Construction of periodic solutions of the nonlinear wave equation under strong irrationality conditions by the Lindstedt series method

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Abstract. *We prove the existence of oscillatory solutions of the nonlinear wave equation, under irrationality conditions stronger than the usual Diophantine one, by perturbative techniques inspired by the Lindstedt series method originally introduced in classical mechanics to study the existence of invariant tori in quasi-integrable Hamiltonian systems.*

Keywords: Nonlinear wave equation; Periodic solutions; Lindstedt series method; Tree formalism; Perturbation theory

1. Statement of the result

1.1. *Introduction.* We consider the nonlinear wave equation in dimension $d = 1$ given by

$$\begin{cases} u_{tt} - u_{xx} + \mu u = f(u), \\ u(0,t) = u(\pi,t) = 0, \end{cases} \tag{1.1}$$

where μ is a positive real parameter (*mass*) and $f(u)$ is an odd analytic function (*perturbation*) of order at least 3:

$$f(u) = \sum_{\nu=1}^{\infty} f_{2\nu+1} u^{2\nu+1}, \qquad |f_\nu| \le F\, e^{\xi\nu}, \tag{1.2}$$

for some real constants F and ξ. Let us call

$$M = \inf\{\nu \ge 1 \ : \ f_{2\nu+1} \ne 0\}. \tag{1.3}$$

Dirichlet boundary conditions allow us to use as a basis in $L^2([0,\pi])$ the set of functions $\{e_n\} = \{\sin nx, n \in \mathbb{N}\}$.

If $f = 0$ the generic solution of (1.1) has the form

$$u(x,t) = \sum_{n=1}^{\infty} [A_n \cos\omega_n t + B_n \sin\omega_n t] \sin nx, \tag{1.4}$$

where $\omega_n = \sqrt{n^2 + \mu}$. If only the first mode is present one has (for a suitable choice of the initial time)

$$u(x,t) = \varepsilon \cos\omega_1 t \sin x, \tag{1.5}$$

which is periodic in t with period $T = 2\pi/\omega_1$; the variable ε will play the rôle of the *perturbative parameter.*

Are there still oscillatory solutions when the nonlinear part is present, i.e. when $f \ne 0$? To answer to such a question is in general quite difficult. There are many deep results in the literature, which use KAM theory or Nash-Moser's implicit function theorem, see for instance [2], [3] or [4]; a major source of technical problems is due to the presence of small divisors in a system with an infinite number of degrees of freedom. It was however noted in [1] that in some special case there is no small divisors problem; in particular it was shown by a very simple analysis the existence of $(2\pi/\omega_\varepsilon)$-periodic solutions of the equation (1.1) of the form

$$u(x,t;\varepsilon) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \tilde{u}_{n,m}(\varepsilon)\cos(n\omega_\varepsilon t)\sin(mx). \tag{1.6}$$

with ω_ε close to ω_1 and satisfying a quite special Diophantine condition (see (2.10) below). Of course this follows also from the more general results of [3] and [2]; the interest of [1] is however that under proper assumptions the result is quite simple to prove.

As the problem of the existence of oscillatory solutions of nonlinear wave equations is naturally related to KAM theory, it is tempting to try to study it by the *Lindstedt series* method. The invariant tori of finite dimensional quasi-integrable Hamiltonian systems can be parameterized as power series in the perturbation parameter (Lindstedt series) which can be proven to be convergent (see for instance [5], [6], [7] and many other papers). The Lindstedt series can be written as a "sum over trees" and convergence can be proved by exploiting suitable cancellations between the numerical values associated to the trees. It is naural to apply a similar method to the search of periodic or quasi-periodic solutions of partial differential equations. As a first step we start by studying the case considered in [1], and we hope to prove similar results under the more general conditions in [3] and [2].

To find a power series expansion in ε seems at first sight hopeless, as the solution of (1.6) is not expected to be analytic in ε; however under a quite special Diophantine condition on ω_ε considered in [1], see (2.10) below, considering ω_ε as a parameter, we shall be able to write indeed (1.6) as a power series expansion.

We shall prove the following result (see [1]).

1.2. Theorem. *For μ small enough there exist a set $\mathcal{E} \subset \mathbb{R}$ having zero as an accumulation point and, for for any $\varepsilon \in \mathcal{E}$, a number ω_ε close to ω_1 within $O(\varepsilon^{M-1})$ such that there is a periodic solution of (1.1) of the form (1.6) which is analytic in x, t.*

1.3. *Remarks.* (1) The solutions in [1], in a more abstract formulation and with some weaker assumptions, are solutions in a *weak* sense and belong to $H^1([0, 2\pi/\omega_\varepsilon], \ell^2_s)$. Instead, we are interested in proving the existence of smooth solutions: this accounts for the parity assumptions on f. (2) Along the proof we shall see that the solution $u(x, t; \varepsilon)$ will be obtained through two steps: first, by constructing a suitable function $\tilde{u}(x, t; \omega, \varepsilon)$ which will be analytic in the arguments x, t, ε, while defined only on a non-numerable set Ω of values ω; then by fixing, for all $\omega \in \Omega$ close enough to ω_1, a value $\varepsilon = \varepsilon(\omega)$ such that $\tilde{u}(x, t; \omega, \varepsilon(\omega))$ is a solution of (1.1). As the correspondence between ε and ω will be shown to be one-to-one (for $\omega \in \Omega$ and $\varepsilon \in \mathcal{E}$ for a suitable subset $\mathcal{E}$ of the real line having zero as an accumulation point), we can write $u(x, t; \varepsilon) = \tilde{u}(x, t; \omega_\varepsilon, \varepsilon)$, for ω_ε such that $\varepsilon(\omega_\varepsilon) = \varepsilon$.

2. Proof of the theorem

2.1. *Perturbative expansion.* Let us replace ω_ε in (1.6) with ω, i.e with a quantity that we shall consider as independent of ε: this means that we shall deal with a function

$$u(x, t; \omega, \varepsilon) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \tilde{u}_{n,m}(\varepsilon, \omega) \cos(n\omega t) \sin(mx), \tag{2.1}$$

where ω and ε are seen as independent parameters. In the following we shall let drop the arguments ω, ε in the coefficients $\tilde{u}_{m,n}$.

Inserting (2.1) into (1.1) gives

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \tilde{u}_{n,m} \cos n\omega t \sin(mx) \left[-n^2\omega^2 + \omega_m^2 \right] = f(u), \tag{2.2}$$

which can be also written

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \hat{u}_{n,m}\, e^{in\omega t} e^{imx} \left[-n^2\omega^2 + \omega_m^2 \right] = f(u), \tag{2.3}$$

where the Fourier coefficients $\hat{u}_{n,m}$ are trivially related to the coefficients $\tilde{u}_{n,m}$. By writing

$$f(u)=\sum_{\nu=1}^{\infty} f_{2\nu+1} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{\substack{n_1+\ldots+n_{2\nu+1}=n \\ m_1+\ldots+m_{2\nu+1}=m}} \hat{u}_{n_1,m_1} \ldots \hat{u}_{n_{2\nu+1},m_{2\nu+1}} e^{in\omega t} e^{imx}, \tag{2.4}$$

one obtains for the Fourier coefficients the equation

$$\hat{u}_{n,m}\left[-n^2\omega^2+\omega_m^2\right]=\sum_{\nu=1}^{\infty} f_{2\nu+1} \sum_{\substack{n_1+\ldots+n_{2\nu+1}=n \\ m_1+\ldots+m_{2\nu+1}=m}} \hat{u}_{n_1,m_1} \ldots \hat{u}_{n_{2\nu+1},m_{2\nu+1}}. \tag{2.5}$$

We shall set $\hat{u}_{\pm 1,1}=-\hat{u}_{\pm 1,-1}=\varepsilon/4i$, and we shall write, for $(|n|,|m|)\neq(1,1)$,

$$\hat{u}_{n,m}=\sum_{k=1}^{\infty} \hat{u}_{n,m}^{(k)} \varepsilon^k; \tag{2.6}$$

which can be extended to the case $(|n|,|m|)=(1,1)$ by setting $\hat{u}_{\pm 1,\pm 1}^{(1)}=\pm 1/4i$ and $\hat{u}_{\pm 1,\pm 1}^{(k)}=0$ $\forall k\geq 2$.

By inserting (2.5) into (2.3) we obtain for $k\geq 2$ the recursive equations

$$\hat{u}_{n,m}^{(k)}=\frac{1}{-n^2\omega^2+\omega_m^2}\sum_{\nu=1}^{\infty} f_{2\nu+1} \sum_{k_1+k_2+\ldots k_{2\nu+1}=k} \sum_{\substack{n_1+n_2+\ldots n_{2\nu+1}=n \\ m_1+m_2+\ldots m_{2\nu+1}=m}} \prod_{j=1}^{2\nu+1} \hat{u}_{n_j,m_j}^{(k_j)}; \tag{2.7}$$

then it is straightforward to prove the parity and conjugacy properties

$$\hat{u}_{n,m}^{(k)}=-\hat{u}_{-n,m}^{(k)}=-\hat{u}_{-n,-m}^{(k)}, \qquad \hat{u}_{n,m}^{(k)*}=-\hat{u}_{n,m}^{(k)}, \tag{2.8}$$

by induction on k.

For $(n,m)=(1,1)$ we obtain from (2.5)

$$\left(-\omega^2+\omega_1^2\right)\varepsilon=4i\sum_{\nu=1}^{\infty} f_{2\nu+1} \sum_{\substack{n_1+n_2+\ldots n_{2\nu+1}=1 \\ m_1+m_2+\ldots m_{2\nu+1}=1}} \hat{u}_{n_1,m_1} \ldots \hat{u}_{n_{2\nu+1},m_{2\nu+1}}, \tag{2.9}$$

which will be regarded as an equation for fixing $\omega=\omega_\varepsilon$ as a suitable function of ε.

First we solve (2.7). We shall consider ω as a parameter (to be fixed later) verifying the following Diophantine condition:

$$|-n\omega+\omega_m|\geq\frac{\gamma}{n} \qquad \text{for all } n\geq 1, \quad m\geq 2, \tag{2.10}$$

for some positive (ε-independent) constant γ. Then (2.7) is formally soluble only if f is odd in u: otherwise the condition $\hat{u}_{n,m}=-\hat{u}_{n,-m}$ in (2.8) can not be verified and no C^∞ function can be smoothly expanded in the basis $\{e_n\}$.

Afterwards we shall pass to (2.9), and we shall see that it will be possible to fix ε, and ω_ε satisfying (2.10), in such a way that (2.9) is satisfied.

2.2. *Tree formalism.* Recall that a *(rooted) tree* is a connected set of point and lines such that the lines are oriented towards a point which is called the *root* of the tree [8]; the orientation introduces a partial oredering relation between the nodes (and lines). Define $\Theta^k_{n,m}$ as the set of trees obtained in the following way.
(1) For each tree θ, the set of end-points $E(\theta)$ contains k elements.
(2) To each end-point v one associates two integers $(n_v, m_v) = (\pm 1, \pm 1)$ such that

$$\sum_{v\in E(\theta)} n_v = n \qquad \text{and} \qquad \sum_{v\in E(\theta)} m_v = m, \tag{2.11}$$

and a factor $m_v/4i$.
(3) If $V(\theta)$ denotes the set of nodes which are not end-points, for each node $v \in V(\theta)$ one has s_v entering lines, with $s_v = 2\nu_v + 1 \geq M$ odd. To each node $v \in V(\theta)$ one associates a factor f_{s_v}.
(4) If $L(\theta)$ denotes the set of lines exiting from nodes in $V(\theta)$, to each line $\ell \in L(\theta)$ one associates a *propagator*

$$G_\ell \equiv g(n_\ell, m_\ell) = \frac{1}{-n_\ell^2\omega^2 + \omega_{m_\ell}^2}, \tag{2.12}$$

with *momentum* (n_ℓ, m_ℓ), where one has

$$n_\ell = \sum_{w\in E_\ell} n_w \qquad \text{and} \qquad m_\ell = \sum_{w\in E_\ell} m_w, \tag{2.13}$$

if we denote by E_ℓ the set of end-points of the subtree with root line ℓ.

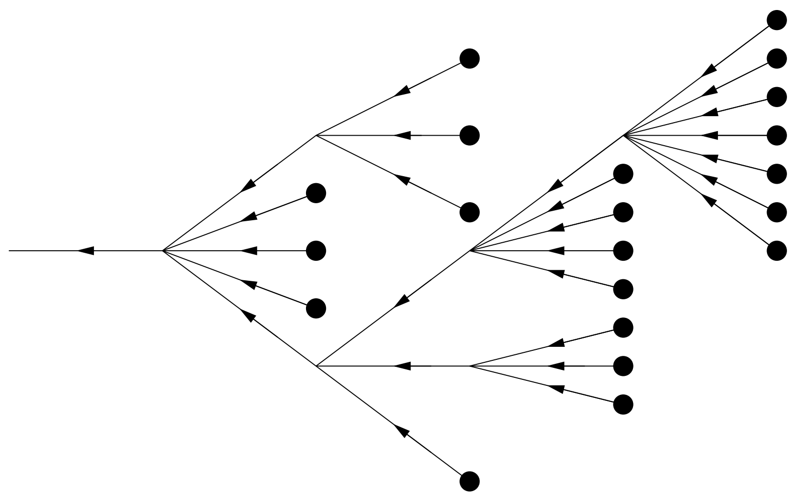

FIGURA 2.1. A tree θ belonging to the set $\Theta_{k,n,m}$ with $k = 21$. One has $|E(\theta)| = 21$, $|V(\theta)| = 6$ and $|L(\theta)| = 6$. The end-points are indicated by bullets.

Note that items (2) and (4) imply that to the root line a momentum (n, m) is associated for all trees $\theta \in \Theta^k_{n,m}$.

It is easy to see that, with the just stated rules, one has

$$\hat{u}^{(k)}_{n,m} = \sum_{\theta \in \Theta^k_{n,m}} \mathrm{Val}(\theta), \tag{2.14}$$

where

$$\mathrm{Val}(\theta) = \left(\prod_{\ell \in L(\theta)} G_\ell \right) \left(\prod_{v \in V(\theta)} f_{\nu_v} \right) \left(\prod_{v \in E(\theta)} \frac{m_v}{4i} \right). \tag{2.15}$$

By such definitions and the Diophantine condition (2.10) it follows that

$$|g(n_\ell, m_\ell)| \le \frac{1}{\omega\gamma} \tag{2.16}$$

for all lines $\ell \in L(\theta)$. Moreover one has

$$\sup_{\theta \in \Theta^k_{n,m}} \sup_{\ell \in L(\theta)} \{|n_\ell|, |m_\ell|\} \le k, \tag{2.17}$$

as one needs at least k end-points to have a momentum (n, m) with $\max\{|n|, |m|\} = k$ (by item (4) above); hence

$$\sup_{\theta \in \Theta^k_{n,m}} |\mathrm{Val}(\theta)| \le C^k, \tag{2.18}$$

with $C = \sqrt{F} e^{\xi} / 4\sqrt{\omega\gamma}$, where we have used

$$|V(\theta)| \le \frac{k}{2}, \tag{2.19}$$

and

$$\sum_{\theta \in \Theta^k_{n,m}} |\mathrm{Val}(\theta)| \le 4^k D^k C^k, \tag{2.20}$$

where D^k is a bound on the number of trees with k end-points (one can take $D = 4^{3/2}$) and 4^k takes into account the possible assignments of the labels (n_v, m_v) for all $v \in E(\theta)$.

Therefore (assuning for simplicity $4CD \ge 1$) we have

$$\sum_{k=1}^{\infty} \sum_{|n|,|m| \le k} \left| \hat{u}^{(k)}_{n,m} \varepsilon^k \right| \le \sum_{k=1}^{\infty} \sum_{|n|,|m| \le k} (4CD)^k \varepsilon^k \le \sum_{k=1}^{\infty} k^2 (4CD\varepsilon)^k, \tag{2.21}$$

which is convergent for $|\varepsilon| < \varepsilon_0 \equiv (4CDe^2)^{-1}$ for all ω satisfying the condition (2.10). Then (2.1), (2.6) and (2.21) will define an analytic function of ε, which we shall call $\tilde{u}(x, t; \omega, \varepsilon)$, as far as ω is taken as an independent parameter.

2.3. *Final step.* We consider then (2.9). The first nontrivial term is $c\varepsilon^{M-1}$, with a nonvanishing (explicitly computable) constant c, so that one can write

$$\omega_1^2 - \omega^2 = F(\varepsilon, \omega) \equiv c\varepsilon^{M-1} + G(\varepsilon, \omega), \tag{2.22}$$

with

$$G(\varepsilon,\omega)=\sum_{\nu=1}^{\infty} f_{2\nu+1}\sum_{k=2\nu+1}^{\infty}\varepsilon^{k-1}\sum_{k_1+\ldots+k_{2\nu+1}=k}\ \sum_{\substack{n_1+\ldots+n_{2\nu+1}=n\\ m_1+\ldots+m_{2\nu+1}=m}}{}^{\prime}\ \prod_{j=1}^{2\nu+1}\hat{u}_{n_j,m_j}^{(k_j)}, \tag{2.23}$$

where the prime means that the terms with all (m_j,n_j) equal to $(\pm1,\pm1)$ have to be discarded (as already taken into account by the term $c\varepsilon^{M-1}$ in (2.22)). The function $G(\varepsilon,\omega)$ is analytic for $|\varepsilon|<\varepsilon_0$ and of order M in ε, and it admits the bound

$$|G(\varepsilon,\omega)|\leq A\sum_{k=M}^{\infty}\left(\frac{\varepsilon}{\varepsilon_0}\right)^k, \tag{2.24}$$

for some positive constant A and for all ω satisfying the Diophantine condition (2.10).

This means that we can fix ε small enough (say $|\varepsilon|<\varepsilon_1$ for a suitable $\varepsilon_1<\varepsilon_0$) so that

$$|G(\varepsilon,\omega)|\leq\left|\frac{c\varepsilon^{M-1}}{2}\right|,\qquad\left|\frac{\partial}{\partial\varepsilon}G(\varepsilon,\omega)\right|\leq\left|\frac{c\,(M-1)\,\varepsilon^{M-2}}{2}\right|, \tag{2.25}$$

which implies that the following scenario arises. We can fix ω satisfying (2.10) and such that $|\omega^2-\omega_1^2|<C\varepsilon_1^{M-1}$ for a suitable constant C (one can take $C=|c|/2$): such values ω exist because, as it is proved in [1], there exists a family of ω verifying (2.10) and accumulating to ω_1 (see Proposition 3.7 and Corollary 3.8 of [1]): this gives the condition on μ to be small enough in the statement of the theorem.

Hence, by the first inequality in (2.25), the modulus of the function $F(\varepsilon,\omega)$ is larger than $|c\varepsilon^{M-1}/2|$ in $(0,\varepsilon_1)$, so that, for the fixed ω, there exists $\varepsilon=\varepsilon(\omega)$ verifying

$$\omega_1^2-\omega^2=F(\varepsilon(\omega),\omega). \tag{2.26}$$

If Ω denotes the set of values ω considered above, and $\mathcal{E}$ the set of values ε fixed by the above procedure, as $F(\varepsilon,\omega)$ is monotonic in $(0,\varepsilon_1)$ by the second inequality in (2.25) we deduce that there is a one-to-one correspondence between Ω and $\mathcal{E}$, so that we can fix $\varepsilon\in\mathcal{E}$ and choose the corresponding $\omega=\omega_\varepsilon$ in Ω: then the proof of the theorem in Section 1 is complete.

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