{k})$ reaches is unique minimum in the square $-\pi < k_i \leq \pi$ $i = 2, \ldots, d$ at the point $\vec{k} = 0$. This occurs at least if one chooses λ strictly positive, which of course is always the case.
Namely we will show the following proposition

Proposition 4.2 *For all $\lambda \in \mathbb{R}^+$ such that $0 < \lambda < r_3$, the following equalities hold.*

$$\frac{\partial \omega(\vec{k})}{\partial k_i} = 2 \sin k_i \lambda^{2d-2} O(1) \quad \text{for } -\pi < k_i < \pi \ i = 2, \ldots, d \tag{4.9}$$

$$\left. \frac{\partial^2 \omega(\vec{k})}{\partial k_i \partial k_j} \right|_{\vec{k}=0} = 2\lambda^{2d-2} O(1) \delta_{ij} \tag{4.10}$$

and hence $\vec{k} = 0$ is a global minimum of $\omega(\vec{k})$ for $\vec{k} \in [-\pi, \pi]^{d-1}$.

Proof. Recalling (4.1), by differentiating the equation $\hat{M}(i\omega(\vec{k}), \vec{k}) = 0$ with respect to $\vec{k}_j$, for $j \neq 1$, we obtain the identity

$$\frac{\partial i\omega(\vec{k})}{\partial k_j} \frac{\partial \hat{M}(i\omega(\vec{k}), \vec{k})}{\partial k_1} + \frac{\partial \hat{M}(i\omega(\vec{k}), \vec{k})}{\partial k_j} = 0 \tag{4.11}$$

Observe now that,

$$\left. \frac{\partial \hat{M}(k_1, \vec{k})}{\partial k_1} \right|_{k_1 = i\omega(\vec{k})} = -i \left\{ \sinh[\omega(\vec{k})] g_1(\vec{k}) + \sum_{n=2}^{\infty} \sinh[n\omega(\vec{k})] g_n(\vec{k}) \right\} = +iO(1) \tag{4.12}$$

since, recalling that $\sinh[\omega(\vec{k})] = \lambda^{-(2d-2)} O(1)$ and (4.3), we have that $\sinh[\omega(\vec{k})] g_1(\vec{k}) = -O(1)$, while, using (4.5), we get $\sum_{n=2}^{\infty} \sinh[n\omega(\vec{k})] g_n(\vec{k}) = O(\lambda)$.
On the other hand

$$\left. \frac{\partial \hat{M}(k_1, \vec{k})}{\partial k_i} \right|_{k_1 = i\omega(\vec{k})} = \left\{ \frac{\partial g_0(\vec{k})}{\partial k_i} + \sum_{n=1}^{\infty} \cosh[n\omega(\vec{k})] \frac{\partial g_n(\vec{k})}{\partial k_i} \right\} \tag{4.13}$$

Now, for $n \geq 1$ that, we have

$$\frac{\partial g_n(\vec{k})}{\partial k_i} = -\sum_{\vec{x} \in \mathbb{Z}^{d-1}} x_i \sin(\vec{k} \cdot \vec{x}) M_{n,\vec{x}} = -2 \sum_{s=1}^{\infty} s \sin(sk_i) \sum_{\substack{\vec{x} \in \mathbb{Z}^{d-1} \\ x_i = s}} \cos(\vec{k} \cdot \vec{x} - k_i x_i) M_{n,\vec{x}} =$$

$$= -2 \sin k_i \sum_{s=1}^{\infty} s \frac{\sin(sk_i)}{\sin k_i} \sum_{\substack{\vec{x} \in \mathbb{Z}^{d-1} \\ x_i = s}} \cos(\vec{k} \cdot \vec{x} - k_i x_i) M_{n,\vec{x}} = \sin k_i O(\lambda^{(2d-2)(n+1)+1})$$

where in the last line we have used the bound (3.32) and the fact that $|\sin(sk_i)/\sin k_i| \leq s$. Hence the function $F(\lambda) = \sum_{n=1}^{\infty} \cosh[n\omega(\vec{k})](\partial g_n(\vec{k})/\partial k_i)$ which we know to be analytic in λ at least in the disc $|\lambda| < r_3$, starts with a power which is at least of the order λ^{2d-1} with a coefficient proportional to $\sin k_i$. Hence $F(\lambda) = \sin k_i O(\lambda^{2d-1})$ (here the signal of $O(\lambda^{2d-1})$ is out of control).
On the other hand, recalling that $M_1 = -\lambda^{2d-2}O(1)$ and using again bound (3.32), we have

$$\frac{\partial g_0(\vec{k})}{\partial k_i} = -2\sin k_i \Bigg[M_1 + \sum_{\substack{\vec{x}\in\mathbb{Z}^{d-1} \\ x_i=1,\, |\vec{x}|>1}} \cos(\vec{k}\cdot\vec{x} - k_i x_i) M_{0,\vec{x}} +$$

$$+ \sum_{s=2}^{\infty} s\frac{\sin sk_i}{\sin k_i} \sum_{\substack{\vec{x}\in\mathbb{Z}^{d-1} \\ x_i=s,}} \cos(\vec{k}\cdot\vec{x} - k_i x_i) M_{0,\vec{x}} \Bigg] \;=\; 2\sin k_i \lambda^{2d-2} O(1)$$

This means that

$$\left.\frac{\partial \hat{M}(k_1, \vec{k})}{\partial k_i}\right|_{k_1 = i\omega(\vec{k})} = 2\sin k_i \lambda^{2d-2} O(1)$$

and thus using (4.11)

$$\frac{\partial \omega(\vec{k})}{\partial k_j} = 2\sin k_i \lambda^{2d-2} O(1)$$

and (4.9) is proved.
Concerning the identity (4.10), since $\omega(\vec{k})$ is even, then $(\partial^2\omega(\vec{k})/\partial k_i \partial k_j)|_{\vec{k}=0} = 0$ if $i \neq j$. In the case $i = j$, by (4.11) we have that $(\partial\omega(\vec{k})/\partial k_j)|_{\vec{k}=0} = 0$, and thus

$$\left.\frac{\partial^2 i\omega(\vec{k})}{\partial^2 k_i}\right|_{\vec{k}=0} \left.\frac{\partial \hat{M}(i\omega(\vec{k}), \vec{k})}{\partial k_1}\right|_{\vec{k}=0} + \left.\frac{\partial^2 \hat{M}(i\omega(\vec{k}), \vec{k})}{\partial^2 k_i}\right|_{\vec{k}=0} = 0. \tag{4.14}$$

and calculating $(\partial^2 \hat{M}(i\omega(\vec{k}), \vec{k})/\partial^2 k_i)|_{\vec{k}=0}$

$$\left.\frac{\partial^2 \hat{M}(i\omega(\vec{k}), \vec{k})}{\partial^2 k_i}\right|_{\vec{k}=0} = \left.\frac{\partial^2 g_0(\vec{k})}{\partial^2 k_i}\right|_{\vec{k}=0} + \sum_{n=1}^{\infty} \cosh[n\omega(\vec{k})] \left.\frac{\partial^2 g_n(\vec{k})}{\partial^2 k_i}\right|_{\vec{k}=0}$$

By the same reasoning seen before it is easy to check that $\partial g_n(\vec{k})/\partial k_i$ is at least of the order $\lambda^{(2d-2)(n+2)+1}$ and hence that $G(\lambda) = \sum_{n=1}^{\infty} \cosh[n\omega(\vec{k})](\partial^2 g_n(\vec{k})/\partial^2 k_i)|_{\vec{k}=0}$ is at least of the order λ^{2d}, while $(\partial^2 g_0(\vec{k})/\partial^2 k_i)|_{\vec{k}=0} = +2\lambda^{2d-2}O(1)$ hence

$$\left.\frac{\partial^2 \hat{M}(i\omega(\vec{k}), \vec{k})}{\partial^2 k_i}\right|_{\vec{k}=0} = 2\lambda^{2d-2} O(1) \tag{4.15}$$

Thus, putting (4.12) and (4.15) in (4.14) we obtain (4.10). $\square$

§ 5. *Proof of theorem 1.1.*

¿From the definition of the Fourier transform, we have

$$\tau^{\mathrm{f}}_{(x_1,0)} = \frac{1}{(2\pi)^d}\int_{V_\pi} d\vec{k}\int_{-\pi}^{+\pi} dk_1 e^{ik_1x_1}\widehat{\tau}(k_1,\vec{k})$$

where $V_\pi = \{\vec{k}\in\mathbb{Z}^{d-1} : -\pi\le k_i\le\pi, \text{ for } i=2,\dots,d\}$. Suppose, without loss of generality, that $x_1>0$. Let $\epsilon>0$ such that $\tilde{m}_2(\lambda)>\omega(\vec{k})+\epsilon$ (by theorem 4.1 this ϵ always exists) and consider the rectangle in the complex plane $R=\{k_1\in\mathbb{C} : \; -\pi\le \mathrm{Re}k_1\le\pi,\ 0\le \mathrm{Im}k_1\le\omega(\vec{k})+\epsilon\}$ with boundary ∂R (if x_1 is negative consider the symmetric of R with respect to the real axis). By theorem 4.1 and due to the choice of ϵ, the function $\widehat{\tau}(k_1,\vec{k})$ is meromorphic in R with a unique pole at $k_1=i\omega(\vec{k})$. Thus, applying the residue theorem and using that $\widehat{\tau}(k_1,\vec{k})=\tau^{\mathrm{f}}_0(\lambda)/\hat{M}(k_1,\vec{k}))$ and (4.12) we get

$$\int_{\partial R} e^{ik_1x_1}\widehat{\tau}(k_1,\vec{k})dk_1 = 2\pi i\tau^{\mathrm{f}}_0(\lambda)e^{-\omega(\vec{k})x_1}r(\vec{k})$$

where, by equation (4.12)

$$r(\vec{k}) = \left[\left.\frac{\partial\hat{M}(k_1,\vec{k})}{\partial k_1}\right|_{k_1=i\omega(\vec{k})}\right]^{-1} = \frac{1}{i}O(1)$$

On the other hand, by direct calculation

$$\int_{\partial R} e^{ik_1x_1}\widehat{\tau}(k_1,\vec{k})dk_1 = \int_{-\pi}^{\pi} e^{ik_1x_1}\widehat{\tau}(k_1,\vec{k})dk_1 - C_\epsilon(x_1,\vec{k})e^{-(\tilde{\omega}(\vec{k})+\epsilon)x_1}$$

where $C_\epsilon(x_1,\vec{k})=\int_{-\pi}^{\pi} e^{ik_1x_1}\widehat{\tau}(k_1+i(\omega(\vec{k})+\epsilon),\vec{k})dk_1$ and note that $|C_\epsilon(x_1,\vec{k})|\le C_\epsilon$ where

$$C_\epsilon = \sup_{\vec{k}\in\mathbb{R}^{d-1}}\int_{\pi}^{\pi} |\widehat{\tau}(k_1+i(\omega(\vec{k})+\epsilon),\vec{k})|dk_1$$

Thus, using also that $\omega(\vec{k})\ge\omega(0)$ (by proposition 4.2) we obtain

$$\int_{-\pi}^{\pi} e^{ik_1x_1}\widehat{\tau}(k_1,\vec{k}) = 2\pi\tau^{\mathrm{f}}_0(\lambda)e^{-\omega(\vec{k})}r(\vec{k}) + O(e^{-[\omega(0)+\epsilon]x_1})$$

Then we can write

$$\tau(x_1,0) = \frac{\tau^{\mathrm{f}}_0(\lambda)}{(2\pi)^{d-1}}\frac{e^{-\omega(0)x_1}}{x_1^{\frac{d-1}{2}}}\left[x_1^{\frac{d-1}{2}}\int_{V_\pi} r(\vec{k})e^{-\Delta\omega(\vec{k})x_1}d\vec{k}\right] + O(e^{-[\omega(0)+\epsilon]x_1}).$$

where $\Delta\omega(\vec{k}) = \omega(\vec{k}) - \omega(0)$. Note that the second term in the above identity decays to zero faster than $e^{-\omega(0)|x_1|}$. Now, changing the variable $\vec{k} = |x_1|^{1/2}\vec{q}$, we get

$$|x_1|^{\frac{d-1}{2}} \int_{V_\pi} r(\vec{k}) e^{-\Delta\omega(\vec{k})|x_1|} d\vec{k} = \int_{V_{\pi|x_1|^{1/2}}} r(\vec{q}|x_1|^{-1/2}) e^{-\Delta\omega(\frac{\vec{q}}{\sqrt{|x_1|}})|x_1|} d\vec{q}$$

By (4.10) we have that $\lim_{\lambda\to 0} \Delta\omega(\lambda\vec{q})/\lambda^2 = 2O(1)\vec{q}^2$, hence

$$\lim_{x_1\to\infty} \int_{V_{\pi|x_1|^{\frac{1}{2}}}} r\left(\frac{\vec{q}}{|x_1|^{\frac{1}{2}}}\right) e^{-\Delta\omega(\frac{\vec{q}}{\sqrt{|x_1|}})|x_1|} d\vec{q} = r(0) \int_{\mathbb{R}^{d-1}} e^{-2\lambda^{2d-2}O(1)\vec{q}^2} d\vec{q} = \frac{(2\pi)^{\frac{d-1}{2}}}{\lambda^{(d-1)^2}} O(1) \tag{5.1}$$

which is non-zero. Note that the function $C(\lambda, x_1)$ appearing in theorem 1.1 is given explicitly by

$$C(\lambda, x_1) = \frac{\tau_0^{\mathrm{f}}(\lambda)}{(2\pi)^{d-1}} \int_{V_{\pi|x_1|^{1/2}}} r(\vec{q}|x_1|^{-1/2}) e^{-\Delta\omega(\frac{\vec{q}}{\sqrt{|x_1|}})|x_1|} d\vec{q} + x_1^{\frac{d-1}{2}} O(e^{-\epsilon x_1})$$

and, by (5.1) and recalling that $\tau_0^{\mathrm{f}}(\lambda) = \lambda^{2d}O(1)$, equation (1.3) follows. $\square$

Remark. Note that the leading exponential decay of $\tau_{x_1,0}^{\mathrm{f}}(\lambda)$ occurs with a rate $m(\lambda) = \omega(0)$. In particular, since $\exp\{-\omega(0)\} = \lambda^{2d-2}O(1)$ we recover a result stated in [20], saying that $m(\lambda) = 2(d-2)|\ln\lambda| + O(\lambda)$.

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