Contemporary Mathematics

On the dynamics of interfaces in the ferromagnetic XXZ chain under weak perturbations

Bruno Nachtergaele, Wolfgang L Spitzer, and Shannon Starr

Abstract. We study the time evolution of interfaces of the ferromagnetic XXZ chain in a magnetic field. A scaling limit is introduced where the strength of the magnetic field tends to zero and the microscopic time to infinity while keeping their product constant. The leading term and its first correction are determined and further analyzed in more detail for the case of a uniform magnetic field.

1. Introduction

The quantum ferromagnetic Heisenberg model has received a great deal of interest in recent years. We shall concentrate here on the one-dimensional model. The starting point was the discovery of interface ground states in the anisotropic XXZ Heisenberg chain by Alcaraz, Salinas and Wreszinski [**2**] and independently by Gottstein and Werner [**10**]. These interface ground states display two regions of opposite magnetization separated by a domain wall whose thickness depends on the anisotropy. Explicit expressions for the ground states exist for all values of the spin, see Section 2.

The next important contribution was the proof of completeness of the infinite volume ground states (in the sense of local stability) and the proof of a positive gap above the ground states for any amount of anisotropy and any value of the spin. The spin-$\frac{1}{2}$ model is $SU_q(2)$ symmetric which was used to determine the exact value of the gap. This was accomplished by Koma and Nachtergaele [**12, 13**] and Matsui [**16**].

The existence of a spectral gap for general j was proven by Koma, Nachtergaele and Starr [**14**]. Recently, it was proven by Caputo and Martinelli [**8**] that the gap for spin j scales asymptotically like j, however, no precise values are known. This verifies a conjecture stated in [**14**], and comes as a surprise since the spin-j Ising chain (which can be considered as the XXZ chain with infinite anisotropy) always has a gap equal to 1.

1991 *Mathematics Subject Classification.* 82C10;82C24.

Key words and phrases. XXZ chain, interfaces, dynamics, kink states, Heisenberg ferromagnet.

This material is based on work supported by the National Science Foundation under Grant No. DMS0070774. Shannon Starr is a National Science Foundation Postdoctoral Fellow.

There has been a lot of progress over the last two decades on the dynamics of classical spin systems, or lattics gas systems. For instance, for the Ising model the Glauber and Kawasaki dynamics have been studied extensively. Here, we only want to mention the work by Schonmann and Shlosman [**19**] on the relaxation in the Ising model which is similar to what we want to study in the quantum XXZ chain. A quantum model which has been studied in detail is the one-dimensional XY model, by Antal, Rácz, Rákos, and Schütz [**3**] at zero temperature and by Ogata [**18**] at non-zero temperature. More recently, Aschbacher and Pillet [**5**] studied non-equilibrium steady states in the same model at non-zero temperature. This is of great interest also in the context of entropy production. The XY model (in one dimension) is easier to deal with than the XXZ model since it can be mapped to a model of free fermions (see [**15**] and [**4**]).

Interfaces and their response to magnetic fields are of great physical and technological interest, for instance in storing and reading information on magnetic devices. The concept of a magnetic domain was first introduced by Pierre Weiss, already in 1906. It was in 1935 in a seminal paper that Landau and Lifshitz presented a theoretical treatment of domain walls and their motion when a magnetic field is applied. Although the Landau-Lifshitz equation was rigorously derived in the mean-field limit by Moser, Prets, and Spitzer [**17**], its predictions do not universally hold true for all magnetic materials. Thus we want to understand the dynamical behavior of domain walls in a quantum model, in which they occur naturally as ground states, namely in the XXZ model.

The dynamics we consider is generated by the XXZ kink Hamiltonian perturbed by a magnetic field. As in many similar cases, we introduce a scaling where the strength of the perturbation, say the coupling constant λ for a magnetic field, tends to 0 and the microscopic time scale to infinity such that their product, which we denote by τ, is kept fixed. We obtain the following results about this limit: (i) For external magnetic fields of bounded spatial support that are sufficiently regular functions of time, the leading contribution to the time evolution is identified as the reduced dynamics determined by the time averaged field projected onto the ground state space of the unperturbed model with an error term of order $\lambda^{1-\delta}, \delta \in (0,1)$ (Theorem 3.3); (ii) If the spectrum of the unperturbed model has a gap between the ground states and the rest of the spectrum, and the rest of the spectrum is absolutely continuous, the next-to-leading order term is of order λ (Theorem 3.5); (iii) Analysis of the reduced dynamics for uniform, time-independent magnetic fields, reveals a markedly different behavior depending on whether or not the field has a non-vanishing component in the z direction. If the z component vanishes, the reduced dynamics for the magnetization profile is “ballistic”, while if there is a non-vanishing component of the field in the z direction, the magnetization profile evolves periodically.

In Section 2 we define the model and state the assumptions on the magnetic field. We derive the leading dynamics and its first correction in Section 3. In Section 4 we analyze the leading dynamical behavior in more detail for a uniform field and comment on small perturbations thereof. Two appendices provide some auxiliary results on the magnetization profile in the kink ground states of the XXZ chain and on the spectrum of the Stark-Jacobi operator.

2. The model

2.1. Finite chain Hamiltonian. In this paper we shall only consider the spin-$\frac{1}{2}$ chain. Unless stated otherwise, our results extend to higher values of the spin. As usual, let us denote by $\Delta > 1$ the anisotropy parameter. The kink Hamiltonian on the chain $[a, b] \cap \mathbb{Z}$ is defined as

(1)
$$H[a,b] = -\sum_{x=a}^{b-1} \left[\frac{1}{\Delta}(S_x^1 S_{x+1}^1 + S_x^2 S_{x+1}^2) + S_x^3 S_{x+1}^3 - \frac{1}{4}\mathbf{1} \right] - \frac{1}{2}\sqrt{1-\Delta^{-2}}(S_a^3 - S_b^3)\,,$$

where the spin operators at position x, S_x^1, S_x^2, S_x^3 satisfy the usual commutation relations, $[S_x^\alpha, S_x^\beta] = i\,\varepsilon^{\alpha\beta\gamma} S_x^\gamma$. We have added the kink boundary condition, $-\frac{1}{2}\sqrt{1-\Delta^{-2}}(S_a^3 - S_b^3)$ which makes the model $SU_q(2)$ invariant and displays the full set of kink ground states which we derive next. It is convenient to introduce another parameter, $0 < q < 1$, such that $q + q^{-1} = 2\Delta$. One can show that

$$H = \sum_{x=a}^{b-1} P_{xx+1}^q\,,$$

where P_{xx+1}^q is the orthogonal projection onto the vector $q|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$. By introducing the q-deformed lowering operator

$$S^- = \sum_{x=a}^{b} \mathbf{1} \otimes \cdots \otimes S_x^- \otimes t_{x+1} \otimes \cdots \otimes t_b$$

with $t = q^{2S^3}$, we can express all $b - a + 2$ ground states as $(S^-)^k|\uparrow\cdots\uparrow\rangle, k = 0, 1, \ldots, b-a+1$, since $[H, S^-] = 0$. Clearly, these vectors run through all possible eigenspaces of the total S^3 component, which obviously commutes with $H[a,b]$. The uniqueness of the ground states follows, for instance, by a Perron-Frobenius argument.

More explicitly, let us denote the eigenvectors of the total S^3 component by $|(m_x)\rangle$, where $(m_x) \in \{\pm\frac{1}{2}\}^{b-a+1}$ and $S_y^3|(m_x)\rangle = m_y|(m_x)\rangle$. Then, as was found by Alcaraz, Salinas and Wreszinski [**2**], these ground states are (up to normalization)

$$\psi_m = \sum_{(m_x)} \prod_{x=a}^{b} q^{-x(1-m_x)/2}|(m_x)\rangle\,, \tag{2}$$

where the sum runs over all sets (m_x) such that $\sum_x m_x = m$, which is the total S^3 eigenvalue. As the ψ_m are not normalized, we also define $|m\rangle = \psi_m/\|\psi_m\|$. If $m = \pm(b-a+1)$ is maximal/minimal, then we have the all spin-up/down state, i.e., the magnetization profile in the z direction equals $\langle m|S_x^3|m\rangle = \pm\frac{1}{2}$ for all x.

2.2. Infinite chain kink Hamiltonian. The infinite volume kink Hamiltonian is defined in the standard way via the generator of the Heisenberg dynamics on the algebra of quasi-local observables, $\mathcal{A}$. This algebra is the norm closure of the algebra of local observables,

$$\mathcal{A}_{\text{loc}} = \bigcup_{\Lambda\subset\mathbb{Z}, |\Lambda|<\infty} \mathcal{A}_\Lambda\,, \quad \mathcal{A}_\Lambda = \bigotimes_{x\in\Lambda} \text{Mat}_{\mathbb{C}}(2)\,.$$

We prefer to work in the GNS Hilbert space representation of the infinite volume kink states. To this end we introduce the incomplete tensor product Hilbert space

$$\mathcal{H} = \overline{\bigcup_{\Lambda, |\Lambda|<\infty} \left(\bigotimes_{x\in\Lambda} \mathbb{C}^2 \otimes \bigotimes_{y\in\mathbb{Z}\setminus\Lambda} \Omega(y) \right)}, \tag{3}$$

where

$$\Omega(y) = \begin{cases} |\uparrow\rangle & \text{if } y \le 0 \\ |\downarrow\rangle & \text{if } y > 0 \end{cases} .$$

Let us define the vector

$$\Omega = \bigotimes_{y\in\mathbb{Z}} \Omega(y) , \tag{4}$$

which is a vector in $\mathcal{H}$. We also define the (unnormalized) GNS vector

$$\Omega^{\mathsf{GNS}} = \sum_{k\ge 0} \sum_{x_1<\cdots<x_k\le 0<y_1<\cdots<y_k} q^{\sum_{j=1}^k (y_j - x_j)} \prod_{j=1}^k S^1_{x_j} S^1_{y_j} \Omega .$$

The set of vectors $\psi = A\Omega^{\mathsf{GNS}}$, where A is a local observable, constitute a dense subspace of $\mathcal{H}$. In this representation, the kink Hamiltonian is now defined by

$$HA\Omega^{\mathsf{GNS}} = \lim_{[a,b]\to\mathbb{Z}} [H[a,b], A]\Omega^{\mathsf{GNS}} . \tag{5}$$

Next we state the assumptions on the magnetic field by which we perturb the kink Hamiltonian.

Assumption 2.1. *Let $\mathbb{Z}\times\mathbb{R} \ni (x,t) \mapsto \vec{B}(x,t) \in \mathbb{R}^3$ denote a magnetic field which has spatial support in some finite set $\Lambda \subset \mathbb{Z}$, and is absolutely continuous with bounded derivative (as a function of time). Let $V(t) = \sum_{x\in\mathbb{Z}} \vec{B}(x,t)\vec{S}_x$. We further impose a scaling on the strength and the variation of V, and define*

$$V^\lambda(t) = \lambda V(\lambda t) . \tag{6}$$

Clearly, $H + V^\lambda(t)$ is a selfadjoint operator with domain $\mathcal{D}(H)$. The reason for our assumption on the support of B is that if we want to consider, for simplicity, a time independent and translation invariant magnetic field, then, in general, $e^{-it(H+V)}$ is not a strongly continuous group of unitaries on $\mathcal{D}(H)$. The application of V on a ground state will no longer be in the Hilbert space $\mathcal{H}$. This is no contradiction to the fact that the unperturbed time evolution is strongly continuous in the Heisenberg picture, i.e., on the algebra of (quasi-local) observables $\mathcal{A}$, since V would no longer be quasi-local.

We could extend the class in (2.1) to perturbations relatively bounded to H (with relative bound less than 1 to ensure selfadjointness according to the Theorem by Kato-Rellich) but this does also not allow to consider uniform fields, and we therefore stick to bounded perturbations.

We shall first find the leading dynamics, and then consider the limit as the spatial support of $\vec{B}(x,t)$ tends to infinity.

3. Dynamics

LEMMA 3.1 (Dyson series). *Let $U^\lambda(t,t_0)$ be the solution to the time dependent Schrödinger equation*

$$i\frac{d}{dt}U^\lambda(t,t_0) = (H + V^\lambda(t))U^\lambda(t,t_0)$$

with V^λ satisfying (2.1). Then for $\phi \in \mathcal{D}(H)$ and $N \in \mathbb{N}$

$$\begin{aligned} e^{itH}U^\lambda(t,t_0)\phi &= \phi + \sum_{0<n<N}(-i)^n \int_{t_0\le t_n\le\ldots\le t_1\le t} d\mathbf{t}\, V^\lambda(t_1,t_1)\cdots V^\lambda(t_n,t_n)\phi \\ &\quad + \mathbf{E}_N\phi\,, \end{aligned} \tag{7}$$

where we used the notation $d\mathbf{t} = dt_1\cdots dt_n$, and $V(t,t) = e^{itH}V(t)e^{-itH}$. The error term, $\mathbf{E}_N$, is bounded by $\frac{|t-t_0|^N}{N!}(\lambda\|V\|)^{N-1}$, which converges to 0 as $N\to\infty$ uniformly for t on compacts; we use $\|V\| = \sup\{\|V(s)\|; s\ge t_0\}$.

PROOF. We may assume that $t_0 = 0$ and $\lambda = 1$. Setting $\Omega(t) = e^{itH}U(t)$, then for $\phi\in\mathcal{D}(H)$ we have $\frac{d}{dt}\Omega(t)\phi = -iV(t,t)\Omega(t)\phi$. Integrating over t and iterating we obtain

$$\begin{aligned} \Omega(t)\phi &= \phi + \sum_{0<n<N}(-i)^n\int_{0\le t_n\le\ldots\le t_1\le t} d\mathbf{t}\, V(t_1,t_1)\cdots V(t_n,t_n)\phi \\ &\quad + (-i)^N\int_{0\le t_{N-1}\le\ldots\le t_1\le t} d\mathbf{t}\, V(t_1,t_1)\cdots V(t_{N-1},t_{N-1})\Omega(t_{N-1})\phi\,. \end{aligned}$$

The error estimate follows then easily. □

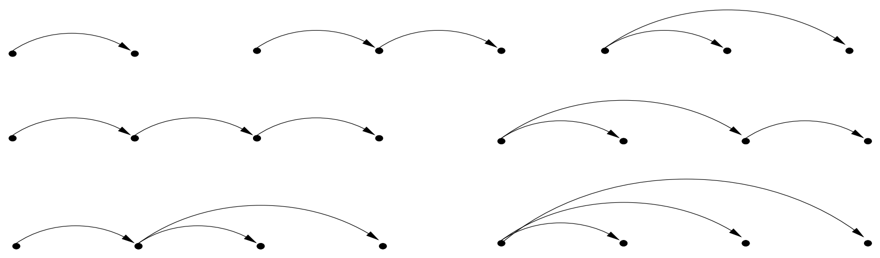

FIGURE 1. The first 7 graphs G, comprising $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$.

We discuss next a formula for the iterated t integral allowing us to decide which energy terms contribute at what rate. To this end we consider the set $\mathcal{G}_n$ of directed graphs G with vertex set equal to $\{n+1,n,\ldots,1\}$ and bonds satisfying the following rules. 1) The direction of a bond is from a higher to a lower value of n. 2) There is exactly one bond for every site $1\le k\le n$, leading to it from a site $p(k) > k$, although many bonds can originate from a certain site. We call the bond $b(k) = (p(k),k)$, and define $\ell(b(k)) = p(k) - k$ the length of this bond, and $\ell(G) = \sum \ell(b(k))$ the length of the graph G. 3) If there is a bond between i and j, then there is no bond possible between k and l if ($k > b(i)$ and $l < b(i)$) or ($k > i$ and $l < i$) or ($k < b(i)$, $l > i$). In other words, there are no crossing bonds, and no bonds properly nested. (With these rules, the digraphs G are equivalent to the set of all compositions, i.e., all finite ordered sequences of positive integers. To a digraph $G = \{b(1), \ldots, b(n)\}$, we associate the composition $[p(1)-1, p(p(1))-p(1), p(p(p(1)))-p(p(1)), \ldots]$. $\mathcal{G}_n$ is the set of all compositions which sum to n.)

We can show that $|\mathcal{G}_n| = 2^{n-1}$. It is clear that $|\mathcal{G}_1| = 1$. If we have a graph $G \in \mathcal{G}_n$, then we may construct two new graphs in $\mathcal{G}_{n+1}$. Let us denote the $n+2$nd vertex by 0, then 0 may be connected to 1 whose graph we denote by G_1. The only other possibility is to join 0 with $p(1)$ and we call the graph G_2. We define signs for the graphs, inductively. Let $\curvearrowright$ be the unique graph with vertex set $\{2, 1\}$ and bond from 2 (left) to 1 (right), then we set $\sigma(\curvearrowright) = 1$. Using the above construction, we define the sign by $\sigma(G_1) = -\sigma(G)$, and $\sigma(G_2) = \sigma(G)$. Some graphs are shown in Figure 1.

Given a sequence of numbers $k_j : 1 \leq j \leq n$, for any graph $G \in \mathcal{G}_n$ and vertex i we define $k(i; G) = \sum_{i \leq j \leq i+\ell(b(i))-1} k_j$.

LEMMA 3.2. *Let $E_j : 1 \leq j \leq n+1$ and $k_j : 1 \leq j \leq n$ be two sequences of numbers, then*

$$\int_{0 \leq t_n \leq \ldots \leq t_1 \leq t} d\mathbf{t} \prod_{j=1}^{n} e^{-it_j(E_{j+1}-E_j+i\lambda k_j)} =$$

$$\sum_{G \in \mathcal{G}_n} \sigma(G) \left(\prod_{j=1}^{n} \frac{i}{E_{p(j)} - E_j + i\lambda k(j;G)} \right) \left(e^{-it(E_{p(1)}-E_1+i\lambda k(1;G))} - 1 \right). \tag{8}$$

PROOF. We can easily check this by induction. The formula is obviously correct for $n = 1$. Assuming that the formula is true for $n-1$, we may write the n fold integral as

$$\sum_{G \in \mathcal{G}_{n-1}} \sigma(G) \left(\prod_{j=2}^{n} \frac{i}{E_{p(j)} - E_j + i\lambda k(j;G)} \right)$$
$$\times \int_0^t dt_1 \left(e^{-it_1(E_{p(2)}-E_1+i\lambda k(2;G)+i\lambda k_1)} - e^{-it_1(E_2-E_1+i\lambda k_1)} \right).$$

The vertex set of the graphs $G \in \mathcal{G}_{n-1}$ is here $\{n+1, \ldots, 2\}$. Now, in the first term of the integrand we have that $p(2;G) = p(1;H), k(1;H) = k(2;G) + k_1, \sigma(H) = \sigma(G)$, where H is the graph on $\{n+1, \ldots, 1\}$ which agrees on $n+1, \ldots, 2$ with G plus a bond connecting 1 with the same site as 2. In the second term we have a bond between 1 and 2, and by our rules hence a change of the sign. We can combine this into a sum over all graphs $H \in \mathcal{G}_n$, which verifies the formula. □

In the following theorem we identify the leading term in the limit when $\lambda \to 0$, and $t \to \infty$, such that $\tau = \lambda t$ is constant. Here, τ can be interpreted as a macroscopic time scale. We shall use the spectral decomposition of the Hamiltonian

$$H = \int_0^\infty E \, dP(E) \ , \tag{9}$$

such that for any $\phi \in \mathcal{H}$, we have

$$\phi = \int_0^\infty \ dP(E)\phi \ .$$

THEOREM 3.3 (Leading dynamics). *Let $\phi \in \mathcal{H}$, and let V satisfy Assumption (2.1). Then, for all δ, $1 > \delta > 0$, there exists a λ_0 depending on $\|V\|, \tau, \|\phi\|$ and δ such that for $0 < \lambda < \lambda_0$*

$$\left\| e^{i\lambda^{-1}\tau H} U^{\lambda}(\lambda^{-1}\tau)\phi - \int_0^{\infty} \mathbb{T}\left(e^{-iP(E)\int_0^{\tau} dt\, V(t) P(E)}\right) dP(E)\phi \right\| < \lambda^{1-\delta}\,. \tag{10}$$

$\mathbb{T}$ means time ordering, i.e., $\mathbb{T}(V(s)V(t)) = V(s)V(t)$ if $s > t$, and zero otherwise.

PROOF. We may assume that $\|\phi\| = 1$. Let N be an integer which we choose later. We may write ($t = \lambda^{-1}\tau$)

$$\begin{aligned}&\left\| e^{itH} U^{\lambda}(\lambda^{-1}\tau)\phi - \int_0^{\infty} \mathbb{T}\left(e^{-iP(E)\int_0^{\tau} dt\, V(t) P(E)}\right) dP(E)\phi \right\| \\ &\qquad \le |\mathbf{E}_N| + |\Sigma_N - \Sigma_N'| + |\mathbf{E}_N'|\,.\end{aligned}$$

Here, $\mathbf{E}_N$ is the error term from equation (7), and $\mathbf{E}_N'$ the remainder term of the power series for the exponential function in $\mathbb{T}\left(e^{-iP(E)\int_0^{\tau} dt\, V(t)P(E)}\right)\phi$ to the order $N-1$, whereas Σ_N and Σ_N' refers to their finite sums.

First we take care of the error terms $\mathbf{E}_N$ and $\mathbf{E}_N'$ by choosing $N = N(\lambda; \|V\|, \tau)$ such that

$$\frac{1}{\|V\|}\frac{(\tau\,\|V\|)^N}{N!} < \tfrac{1}{3}\lambda^{1-\delta}.$$

This can be accomplished with $N = -C\ln\lambda$ and the constant C depending on $\|V\|$ and τ.

Second, we investigate the limit $(\lambda \to 0, t \to \infty, \lambda t \to \tau)$ for the n-th term in the finite Dyson series. By inserting the spectral resolution for H (9), the n-th term in $e^{itH}U^{\lambda}(t)\phi$ equals

$$(-i\lambda)^n \int_{[0,t]^n} d\mathbf{t} \int_{\mathbb{R}_+^{n+1}} d\mathbf{E}_n \prod_{j=1}^{n} e^{-it_j(E_{j+1}-E_j)}\,\mathbb{T}\left(\prod_{j=1}^{n} P(E_j)V(\lambda t_j)\right) dE_{n+1}\phi\,. \tag{11}$$

In order to perform the t integration we write V as its inverse Laplace transform. Let $c_1 < c_2 < \cdots < c_n < 0$, then we obtain

$$\begin{aligned}&(-i\lambda)^n \frac{1}{(2\pi i)^n} \int_{c_1-i\infty}^{c_1+i\infty} \cdots \int_{c_n-i\infty}^{c_n+i\infty} d\mathbf{k} \int_{\mathbb{R}_+^{n+1}} d\mathbf{E}_n \\ &\quad\times \int_{[0,t]^n} d\mathbf{t} \prod_{j=1}^{n} e^{-it_j(E_{j+1}-E_j+i\lambda k_j)}\,\mathbb{T}\left(\prod_{j=1}^{n} P(E_j)\hat{V}(k_j)\right) dP(E_{n+1})\phi\,.\end{aligned}$$

By using formula (8) we can do the t integration, and obtain

$$\begin{aligned}&\frac{1}{(2\pi i)^n} \int_{c_1-i\infty}^{c_1+i\infty} \cdots \int_{c_n-i\infty}^{c_n+i\infty} d\mathbf{k} \int_{\mathbb{R}_+^{n+1}} d\mathbf{E} \int_{[0,t]^n} d\mathbf{t} \prod_{j=1}^{n} e^{-it_j(E_{j+1}-E_j+i\lambda k_j)} \\ &\quad \sum_{G\in\mathcal{G}_n} \sigma(G) \left(\prod_{j=1}^{n} \frac{\lambda}{E_{p(j)} - E_j + i\lambda k(j;G)}\right)\left(e^{-it(E_{p(1)}-E_1+i\lambda k(1;G))} - 1\right) \\ &\quad \times \mathbb{T}\left(\prod_{j=1}^{n} P(E_j)\hat{V}(k_j)\right) dP(E_{n+1})\phi\,.\end{aligned}$$

Here, it is convenient to have c_i's such that there are no zeros in the denominator. We observe here the crucial fact that in order to find a term of the order 1, all E_j have to be equal. We may now go back to (11), change coordinates and conclude that in this limit we get

$$\begin{aligned}(-i\lambda)^n \int_0^\infty \int_{[0,t]^n} d\mathbf{t}\, \mathbb{T}\left(\prod_{j=1}^n P(E)V(\lambda t_j)\right) dP(E)\phi \\ = \quad (-i)^n \int_0^\infty \mathbb{T}\left(\int_0^\tau dt P(E)V(t)\right)^n dP(E)\phi\,.\end{aligned}$$

Finally, we have to bound the sum of all other terms in the finite Dyson series which individually tend to 0 as $\lambda \to 0$, and are contained in $|\Sigma_N - \Sigma'_N|$. As there are 2^{n-1} terms in formula (8), we immediately obtain the bound

$$|\Sigma_N - \Sigma'_N| \le \frac{\lambda}{2} \sum_{n=1}^{N-1} \frac{(2\tau\|V\|)^n}{n!}\,,$$

which in turn is bounded by $\frac{1}{3}\lambda^{1-\delta}$ if we choose $\lambda < \lambda_0$ with λ_0 such that $e^{2\tau\|V\|} < \frac{2}{3}\lambda_0^{-\delta}$. □

If $H\phi = 0$ is a kink ground state of H, then the leading term equals

$$\mathbb{T}\left(e^{-i\int_0^\tau P(0)V(t)P(0)}\right)\phi\,. \tag{12}$$

This expression will be further analyzed for a specific choice of V in Section 4.

Note that so far we did not use any specific spectral properties of the Hamiltonian, and thus Theorem 3.3 is valid, for instance, for any value of the spin.

We know that $\sigma(H) \subset \{0\} \cup [1-\Delta^{-1}, \infty)$, using the theorem of [**12**] that there is a gap of $1-\Delta^{-1}$ above the infinite volume ground states. If we decompose the kink Hamiltonian on $\mathbb{Z}$ as the sum of a kink plus a droplet Hamiltonian, i.e., $H^{+-}_{\mathbb{Z}} = H^{+-}_{(-\infty,y]} + H^{--}_{(y,\infty)}$ with $y \in \mathbb{Z}$, then we get approximate eigenvectors by tensoring a kink ground on $(-\infty, y]$ (and with interface far to the left of y) with eigenvectors of $H^{--}_{(y,\infty)}$. In the case $y = -\infty$ the spectrum of the latter has been shown by Babbitt and Gutkin [**6**] to be $\{0\} \cup [1-\Delta^{-1}, \infty)$. Their analysis follows the earlier work by Babbitt and Thomas [**7**] for $\Delta = 1$ proving the completeness of the Bethe ansatz eigenfunctions in the infinite volume limit and thereby also showing that the spectrum is absolutely continuous.

This way, one obtains $\sigma(H) = \{0\} \cup [1-\Delta^{-1}, \infty)$. But this construction does not give the nature of the spectrum in the interval in $[1-\Delta^{-1}, \infty)$. We plan to address this issue in the future; for now we state the following hypothesis.

HYPOTHESIS 3.4. *The spectrum of the spin-$\frac{1}{2}$ kink Hamiltonian H with $\Delta > 1$ has no other point spectrum than 0, and the rest is absolutely continuous, equal to $[1-\Delta^{-1}, \infty)$.*

By assuming this information we can identify the first correction to the leading term given in Theorem 3.3 for ground states, and we show that it is $\mathcal{O}(\lambda)$. For simplicity, we state the result only for time independent fields and leave the general case to the reader. The main argument in the proof below, however, is carried out for a time dependent field.

THEOREM 3.5 (First order correction). *Let ϕ be a (normalized) ground state of the spin-$\frac{1}{2}$ kink Hamiltonian, V a perturbation as in Assumption (2.1) and H satisfying Hypothesis (3.4). Then,*

$$\Big\| e^{i\lambda^{-1}\tau H} e^{-i\lambda^{-1}\tau(H+\lambda V)}\phi - e^{-i\tau P(0)V}\phi - \lambda\int_{1-\Delta^{-1}}^{\infty}\frac{dE}{E}P(E)Ve^{-i\tau P(E)V}\phi$$

$$- P(0)V\left(e^{-i\tau[P(0)V+\lambda H^{-1}(\mathbf{1}-P(0))V]} - e^{-i\tau P(0)V}\right)\phi\Big\| = o(\lambda)\,. \tag{13}$$

PROOF. We analyze the n-th term in the finite Dyson series and treat the error terms as before by choosing first $N = N(\lambda, \|V\|, \tau)$. The spectral resolution of H has now the form $\mathbf{1} = P(0) + \int_{1-\Delta^{-1}}^{\infty} dP(E)$. We have seen above that to leading order all energies E_i have to be equal to 0. According to (8), terms of the order λ come from sequences $(E_m - E_l) : \{m < l\} \in G$ such that all but one pair is non-zero. We represent such sequences graphically by polygons with $n+1$ vertices. We draw a horizontal line between the vertices k and $k+1$ whenever $E_k = E_{k+1}$, an upward line if $E_k > E_{k+1} = 0$, and a downward line if $0 = E_k < E_{k+1}$.

By our rules for the graphs $G \in \mathcal{G}_n$, the only possible polygons contributing to the order $\mathcal{O}(\lambda)$ are of the form displayed in Figure 2.

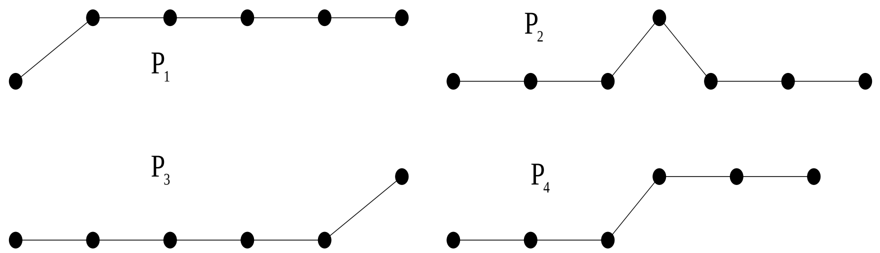

FIGURE 2. Sequences of energies satisfying the condition that their corresponding differences is non-zero for exactly one pair. The bump in polygon P_2 can be at any site such that the starting and ending point of the polygon lies at the horizontal line.

Using Hypothesis 3.4 we shall show that, in fact, only P_1 and P_2 contribute to $\mathcal{O}(\lambda)$ while P_3 and P_4 are of lower order. We shall now consider the contributions from P_1, P_2, P_3 and P_4 to formula (11). Let us start with polygon P_1, i.e., we have a sequence $0 = E_{n+1} < E_n = \ldots = E_1 = E$. In

$$\int_0^{t_{n-1}} dt_n\, e^{it_nE}V(\lambda t_n)\phi =$$

$$\frac{1}{iE}\left(e^{it_{n-1}E}V(\lambda t_{n-1}) - V(0)\right)\phi - \frac{1}{iE}\int_0^{t_{n-1}} dt_n\, e^{it_nE}V(\lambda t_n)\phi$$

we can use that, by our assumption on the spectrum of H and the Riemann-Lebesgue Lemma, the leading term is $\frac{i}{E}V(0)$, and the rest is $o(1)$, where $o(1)$ depends on the inverse gap $\Delta/(\Delta-1)$ and $\sup\{\|V(s)\| : s \in [0,\tau]\}$. Then, up to

$o(1)$ we have

$$\begin{aligned}
&\lambda \int_{1-\Delta^{-1}}^{\infty} dE\, \frac{1}{E}(-i\lambda)^{n-1} \int_{0\le t_1\le \cdots \le t_{n-1}} d\mathbf{t} \prod_{j=1}^{n-1} P(E)V(\lambda t_j)P(E)V(0)\phi \\
&= \lambda \int_{1-\Delta^{-1}}^{\infty} dE\, \frac{1}{E}(-i)^{n-1}\mathbb{T}\left(\int_0^{\tau} d\tau_1\, P(E)V(\tau_1)\right)^{n-1} P(E)V(0)\phi\,,
\end{aligned}$$

where we have used a change of variables as in the proof of Theorem 3.3. This gives the first correction term in (13).

Next, we consider the polygon P_2 with all energies equal to 0 except for $E_m = E > 0$ with $2 \le m \le n$. Then, the contribution of P_2 to the n-th term in the Dyson series is equal to

$$(-i\lambda)^{m+1} \int_{1-\Delta^{-1}}^{\infty} dE \int_{0\le t_m\le\cdots\le t_1\le t} d\mathbf{t}_m e^{-it_{m-1}E+it_mE} \left[\prod_{j=1}^{m-1} P(0)V(\lambda t_j)\right] A(\lambda t_m)$$

with

$$A(\lambda t_m) = P(E)V(\lambda t_m)(-i\lambda)^{n-m-1} \int_{0\le t_n\le\cdots\le t_m} d\mathbf{t}_{n-m} \prod_{j=m+1}^{n} P(0)V(\lambda t_j)\phi\,.$$

Note that $A(0) = 0$. In order to simplify matters, we let V be time independent and perform the time integration. Hence, we get (to leading order)

$$\begin{aligned}
&-\lambda\frac{(-i\lambda t)^{n-1}}{(n-1)!} \int_{1-\Delta^{-1}}^{\infty} dE\, \frac{1}{E}\Big(P(0)V\Big)^{m-1} P(E)V\left(P(0)V\right)^{n-m}\phi \\
&= \lambda\frac{(-i\tau)^{n}}{(n-1)!}\Big(P(0)V\Big)^{m-1} H^{-1}(\mathbf{1}-P(0))V\left(P(0)V\right)^{n-m}\phi\,.
\end{aligned}$$

Now we can easily sum up the contributions of all $n-2$ polygons of the form P_2, and obtain the n-th term in the power series of the exponential in the second correction term in (13) up to errors $o(\lambda)$.

Finally, we prove that the other two kinds of polygons contribute of lower order. For the polygon P_3, (11) equals

$$(-i\lambda)\int_0^t dt_1\, e^{it_1E}P(E)B(\lambda t_1)\,,$$

with

$$B(\lambda t_1) = (-i\lambda)^{n-1}\int_{0\le t_n\le\cdots\le t_1} d\mathbf{t}\, V(\lambda t_1)\prod_{j=2}^{n} P(0)V(\lambda t_j)\phi\,.$$

$B(0) = 0$, and by integration by parts we get

$$(-i\lambda)\int_0^t dt_1\, e^{it_1E}P(E)B(\lambda t_1) = -\frac{\lambda}{E}e^{itE}P(E)B(\lambda t) + o(\lambda)$$

by our assumption on H.

For a sequence of the form P_4, with $E = E_1 = \ldots = E_m > E_{m+1} = \ldots = E_{n+1} = 0$. Then, the expression in formula (11) equals

$$(-i\lambda)^m \int_{0\le t_m\le\cdots\le t} d\mathbf{t}\, e^{it_mE}\prod_{j=1}^{m} P(E)V(\lambda t_j)C(\lambda t_m)\,,$$

with

$$C(\lambda t_m) = (-i\lambda)^{n-m} \int_{0\le t_n \le \cdots \le t_m} d\mathbf{t} \prod_{j=m+1}^{n} P(0)V(\lambda t_j)\phi\,.$$

Integration by parts in the first integral shows that this term is $o(\lambda)$; in fact, if we assumed V to be m times differentiable, then it would be $o(\lambda^m)$. $\square$

Notice also that the first correction term of the order λ contains the information on the whole spectrum while the leading term only depends on the highly reduced space of ground states, which we analyze in the next Section.

4. Leading dynamics

In this Section we consider the leading dynamics generated by $P(0)VP(0)$. We start with a uniform magnetic field V_Λ for which we determine an explicit expression as $\Lambda \to \infty$.

4.1. Uniform time-independent magnetic field. Let $\vec{B}(x,t) = \vec{B}$ be a constant vector, and thus $V_\Lambda = \sum_{x\in\Lambda} \vec{B}\cdot\vec{S}_x$. We shall consider $K_\Lambda = P(0)V_\Lambda P(0)$ as a Jacobi operator (i.e., tridiagonal) on $\ell^2(\mathbb{Z})$. Since H from (1) commutes with global rotations in the xy plane, we choose a new basis $|n,U\rangle = U^{-1}|n\rangle$ of $P(0)\mathcal{H}$, where U satisfies $UV_\Lambda U^{-1} = \sum_{x\in\Lambda}\sqrt{B_1^2+B_2^2}S_x^1 + B_3 S_x^3$. In other words, $U = \prod_x e^{i\theta S_x^3}$, where $(B_1,B_2) = (\sqrt{B_1^2+B_2^2}\cos\theta, \sqrt{B_1^2+B_2^2}\sin\theta)$.

When taking the infinite volume limit we have to renormalize the total S^3 component, and define

$$S^3 = \lim_{n\to\infty} \sum_{|x|\le n} S_x^3\,. \tag{14}$$

In the new basis, we have

$$\begin{aligned}
&\langle m,U|K_\Lambda|n,U\rangle \\
&\quad= \langle m|P(0)UV_\Lambda U^{-1}P(0)|n\rangle \\
&\quad= \sum_{x\in\Lambda}\sqrt{B_1^2+B_2^2}\langle m|[S_x^+ + S_x^-]|n\rangle + B_3\langle m|S_x^3|n\rangle \\
&\quad= \sqrt{B_1^2+B_2^2}\,a_\Lambda(n)\,(\delta(m,n-1)+\delta(m,n+1)) + B_3 b_\Lambda(n)\delta(n,m)\,,
\end{aligned}$$

where the index $n\in\mathbb{Z}$ denotes the eigenvalue of the renormalized total S^3 component. In Appendix A we calculate $a = \lim_{\Lambda\to\mathbb{Z}} a_\Lambda(n)$, see equation (22). Notice that the limit is independent of n by translation invariance. The function $b_\Lambda(n)$ satisfies $|b_\Lambda(n)| \le |\Lambda|$, and $\lim_{\Lambda\to\mathbb{Z}} b_\Lambda(n) = n$ for all n.

For finite Λ, K_Λ is a trace class operator, and the spectrum is pure point, accumulating to zero. We now discuss the case when $\Lambda = \mathbb{Z}$.

Let Δ be the lattice Laplacian with unit off-diagonal entries and zero diagonal terms on $\ell^2(\mathbb{Z})$, and let $W(n) = n$ be the Stark potential. Then we define the Stark-Jacobi operator,

$$K_0 = \alpha\Delta + \gamma W \tag{15}$$

with $\alpha = \sqrt{B_1^2+B_2^2}a$ and $\gamma = B_3$.

The operator K_0 is selfadjoint with dense domain $\mathcal{D}(K_0) = \{f \in \ell^2(\mathbb{Z}) : (nf(n))_n \in \ell^2(\mathbb{Z})\}$. This follows from the fact that W is selfadjoint on $\mathcal{D}(K_0)$, and that Δ is bounded and selfadjoint.

As for the continuous case, one can find the spectrum explicitly but it turns out to have rather different properties. Only in the case $\gamma = 0$ we shall have pure absolutely continuous spectrum which is equal to $[-2\alpha, 2\alpha]$. As soon as $\gamma \neq 0$, the spectrum is pure point. The latter is analyzed in the following lemma, and the generalization to higher dimensions is given in Appendix B.

LEMMA 4.1. *Let $K_0 = \alpha\Delta + \gamma W$ be the Stark-Jacobi operator (15), and $\gamma \neq 0$. Then, spec$(K_0) = \gamma\mathbb{Z}$ is pure point spectrum with simple eigenvectors $\phi_m(n) = J_{m-n}(2\alpha/\gamma)$; J_n denotes the (ordinary) Bessel function of order n.*

PROOF. By Fourier transforming the eigenvalue equation $\alpha\phi(n-1) + \alpha\phi(n+1) + \gamma n\phi(n) = \lambda\phi(n)$, we obtain

$$\left(2\alpha\cos k + i\gamma\frac{d}{dk}\right)\phi(k) = \lambda\phi(k),$$

where $k \in [-\pi, \pi]$. This is solved by $\phi(k) = e^{i(2\alpha\sin k - \lambda k)/\gamma}$. The periodicity condition $\phi(k+2\pi) = \phi(k)$ shows that $\lambda \in \gamma\mathbb{Z}$. Note, that $\sum_{m\in\mathbb{Z}}\phi_m(k) = 2\pi\delta(k)$ which proves that these (mutually orthogonal) eigenfunctions are complete. In configuration space we thus have

$$\phi_m(n) \quad = \quad \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos\left(\tfrac{2\alpha}{\gamma}\sin k + k(n-m)\right)dk = J_{n-m}\left(\tfrac{2\alpha}{\gamma}\right).$$

□

LEMMA 4.2. *The kernel for the reduced dynamics generated by $P(0)VP(0)$, and hence of K_0, has the following form:*

$$\begin{aligned}\langle x, U|e^{-itP(0)V_{\mathbb{Z}}P(0)}|n, U\rangle &= \left[e^{-iK_0t}\delta_n\right](x)\\ &= J_{n-x}\left(\tfrac{4\alpha}{\gamma}\sin\left(\tfrac{\gamma t}{2}\right)\right)\exp\left[-i\left(\frac{\gamma t-\pi}{2}x + \frac{\gamma t+\pi}{2}n\right)\right].\end{aligned} \tag{16}$$

PROOF.

$$\begin{aligned}e^{-iK_0t}\delta_n(x) &= \sum_{m\in\mathbb{Z}} e^{-i\gamma mt}\phi_m(x)\langle\phi_m|\delta_n\rangle\\ &= \sum_{m\in\mathbb{Z}} e^{-i\gamma mt}J_{x-m}(\tfrac{2\alpha}{\gamma})J_{n-m}(\tfrac{2\alpha}{\gamma})\\ &= \sum_{m\in\mathbb{Z}} e^{-i\gamma nt}e^{i\gamma mt}J_{m-n+x}(\tfrac{2\alpha}{\gamma})J_m(\tfrac{2\alpha}{\gamma})\\ &= e^{-i\gamma nt}e^{i(\pi-\gamma t)(x-n)/2}J_{n-x}\left(\tfrac{4\alpha}{\gamma}\sin\left(\tfrac{\gamma t}{2}\right)\right).\end{aligned}$$

The last step is content of Graf's addition theorem, see [**1**], formula (9.1.79). □

For future ease of notation, we define $w = \frac{4\alpha}{\gamma}\sin\left(\frac{\gamma t}{2}\right)$, and $\chi = \frac{\pi-\gamma t}{2}$.

We now inspect closer the reduced time evolution of the magnetization profile of a kink ground state localized at 0. Defining

$$m(\hat{\Omega}, x, t) = \langle 0, U|e^{itP(0)VP(0)}(\hat{\Omega}\cdot\vec{S}_x)e^{-itP(0)VP(0)}|0, U\rangle$$

for any vector $\hat{\Omega} \in \mathbb{S}^2$, we calculate

$$\begin{aligned} & m(\hat{\Omega}, x, t) \\ = & \sum_{m,m' \in \mathbb{Z}} \langle 0, U | e^{itP(0)VP(0)} | m, U \rangle \langle m, U | \hat{\Omega} \cdot \vec{S}_x^3 | m', U \rangle \langle m', U | e^{-itP(0)VP(0)} | 0, U \rangle \\ = & \sum_{m,m' \in \mathbb{Z}} e^{i\chi(m'-m)} J_m(w) J_{m'}(w) \langle m | R_{-\theta}(\hat{\Omega}) \cdot \vec{S}_x | m' \rangle , \end{aligned}$$

where $R_{-\theta}$ is rotation about the z axis by an angle $-\theta$. (Note this is the opposite rotation as that used to go from $P(0)VP(0)$ to K_0.) Taking $\hat{\Omega} = \hat{e}_3$, and defining $m^3(x,t) = m(\hat{e}_3, x, t)$, we have

$$\begin{aligned} m^3(x,t) &= \sum_{m,m' \in \mathbb{Z}} e^{i\chi(m'-m)} J_m(w) J_{m'}(w) \langle m | S_x^3 | m' \rangle \\ &= \sum_{m \in \mathbb{Z}} J_m^2(\tfrac{4\alpha}{\gamma} \sin(\tfrac{\gamma t}{2}))\, \langle m | S_x^3 | m \rangle . \end{aligned}$$

Similarly, defining $m^1(x,t) = m(\hat{e}_1, x, t)$, and observing that $R_{-\theta}(\hat{e}_1) = a\hat{e}_1 - b\hat{e}_2$, where $a = \cos\theta = B_1/\sqrt{B_1^2 + B_2^2}$ and $b = \sin\theta = B_2/\sqrt{B_1^2 + B_2^2}$, we have

$$\begin{aligned} m^1(x,t) &= \sum_{m,m' \in \mathbb{Z}} e^{i\chi(m'-m)} J_m(w) J_{m'}(w) \langle m | [a S_x^1 - b S_x^2] | m' \rangle \\ &= \sum_{m,m' \in \mathbb{Z}} e^{i\chi(m'-m)} J_m(w) J_{m'}(w) \langle m | \left[\frac{a+ib}{2} S_x^+ + \frac{a-ib}{2} S_x^- \right] | m' \rangle . \end{aligned}$$

We now observe a phase condition on $|m\rangle$ that guarantees $\langle m | S^- | m' \rangle$ is real. This means that

$$\begin{aligned} & m^1(x,t) \\ = & \sum_{m,m' \in \mathbb{Z}} e^{i\chi(m'-m)} J_m(w) J_{m'}(w) \left(\frac{a-ib}{2} \langle m | S_x^- | m' \rangle + \frac{a+ib}{2} \langle m' | S_x^- | m \rangle \right) \\ = & \left[\frac{e^{i\chi}(a-ib)}{2} + \frac{e^{-i\chi}(a+ib)}{2} \right] \sum_{m \in \mathbb{Z}} J_m(w) J_{m+1}(w) \langle m | S_x^- | m+1 \rangle \\ = & [a\cos(\chi) + b\sin(\chi)] \sum_{m \in \mathbb{Z}} J_m(w) J_{m+1}(w) \langle m | S_x^- | m+1 \rangle \\ = & [a\sin(\tfrac{\gamma t}{2}) + b\cos(\tfrac{\gamma t}{2})] \sum_{m \in \mathbb{Z}} J_m(\tfrac{4\alpha}{\gamma}\sin(\tfrac{\gamma t}{2})) J_{m+1}(\tfrac{4\alpha}{\gamma}\sin(\tfrac{\gamma t}{2})) \langle m | S_x^- | m+1 \rangle . \end{aligned}$$

Notice that for $\gamma \neq 0$ the motion is periodic whose period $2\pi\gamma^{-1}$ is solely determined by the field in the z direction. The profile $m^3(x,t)$ is given by an absolutely convergent series. Therefore, pointwise convergence implies convergence on compacts. As the function is periodic in t and thus the image is a compact interval, the profile stays exponentially localized for all times.

However, as $\gamma \to 0$

$$m^3_{\gamma=0}(x,t) = \sum_{m \in \mathbb{Z}} J_m^2(2\alpha t)\, \langle m | S_x^3 | m \rangle , \tag{17}$$

$$m^1_{\gamma=0}(x,t) = b \sum_{m \in \mathbb{Z}} J_m(2\alpha t) J_{m+1}(2\alpha t)\, \langle m | S_x^+ | m-1 \rangle . \tag{18}$$

Next, we analyze $m^3_{\gamma=0}(x,t)$ in more detail. A very similar situation has been studied by Antal et al [**3**] in the context of the time evolution of the XX model. We follow their analysis. Although we have just derived explicit formulas for the profiles, which can be analyzed numerically, we prefer to have simpler expressions. As we shall show, for large x and t, $m^3(x,t)$ will be a function of the velocity $v = \frac{x}{t}$, only. To this end we define the discrete derivative of $m^3(x,t)$ as

$$\begin{aligned}\phi'_x(v) &:= t\Big[m^3_{\gamma=0}(x+1,t) - m^3_{\gamma=0}(x,t)\Big]_{\frac{x}{t}=v}\\ &= \tfrac{x}{v}\sum_{m\in\mathbb{Z}} J^2_m(\tfrac{x\alpha}{v})\,\langle 0|S^3_{x-m-1} - S^3_{x-m}|0\rangle\,,\end{aligned}$$

and study the limit $\lim_{x\to\infty}\phi'_x(v)$. Note that $\langle 0|S^3_{x-m-1} - S^3_{x-m}|0\rangle \geq 0$ for all x and m because the profile of the kinks is a decreasing function. If we define $p(m) = \langle 0|S^3_{m-1} - S^3_m|0\rangle$, then since it is a telescoping sum

$$\sum_{m=-\infty}^{\infty} p(m) = \lim_{m\to\infty}\langle 0|S^3_m|0\rangle - \lim_{m\to-\infty}\langle 0|S^3_m|0\rangle = 1\,,$$

i.e., $p(m)$ is a probability distribution. All moments of p are finite and the first moment of p is $\frac{1}{2}$. In fact $p(m)\sim e^{-c|m|}$ for some positive c depending on $\Delta > 1$ (see Lemma A.1.1). Thus,

$$\phi'_x(v) = -\sum_{m\in\mathbb{Z}} \tfrac{x}{v} J^2_{m+x}(\tfrac{x\alpha}{v})p(m)\,.$$

We used the fact that $J_{-n}(x)^2 = J_n(-x)^2 = J_n(x)^2$. One may easily bound $\frac{x}{v}J^2_{x-m}(\frac{x\alpha}{v})$ uniformly in x and m, which our asymptotics will show (see below). Hence, we may interchange the limits $x\to\infty$ and the sum, to calculate

$$\lim_{x\to\infty}\phi'_x(v) = \sum_{m\in\mathbb{Z}} p(m)\lim_{x\to\infty}\tfrac{x}{v}J^2_{m+x}(\tfrac{x\alpha}{v})\,.$$

We distinguish between the two cases $v > 2\alpha$, and $v < 2\alpha$. The asymptotic behavior will be very different in these two cases. Let us start with $v > 2\alpha$. Then by (9.3.2) in [**1**], asymptotically for large x

$$\begin{aligned}2\tfrac{x\alpha}{v}J^2_{m+x}(\tfrac{2x\alpha}{v}) &= \tfrac{x}{m+x}\,\tfrac{2(m+x)\alpha}{x}J^2_{m+x}\left((m+x)\tfrac{2x\alpha}{(m+x)v}\right)\\ &\sim \frac{1}{2\pi\sqrt{\frac{v^2}{4\alpha^2}-1}}e^{-\frac{2x\alpha}{v}(1-\sqrt{\frac{v^2}{4\alpha^2}-1})}\,.\end{aligned}$$

Since $p(m)$ is a probability measure, which decays exponentially, the bound above shows that $\phi'_x(v)$ is exponentially small, and $\lim_{x\to\infty}\phi'_x(v) = 0$, which implies that $\phi(v) := \lim_{x\to\infty}\phi_x(v) = -\frac{1}{2}$. This behavior is not surprising for it just means a finite speed of propagation, which has to be less than 2α.

However for $0\leq v < 2\alpha$, the behavior of the Bessel functions changes, cf (9.3.3), [**1**]:

$$\tfrac{2x\alpha}{v}J^2_{m+x}(\tfrac{2x\alpha}{v}) \sim \frac{2}{\pi\sqrt{1-\frac{v^2}{4\alpha^2}}}\cos^2\left(x\sqrt{\tfrac{\alpha^2}{4v^2}-1} - x\arccos(\tfrac{v}{2\alpha}) - \tfrac{\pi}{4}\right)\,.$$

This implies that for large x

$$\phi'_x(v) \quad \sim \quad \frac{1}{\alpha\pi\sqrt{1-\frac{v^2}{4\alpha^2}}}\cos^2\left(x\sqrt{\tfrac{4\alpha^2}{v^2}-1}-x\arccos\left(\tfrac{v}{2\alpha}\right)-\tfrac{\pi}{4}\right).$$

For $v<2\alpha$ we recover the function $\phi(v)$ by integration, (we switch the limit and integral by dominated convergence, since $\phi'_x(v)$ is uniformly bounded in x and v for $v\in[0,\frac{2}{\alpha}-\epsilon]$), i.e.,

$$\phi(v) \quad := \quad \lim_{x\to\infty}\phi_x(v)=\int_0^v dy\ \lim_{x\to\infty}\phi'_x(y)=-\frac{2}{\pi}\arcsin\left(\tfrac{v}{2\alpha}\right).$$

The oscillating cosine contributes an average factor $\frac{1}{2}$ to the integral. Using symmetry, and collecting terms, we have shown that

$$\lim_{t\to\infty:x=vt} m^3_{\gamma=0}(x,t)=\phi(v)=\begin{cases} +\frac{1}{2} & \text{for} \quad v<-2\alpha \\ -\frac{2}{\pi}\arcsin\left(\frac{v}{2\alpha}\right) & \text{for} \quad -2\alpha<v<2\alpha \\ -\frac{1}{2} & \text{for} \quad v>2\alpha \end{cases}. \tag{19}$$

The importance behind a further analysis of the profile perpendicular to the z direction is that it allows us to decide whether the state is rotated by the perturbation into the xy plane or whether there is ballistic diffusion of the interface. Comparing equations (17) and (18), we see that the main difference is a replacement of the probability measure $p(m)=\langle 0|S^3_{m-1}-S^3_m|0\rangle$ by the signed measure $\tilde{p}(m)=\langle 1|S^+_m-S^+_{m-1}|0\rangle$. One can easily determine that $|\tilde{p}(m)|<C_1e^{-c|m|}$ (see Lemma A.1.2). However, now $\tilde{p}(m)$ oscillates, and again because of a telescoping sum,

$$\sum_{m=-\infty}^{\infty}\tilde{p}(m)=\lim_{m\to\infty}\langle 1|S^+_m|0\rangle-\lim_{m\to-\infty}\langle 1|S^+_m|0\rangle=0-0=0\,.$$

Since we have determined that for any fixed finite m, $\lim_{x\to\infty}\frac{x}{v}J^2_{m+x}(\frac{2x\alpha}{v})$ is a fixed function of $v/2\alpha$ independent of m, and in particular this implies the same limit for $\lim_{x\to\infty}\frac{x}{v}J_{m+x}(\frac{2x\alpha}{v})J_{m+x+1}(\frac{2x\alpha}{v})$, the oscillation of $\tilde{p}$ implies that

$$\begin{aligned}\lim_{x\to\infty,x=vt}\psi'_x(v) \quad &:= \quad t\Big[m^1_{\gamma=0}(x+1,t)-m^1_{\gamma=0}(x,t)\Big]_{\frac{x}{t}=v}\\ &= \quad \lim_{x\to\infty} b\sum_{m\in\mathbb{Z}}\tilde{p}(m)\lim_{x\to\infty}\tfrac{x}{v}J_{m+x}(\tfrac{2x\alpha}{v})J_{m+x+1}(\tfrac{2x\alpha}{v})\\ &= \quad 0\,.\end{aligned}$$

This does not say that the profile in the xy plane is vanishing. Quite on the contrary, it means that it stays exponentially concentrated around the kink at 0, and does not follow the ballistic motion of the third component.

Let us shortly comment on time dependent uniform fields with time dependent Stark-Jacobi operator $K(t)=\alpha(t)\Delta+\gamma(t)W$. It is easy to see that if $B_3(t)=0$ then we just have to replace αt by $\int_0^t\alpha(s)\,ds$. E.g.,

$$m^3(x,t)=\sum_{m\in\mathbb{Z}}J^2_m(2\textstyle\int_0^t\alpha(s)\,ds)\,\langle m|S^3_x|m\rangle\,.$$

We have not been able to derive explicit expressions in the general case, but one can show, for instance, by repeated use of Graf's addition theorem, that there is a function F such that

$$m^3(x,t) = \sum_{m\in\mathbb{Z}} J_m^2(F(\alpha(t),\gamma(t))\,\langle m|S_x^3|m\rangle\,.$$

In these time dependent cases, a scaling limit as before does not exist.

4.2. General infinitely extended fields. Since a thorough discussion would be out of the scope of this paper, we only want to convince the reader that the leading dynamics of the "many-body" Hamiltonian leads to a well studied one-body problem which has seen great progress in recent years.

Let us start with the simplest case when the magnetic field $\vec{B}(n)$ is asymptotically uniform with non-zero third component. I.e., we assume that there is a vector $\vec{B}$ and some $p > 1$ such that $(B_3(n) - B_3)n \in \ell^p(\mathbb{Z})$, and $(B_{1,2}(n) - B_{1,2}) \in \ell^p(\mathbb{Z})$. Let K denote the Jacobi operator corresponding to this vector field. Then K is a compact perturbation of the Stark-Jacobi operator K_0 in (15). By Weyl's Theorem, K has pure point spectrum and the motion is quasi-periodic and the interface stays localized at 0.

The case of a perturbation of a uniform field with $B_3 = 0$ is much more complicated. If we perturb by a field in the z direction, which corresponds to adding a potential to the (discrete) Laplace operator, then we are in the widely studied case of a (discrete) Schrödinger operator.

The effect of a perturbation of $\vec{B}$ in the xy plane, however, is non-local. I.e., if we change B_1 at 0 into $B_1(0)$, then the off-diagonal matrix element computed from

$$\langle n|\sum_{x\in\mathbb{Z}} B_1(x)S_x^+|n-1\rangle = B_1\langle n|\sum_{x\in\mathbb{Z}} S_x^+|n-1\rangle + (B_1(0) - B_1)\langle n|S_0^+|n-1\rangle$$

is effected for all n, although exponentially localized at the impurity at 0. By a rotation, one can map this Jacobi operator into a Schrödinger operator. Any non-uniform perturbation of the uniform case by a bounded perturbation will modify the spectrum of the reduced Jacobi operator, as was recently proven by Killip and Simon [**11**].

There are many more interesting fields for which we would like to understand the time evolution of interfaces. A particularly simple case is a sharply localized field at a single site y, i.e., $V = \vec{B}\cdot\vec{S}_y$. Here, one should rather approximate the time evolution by the projection of $H + \vec{B}\cdot\vec{S}_y$ onto its low energy spectrum which seems to be separated by a gap from the rest of the spectrum, and which is similar to the spectrum of the unperturbed kink Hamiltonian. So far only the ground state and the gap above the ground state of this Hamiltonian have been studied by Contucci, Nachtergaele and Spitzer [**9**]. We plan to pursue this in the future.

Appendix A. Magnetization profiles

The exact formula for the spin-$\frac{1}{2}$ magnetization profile of kink states in the z direction was derived in [**20**], pp 55–58. Note that by rotational symmetry, $\langle n|S_x^1|n\rangle = \langle n|S_x^2|n\rangle = 0$. We introduce the function

$$f(z) = \sum_{k\geq 0}(-1)^k z^k q^{k(k-1)}. \tag{20}$$

LEMMA A.1. *For the spin-$\frac{1}{2}$ infinite chain kink Hamiltonian we have the following profiles of the normalized kink state, $|0\rangle$ (which has total z component 0):*

(1) *If $x > 0$, then*

$$\langle 0|S_x^3|0\rangle = -\frac{1}{2} + q^{2x}\sum_{k=0}^{\infty}(-1)^k q^{k(k+2x+1)}\,, \tag{21}$$

while for $x \leq 0$ we have $\langle 0|S_x^3|0\rangle = -\langle 0|S_{1-x}^3|0\rangle$. Further, $p(m) = \langle 0|S_{m-1}^3 - S_m^3|0\rangle \leq e^{-c|m|}$ with c depending on q.

(2) *$\langle n|S_0^-|n-1\rangle = q^{|n|}f(q^{2|n|+2})$, and $\tilde{p}(m) = \langle 1|S_m^+ - S_{m-1}^+|0\rangle$ satisfies $|\tilde{p}(m)| < C_1 e^{-c|m|}$ with c depending on q.*

(3) *The number a in the Stark-Jacobi operator (15) is equal to*

$$a = \langle n|\sum_{x\in\mathbb{Z}} S_x^1|n-1\rangle = \frac{1}{2}\sum_{k\geq 0}(-1)^k q^{k(k+1)}\frac{1+q^{1+2k}}{1-q^{1+2k}}\,. \tag{22}$$

Proof. Here, we only show the key steps for Part 2 and 3; the details can be found in [**20**]. The exponential localization of the measures $p(m)$ and $\tilde{p}(m)$ can be deduced from the explicit formulas but it can also be done much easier by using the following estimate

$$\langle\psi_0|[\tfrac{1}{2} + \operatorname{sgn}(x)S_x^3]|\psi_0\rangle \leq Cq^{|x|} \tag{23}$$

for some $C = C(q)$ independent of x. Let $\Omega^{+-} = \bigotimes_{x\leq 0}\binom{1}{0}\otimes\bigotimes_{x>0}\binom{0}{1}$. We define the so-called grand-canonical states

$$\psi(z) = \prod_{x=-\infty}^{0}(1 + z^{-1}q^{-x}S_x^-)\prod_{x=1}^{\infty}(1 + zq^x S_x^+)\Omega^{+-}. \tag{24}$$

The reason for this definition is that

$$\psi(z) = \sum_{n\in\mathbb{Z}}\psi_n z^n. \tag{25}$$

It is clear that in the grand canonical ground state, for real $z = q^{-\mu}$,

$$\frac{\langle\psi(z)|S_x^3|\psi(z)\rangle}{\langle\psi(z)|\psi(z)\rangle} = \frac{1}{2}\cdot\frac{q^{(x-\mu)/2} - q^{(\mu-x)/2}}{\sqrt{q^{x-\mu}+q^{\mu-x}}}\,,$$

which proves exponential localization of the interface in the grand canonical ground state. E.g., setting $\mu = 0$, one has for $x > 0$,

$$\begin{aligned} -\frac{1}{2} \leq \frac{1}{2}\cdot\frac{q^{x/2}-q^{-x/2}}{\sqrt{q^x+q^{-x}}} &= -\frac{1}{2} + \frac{1}{2}\cdot\frac{\sqrt{1+q^{2x}} - 1 + q^x}{\sqrt{1+q^{2x}}} \\ &\leq -\frac{1}{2} + \frac{1}{2}[1 + \frac{1}{2}q^{2x} - 1 + q^x] \\ &\leq -\frac{1}{2} + \frac{3}{4}q^x\,, \end{aligned} \tag{26}$$

and a similar inequality holds for $x < 0$. Setting $\mu = 0$, (i.e. $z = 1$) we see that

$$\langle\psi(1)|(\tfrac{1}{2}\pm S_x^3)|\psi(1)\rangle = \sum_{n\in\mathbb{Z}}\langle\psi_n|(\tfrac{1}{2}\pm S_x^3)|\psi_n\rangle \geq \langle\psi_0|(\tfrac{1}{2}\pm S_x^3)|\psi_0\rangle\,,$$

using the fact that $(\frac{1}{2}\pm S_x^3)$ is a nonnegative operator. Hence,

$$\frac{\langle\psi_0|[\frac{1}{2}\pm S_x^3]|\psi_0\rangle}{\langle\psi_0|\psi_0\rangle} \leq \frac{\|\psi(z)\|^2}{\|\psi_0\|^2}\cdot\frac{\langle\psi(1)|[\frac{1}{2}\pm S_x^3]|\psi(1)\rangle}{\langle\psi(1)|\psi(1)\rangle}\,.$$

Since $(\|\psi(z)\|^2/\|\psi_0\|^2) < \infty$, and using equation (26) we derive equation (23). This proves Part 1.

Next, using the q-binomial product theorem we get that

$$\begin{aligned}\langle\psi(z)|S_0^+|\psi(z)\rangle &= z^{-1}\prod_{x=-\infty}^{-1}(1+z^{-2}q^{-2x})\prod_{x=1}^{\infty}(1+z^2q^{2x})\\ &= z^{-1}\left(\sum_{n=0}^{\infty}|z|^{-2n}\frac{q^{n(n+1)}}{(q^2;q^2)_n}\right)\left(\sum_{n=0}^{\infty}|z|^{2n}\frac{q^{n(n+1)}}{(q^2;q^2)_n}\right)\\ &= z^{-1}\sum_{n=-\infty}^{\infty}|z|^{2n}\frac{q^{|n|(|n|+1)}f(q^{2(|n|+1)})}{(q^2;q^2)_\infty}\,.\end{aligned}$$

Comparing powers of z in (25) we have

$$\begin{aligned}\langle n|S_0^+|n-1\rangle &= \frac{\langle\psi_n|S_0^+|\psi_{n-1}\rangle}{\|\psi_{n-1}\|\|\psi_n\|}\\ &= q^{|n|}f(q^{2|n|+2})\,.\end{aligned}$$

We have used that $\|\psi_n\|^2 = q^{n(n+1)}/(q^2;q^2)_\infty$. Finally,

$$\begin{aligned}\sum_{x\in\mathbb{Z}}\langle n|S_x^+|n-1\rangle &= \sum_{x\in\mathbb{Z}}\langle x|S_0^+|x-1\rangle\\ &= \sum_{x\in\mathbb{Z}}q^{|x|}\sum_{k\geq 0}(-1)^k q^{2(|x|+1)k+k(k-1)}\\ &= \sum_{k\geq 0}(-1)^k q^{k(k+1)}\frac{1+q^{1+2k}}{1-q^{1+2k}}\,.\end{aligned}$$

□

Appendix B. Stark-Jacobi Operator on $\mathbb{Z}^d$

LEMMA B.1. *Let $Kf(\vec{n}) = (\alpha\Delta + \vec{\gamma}\cdot\vec{n})f(\vec{n})$ be selfadjointly defined on $\mathcal{D}(K) = \{f \in \ell(\mathbb{Z}^d) : (\vec{\gamma}\cdot\vec{n})f(\vec{n}) \in \ell^2(\mathbb{Z}^d)\}$. If all γ_j are non-zero and there is some γ_0 such that $\gamma_j = a_j\gamma_0$ with $a_j \in \mathbb{Z}$ for all $j = 1,\ldots,d$, i.e. if $(\gamma_1,\ldots,\gamma_d)$ are commensurable, then*

$$spec(K) = \frac{\|\gamma\|^2}{\mathrm{lcm}(a_1,\ldots,a_d)\gamma_0}\mathbb{Z}\,,$$

where $\mathrm{lcm}(a_1,\ldots,a_d)$ is the least common multiple. If all g_j are nonzero, but they are not commensurable, i.e., there exists a pair γ_j, γ_k such that γ_j/γ_k is irrational, then $spec(K) = \mathbb{R}$ and the spectrum is dense pure point spectrum. In this case there is an eigenvalue at every point of the lattice

$$\sum_{j=1}^{d}\frac{\|\gamma\|^2}{\gamma_j}\mathbb{Z} = \{\sum_{j=1}^{d}a_j\|\gamma\|^2/\gamma_j : a_1,\ldots,a_d\in\mathbb{Z}\}\,.$$

However, if there are $1 \leq l < d$ non-zero components of $\vec{\gamma}$, say the first l, then

$$spec(K) = \mathrm{cl}\Big([-2\alpha(d-l), 2\alpha(d-l)] + \sum_{j=1}^{l}(\frac{\|\gamma\|^2}{\gamma_j}\mathbb{Z})\Big)\,,$$

where cl(.) *is norm closure in* $\mathbb{R}$, *and there is essential spectrum even when all* γ_j *are commensurable.*

PROOF. Let us assume first that all components are non-zero. In Fourier space we have the eigenvalue equation

$$\left(2\alpha \sum_{j=1}^{d} \cos(k_j) + i\|\gamma\| \sum_{j=1}^{d} \frac{\gamma_j}{\|\gamma\|} \frac{\partial}{\partial k_j} - \lambda\right) \phi(k) = 0\,.$$

Let $R = R(\gamma)$ be an orthogonal matrix such that $\sum_{j=1}^{d} R_{ij} \frac{\gamma_j}{\|\gamma\|} = \delta(i,1)$. (In particular this means $R_{1j} = \frac{\gamma_j}{\|\gamma\|}$.) Setting $\tilde{k} = Rk$, we get

$$\frac{\partial}{\partial \tilde{k}_1} \phi(\tilde{k}) = -\frac{i}{\|\gamma\|} \left(\lambda - 2\alpha \sum_{j=1}^{d} \cos\Big(\sum_{m=1}^{d} R_{mj} \tilde{k}_m\Big)\right) \phi(\tilde{k})\,.$$

The solution is thus

$$\phi(\tilde{k}) = C e^{-\frac{i}{\|\gamma\|}\left(\lambda \tilde{k}_1 - 2\alpha \sum_{j=1}^{d} (R_{1j})^{-1} \sin\left(\sum_{m=1}^{d} R_{mj} \tilde{k}_m\right)\right)}\,.$$

Using that $R_{1j} = \frac{\gamma_j}{\|\gamma\|}$ we have

$$\phi(k) = C e^{-\frac{i}{\|\gamma\|}\left(\lambda \sum_{j=1}^{d} \frac{\gamma_j}{\|\gamma\|} k_j - 2\alpha \sum_{j=1}^{d} \frac{\|\gamma\|}{\gamma_j} \sin(k_j)\right)}\,. \tag{27}$$

Finally, the periodicity condition, $\phi(k_j + 2\pi) = \phi(k_j), j = 1, \ldots, d$ requires that

$$\lambda \in \frac{\|\gamma\|^2}{\gamma_j} \mathbb{Z}\,.$$

Now suppose that the first l components of $\vec{\gamma}$ are non-zero and the rest are zero. According to the decomposition $\ell^2(\mathbb{Z}^d) = \ell^2(\mathbb{Z}^l) \bigoplus \ell^2(\mathbb{Z}^{d-l})$, we write K as the sum of Stark-Jacobi operator $K^{(l)}$ acting on $\ell^2(\mathbb{Z}^l)$ and $\alpha\Delta^{(l)}$ acting on $\ell^2(\mathbb{Z}^{d-l})$. Hence the result follows. □

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Department of Mathematics, University of California, Davis, One Shields Avenue, Davis 95616-8366, USA

E-mail address: `bxn@math.ucdavis.edu`

Department of Mathematics, University of California, Davis, One Shields Avenue, Davis 95616-8366, USA

E-mail address: `spitzer@math.ucdavis.edu`

Department of Physics, Princeton University, Princeton, NJ 08544, USA

E-mail address: `sstarr@math.princeton.edu`