Quasi-periodic solutions for two-level systems

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ABSTRACT. *We consider the Schrödinger equation for a class of two-level atoms in a quasi-periodic external field in the case in which the spacing 2ε between the two unperturbed energy levels is small. We prove the existence of quasi-periodic solutions for a Cantor set $\mathcal{E}$ of values of ε around the origin which is of positive Lebesgue measure: such solutions can be obtained from the formal power series by a suitable resummation procedure. The set $\mathcal{E}$ can be characterized by requesting infinitely many Diophantine conditions of Mel'nikov type.*

1. Introduction

Consider the Hamiltonian describing a two-level system in a quasi-periodic external field

$$H(t) = \varepsilon\sigma_3 - f(t)\sigma_1, \tag{1.1}$$

where $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices and $f(t)$ is a quasi-periodic function; the real parameter ε measures half the spacing between the unperturbed energy levels.

The model has been widely studied in physics (for an introduction to the subject we refer to classical textbooks as [5] and [10]), and it was recently considered in [12], in connection with the problem of studying the existence of pure point spectrum for the quasi-energy operator.

In [1] the problem was investigated of studying quasi-periodic solutions of the corresponding Schrödinger equation

$$i\frac{\partial}{\partial t}\psi(t) = H(t)\psi(t). \tag{1.2}$$

In particular it was found that quasi-periodic solutions exist in the form of formal power series, but such series were argued to be in general divergent. Here we prove that quasi-periodic solutions exist indeed, and that they are not analytic in ε, according to the conjecture proposed in [1]; in fact they are defined on a set of values of the perturbative parameter ε centered around the origin and with a dense set of holes.

We use a version of the techniques introduced in classical mechanics by Eliasson, [4], in order to study KAM-type problems. Such techniques were further developed (see [6], [8], [9], [7] and papers quoted therein), by emphasizing the analogy with the methods of quantum field theory. In particular in [9] a resummation procedure was introduced which was reminiscent of the mass graphs resummation in field theory. Here we follow the same approach but slightly changing the resummation procedure in a form which is more suitable to deal with the small divisors in the present case. With respect to [9] we have the extra difficulty that the resummation produces new small divisors which can be vanishing for certain values of ε: so we have to perform the resummation in an iterative way by being careful to exclude more and more values of ε at each step. Such an approach shows also analogies with that followed in [3] for the study of low-dimensional elliptic tori in certain systems of PDE's.

A quasi-periodic solution of the Schrödinger equation for the system (1.1) was shown in [1] to be expressible in terms of a particular solution of a generalized Riccati equation (see next section), so we start directly from the latter equation.

For simplicity we assume a nondegeneracy condition of the external field (which corresponds to the condition of case (1) of Theorem 2.2. in [1]), but we think that our methods can be successfully applied in order to deal also with the condition of case (2); we refer to Section 2 for a more technical discussion.

We first show that it is possible, by a suitable choice of the initial conditions, to eliminate the secular terms and to obtain a formal solution which is given by a formal power series in ε quasi-periodic in time, so recovering the case (1) of Theorem 2.2. in [1]. The main difference with respect to [1] is that we use a graphic representation of the solution in terms of trees which allows us to obtain a very simple proof of existence of the formal solution.

Then we introduce a suitable resummation which leads to the proof of existence of a solution which is quasi-periodic in time and non-analytic in the perturbative parameter. This represents the main interest of the present paper, and the main novelty with respect to the existing literature.

2. Existence of formal solutions

Consider the *generalized Riccati equation*

$$\dot{G} - iG^2 - 2ifG + i\varepsilon^2 = 0, \tag{2.1}$$

where $\dot{G} = \mathrm{d}G/\mathrm{d}t$,

$$f = f(t) = \sum_{\underline{\nu}\in\mathbb{Z}^{d'}} e^{i\underline{\nu}\cdot\underline{\omega}t} f_{\underline{\nu}} \tag{2.2}$$

is a real analytic quasi-periodic function, and $d' \geq 1$ is an integer.

Let us look for a solution of the form

$$G = -i\varepsilon Q u, \qquad Q(t) = e^{2iF(t)}, \qquad F(t) = \int_0^t \mathrm{d}t'\, f(t'). \tag{2.3}$$

Then, for $\varepsilon \neq 0$, (2.1) implies for u the following equation:

$$\dot{u} = \varepsilon\left(Qu^2 + Q^{-1}\right). \tag{2.4}$$

Define

$$d = \begin{cases} d', & \text{if } f_{\underline{0}} = 0, \\ d'+1, & \text{if } f_{\underline{0}} \neq 0, \end{cases} \qquad \boldsymbol{\omega} = \begin{cases} \underline{\omega}, & \text{if } f_{\underline{0}} = 0, \\ (f_{\underline{0}}, \underline{\omega}), & \text{if } f_{\underline{0}} \neq 0, \end{cases} \tag{2.5}$$

and assume that $\boldsymbol{\omega}$ is a *Diophantine vector*, i.e. a vector satisfying the Diophantine condition

$$|\boldsymbol{\omega} \cdot \boldsymbol{\nu}| > C_0 |\boldsymbol{\nu}|^{-\tau} \qquad \forall \boldsymbol{\nu} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}, \tag{2.6}$$

with C_0, τ two positive constants and $|\boldsymbol{\nu}| = |\nu_1| + \ldots + |\nu_d|$ for $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_d)$.

Given any function g of the form

$$g(t) = \sum_{\boldsymbol{\nu} \in \mathbb{Z}^d} e^{i\boldsymbol{\nu}\cdot\boldsymbol{\omega}t} g_{\boldsymbol{\nu}}, \tag{2.7}$$

let us denote by $\langle g \rangle = g_{\mathbf{0}}$ the constant term in its Fourier expansion.

Let us suppose that one has

$$\langle Q \rangle \neq 0; \tag{2.8}$$

this corresponds to the assumption (1) of the theorem 2.2 in [1].

By the analyticity assumptions on f one has

$$|Q_{\boldsymbol{\nu}}| \le \mathcal{Q} e^{-\kappa|\boldsymbol{\nu}|}, \qquad |(Q^{-1})_{\boldsymbol{\nu}}| \le \mathcal{Q} e^{-\kappa|\boldsymbol{\nu}|}, \tag{2.9}$$

for two suitable positive constants $\mathcal{Q}$ and κ. Moreover one has $\langle Q^{-1} \rangle \neq 0$ if and only if (2.8) holds.

Then we have the following result, [1].

Theorem 1. *The generalized Riccati equation (2.1), with f a real analytic quasi-periodic function of the form (2.2), under the hypotheses (2.6) and (2.8), admits a formal power series*

$$g(t;\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \sum_{\boldsymbol{\nu} \in \mathbb{Z}^d} e^{i\boldsymbol{\nu}\cdot\boldsymbol{\omega}t} g_{\boldsymbol{\nu}}^{(k)}, \tag{2.10}$$

which represents a formal particular solution.

The proof consists in determining the initial conditions in a suitable way in order to eliminate the secular terms, and it is performed in Sections 4 and 5.

3. Existence of solutions

The result of the previous section can be improved into the following one.

Theorem 2. *Consider the generalized Riccati equation (2.1), with f a real analytic quasi-periodic function of the form (2.2), and assume that the hypotheses (2.6) and (2.8) are fulfilled. There exist three positive constants ε_0, b and ξ and a set $\mathcal{E} \subset (-\varepsilon_0, \varepsilon_0)$ of Lebesgue measure* $\mathrm{meas}(\mathcal{E}) \ge 2\varepsilon_0(1 - b\varepsilon_0^{\xi})$ *such that (2.1) admits a particular solution of the form*

$$\overline{g}(t;\varepsilon) = \sum_{\boldsymbol{\nu} \in \mathbb{Z}^d} e^{i\boldsymbol{\nu}\cdot\boldsymbol{\omega}t} \overline{g}_{\boldsymbol{\nu}}(\varepsilon) \tag{3.1}$$

for all $\varepsilon \in \mathcal{E}$.

The proof of the above statements will be performed in Sections 6 to 8.

Therefore the solution (3.1) will turn out to be not analytic in ε; in fact it can be obtained by the formal power expansion (2.10) through a suitable resummation procedure.

The set of values of ε which have to be excluded from $(-\varepsilon_0, \varepsilon_0)$ is dense, and it depends on the external field $f(t)$. From a technical point of view such values arise by imposing infinitely many Diophantine conditions of Mel'nikov type of the form

$$\left| i\boldsymbol{\omega} \cdot \boldsymbol{\nu} - \mathcal{M}^{[n]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}; \varepsilon) \right| \geq C_0 |\nu|^{-\tau_1} \qquad \forall \boldsymbol{\nu} \in \mathbb{Z}^d \setminus \{\mathbf{0}\} \text{ and } \forall n \geq -1, \tag{3.2}$$

where $\tau_1 > \tau + d$ and $\mathcal{M}^{[n]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}; \varepsilon)$ are suitable functions which will be constructed recursively along the iterative resummation of the formal solution. For instance one has $\mathcal{M}^{[-1]}(x; \varepsilon) = 0$, $\mathcal{M}^{[0]}(x; \varepsilon) = 2\varepsilon \langle Q \rangle c^{[0]} + O(\varepsilon^2)$, with $c^{[0]} = i\sqrt{\langle Q^{-1} \rangle / \langle Q \rangle}$, and so on.

Of course the relevance of the conditions (3.2) depends on the value of the imaginary parts of the functions $\mathcal{M}^{[n]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}; \varepsilon)$.

We do not study this problem in general, but we can immediately realize that we can have easily nontrivial situations. Consider for instance the case of an external field f which is an even function with vanishing average and of order μ, with $\mu \ll 1$: then $F(t)$ in (2.4) is odd, and one has that $\langle Q \rangle = 1 + O(\mu)$ and $\langle Q^{-1} \rangle = 1 + O(\mu)$ are both real, so that $c^{[0]} = i(1 + O(\mu))$, hence $\mathcal{M}^{[0]}(x; \varepsilon) = 2i\varepsilon(1 + O(\mu)) + O(\varepsilon^2)$. Therefore the conditions (3.2) give nontrivial results at least for such a case for $n = 0$; furthermore we shall see that one has $\mathcal{M}^{[n]}(x; \varepsilon) = \mathcal{M}^{[0]}(x; \varepsilon) + O(\varepsilon^2)$ for all $n \geq 1$ and that the difference between two functions $\mathcal{M}^{[n+1]}(x; \varepsilon)$ and $\mathcal{M}^{[n]}(x; \varepsilon)$ tends exponentially to zero as $n \to \infty$.

4. Graphical representation and tree formalism

We look for a formal solution of (2.4) of the form

$$u(t) = \sum_{k=1}^{\infty} \varepsilon^k u^{(k)}(t) = \sum_{k=1}^{\infty} \varepsilon^k \sum_{\boldsymbol{\nu} \in \mathbb{Z}^d} e^{i\boldsymbol{\nu} \cdot \boldsymbol{\omega} t} u_{\boldsymbol{\nu}}^{(k)}; \tag{4.1}$$

by setting, for all $k \geq 0$,

$$u_{\mathbf{0}}^{(k)} = c^{(k)}, \tag{4.2}$$

we can write

$$u^{(k)}(t) = c^{(k)} + U^{(k)}, \qquad \langle U^{(k)} \rangle = 0 \qquad \forall k \geq 0. \tag{4.3}$$

By inserting (4.1) into (2.4) we obtain

$$\begin{aligned} \dot{u}^{(0)} &= 0, \\ \dot{u}^{(1)} &= Q^{-1} + Q(u^{(0)})^2, \\ \dot{u}^{(k)} &= Q \sum_{k_1 + k_2 = k-1} u^{(k_1)} u^{(k_2)} \qquad \forall k \geq 2, \end{aligned} \tag{4.4}$$

which, expressed in Fourier space, becomes, for all $\boldsymbol{\nu} \neq \mathbf{0}$,

$$\begin{aligned} u_{\boldsymbol{\nu}}^{(0)} &= 0, \\ (i\boldsymbol{\omega} \cdot \boldsymbol{\nu})\, u_{\boldsymbol{\nu}}^{(1)} &= \left(Q^{-1}\right)_{\boldsymbol{\nu}} + Q_{\boldsymbol{\nu}} (c^{(0)})^2, \\ (i\boldsymbol{\omega} \cdot \boldsymbol{\nu})\, u_{\boldsymbol{\nu}}^{(k)} &= \sum_{k_1 + k_2 = k-1} \sum_{\boldsymbol{\nu}_0 + \boldsymbol{\nu}_1 + \boldsymbol{\nu}_2 = \boldsymbol{\nu}} Q_{\boldsymbol{\nu}_0} u_{\boldsymbol{\nu}_1}^{(k_1)} u_{\boldsymbol{\nu}_2}^{(k_2)} \qquad \forall k \geq 2, \end{aligned} \tag{4.5}$$

provided that the right hand side of (4.4) has vanishing average. This requires

$$
\begin{aligned}
0 &= (Q^{-1})_{\mathbf{0}} + Q_{\mathbf{0}}(c^{(0)})^2, \\
0 &= \sum_{k_1+k_2=k-1} \sum_{\boldsymbol{\nu}_0+\boldsymbol{\nu}_1+\boldsymbol{\nu}_2=\mathbf{0}} Q_{\boldsymbol{\nu}_0} u_{\boldsymbol{\nu}_1}^{(k_1)} u_{\boldsymbol{\nu}_2}^{(k_2)} \qquad \forall k \geq 2.
\end{aligned}
\tag{4.6}
$$

The first equation in (4.6) fixes $c^{(0)}$ to a value such that

$$
(c^{(0)})^2 = -\frac{\langle Q^{-1}\rangle}{\langle Q\rangle},
\tag{4.7}
$$

which is well defined and different from zero by the hypothesis (2.8).

The second equation in (4.6) can be written as

$$
0 = 2Q_{\mathbf{0}} c^{(0)} c^{(k-1)} + \text{other terms depending on } c^{(0)}, \ldots, c^{(k-2)},
\tag{4.8}
$$

which allows us, in principle, to fix iteratively the coefficients $\{c^{(k)}\}_{k=1}^{\infty}$.

We can represent graphically the functions $u_{\boldsymbol{\nu}}^{(0)}$ and $u_{\boldsymbol{\nu}}^{(1)}$ as in Figure 4.1.

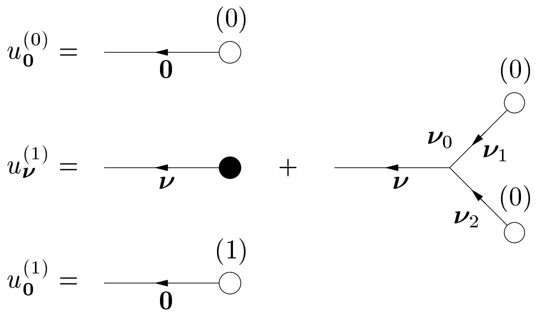

FIGURA 4.1. Graphical representation of $u_{\boldsymbol{\nu}}^{(0)}$ and $u_{\boldsymbol{\nu}}^{(1)}$. The function $u_{\boldsymbol{\nu}}^{(0)}$ is represented by a graph formed by a line and an endpoint (white bullet): we associate to the line a momentum $\boldsymbol{\nu} = \mathbf{0}$ and a propagator 1, and to the white bullet an order label $k = 0$, a mode label $\boldsymbol{\nu} = \mathbf{0}$ and a node factor $c^{(0)}$, so that one has $u_{\boldsymbol{\nu}}^{(0)} = 0$ for $\boldsymbol{\nu} \neq \mathbf{0}$, while $u_{\mathbf{0}}^{(0)} = c^{(0)}$. The function $u_{\boldsymbol{\nu}}^{(1)}$, for $\boldsymbol{\nu} \neq \mathbf{0}$, is represented by the sum of two graphs. We associate to the line with momentum $\boldsymbol{\nu}$ a propagator $1/(i\boldsymbol{\omega} \cdot \boldsymbol{\nu})$. In the first graph we associate to the endpoint (black bullet) a mode label $\boldsymbol{\nu}$ and a node factor $(Q^{-1})_{\boldsymbol{\nu}}$, while in the second graph we associate to the point (vertex) carrying the mode $\boldsymbol{\nu}_0$ a node factor $Q_{\boldsymbol{\nu}_0}$ and to the two white bullets order labels $k_1 = k_2 = 0$, modes $\boldsymbol{\nu}_1 = \boldsymbol{\nu}_2 = \mathbf{0}$ and node factors $c^{(0)}$. In the second graph one has the constraint $\boldsymbol{\nu} = \boldsymbol{\nu}_0 + \boldsymbol{\nu}_1 + \boldsymbol{\nu}_2$. The function $u_{\mathbf{0}}^{(1)}$ is represented as $u_{\mathbf{0}}^{(0)}$, with the only difference that now the white bullet carries an order label $k = 1$.

More generally we can represent $u_{\boldsymbol{\nu}}^{(k)}$ for all $k \geq 0$ as in Figure 4.2, where the graphical representation has to be interpreted as in Figure 4.1 when either $k = 0$ or $k = 1$, while it can be developed

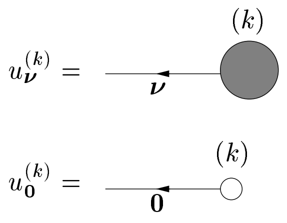

FIGURA 4.2. Graphical representation of $u_{\boldsymbol{\nu}}^{(k)}$. The line carries a momentum $\boldsymbol{\nu}$: we associate to it a propagator $1/(i\boldsymbol{\omega}\cdot\boldsymbol{\nu})$ if $\boldsymbol{\nu}\neq\mathbf{0}$, and a propagator 1 if $\boldsymbol{\nu}=\mathbf{0}$.

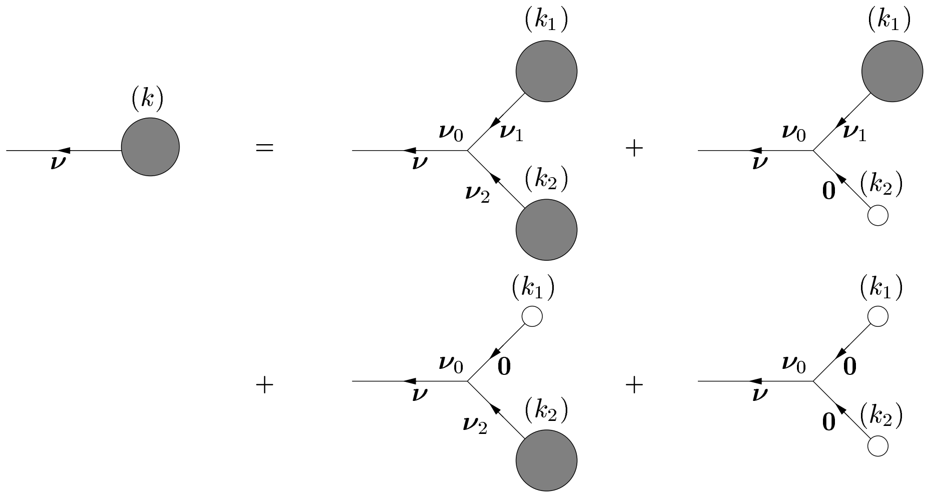

FIGURA 4.3. Graphical representation of $u_{\boldsymbol{\nu}}^{(k)}$ in terms of $u_{\boldsymbol{\nu}'}^{(k')}$, with $k'<k$, for $\boldsymbol{\nu}\neq\mathbf{0}$. One has the constraints $\boldsymbol{\nu}=\boldsymbol{\nu}_0+\boldsymbol{\nu}_1+\boldsymbol{\nu}_2$ and $k=1+k_1+k_2$. To the vertex with mode $\boldsymbol{\nu}_0$ we associate a node factor $Q_{\boldsymbol{\nu}_0}$.

iteratively as shown in Figure 4.3 when $k\geq 2$ when $\boldsymbol{\nu}\neq\mathbf{0}$. If $\boldsymbol{\nu}=\mathbf{0}$ one has $u_{\mathbf{0}}^{(k)}=c^{(k)}$, with $c^{(k)}$ to be recursively defined, as it will be explained below.

For instance when $k=2$ and $\boldsymbol{\nu}\neq\mathbf{0}$ we obtain the graphical representation of Figure 4.4.

Therefore we can see that, iterating the graphical procedure described above, we can give a graphical representation of $u_{\boldsymbol{\nu}}^{(k)}$, for all $k\in\mathbb{Z}_+$ and for all $\boldsymbol{\nu}\in\mathbb{Z}^d$, in terms of trees.

A tree θ is a connected set of points and lines such that the lines are oriented toward a point which is called the *root* of the tree. We call *nodes* all the points of the tree other than the root. The orientation induces a partial ordering relation between the nodes (and the lines), which will shall denote by $\preceq$: given two nodes v and w we shall write $w\preceq v$ if v is along the path (of lines) connecting v to the root, and $w=v$ means that the two nodes coincide. We shall be interested only in trees such that for all nodes there are only either two or zero entering lines (keep in mind Figure 4.4 as an example). Note that for the root there is by construction one and only one line entering it: we shall call *root line* such a line.

Given a tree θ let us distinguish between the set $E(\theta)$ of nodes such that no line enter them and the set $V(\theta)$ of nodes such that there is at least a line (hence two lines) entering them: we

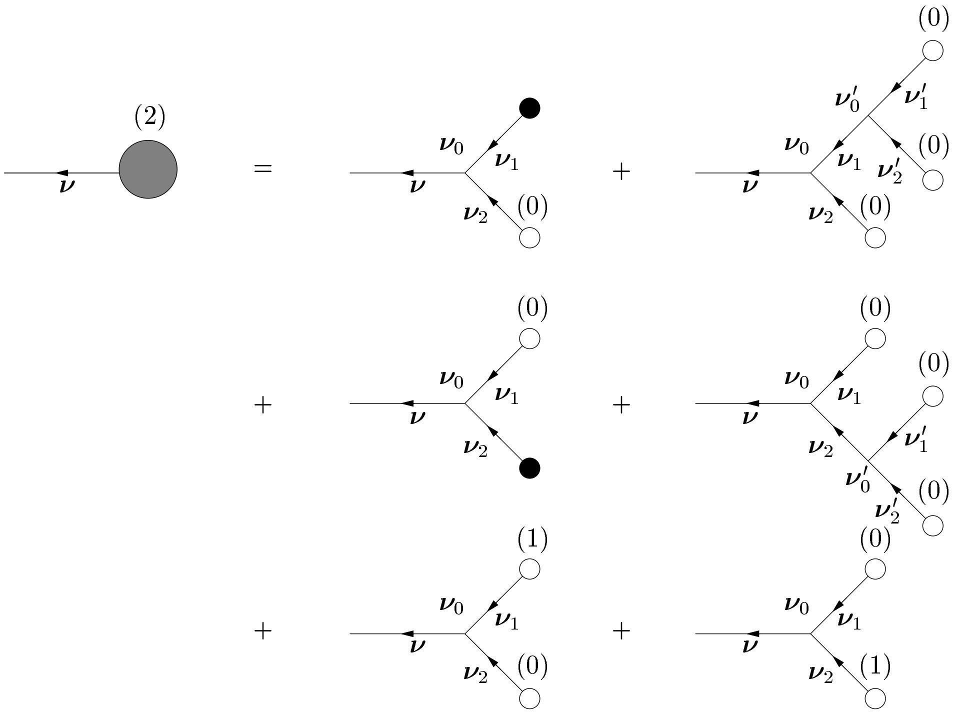

FIGURA 4.4. Graphical representation of $u^{(2)}_{\boldsymbol{\nu}}$ for $\boldsymbol{\nu} \neq \mathbf{0}$. The symbols and the labels have the same meaning as in the previous Figures. One has the constraints $\boldsymbol{\nu} = \boldsymbol{\nu}_0 + \boldsymbol{\nu}_1 + \boldsymbol{\nu}_2$ in all graphs, $\boldsymbol{\nu}_1 = \boldsymbol{\nu}'_0 + \boldsymbol{\nu}'_1 + \boldsymbol{\nu}'_2$ in the second graph, and $\boldsymbol{\nu}_2 = \boldsymbol{\nu}'_0 + \boldsymbol{\nu}'_1 + \boldsymbol{\nu}'_2$ in the fourth graph. The lines coming out from the white endpoints must have momentum $\boldsymbol{\nu} = \mathbf{0}$; to each white endpoint we associate a node factor $c^{(k)}$ if $k = 0, 1$ is the order label of the endpoint, to each black endpoint we associate a node factor $(Q^{-1})_{\boldsymbol{\nu}}$ if $\boldsymbol{\nu}$ is the momentum of the line coming out from it (which is equal to the mode of the endpoint itself), and to each node v which is not an endpoint (vertex) we associate a mode $\boldsymbol{\nu}_v$ and a node factor $Q_{\boldsymbol{\nu}_v}$.

call *endpoints* the first ones and *vertices* the latter. Graphically the endpoints will be depicted as bullets which can be black or white (see for instance Figure 4): we denote by $E_B(\theta)$ and $E_W(\theta)$ the two sets, respectively.

Define $W(\theta)$ as the set of the endpoints represented by white bullets, and $B(\theta)$ as the set of vertices and of endpoints represented by black bullets; of course one has $W(\theta) = E_W(\theta)$ and $B(\theta) = E_B(\theta) \cup V(\theta)$.

To each vertex $v \in V(\theta)$ we associate a *mode* label $\boldsymbol{\nu}_v \in \mathbb{Z}^d$ and a *node factor* $F_v = Q_{\boldsymbol{\nu}_v}$, to each endpoint $v \in E_B(\theta)$ we associate a *mode* label $\boldsymbol{\nu}_v \in \mathbb{Z}^d$ and a *node factor* $F_v = (Q^{-1})_{\boldsymbol{\nu}_v}$, and to each endpoint $v \in E_W(\theta)$ we associate a *mode* label $\boldsymbol{\nu}_v = \mathbf{0}$, an *order* label $k_v \in \mathbb{Z}_+$ and a *node factor* $F_v = c^{(k_v)}$.

Define $L(\theta)$ as the set of lines in θ. Each line ℓ comes out from a point and enters another point; if we denote by v the first one we shall denote by v' the latter, and we shall call it the "point immediately following v"; as the line is uniquely identified by the node v we shall write also $\ell = \ell_v$.

To each line we associate a *momentum* label $\boldsymbol{\nu}_\ell \in \mathbb{Z}^d$ and a *propagator* which is $g_\ell = 1/(i\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell)$ if $\boldsymbol{\nu}_\ell \neq \mathbf{0}$ and $g_\ell = 1$ if $\boldsymbol{\nu}_\ell = \mathbf{0}$. If the line ℓ comes out from a node $v \in W(\theta)$ one has necessarily $\boldsymbol{\nu}_\ell = \mathbf{0}$, while if the line ℓ comes out from a node $v \in B(\theta)$ all values of $\boldsymbol{\nu}_\ell$ are possible. We say that $\boldsymbol{\nu}_\ell$ "flows" through the line ℓ.

The modes and the momenta are related by the following relation: if $\ell = \ell_v$ and ℓ' and ℓ'' are the lines entering v one has

$$\boldsymbol{\nu}_\ell = \boldsymbol{\nu}_v + \boldsymbol{\nu}_{\ell'} + \boldsymbol{\nu}_{\ell''} = \sum_{\substack{w\in B(\theta)\\ w\preceq v}} \boldsymbol{\nu}_w, \tag{4.9}$$

which represents a sort of conservation law.

We call *equivalent* two trees which can be transformed into each other by permuting any two lines entering the same point.

Finally we define $\mathcal{T}_{k,\boldsymbol{\nu}}$ as the set of inequivalent trees θ such that
(1) for each vertex $v \in V(\theta)$ there are exactly two entering lines;
(2) the endpoints $v \in E(\theta)$ can be either white or black;
(3) the number of black endpoints, the number of vertices and the order labels of the white endpoints are such that, by setting $|B(\theta)| = |E_B(\theta)| + |V(\theta)| = k_1$ and $\sum_{v\in E_W(\theta)} k_v = k_2$, one has $k_1 + k_2 = k$;
(4) the momentum flowing through the line entering the root (root line) is $\boldsymbol{\nu}$.

We shall call $\mathcal{T}_{k,\boldsymbol{\nu}}$ the *set of trees of order k and with total momentum $\boldsymbol{\nu}$*.

With the above notations we can write, for $\boldsymbol{\nu} \neq \mathbf{0}$,

$$\begin{aligned} u_{\boldsymbol{\nu}}^{(k)} &= \sum_{\theta\in\mathcal{T}_{k,\boldsymbol{\nu}}} \mathrm{Val}(\theta), \\ \mathrm{Val}(\theta) &= \left(\prod_{\ell\in L(\theta)} g_\ell\right)\left(\prod_{v\in E(\theta)\cup V(\theta)} F_v\right), \end{aligned} \tag{4.10}$$

where $\mathrm{Val}(\theta)$ is called the *value* of the tree θ, and

$$g_\ell = \begin{cases} \dfrac{1}{i\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell}, & \text{if } \boldsymbol{\nu}_\ell \neq \mathbf{0}, \\ 1, & \text{if } \boldsymbol{\nu}_\ell = \mathbf{0}, \end{cases} \qquad F_v = \begin{cases} Q_{\boldsymbol{\nu}_v}, & \text{if } v \in V(\theta), \\ (Q^{-1})_{\boldsymbol{\nu}_v}, & \text{if } v \in E_B(\theta), \\ c^{(k_v)}, & \text{if } v \in E_W(\theta), \end{cases} \tag{4.11}$$

while, for $\boldsymbol{\nu} = \mathbf{0}$, one can easily write the contribution $c^{(k)} = u_{\mathbf{0}}^{(k)}$ of order k to the initial condition by imposing (4.8). This yields

$$c^{(k)} = -\frac{1}{2c^{(0)}\langle Q\rangle} \sum_{\theta\in\mathcal{T}^*_{k,\mathbf{0}}} \mathrm{Val}(\theta), \tag{4.12}$$

where $\mathcal{T}^*_{k,\mathbf{0}}$ is defined as $\mathcal{T}_{k,\mathbf{0}}$, with the constraint that one has to discard the two trees of the form represented in Figure 4.5 such that the three represented lines carry vanishing momenta and the mode label associated to the represented vertex is zero.

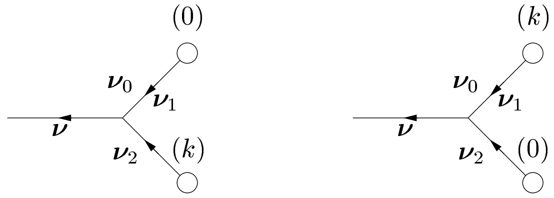

FIGURA 4.5. Trees that does not appear in $\mathcal{T}^*_{k,\mathbf{0}}$: besides of having $\boldsymbol{\nu}_1 = \boldsymbol{\nu}_2 = \mathbf{0}$, by definition of white endpoint, one requires also $\boldsymbol{\nu} = \boldsymbol{\nu}_0 = \mathbf{0}$, so that the value of both such trees is $Q_{\mathbf{0}} c^{(0)} c^{(k)} = \langle Q \rangle c^{(0)} c^{(k)}$.

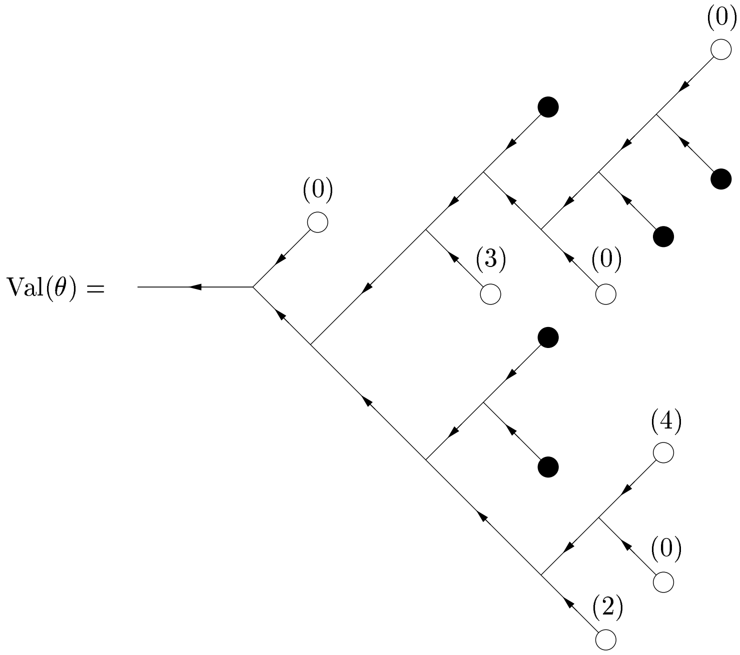

FIGURA 4.6. An example of tree with 11 vertices, 5 black endpoints and 7 white endpoints. Unlike the order labels, the modes of the nodes and the momenta of the lines are not explicitly showed. The order of the tree is $k = 11 + 5 + (3 + 2 + 4) = 25$, while the total momentum $\boldsymbol{\nu}$ is the momentum flowing through the root line, which is the leftmost one.

Note that in (4.12), as well as in (4.10), the values $\mathrm{Val}(\theta)$ will depend on the node factors $c^{(k')}$, with $k' < k$, so that (4.12) provides a recursive definition of the coefficients $c^{(k)}$.

An example of tree of order $k = 25$ is given in Figure 4.6.

5. Multiscale decomposition

In this section we introduce a multiscale decomposition of the propagators: with respect to [1] this will allow us to obtain better estimates on some contributions to the coefficients $u^{(k)}_{\boldsymbol{\nu}}$. Moreover this will be the first step in order to prove theorem 2 in section 3.

Let $\psi(x)$ a C^∞ non-decreasing compact support function defined on $\mathbb{R}^+$ such that

$$\psi(x) = \begin{cases} 1, & \text{for } x \geq C_0, \\ 0, & \text{for } x \leq C_0/2, \end{cases} \tag{5.1}$$

where C_0 is the Diophantine constant appearing in (2.6), and set $\chi(x) = 1 - \psi(x)$; see Figure 5.1. Define also $\psi_n(x) = \psi(2^n x)$ and $\chi_n(x) = \chi(2^n x)$ for all $n \geq 0$; of course $\psi_0 = \psi$ and $\chi_0 = \chi$.

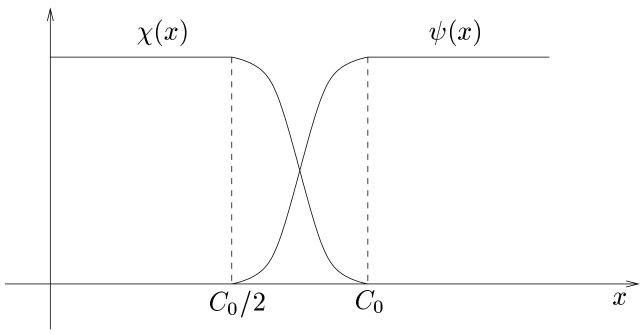

FIGURA 5.1. Possible graphs of the C^∞ compact support functions $\psi(x)$ and $\chi(x)$.

Then, for any line $\ell \in L(\theta)$ with $\boldsymbol{\nu}_\ell \neq \mathbf{0}$, set

$$g_\ell \equiv \frac{1}{i\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell} = \frac{\psi_0(|\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell|)}{i\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell} + \sum_{n=1}^{\infty} \frac{\psi_n(|\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell|)\chi_{n-1}(|\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell|)}{i\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell}, \tag{5.2}$$

which can be rewritten as

$$\begin{aligned} g_\ell &= \sum_{n=0}^{\infty} g_\ell^{(n)}, \\ g_\ell^{(0)} &= \frac{\psi_0(|\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell|)}{i\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell}, \\ g_\ell^{(n)} &= \frac{\psi_{n+1}(|\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell|)\chi_n(|\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell|)}{i\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell} \qquad \forall n \geq 1. \end{aligned} \tag{5.3}$$

We shall call $g_\ell^{(n)} \equiv g^{(n)}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell)$ a propagator on scale n.

We shall assign to each line $\ell \in L(\theta)$ with $\boldsymbol{\nu}_\ell \neq 0$ also a new label $n_\ell = 0, 1, 2, \ldots$, which will be called the *scale label* the line ℓ; we can associate a scale label also to a line ℓ with $\boldsymbol{\nu}_\ell = \mathbf{0}$, by setting $n_\ell = -1$. Then we shall define $\Theta_{k,\boldsymbol{\nu}}$ as the set of trees which differ from those in $\mathcal{T}_{k,\boldsymbol{\nu}}$ just because of the newly introduced scale labels, so that (4.10) can be replaced with

$$\begin{aligned} u_{\boldsymbol{\nu}}^{(k)} &= \sum_{\theta \in \Theta_{k,\nu}} \mathrm{Val}(\theta), \\ \mathrm{Val}(\theta) &= \left(\prod_{\ell \in L(\theta)} g_\ell^{(n_\ell)} \right) \left(\prod_{v \in E(\theta) \cup V(\theta)} F_v \right), \end{aligned} \tag{5.4}$$

and an expression analogous to (4.11) holds for $c^{(k)}$, provided that $\mathcal{T}^*_{k,\mathbf{0}}$ is replaced with $\Theta^*_{k,\mathbf{0}}$, with obvious meaning of the symbols.

Note that, for fixed $x = \boldsymbol{\omega}\cdot\boldsymbol{\nu}$, one can have $g^{(n)}(x) \neq 0$ only for two values of n, so that the series (5.3) is in fact a finite sum.

Note also that $g^{(n)}(x) \neq 0$ only if $2^{-n-1}C_0 < |x| < 2^{-n+1}C_0$ for $n \geq 1$ and only if $|x| > 2^{-1}C_0$ for $n = 0$. This means that for any line ℓ on scale n such that $g^{(n)}_\ell \neq 0$ we can bound $|g^{(n)}_\ell| \leq C_0^{-1}2^{n+1}$. Hence, if $N_n(\theta)$ denotes the number of lines on scale n in θ and $|\boldsymbol{\nu}(\theta)| = \sum_{v\in B(\theta)}|\boldsymbol{\nu}_v|$, we can bound, for each tree $\theta \in \Theta_{k,\boldsymbol{\nu}}$ and for any integer n_0,

$$|\mathrm{Val}(\theta)| \leq C_0^{-k_1}2^{2k_1}2^{n_0k_1}\mathcal{Q}^{k_1}e^{-\kappa|\boldsymbol{\nu}(\theta)|}\Big(\prod_{v\in E_W(\theta)}|c^{(k_v)}|\Big)\Big(\prod_{n=n_0}^{\infty}2^{nN_n(\theta)}\Big), \tag{5.5}$$

where $k_1 = |B(\theta)|$ and $\sum_{v\in E_W(\theta)}k_v = k_2 = k - k_1$.

A *cluster* T on scale n is a maximal set of points and lines connecting them such that all the lines have scales $n' \leq n$ and there is at least one line with scale n. The $m_T \geq 0$ lines entering the cluster T and the possible line coming out from it (unique if existing at all) are called the *external lines* of the cluster T. Given a cluster T on scale n, we shall denote by $n_T = n$ the scale of the cluster.

Given a cluster T in a tree θ call $V(T)$, $E(T)$, $E_W(T)$, $E_B(\theta)$, $B(T)$, and $L(T)$ the set of vertices, of endpoints, of white endpoints, of black endpoints, of vertices plus black endpoints, and of lines of T, respectively. Let us define also $\boldsymbol{\nu}_T = \sum_{v\in B(T)}\boldsymbol{\nu}_v$.

We call *self-energy graph* of a tree θ any cluster T such that
(1) T has only one entering line ℓ_T^2 and one exiting line ℓ_T^1,
(2) one has

$$\boldsymbol{\nu}_T \equiv \sum_{v\in B(T)}\boldsymbol{\nu}_v = \mathbf{0}. \tag{5.6}$$

We say that the line ℓ_T^1 exiting a self-energy graph T is a *self-energy line*; we call *normal line* any line of the tree which is not a self-energy line.

Note that the two external lines of a self-energy graph have not necessarily the same scale: if n_1 and n_2 are the scale of the lines ℓ_T^1 and ℓ_T^2, and $n = n_T$ is the scale of the self-energy graph as a cluster, one must have $n+1 \leq \min\{n_1, n_2\}$.

An example of tree with self-energy graphs is depicted in Figure 5.2; one can immediately realize that because of the presence of self-energy graphs one can have accumulation of small divisors.

It is not difficult to prove (see for instance [9]) that, if we denote by $N_n^{\mathrm{norm}}(\theta)$ the number of normal lines in θ, then there exists a positive constant c such that

$$N_n^{\mathrm{norm}}(\theta) \leq c\,2^{-n/\tau}\sum_{v\in B(\theta)}|\boldsymbol{\nu}_v|, \tag{5.7}$$

so that, if we could neglect the self-energy lines, i.e if we could replace $N_n(\theta)$ with $N_n^{\mathrm{norm}}(\theta)$ in (5.5), we would obtain, for some constant C,

$$\begin{aligned}|\mathrm{Val}(\theta)| &\leq C_0^{-k_1}2^{2k_1}2^{n_0k_1}\mathcal{Q}^{k_1}e^{-\kappa|\boldsymbol{\nu}(\theta)|}\Big(\prod_{v\in E_W(\theta)}|c^{(k_v)}|\Big)\Big(\prod_{n=n_0}^{\infty}\exp\Big[|\boldsymbol{\nu}(\theta)|c\log 2\sum_{n=n_0}^{\infty}n2^{-n/\tau}\Big]\Big)\\ &\leq C^{k_1}e^{-\kappa'|\boldsymbol{\nu}(\theta)|}\Big(\prod_{v\in E_W(\theta)}|c^{(k_v)}|\Big),\end{aligned} \tag{5.8}$$

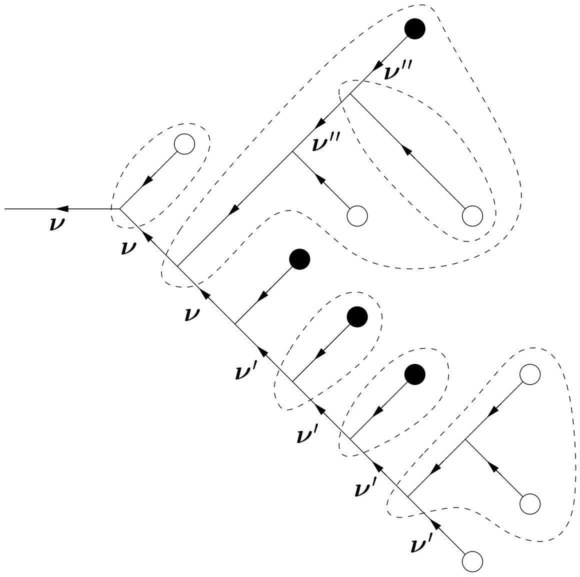

FIGURA 5.2. Example of trees containing self-energy graphs. All the lines contained inside a self-energy graph have scales strictly less than the scales of the external lines of the self-energy graph. One can have self-energy graphs containing other self-energy graphs on lower scales; in the Figure one has a self-energy graph with external lines carrying a momentum $\boldsymbol{\nu}''$ contained inside a self-energy graph with external lines carrying a momentum $\boldsymbol{\nu}$: the first one will be on a scale $n'' \leq n-1$ if n is the scale of the latter.

for some $\kappa' < \kappa$ and n_0 suitably chosen (see [8] or [9]).

Unfortunately the bound (5.8) is in general false, as it can apply only to trees without self-energy graphs. Therefore the above analysis is sufficient to prove theorem 1, but not to prove theorem 2.

Indeed, as we can not use the bound (5.7) for all trees, all we can do in general for a tree $\theta \in \Theta_{k,\boldsymbol{\nu}}$ is to estimate the product of propagators in (5.4) by

$$\Big| \prod_{\ell \in L(\theta)} g_\ell^{(n_\ell)} \Big| \leq C_0^{-k_1} \prod_{\ell \in L(\theta)} |\boldsymbol{\nu}_\ell|^\tau, \tag{5.9}$$

so that, by writing

$$\prod_{v \in B(\theta)} e^{-\kappa |\boldsymbol{\nu}_v|} \leq \Big(\prod_{v \in B(\theta)} e^{-\kappa |\boldsymbol{\nu}_v|/2} \Big) \Big(\prod_{\ell \in L(\theta)} e^{-\kappa |\boldsymbol{\nu}_\ell|/2k_1} \Big), \tag{5.10}$$

one obtain for each line $\ell \in L(\theta)$ the bound $|\boldsymbol{\nu}_\ell|^\tau e^{-\kappa|\boldsymbol{\nu}_\ell|/2k_1} \leq \tau!(2k_1/\kappa)^\tau$. Therefore we can bound

$$\Big| \prod_{\ell \in L(\theta)} g_\ell^{(n_\ell)} \Big| \Big(\prod_{\ell \in L(\theta)} e^{-\kappa |\boldsymbol{\nu}_\ell|/2k_1} \Big) \leq C_0^{-k_1} \tau!^{k_1} \left(\frac{2k_1}{\kappa} \right)^{k_1}, \tag{5.11}$$

and for all trees in $\Theta_{k,\boldsymbol{\nu}}$ we have

$$|\mathrm{Val}(\theta)| \leq C_1^{k_1} (k_1!)^\alpha \Big(\prod_{v \in B(\theta)} e^{-\kappa|\boldsymbol{\nu}_v|/2} \Big) \Big(\prod_{v \in E_W(\theta)} |c^{(k_v)}| \Big), \tag{5.12}$$

for two positive constants C_1 and α. Therefore, by using that the number of trees of fixed order and fixed mode labels is bounded by C_2^k for some positive constant C_2 (taking into account the number of shapes of trees and the number of ways of assigning the scale labels in such a way that the corresponding tree value is not vanishing) and expanding each $c^{(k_v)}$ in terms of trees according to (4.12), as in [9], we obtain at the end, for suitable positive constants $\kappa' < \kappa$ and C_3, a bound $|u_{\boldsymbol{\nu}}^{(k)}| \le e^{-\kappa' |\boldsymbol{\nu}|} C_3^k k!^{\alpha}$, which reproduces the result in [1].

The only case in which we obtain bounds containing no factorial and hence we can deduce the convergence of the perturbative series is the case $d = 1$, where there are no small divisors (one can bound $|\boldsymbol{\omega} \cdot \boldsymbol{\nu}| = |\omega \nu| \ge |\omega|$), as it was already pointed out in [2].

6. Renormalized expansion

To prove theorem 2 we need a different tree expansion, that we envisage by starting from the present section.

We shall define new propagators $g_\ell^{[n_\ell]}$ iteratively. First some notations are needed.

Suppose that the node factors $c^{[k_v]}$ and the propagators $g_\ell^{[n_\ell]}$ are assigned. Given a self-energy graph T which does not contain any other self-energy graphs, define the *self-energy value* as

$$\mathcal{V}_T(\boldsymbol{\omega} \cdot \boldsymbol{\nu}; \varepsilon) = \varepsilon^{k_T} \Big(\prod_{\ell \in L(T)} g_\ell^{[n_\ell]} \Big) \Big(\prod_{v \in E(T) \cup V(T)} F_v \Big), \tag{6.1}$$

where F_v is defined as in (4.11) except for the white endpoints for which one has $F_v = c^{[k_v]}$, and $k_T = |B(T)| + \sum_{v \in E_W(\theta)} k_v \ge 1$ is called the *self-energy order* and represents the number of vertices and black endpoints in T plus the sum of the orders of the white endpoints in T; of course $\mathcal{V}_T(\boldsymbol{\omega} \cdot \boldsymbol{\nu}; \varepsilon)$ depends on $\boldsymbol{\omega} \cdot \boldsymbol{\nu}$ through the propagators of the lines in $L(T)$.

Define $\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu}}$ as the set of trees which do not contain any self-energy graphs (*renormalized trees*), and $\mathcal{S}^{\mathcal{R}}_{k,n}$ as the set of self-energy graphs of order k which do not contain any other self-energy graph and such that the maximum of the scales of the lines in T is exactly n (*renormalized self-energy graphs on scale* n).

Then we can define the *renormalized propagators* $g_\ell^{[n]} \equiv g^{[n]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell; \varepsilon)$ and the quantities $M^{[n]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell; \varepsilon)$ recursively as follows.

We set

$$\begin{aligned} & g^{[-1]}(x;\varepsilon) = 1, \qquad M^{[-1]}(x;\varepsilon) = 0, \\ & g^{[0]}(x;\varepsilon) = \frac{\psi_0(|x|)}{ix}, \qquad M^{[0]}(x;\varepsilon) = 2\varepsilon Q_{\mathbf{0}} c^{[0]} + \sum_{k=1}^{\infty} \sum_{T \in \mathcal{S}^{\mathcal{R}}_{k,0}} \mathcal{V}_T(x;\varepsilon), \end{aligned} \tag{6.2}$$

with $c^{[0]} = c^{(0)}$, as given by (4.7), while, for $n \ge 1$, we define

$$\begin{aligned} g^{[n]}(x;\varepsilon) &= \frac{\chi_0(|x|)\chi_1(|ix - \mathcal{M}^{[0]}(x;\varepsilon)|) \dots \chi_{n-1}(|ix - \mathcal{M}^{[n-2]}(x;\varepsilon)|)\psi_n(|ix - \mathcal{M}^{[n-1]}(x;\varepsilon)|)}{ix - \mathcal{M}^{[n-1]}(x;\varepsilon)}, \\ \mathcal{M}^{[n]}(x;\varepsilon) &= \mathcal{M}^{[n-1]}(x;\varepsilon) + M^{[n]}(x;\varepsilon)\chi_0(|x|)\chi_1(|ix - \mathcal{M}^{[0]}(x;\varepsilon)|) \dots \chi_n(|ix - \mathcal{M}^{[n-1]}(x;\varepsilon)|), \\ &= \sum_{j=0}^{n} M^{[j]}(x;\varepsilon)\chi_0(|x|)\chi_1(|ix - \mathcal{M}^{[0]}(x;\varepsilon)|) \dots \chi_j(|ix - \mathcal{M}^{[j-1]}(x;\varepsilon)|), \end{aligned} \tag{6.3}$$

$$M^{[n]}(x;\varepsilon)=\sum_{k=1}^{\infty}\sum_{T\in\mathcal{S}^{\mathcal{R}}_{k,n}}\mathcal{V}_T(x;\varepsilon),$$

where $\mathcal{V}_T(x;\varepsilon)$ is defined as in (6.1). Note that for all $n\geq 0$ and for all $T\in\mathcal{S}^{\mathcal{R}}_{k,n}$ one has $k_T\geq 2$, so that $M^{[0]}(x;\varepsilon)=2\varepsilon Q_{\mathbf{0}}c^{(0)}+O(\varepsilon^2)$ and $M^{[n]}(x;\varepsilon)=O(\varepsilon^2)$ for $n\geq 1$.

For instance one has

$$\begin{aligned} g^{[1]}(x;\varepsilon) &= \frac{\chi_0(|x|)\psi_1(|ix-\mathcal{M}^{[0]}(x;\varepsilon)|)}{ix-\mathcal{M}_0(x;\varepsilon)}, \\ g^{[2]}(x;\varepsilon) &= \frac{\chi_0(|x|)\chi_1(|ix-\mathcal{M}^{[0]}(x;\varepsilon)|)\psi_2(x-\mathcal{M}^{[1]}(x;\varepsilon))}{ix-\mathcal{M}^{[1]}(x;\varepsilon)}, \end{aligned} \tag{6.4}$$

with

$$\begin{aligned} &\mathcal{M}^{[0]}(x;\varepsilon)=M^{[0]}(x;\varepsilon)\chi_0(|x|), \\ &\mathcal{M}^{[1]}(x;\varepsilon)=\mathcal{M}^{[0]}(x;\varepsilon)+M^{[1]}(x;\varepsilon)\chi_0(|x|)\chi_1(|ix-\mathcal{M}^{[0]}(x;\varepsilon)|), \end{aligned} \tag{6.5}$$

and so on.

Note that if a line ℓ is on scale n and, by setting $x=\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell$, one has $g^{[n]}(x;\varepsilon)\neq 0$, this requires $\chi_0(|x|)\neq 0$, $\chi_1(|ix-\mathcal{M}^{[0]}(x;\varepsilon)|)\neq 0$, ..., $\chi_{n-1}(|ix-\mathcal{M}^{[n-2]}(x;\varepsilon)|)\neq 0$ and $\psi_n(|ix-\mathcal{M}^{[n-1]}(x;\varepsilon)|)\neq 0$, which means

$$\begin{aligned} &|x|\leq C_0, \\ &|ix-\mathcal{M}^{[0]}(x;\varepsilon)|\leq 2^{-1}C_0, \\ &|ix-\mathcal{M}^{[1]}(x;\varepsilon)|\leq 2^{-2}C_0, \\ &\cdots\cdots\cdots \\ &|ix-\mathcal{M}^{[n-2]}(x;\varepsilon)|\leq 2^{-(n-1)}C_0, \\ &|ix-\mathcal{M}^{[n-1]}(x;\varepsilon)|\geq 2^{-(n+1)}C_0, \end{aligned} \tag{6.6}$$

so that, in particular, if a line ℓ is on scale n, then one has $|g^{[n]}_\ell|\leq C_0^{-1}2^{n+1}$.

Then we define, formally, for $\boldsymbol{\nu}\neq\mathbf{0}$,

$$\begin{aligned} u^{[k]}_{\boldsymbol{\nu}} &= \sum_{\theta\in\Theta^{\mathcal{R}}_{k,\nu}}\mathrm{Val}(\theta), \\ \mathrm{Val}(\theta) &= \left(\prod_{\ell\in L(\theta)}g^{[n_\ell]}_\ell\right)\left(\prod_{v\in E(\theta)\cup V(\theta)}F_v\right), \end{aligned} \tag{6.7}$$

while, for $\boldsymbol{\nu}=\mathbf{0}$, one has

$$c^{[k]}=-\frac{1}{2c^{[0]}\langle Q\rangle}\sum_{\theta\in\Theta^{\mathcal{R}*}_{k,\mathbf{0}}}\mathrm{Val}(\theta), \tag{6.8}$$

where $\Theta^{\mathcal{R}*}_{k,\mathbf{0}}$ is defined as $\Theta^{*}_{k,\mathbf{0}}$, after (4.12) and (5.4), with the only difference that one has to consider renormalized trees instead of trees; of course this provides a recursive definition of the coefficients $c^{[k]}$, as both (6.7) and (6.8) depend on the values $c^{[k']}$ with $k'<k$.

Then we shall write

$$\overline{u}(t)=\sum_{k=1}^{\infty}\varepsilon^k u^{[k]}(t)=\sum_{k=1}^{\infty}\varepsilon^k\sum_{\boldsymbol{\nu}\in\mathbb{Z}^d}e^{i\boldsymbol{\nu}\cdot\boldsymbol{\omega}t}u^{[k]}_{\boldsymbol{\nu}},\tag{6.9}$$

where the coefficients $u^{[k]}_{\boldsymbol{\nu}}$ are defined through (6.7) and depend on ε (as the propagators do); note that the order k of a renormalized tree θ is still defined as $k=|B(\theta)|$, but it does not correspond anymore to the perturbative order.

Fix ε such that one has

$$\left|i\boldsymbol{\omega}\cdot\boldsymbol{\nu}-\mathcal{M}^{[n]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)\right|\geq C_0|\nu|^{-\tau_1}\qquad\forall\boldsymbol{\nu}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}\text{ and }\forall n\geq-1,\tag{6.10}$$

with Diophantine constants C_0 and τ_1, where C_0 is the same as in (2.6), while $\tau_1>\tau$ is to be fixed later. We call $\mathcal{E}$ the set of ε for which the Diophantine conditions (6.10) are satisfied.

We shall see in next section that for $\varepsilon\in\mathcal{E}$ we shall be able to give a meaning to the (so far formal) renormalized expansion (6.9), hence we shall prove that the set $\mathcal{E}$ has positive Lebesgue measure.

7. Convergence of the renormalized expansion

Now we study the renormalized expansion introduced in Section 6.

First we show that if the propagators satisfy the Diophantine conditions (6.10), with the functions $\mathcal{M}^{[n]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)$ well defined, smooth enough and small enough together their derivatives, then a bound like $|u^{[k]}_{\boldsymbol{\nu}}|\leq C^k e^{-\kappa'|\boldsymbol{\nu}|}$ follows for suitable constants C and κ'. By recalling the discussion in Section 5, we realize that it is sufficient to obtain a bound on the number of lines on fixed scale like (5.7): this will be the content of lemma 1 below. Then we prove inductively that the conditions on the functions $\mathcal{M}^{[n]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}:\varepsilon)$ are satisfied, provided that we exclude some values of the perturbative parameter ε: this will be done in lemma 2. The admissible values of ε are exactly the ones for which the Diophantine conditions (6.10) are satisfied. So we are left with the problem of studying how many values of ε are left, i.e. how large is the set $\mathcal{E}$ of admissible values of ε: through lemma 3 and lemma 4 we shall verify that $\mathcal{E}$ is a set with positive measure.

Lemma 1. *Assume that the set $\mathcal{E}$ has non-zero measure and that for all $\varepsilon\in\mathcal{E}$ the functions $\mathcal{M}^{[n]}(x;\varepsilon)$ are C^1 in x and satisfy the bounds*

$$\left|\mathcal{M}^{[n]}(x;\varepsilon)\right|\leq D|\varepsilon|,\qquad\left|\partial_x\mathcal{M}^{[n]}(x;\varepsilon)\right|\leq D|\varepsilon|,\tag{7.1}$$

for some constant D. Then for any renormalized tree θ such that $\mathrm{Val}(\theta)\neq0$ *the number $N_n(\theta)$ of lines on scale n satisfies the bound*

$$N_n(\theta)\leq c\,2^{-n/\tau_1}\sum_{v\in B(\theta)}|\boldsymbol{\nu}_v|,\tag{7.2}$$

for a suitable positive constant c.

Proof. We prove inductively on the order k of the renormalized trees the bound

$$N_n^*(\theta)\leq\max\{0,2|\boldsymbol{\nu}(\theta)|2^{(3-n)/\tau_1}-1\},\tag{7.3}$$

where $|\boldsymbol{\nu}(\theta)| \equiv \sum_{v\in B(\theta)} |\boldsymbol{\nu}_v|$ and $N_n^*(\theta)$ is the number of lines in $L(\theta)$ on scale $n' \geq n$.

If θ has $k = 1$ one has $B(\theta) = \{v\}$ and $|\boldsymbol{\nu}(\theta)| = |\boldsymbol{\nu}_v|$. In order that the line coming out from v be on scale $\geq n$ one must have $|i\boldsymbol{\omega}\cdot\boldsymbol{\nu}_v - \mathcal{M}^{[n-2]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_v;\varepsilon)| \leq 2^{-n+1}C_0$ (see (6.6)), hence, by the Diophantine conditions (6.10), $|\boldsymbol{\nu}_v| \geq 2^{(n-1)/\tau_1}$, which implies $2|\boldsymbol{\nu}(\theta)|2^{(3-n)/\tau_1} \geq 2 2^{2/\tau_1} \geq 2$. Therefore in such a case the bound (7.3) is trivially satisfied.

If θ is a renormalized tree of order $k > 1$, we assume that the bound holds for all renormalized trees of order $k' < k$. Define $E_n = (2\, 2^{(3-n)/\tau_1})^{-1}$: so we have to prove that $N_n^*(\theta) \leq \max\{0, |\boldsymbol{\nu}(\theta)|E_n^{-1} - 1\}$.

Call ℓ the root line of θ and $\ell_1,\ldots,\ell_m$ the $m \geq 0$ lines on scale $\geq n$ which are the closest to ℓ (i.e. such that no other line along the paths connecting the lines $\ell_1,\ldots,\ell_m$ to the root line is on scale $\geq n$).

If the root line ℓ of θ is on scale $n' < n$, then

$$N_n^*(\theta) = \sum_{i=1}^{m} N_n^*(\theta_i), \tag{7.4}$$

where θ_i is the renormalized subtree with ℓ_i as root line, hence the bound follows by the inductive hypothesis.

If the root line ℓ has scale $\geq n$, then $\ell_1,\ldots,\ell_m$ are the entering lines of a cluster T.

By denoting again with θ_i the renormalized subtree having ℓ_i as root line, one has

$$N_n^*(\theta) = 1 + \sum_{i=1}^{m} N_n^*(\theta_i), \tag{7.5}$$

so that the bound becomes trivial if either $m = 0$ or $m \geq 2$.

If $m = 1$ then one has a cluster T with two external lines ℓ and ℓ_1, which are both with scales $\geq n$; then

$$|i\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell - \mathcal{M}^{[n-2]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell;\varepsilon)| \leq 2^{-n+1}C_0, \qquad |i\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_1} - \mathcal{M}^{[n-2]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_1};\varepsilon)| \leq 2^{-n+1}C_0, \tag{7.6}$$

and $\boldsymbol{\nu}_\ell \neq \boldsymbol{\nu}_{\ell_1}$, otherwise T would be a self-energy graph. Then, by (7.6), one has

$$\begin{aligned} 2^{-n+2}C_0 &\geq \left|i\boldsymbol{\omega}\cdot(\boldsymbol{\nu}_\ell - \boldsymbol{\nu}_{\ell_1}) - \mathcal{M}^{[n-2]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell;\varepsilon) + \mathcal{M}^{[n-2]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_1};\varepsilon)\right| \\ &= |i\boldsymbol{\omega}\cdot(\boldsymbol{\nu}_\ell - \boldsymbol{\nu}_{\ell_1}) + \partial_x\mathcal{M}^{[n-2]}(x_*;\varepsilon)(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_1} - \boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell)| \\ &\geq |\boldsymbol{\omega}\cdot(\boldsymbol{\nu}_\ell - \boldsymbol{\nu}_{\ell_1})| - D\,|\varepsilon|\,\left|\boldsymbol{\omega}\cdot(\boldsymbol{\nu}_\ell - \boldsymbol{\nu}_{\ell_1})\right| \geq \frac{1}{2}\,|\boldsymbol{\omega}\cdot(\boldsymbol{\nu}_\ell - \boldsymbol{\nu}_{\ell_1})|\,, \end{aligned} \tag{7.7}$$

where x_* is a point between $\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell$ and $\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_1}$, and (7.1) has been used. By the Diophantine conditions (6.10), one has $|\boldsymbol{\nu}_\ell - \boldsymbol{\nu}_{\ell_1}| > 2^{(n-3)/\tau_1}$, so that

$$\sum_{v\in B(T)} |\boldsymbol{\nu}_v| \geq |\boldsymbol{\nu}_T| = |\boldsymbol{\nu}_\ell - \boldsymbol{\nu}_{\ell_1}| > 2^{(n-3)/\tau_1} > E_n, \tag{7.8}$$

hence $|\boldsymbol{\nu}(\theta)| - |\boldsymbol{\nu}(\theta_1)| > E_n$, which, inserted into (7.5) with $m = 1$, gives, by using the inductive hypothesis,

$$\begin{aligned} N_n^*(\theta) &= 1 + N_n^*(\theta_1) \leq 1 + |\boldsymbol{\nu}(\theta_1)|E_n^{-1} - 1 \\ &\leq 1 + \Big(|\boldsymbol{\nu}(\theta)| - E_n\Big)E_n^{-1} - 1 \leq |\boldsymbol{\nu}(\theta)|E_n^{-1} - 1, \end{aligned} \tag{7.9}$$

hence the bound is proved also if the root line is on scale $\geq n$. ■

Lemma 2. *For $\varepsilon \in \mathcal{E}$ and for x such that $g^{[n]}(x;\varepsilon) \neq 0$, there exist two constants D and D' such that the functions $\mathcal{M}^{[j]}(x;\varepsilon)$ are smooth functions of x and satisfy the bounds*

$$\begin{aligned} &\left|\mathcal{M}^{[j]}(x;\varepsilon)\right| \leq D|\varepsilon|, \qquad \left|\partial_x \mathcal{M}^{[j]}(x;\varepsilon)\right| \leq D|\varepsilon|, \\ &\left|\mathcal{M}^{[j]}(x;\varepsilon) - \mathcal{M}^{[j-1]}(x;\varepsilon)\right| \leq D|\varepsilon| e^{-D' 2^{j/\tau_1}}, \end{aligned} \tag{7.10}$$

for all $0 \leq j \leq n-1$.

Proof. The proof is by induction on j. For $j=0$ the bounds (7.10) are trivially satisfied; then, assuming that the bounds holds for all $j' < j$, for some $j \leq n-1$, we want to show that they follow also for j.

The quantity $M^{[j]}(x;\varepsilon)$ is given by

$$M^{[j]}(x;\varepsilon) = \sum_{k=1}^{\infty} \sum_{T \in \mathcal{S}^{\mathcal{R}}_{k,j}} \mathcal{V}_T(x;\varepsilon), \tag{7.11}$$

with x satisfying the bounds (6.6) by hypothesis; in particular one has $|ix - \mathcal{M}^{[j-2]}(x;\varepsilon)| < 2^{-j+1}C_0$.

We want to show, by *reductio ad absurdum*, that for all $T \in \mathcal{S}^{\mathcal{R}}_{k,j}$ contributing to $M^{[j]}(x;\varepsilon)$ through the self-energy value $\mathcal{V}_T(x;\varepsilon)$, one must have

$$\sum_{v \in B(T)} |\boldsymbol{\nu}_v| > 2^{(j-4)/\tau_1}. \tag{7.12}$$

By construction all renormalized self-energy graphs in $\mathcal{S}^{\mathcal{R}}_{k,j}$ must contain at least one line ℓ on scale $n_\ell = j$. Therefore for such a line one has (see (6.6))

$$|i\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell - \mathcal{M}^{[j-2]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell;\varepsilon)| \leq 2^{-j+1}C_0. \tag{7.13}$$

Furthermore, by the inductive hypothesis, the quantity $\mathcal{M}^{[j-2]}(x;\varepsilon)$ is smooth in x, so that one can write

$$\mathcal{M}^{[j-2]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell;\varepsilon) = \mathcal{M}^{[j-2]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon) + \partial_x \mathcal{M}^{[j-2]}(x_*;\varepsilon)\,\boldsymbol{\omega}\cdot(\boldsymbol{\nu}_\ell - \boldsymbol{\nu}), \tag{7.14}$$

where x_* is a point between $\boldsymbol{\omega}\cdot\boldsymbol{\nu}$ and $\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell$.

We can write $\boldsymbol{\nu}_\ell = \boldsymbol{\nu}^0_\ell + \sigma_\ell \boldsymbol{\nu}$, where, if we write as usual $\ell = \ell_v$,

$$\boldsymbol{\nu}^0_\ell = \sum_{\substack{w \in B(T) \\ w \preceq v}} \boldsymbol{\nu}_w, \tag{7.15}$$

and $\sigma_\ell = 1$ if the line entering T is comparable with ℓ and $\sigma_\ell = 0$ otherwise.

Note that if (7.12) does not hold then one has $|i\boldsymbol{\omega}\cdot\boldsymbol{\nu}^0_\ell - \mathcal{M}^{[n]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}^0_\ell;\varepsilon)| \geq 2^{4-j}C_0$ for all $n \geq -1$, by the Diophantine conditions (6.10). Then if $\sigma_\ell = 0$ one has $\boldsymbol{\nu}_\ell = \boldsymbol{\nu}^0_\ell$, hence $|i\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell - \mathcal{M}^{[j-2]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell;\varepsilon)| \geq 2^{4-j}C_0$, while if $\sigma_\ell = 1$ one has, by using the inductive hypothesis,

$$\begin{aligned} |i\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell - \mathcal{M}^{[j-2]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell;\varepsilon)| &= |i\boldsymbol{\omega}\cdot\boldsymbol{\nu}^0_\ell + i\boldsymbol{\omega}\cdot\boldsymbol{\nu} - \mathcal{M}^{[j-2]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon) - \partial_x\mathcal{M}^{[j-2]}(x_*;\varepsilon)\,\boldsymbol{\omega}\cdot\boldsymbol{\nu}^0_\ell| \\ &\geq |\boldsymbol{\omega}\cdot\boldsymbol{\nu}^0_\ell| - |i\boldsymbol{\omega}\cdot\boldsymbol{\nu} - \mathcal{M}^{[j-2]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)| - D\,|\varepsilon|\,|\boldsymbol{\omega}\cdot\boldsymbol{\nu}^0_\ell| \\ &\geq \frac{1}{2}|\boldsymbol{\omega}\cdot\boldsymbol{\nu}^0_\ell| - |i\boldsymbol{\omega}\cdot\boldsymbol{\nu} - \mathcal{M}^{[j-2]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)| \geq 2^{-1}2^{4-j}C_0 - 2^{1-j}C_0 > 2^{1-j}C_0, \end{aligned} \tag{7.16}$$

which are both in contradiction with (7.13). Therefore (7.12) follows.

By reasoning as in the proof of lemma 1 one obtains that, if we denote with $N_{j'}(T)$ the number of lines on scale j' contained in $T \in \mathcal{S}^{\mathcal{R}}_{k,j}$, one has

$$N_{j'}(T) \le c\, 2^{-j'/\tau_1} \sum_{v \in B(T)} |\boldsymbol{\nu}_v|. \tag{7.17}$$

More precisely (and more generally), if T is a connected subset of lines and nodes in a tree such that
(1) T has only one exiting line and only one entering line both on scales $\ge j'$,
(2) all the lines in $L(T)$ are on scale $\le j$,
then, by denoting with $N^*_{j'}(T)$ the number of lines on scale $\ge j'$ contained in T and defining $|\boldsymbol{\nu}(T)| = \sum_{v \in B(T)} |\boldsymbol{\nu}_v|$, one can prove inductively on the order of T the bound $N^*_{j'}(T) \le \max\{0, 2|\boldsymbol{\nu}(T)|2^{(3-j')/\tau_1} - 1\}$ for all $j' \le j$, by reasoning as follows. Consider the path $\mathcal{P}$ formed by the lines connecting the entering line with the exiting line of T, and call $V(\mathcal{P})$ the vertices connected by such lines. If all the lines $\ell \in \mathcal{P}$ are on scales $n_\ell < j'$ then one has $N^*_{j'}(T) = \sum_{i=1}^m N^*_{j'}(\theta_i)$, where $\theta_1, \dots, \theta_m$ are the trees inside T with root in a vertex $v \in V(\mathcal{P})$, so that the bound follows from (the proof of) lemma 1, i.e. from the bound (7.3).[1] If there is at least one line $\ell \in \mathcal{P}$ on scale $\ge j'$, call T_1 and T_2 the connected subsets of T such that $L(T) = \{\ell\} \cup L(T_1) \cup L(T_2)$. If both T_1 and T_2 contain at least a line on scale $\ge j'$, then they have the same structure of T, i.e. they are subsets of lines (on scales $\le j$) and nodes with only one exiting line and only one entering line both on scales $\ge j'$, so that by the inductive hypothesis one has $N_{j'}(T) \le 1 + N_{j'}(T_1) + N_{j'}(T_2) \le 1 + (2|\boldsymbol{\nu}(T_1)|2^{(3-j')/\tau_1} - 1) + (2|\boldsymbol{\nu}(T_2)|2^{(3-j')/\tau_1} - 1) \le 2|\boldsymbol{\nu}(T)|2^{(3-j')/\tau_1} - 1$. If only the subset T_2 contains at least a line on scale $\ge j'$, then we can reason as in deriving (7.12) through (7.16) to conclude that one must have $|\boldsymbol{\nu}(T_1)| > 2^{(j'-4)/\tau_1}$, hence $N_{j'}(T) = 1 + N_{j'}(T_2) \le 1 + (2|\boldsymbol{\nu}(T_2)|2^{(3-j')/\tau_1} - 1)$ $\le 2|\boldsymbol{\nu}(T)|2^{(3-j')/\tau_1} - 2|\boldsymbol{\nu}(T_1)|2^{(3-j')/\tau_1} \le 2|\boldsymbol{\nu}(T)|2^{(3-j')/\tau_1} - 1$; analogously one discusses the case in which only the set T_1 contains at least a line on scale $\ge j'$, and the case in which both sets do not contain any line on scale $\ge j'$. Hence the bound on $N^*_{j'}(T)$ is proved also in the case in which there is at least one line $\ell \in \mathcal{P}$ on scale $\ge j'$. In a renormalized self-energy graph $T \in \mathcal{S}^{\mathcal{R}}_{k,j}$ with external lines on scale n all the lines $\ell \in L(T)$ are on scale $j' \le j$, so that for all $j' \le j$ the renormalized self-energy graph $T \in \mathcal{S}^{\mathcal{R}}_{k,j}$ is a subset verifying the properties (1) and (2), and we can apply the above result, so that the bound (7.17) follows.

Therefore we see that (7.12) and (7.17) imply, for all $T \in \mathcal{S}^{\mathcal{R}}_{k,j}$,

$$|\mathcal{V}_T(x;\varepsilon)| \le |\varepsilon|^k A_1 A_2^k e^{-A_3 2^{j/\tau_1}} \prod_{v \in B(T)} e^{-\kappa|\boldsymbol{\nu}_v|/2}, \tag{7.18}$$

for suitable constants A_1, A_2 and A_3; this can be easily obtained from the definitions (6.1) and (6.3), by reasoning as in deducing (5.8) and using the bound (2.9) for the node factors.

[1] Note that, even if in the statement of lemma 1, one requires that the bounds (7.1) hold for all n, what is really needed is that they hold for all n such that $N_{n+1}(\theta) \neq 0$. Therefore we can apply lemma 1 to the trees $\theta_1, \dots, \theta_m$ because for each line $\ell \in L(T)$ one has $n_\ell \le j$ and the bounds (7.1) hold for all $j' < j$ by the inductive hypothesis.

By inserting the bound (7.18) into (7.11) and using the definitions (6.3), we obtain

$$\begin{aligned} \left|\mathcal{M}^{[j]}(x;\varepsilon)-\mathcal{M}^{[j-1]}(x;\varepsilon)\right| &\le \left|M^{[j]}(x;\varepsilon)\right| \le \sum_{k=1}^{\infty} D_1 D_2^k |\varepsilon|^k e^{-D_3 2^{j/\tau_1}}, \\ \left|\mathcal{M}^{[j]}(x;\varepsilon)\right| \le \sum_{i=0}^{j}\left|M^{[i]}(x;\varepsilon)\right| &\le \sum_{k=1}^{\infty} D_1 D_2^k|\varepsilon|^k \sum_{i=0}^{j} e^{-D_3 2^{i/\tau_1}} \le \sum_{k=1}^{\infty}\tilde{D}_1 D_2^k |\varepsilon|^k, \end{aligned} \tag{7.19}$$

for suitable constants D_1, $\tilde{D}_1$, D_2 and D_3; this proves the first and third bounds in (7.10). Note that in the first of (7.19) we can let the sum to start from $k=2$ for all $j\ge 1$ as any renormalized self-energy graph of order ≥ 1 has to contain at least two nodes (see comments after (6.3)).

To prove the second bound in (7.10) we use the second line in the definition of $\mathcal{M}^{[n]}(x;\varepsilon)$ in (6.3), the regularity of the functions χ_n and ψ_n, and the inductive hypothesis. One has

$$\begin{aligned} \partial_x \mathcal{M}^{[j]}(x;\varepsilon) = \sum_{j'=0}^{j} \Big(& \chi_0(|x|)\ldots\chi_{j'}(|ix-\mathcal{M}^{[j'-1]}(x;\varepsilon)|)\partial_x M^{[j']}(x;\varepsilon) \\ & + M^{[j']}(x;\varepsilon)\sum_{i=0}^{j'}\chi_0(|x|)\ldots\partial\chi_i(|ix-\mathcal{M}^{[i-1]}(x;\varepsilon)|)\ldots\chi_{j'}(|ix-\mathcal{M}^{[j'-1]}(x;\varepsilon)|)\Big), \end{aligned} \tag{7.20}$$

where $\partial\chi_i$ denotes the derivative of χ_i with respect to its argument, and

$$\begin{aligned} \partial_x M^{[j']}(x;\varepsilon) &= \sum_{k=1}^{\infty}\sum_{T\in\mathcal{S}^{\mathcal{R}}_{k,j'}} \partial_x \mathcal{V}_T(x;\varepsilon), \\ \partial_x\mathcal{V}_T(x;\varepsilon) &= \varepsilon^{k_T}\Big(\sum_{\ell\in L(T)}\big(\partial_x g^{[n_\ell]}\big)\prod_{\ell'\in L(T)\setminus\ell} g_{\ell'}^{[n_{\ell'}]}\Big)\Big(\prod_{v\in E(T)\cup V(T)} F_v\Big), \end{aligned} \tag{7.21}$$

so that one has to evaluate the derivatives of the propagators.

By using the definition of $g^{[n]}(x;\varepsilon)$ in in (6.3) one has[2]

$$\begin{aligned} \partial_x g^{[n]}(x;\varepsilon) = & -\frac{\chi_0(|x|)\ldots\psi_n(|ix-\mathcal{M}^{[n-1]}(x;\varepsilon)|)}{(ix-\mathcal{M}^{[n-1]}(x;\varepsilon))^2}\partial_x(ix-\mathcal{M}^{[n-1]}(x;\varepsilon)) \\ & +\sum_{j=0}^{n}\frac{\chi_0(|x|)\ldots\partial\chi_j(|ix-\mathcal{M}^{[j-1]}(x;\varepsilon)|)\ldots\psi_n(|ix-\mathcal{M}^{[n-1]}(x;\varepsilon)|)}{ix-\mathcal{M}^{[n-1]}(x;\varepsilon)}\partial_x|ix-\mathcal{M}^{[j-1]}(x;\varepsilon)|, \end{aligned} \tag{7.22}$$

which, by using the fact that $|\partial\chi_j|\le 2^j C_0^{-1}\Xi$ and $|\partial\psi_j|\le 2^j C_0^{-1}$, for some positive constant Ξ, and the inductive hypothesis, can be bounded as

$$\begin{aligned} |\partial_x g^{[n]}(x;\varepsilon)| &\le \tilde{A}\Big(\frac{1}{C_0^2 2^{-2(n+1)}}+\sum_{j=0}^{n}\frac{C_0^{-1}2^j}{C_0 2^{-(n+1)}}\Big) \\ &\le \tilde{A}C_0^{-2}2^{2(n+1)}\Big(1+\sum_{j=0}^{n}2^{j-(n+1)}\Big)\le AC_0^{-2}2^{2(n+1)}, \end{aligned} \tag{7.23}$$

[2] With obvious interpretation of the term with $j=n$ in the last sum.

for some constants $\tilde{A}$ and A.

Therefore we can bound $\partial_x \mathcal{V}_T(x;\varepsilon)$ in (7.21) by

$$
\begin{aligned}
|\partial_x \mathcal{V}_T(x;\varepsilon)| \le \varepsilon^{k_T} \Big(\sum_{\ell \in L(T)} A C_0^{-2} 2^{2(n_\ell+1)} \prod_{\ell' \in L(T)\setminus \ell} C_0^{-1} 2^{n_{\ell'}+1} \Big) \\
\mathcal{Q}^{k_T} \Big(\prod_{v \in B(T)} e^{-\kappa |\boldsymbol{\nu}_v|} \Big) \Big(\prod_{v \in E_W(\theta)} c^{[k_v]} \Big) \le \varepsilon^{k_T} \tilde{A}_1 \tilde{A}_2^{k_T} e^{-A_3 \kappa 2^{j'/\tau_1}},
\end{aligned} \tag{7.24}
$$

where we have used also (7.12), for suitable constants $\tilde{A}_1$ and $\tilde{A}_2$,[3] and a bound $|c^{(k_v)}| < C^{k_v}$ can be inductively assumed.

Then (7.24) implies immediately the bound, for suitable constants $\tilde{D}_1$, $\tilde{D}_2$, $\tilde{D}_3$ and $\tilde{D}$,

$$
|\partial_x M^{[j']}(x;\varepsilon)| \le \sum_{k=2}^{\infty} |\varepsilon|^k \tilde{D}_1 \tilde{D}_2^k e^{-\tilde{D}_3 2^{j'/\tau_1}} \le |\varepsilon|^2 \tilde{D} e^{-\tilde{D}_3 2^{j'/\tau_1}}, \tag{7.25}
$$

which, together with (7.20), yields

$$
|\partial_x \mathcal{M}^{[j]}(x;\varepsilon) \le \sum_{j'=0}^{j} \Big(|\varepsilon|^2 \tilde{D} e^{-\tilde{D}_3 2^{j'/\tau_1}} + |\varepsilon| \tilde{D} e^{-D_3 2^{j'/\tau_1}} \sum_{i=0}^{j'} 2^i \Big) \le |\varepsilon| D, \tag{7.26}
$$

provided that D is large enough, so that the second bound in (7.10) follows. ■

To apply the above results and conclude the proof of theorem 2, we have still to construct the set $\mathcal{E}$ for which the Diophantine conditions (6.10) hold, and to show that such a set has positive measure.

Define recursively the sets $\mathcal{E}^{[n]}$ as follows.

Fix ε_0 such that the series

$$
\begin{aligned}
\sum_{k=1}^{\infty} \varepsilon^k \sum_{\boldsymbol{\nu} \in \mathbb{Z}^d} e^{i \boldsymbol{\omega} \cdot \boldsymbol{\nu} t} \overline{u}_{\boldsymbol{\nu}}^{(k)}, \\
\overline{u}_{\boldsymbol{\nu}}^{(k)} = \sum_{\theta \in \Theta_{k,\nu}^{\mathcal{R}}} \overline{\mathrm{Val}}(\theta),
\end{aligned} \tag{7.27}
$$

obtained from (6.9), with the definitions (6.7), by replacing the propagators $g^{[n_\ell]}$ with $g^{(n_\ell)}$ (which is equal to the series obtained from (4.1), with the definitions (4.10), by discarding all trees containing self-energy graphs), converges for $|\varepsilon| \le \varepsilon_0$. Therefore $\overline{\mathrm{Val}}(\theta)$ is a numerical value satisfying the bound $|\overline{\mathrm{Val}}(\theta)| \le C^k e^{-\kappa' |\boldsymbol{\nu}|}$, for some constant C, as we can prove by reasoning as in Section 5 and bounding the product of the propagators through the bound (7.2) of lemma 1 (equivalently through the bound (5.7)).

[3] With respect to the bound (7.18) we have the extra difficulty that, in order to prove the bound (7.17), when using the inequality like (7.16) with $\ell \in \mathcal{P}$ on scale $\ge j'$, the quantity $\boldsymbol{\omega} \cdot \boldsymbol{\nu}$ has to be replaced with a continuously varying x. Nevertheless, as in the previous case, one has $|ix - \mathcal{M}^{[j'-2]}(x;\varepsilon)| < 2^{-j'+1} C_0$, by the support properties of the functions χ_n, so that (7.16) still applies when needed, and the same conclusions still hold.

Set

$$\mathcal{E}^{[0]} = (-\varepsilon_0, \varepsilon_0), \tag{7.28}$$

and, for $n \geq 1$,

$$\mathcal{E}^{[n]} = \left\{ \varepsilon \in \mathcal{E}^{[n-1]} \;:\; |i\boldsymbol{\omega} \cdot \boldsymbol{\nu} - \mathcal{M}^{[n]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}; \varepsilon)| > C_0 |\nu|^{-\tau_1} \right\}, \tag{7.29}$$

for a suitable Diophantine constant τ_1 (to be fixed later); finally define

$$\mathcal{E} = \bigcap_{n=0}^{\infty} \mathcal{E}^{[n]} = \lim_{n \to \infty} \mathcal{E}^{[n]}. \tag{7.30}$$

Lemma 3. *The function $M^{[n]}(x; \varepsilon)$ is C^1-extendible in the sense of Whitney outside $\mathcal{E}^{[n-1]}$, and for all $\varepsilon, \varepsilon' \in \mathcal{E}^{[n-1]}$ one has*

$$M^{[n]}(x; \varepsilon') - M^{[n]}(x; \varepsilon) = (\varepsilon' - \varepsilon)\, \partial_\varepsilon M^{[n]}(x; \varepsilon) + O((\varepsilon' - \varepsilon)^2), \tag{7.31}$$

where $\partial_\varepsilon M^{[n]}(x; \varepsilon)$ is the formal derivative with respect to ε of $M^{[n]}(x; \varepsilon)$.

Proof. The proof is by induction on n. Both $M^{[n]}(x; \varepsilon)$ and $M^{[n]}(x; \varepsilon')$ can be expressed by the last equation in (6.3): the only difference is that one has to replace ε with ε' for $M^{[n]}(x; \varepsilon')$. This means that there is a correspondence one-to-one between the graphs contributing to $M^{[n]}(x; \varepsilon)$ and those contributing to $M^{[n]}(x; \varepsilon')$, so that we can write

$$\begin{aligned} M^{[n]}(x; \varepsilon') - M^{[n]}(x; \varepsilon) = \sum_{k=1}^{\infty} \sum_{T \in S^{\mathcal{R}}_{k,n}} \varepsilon^{k_T} \Big(\prod_{v \in E(T) \cup V(T)} F_v \Big) \\ \Big[\Big(\prod_{\ell \in L(T)} g^{[n_\ell]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell; \varepsilon') \Big) - \Big(\prod_{\ell \in L(T)} g^{[n_\ell]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell; \varepsilon) \Big) \Big], \end{aligned} \tag{7.32}$$

where of course $k_T = k$.

Let us call $\Lambda(T)$ the set of lines in $L(T)$ coming out from nodes in $B(T)$. We can order the $|B(T)| - 1$ lines in $\Lambda(T)$ and construct a set of $|B(T)|$ subsets $\Lambda_1(T), \ldots, \Lambda_{|B(T)|}(T)$ of $\Lambda(T)$, with $|\Lambda_j(T)| = j$, in the following way. Set $\Lambda_1(T) = \emptyset$, $\Lambda_2(T) = \ell_1$, if ℓ_1 is the root line of θ and, inductively for $2 \leq j \leq |B(T)| - 1$, $\Lambda_{j+1}(T) = \Lambda_j(T) \cup \ell_j$, where the line $\ell_j \in \Lambda(T) \setminus \Lambda_j(T)$ is connected to $\Lambda_j(T)$. Then in (7.32) we have

$$\begin{aligned} &\Big(\prod_{\ell \in \Lambda(T)} g^{[n_\ell]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell; \varepsilon') \Big) - \Big(\prod_{\ell \in \Lambda(T)} g^{[n_\ell]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell; \varepsilon) \Big) \\ &\quad = \sum_{j=1}^{|B(T)|} \Big[\Big(\prod_{\ell \in \Lambda_j(T)} g^{[n_\ell]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell; \varepsilon') \Big) \\ &\qquad \Big(g^{[n_{\ell_j}]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_{\ell_j}; \varepsilon') - g^{[n_{\ell_j}]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_{\ell_j}; \varepsilon) \Big) \Big(\prod_{\ell \in \Lambda(T) \setminus (\Lambda_j(T) \cup \ell_j)} g^{[n_\ell]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell; \varepsilon) \Big) \Big], \end{aligned} \tag{7.33}$$

where, by setting $n_j = n_{\ell_j}$, $x_j = \boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell_j}$, $X_s(\varepsilon) = \chi_s(|ix_j - \mathcal{M}^{[s-1]}(x_j;\varepsilon)|)$ for $s = 1,\ldots,n_j-1$, and $\Psi_{n_j}(\varepsilon) = \psi_{n_j}(|ix_j - \mathcal{M}^{[n_j-1]}(x_j;\varepsilon)|)$, we can write (see the first equation in (6.3))

$$
\begin{aligned}
g^{[n_j]}(x_j;\varepsilon') - g^{[n_j]}(x_j;\varepsilon) &= \frac{\left(\mathcal{M}^{[n_j-1]}(x_j;\varepsilon') - \mathcal{M}^{[n_j-1]}(x_j;\varepsilon)\right)\chi_0(|x_j|)X_1(\varepsilon)\ldots X_{n_j-1}(\varepsilon)\Psi_{n_j}(\varepsilon)}{(ix - \mathcal{M}^{[n_j-1]}(x_j;\varepsilon'))(ix - \mathcal{M}^{[n_j-1]}(x_j;\varepsilon))} \\
&+ \sum_{s=1}^{n_j-1} \frac{\chi_0(|x_j|)\ldots X_{s-1}(\varepsilon')\left(X_s(\varepsilon') - X_s(\varepsilon)\right)X_{s+1}(\varepsilon)\ldots\Psi_{n_j}(\varepsilon)}{ix - \mathcal{M}^{[n_j-1]}(x_j;\varepsilon')} \\
&+ \frac{\chi_0(|x_j|)X_1(\varepsilon')\ldots X_{n_j-1}(\varepsilon')\left(\Psi_{n_j}(\varepsilon') - \Psi_{n_j}(\varepsilon)\right)}{ix - \mathcal{M}^{[n_j-1]}(x_j;\varepsilon')} .
\end{aligned}
\tag{7.34}
$$

By writing "symbolically"

$$
\begin{aligned}
X_s(\varepsilon') - X_s(\varepsilon) &= \partial\chi_s(|ix_j - \mathcal{M}^{[s-1]}(x_j;\varepsilon_*)|)\left(\mathcal{M}^{[s-1]}(x_j;\varepsilon') - \mathcal{M}^{[s-1]}(x_j;\varepsilon)\right), \\
\Psi_{n_j}(\varepsilon') - \Psi_{n_j}(\varepsilon) &= \partial\psi_{n_j}(|ix_j - \mathcal{M}^{[n_j-1]}(x_j;\varepsilon'_*)|)\left(\mathcal{M}^{[n_j-1]}(x_j;\varepsilon') - \mathcal{M}^{[n_j-1]}(x_j;\varepsilon)\right),
\end{aligned}
\tag{7.35}
$$

where ε_* and ε'_* are two suitable values (depending on j and s) between ε and ε', we can use the inductive hypothesis for all differences $M^{[j']}(x_j;\varepsilon') - M^{[j']}(x_j;\varepsilon)$, with $j' \le n-1$, appearing in (7.34) and (7.35), so that (7.31) follows, by defining

$$
\begin{aligned}
\partial_\varepsilon M^{[n]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon) &= \sum_{T\in\mathcal{S}^{\mathcal{R}}_{k,n}} \partial_\varepsilon \mathcal{V}^{[k]}_T(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon), \\
\partial_\varepsilon \mathcal{V}^{[k]}_T(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon) &= k_T\varepsilon^{k_T-1}\Big(\prod_{v\in E(T)\cup V(T)} F_v\Big)\Big(\prod_{\ell\in L(T)} g^{[n_\ell]}_\ell\Big) \\
&\quad + \varepsilon^{k_T}\Big(\prod_{v\in E(T)\cup V(T)} F_v\Big)\sum_{\ell\in\Lambda(T)}\Big[\Big(\partial_\varepsilon g^{[n_\ell]}_\ell\Big)\Big(\prod_{\ell'\in\Lambda(T)\setminus\ell} g^{[n_{\ell'}]}_{\ell'}\Big)\Big],
\end{aligned}
\tag{7.36}
$$

where

$$
\begin{aligned}
\partial_\varepsilon g^{[n]}_\ell &= \frac{g^{[n]}_\ell}{ix - \mathcal{M}^{[n-1]}(x;\varepsilon)}\left(\partial_\varepsilon\mathcal{M}^{[n-1]}(x;\varepsilon)\right) \\
&\quad - g^{[n]}_\ell\Big(\sum_{j=1}^{n-1}\frac{\partial\chi_j(|ix - \mathcal{M}^{[j-2]}(x;\varepsilon)|)}{\chi_j(|ix - \mathcal{M}^{[j-2]}(x;\varepsilon)|)}\partial_\varepsilon\left|ix - \mathcal{M}^{[j-2]}(x;\varepsilon)\right| \\
&\qquad + \frac{\partial\psi_n(|ix - \mathcal{M}^{[n-1]}(x;\varepsilon)|)}{\psi_n(|ix - \mathcal{M}^{[n-1]}(x;\varepsilon)|)}\partial_\varepsilon\left|ix - \mathcal{M}^{[n-1]}(x;\varepsilon)\right|
\end{aligned}
\tag{7.37}
$$

denotes the formal derivative of the propagator. Moreover $M^{[n]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)$ is defined for all $\varepsilon\in\mathcal{E}^{[n-1]}$ and it can be extended by continuity to its closure $\overline{\mathcal{E}^{[n-1]}}$ and hence to the full $\overline{\mathcal{E}^{[0]}}$; its extension is then C^1 in the sense of Whitney, [11], and satisfies the same bounds (7.10). ■

Therefore for all $\varepsilon\in\mathcal{E}^{[n-1]}$ the quantities $\mathcal{M}^{[n]}(x;\varepsilon)$ are well defined and formally differentiable (in the sense of Whitney), so that one has

$$
\frac{B}{2} \le \left|\partial_\varepsilon\mathcal{M}^{[n]}(x;\varepsilon)\right| \le 2B,
\tag{7.38}
$$

for a suitable positive constant B, while the propagators admit the bounds

$$|g_\ell^{[n]}| \le 2^{n+1} C_0^{-1}, \qquad |\partial_\varepsilon g_\ell^{[n]}| \le G 2^{2(n+1)} C_0^{-2}, \tag{7.39}$$

for a suitable constant G, as it can be easily obtained by reasoning as in the proof of lemma 2; the bounds (7.38) and (7.39) follow inductively from the formulae and the analysis performed along the proof of the above lemma, and from the definitions (6.1) and (6.3).

Lemma 4. *There are two positive constants b and ξ such that, for ε_0 small enough, one has*

$$\text{meas}(\mathcal{E}) \ge \varepsilon_0 \left(1 - b\varepsilon_0^{\xi}\right), \tag{7.40}$$

where meas *denotes the Lebesgue measure.*

Proof. Define

$$\begin{cases} \mathcal{I}^{[0]} = \emptyset, \\ \mathcal{I}^{[n]} = \mathcal{E}^{[n-1]} \setminus \mathcal{E}^{[n]}, \quad \text{for } n \ge 1; \end{cases} \tag{7.41}$$

note that $\mathcal{I} \equiv \cup_{n=0}^{\infty} \mathcal{I}^{[n]} = (-\varepsilon_0, \varepsilon_0) \setminus \mathcal{E}$. We shall prove that one has

$$\text{meas}(\mathcal{I}^{[n]}) \le b' \varepsilon_0^{1+\xi'} \qquad \forall n \ge 0, \tag{7.42}$$

for suitable positive constants b' and ξ'.

For all $n \ge 1$ and for all $\boldsymbol{\nu} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$ define

$$I^{[n]}(\boldsymbol{\nu}) = \left\{ \varepsilon \in \mathcal{E}^{[n-1]} \;:\; \left| i\boldsymbol{\omega} \cdot \boldsymbol{\nu} - \mathcal{M}^{[n]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}; \varepsilon) \right| \le C_0 |\boldsymbol{\nu}|^{-\tau_1} \right\}. \tag{7.43}$$

Each set $I^{[n]}(\boldsymbol{\nu})$ has center in a point $\varepsilon^{[n]}(\boldsymbol{\nu})$. We can easily prove that there exist two positive constants B_1 and B_2 such that one has

$$\left| \varepsilon^{[n]}(\boldsymbol{\nu}) - \varepsilon^{[n-1]}(\boldsymbol{\nu}) \right| \le \varepsilon_0 B_1 e^{-B_2 2^{n/\tau_1}} \tag{7.44}$$

for all $n \ge 2$ and for all $\boldsymbol{\nu} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$. By definition one has

$$i\boldsymbol{\omega} \cdot \boldsymbol{\nu} - \mathcal{M}^{[n]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}; \varepsilon^{[n]}(\boldsymbol{\nu})) = 0, \tag{7.45}$$

where we are using the Whitney extension of $\mathcal{M}^{[n]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}; \varepsilon)$ outside $\mathcal{E}^{[n-1]}$, so that, by setting $\delta\varepsilon = \varepsilon^{[n]}(\boldsymbol{\nu}) - \varepsilon^{[n-1]}(\boldsymbol{\nu})$, one obtains

$$\begin{aligned} 0 &= i\boldsymbol{\omega} \cdot \boldsymbol{\nu} - \mathcal{M}^{[n]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}; \varepsilon^{[n]}(\boldsymbol{\nu})) = i\boldsymbol{\omega} \cdot \boldsymbol{\nu} - \mathcal{M}^{[n-1]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}; \varepsilon^{[n-1]}(\boldsymbol{\nu}) + \delta\varepsilon) \\ &\quad - \mathcal{M}^{[n]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}; \varepsilon^{[n]}(\boldsymbol{\nu})) + \mathcal{M}^{[n-1]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}; \varepsilon^{[n]}(\boldsymbol{\nu})) \\ &= -\partial_\varepsilon \mathcal{M}^{[n-1]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}; \varepsilon^{[n-1]}(\boldsymbol{\nu}))\, \delta\varepsilon + O((\delta\varepsilon)^2) \\ &\quad - \left(\mathcal{M}^{[n]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}; \varepsilon^{[n]}(\boldsymbol{\nu})) - \mathcal{M}^{[n-1]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}; \varepsilon^{[n]}(\boldsymbol{\nu})) \right), \end{aligned} \tag{7.46}$$

by lemma 3; therefore one can use that one has

$$\left| \mathcal{M}^{[n]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}; \varepsilon^{[n]}(\boldsymbol{\nu})) - \mathcal{M}^{[n-1]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}; \varepsilon^{[n]}(\boldsymbol{\nu})) \right| \le \varepsilon_0 D_1 e^{-D_2 2^{n/\tau_1}}, \tag{7.47}$$

by lemma 2, and

$$\left|\partial_\varepsilon \mathcal{M}^{[n-1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon^{[n-1]}(\boldsymbol{\nu}))\right| > \frac{B}{2}, \tag{7.48}$$

by (7.38), so that (7.44) follows.

Therefore one has to exclude from the set $\mathcal{E}^{[n-1]}$ all the values ε around $\varepsilon^{[n]}(\boldsymbol{\nu})$ in $I^{[n]}(\boldsymbol{\nu})$, which gives a set of measure

$$\int_{I^{[n]}(\boldsymbol{\nu})} \mathrm{d}\varepsilon = \int_{-1}^{1} \mathrm{d}t \frac{\mathrm{d}\varepsilon(t)}{\mathrm{d}t}, \tag{7.49}$$

where $\varepsilon(t)$ is defined by

$$i\boldsymbol{\omega}\cdot\boldsymbol{\nu} - \mathcal{M}^{[n]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon(t)) = tC_0|\boldsymbol{\nu}|^{-\tau_1}, \tag{7.50}$$

which means

$$\int_{I^{[n]}(\boldsymbol{\nu})} \mathrm{d}\varepsilon \le \int_{-1}^{1} \mathrm{d}t\, C_0|\boldsymbol{\nu}|^{-\tau_1} \frac{1}{|\partial_\varepsilon \mathcal{M}^{[n]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon(t))|} \le \frac{4}{B} C_0|\boldsymbol{\nu}|^{-\tau_1}, \tag{7.51}$$

by (7.38).

This has to be done for all $\boldsymbol{\nu}\in\mathbb{Z}^d$ satisfying $|\boldsymbol{\omega}\cdot\boldsymbol{\nu}| < 2\varepsilon_0 D$, where D is the positive constant such that $\varepsilon_0 D$ is a bound on $|\mathcal{M}^{[n]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)|$, *i.e.* for all $\boldsymbol{\nu}\in\mathbb{Z}^r$ such that

$$|\boldsymbol{\nu}| \ge \left(\frac{C_0}{2\varepsilon_0 D}\right)^{1/\tau} \equiv \mathcal{N}_0. \tag{7.52}$$

This yields that we have to exclude from $\mathcal{E}^{[n-1]}$ a set

$$\mathcal{I}^{[n]} = \bigcup_{|\boldsymbol{\nu}|\ge\mathcal{N}_0} I^{[n]}(\boldsymbol{\nu}) \tag{7.53}$$

of measure bounded by

$$\begin{aligned} \operatorname{meas}(\mathcal{I}^{[n]}) &\le \sum_{|\boldsymbol{\nu}|\ge\mathcal{N}_0} \operatorname{meas}(I^{[n]}(\boldsymbol{\nu})) \le \text{const.} \sum_{|\boldsymbol{\nu}|\ge\mathcal{N}_0} C_0|\boldsymbol{\nu}|^{-\tau_1} \\ &\le \text{const.} C_0 \left(\frac{\varepsilon_0}{C_0}\right)^{(\tau_1-d)/\tau} = \text{const.}\varepsilon_0^{1+\xi'}, \end{aligned} \tag{7.54}$$

where $\xi' = (\tau_1-\tau-d)/\tau$, so that $\xi'>0$ if

$$\tau_1 > \tau + d, \tag{7.55}$$

which fixes the value of τ_1.

For all $|\boldsymbol{\nu}|\ge\mathcal{N}_0$ fix $n_0 = n_0(\boldsymbol{\nu})$ such that

$$\left|\varepsilon^{[n_0+1]}(\boldsymbol{\nu}) - \varepsilon^{[n_0]}(\boldsymbol{\nu})\right| \le C_0|\boldsymbol{\nu}|^{-\tau_1}; \tag{7.56}$$

by (7.44) one can choose

$$n_0 \equiv n_0(\boldsymbol{\nu}) \le \text{const.}\, \tau_1 \log\log|\boldsymbol{\nu}|. \tag{7.57}$$

Then for all $n \le n_0$ define $J^{[n]}(\boldsymbol{\nu})$ as

$$J^{[n]}(\boldsymbol{\nu}) = \left\{ \varepsilon \in \mathcal{E}^{[n-1]} \; : \; \left| i\boldsymbol{\omega} \cdot \boldsymbol{\nu} - \mathcal{M}^{[n]}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}; \varepsilon) \right| < 2C_0 |\boldsymbol{\nu}|^{-\tau_1} \right\} ; \tag{7.58}$$

by construction all the sets $I^{[n]}(\boldsymbol{\nu})$ fall inside $J^{[n_0]}(\boldsymbol{\nu})$ as soon as $n > n_0$.

Then we can bound $\mathrm{meas}(\mathcal{I})$ by the sum of the measures of the sets $J^{[1]}(\boldsymbol{\nu}), \ldots, J^{[n_0]}(\boldsymbol{\nu})$ for all $\boldsymbol{\nu} \in \mathbb{Z}^d$ such that $|\boldsymbol{\nu}|$ verifies (7.52). The condition (7.57) on n_0 implies that such a measure can be bounded by

$$\text{const.} \sum_{|\boldsymbol{\nu}| \ge \mathcal{N}_0} n_0(\boldsymbol{\nu}) C_0 |\boldsymbol{\nu}|^{-\tau_1} \le \text{const.} \varepsilon_0^{1+\xi}, \tag{7.59}$$

with a value ξ smaller than ξ' in order to take into account the logarithmic corrections due to the factor $n_0(\boldsymbol{\nu})$. ■

The above lemmata imply the convergence of the series (6.9) for all values $\varepsilon \in \mathcal{E}$, with $\mathcal{E}$ a Cantor set with positive Lebesgue measure such that

$$\lim_{\varepsilon_0 \to 0} \frac{\mathrm{meas}(\mathcal{E})}{2\varepsilon_0} = 1, \tag{7.60}$$

as it follows immediately from the construction of $\mathcal{E}$ and from the property (7.40).

8. Properties of the renormalized expansion

To complete the proof of theorem 2 we have still to show that the function (6.9), defined through the renormalized expansion (6.7), solves the equation (2.4) for all $\varepsilon \in \mathcal{E}$, i.e. that one has

$$\overline{u} = g \left(Q^{-1} + Q \overline{u}^2 \right) , \tag{8.1}$$

where g is the differential operator with kernel $g(\boldsymbol{\omega} \cdot \boldsymbol{\nu}) = 1/i\boldsymbol{\omega} \cdot \boldsymbol{\nu}$.

We can write

$$\begin{aligned} \overline{u}(t) &= c + \sum_{\boldsymbol{\nu} \in \mathbb{Z}^d} e^{i\boldsymbol{\omega} \cdot \boldsymbol{\nu} t} \overline{u}_{\boldsymbol{\nu}}, \\ \overline{u}_{\boldsymbol{\nu}} &= \sum_{n=0}^{\infty} \overline{u}_{n,\boldsymbol{\nu}}, \\ \overline{u}_{n,\boldsymbol{\nu}} &= \sum_{k=1}^{\infty} \varepsilon^k \sum_{\theta \in \Theta^{\mathcal{R}}_{k,\nu}(n)} \mathrm{Val}(\theta), \end{aligned} \tag{8.2}$$

where $\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu}}(n)$ is the set of trees in $\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu}}$ such that the root line has scale n.

Note that for all $x \neq 0$ one has

$$\begin{aligned} 1 &= \sum_{n=0}^{\infty} \mathcal{X}_n(x; \varepsilon), \\ \mathcal{X}_n(x; \varepsilon) &= \chi_0(|x|) \ldots \chi_{n-1}(|ix - \mathcal{M}^{[n-2]}(x; \varepsilon)|) \psi_n(|ix - \mathcal{M}^{[n-1]}(x; \varepsilon)|), \end{aligned} \tag{8.3}$$

where the term with $n=0$ has to be interpreted as $\psi_0(|x|)$; more generally for all $x \neq 0$ and for all $j \geq 0$ one has

$$1 = \sum_{n=j}^{\infty} \chi_j(|ix - \mathcal{M}^{[j-1]}(x;\varepsilon)|) \dots \chi_{n-1}(|ix - \mathcal{M}^{[n-2]}(x;\varepsilon)|)\psi_n(|ix - \mathcal{M}^{[n-1]}(x;\varepsilon)|), \tag{8.4}$$

where again the term with $n = j$ has to be interpreted as $\psi_j(|ix - \mathcal{M}^{[j-1]}(x;\varepsilon)|)$. Note that both in (8.3) and in (8.4) only a finite number of addends is different from zero, as the analysis of Section 7 implies, so that the two series are well defined.

By using (8.3) one can write, in Fourier space,

$$\begin{aligned} g(\boldsymbol{\omega}\cdot\boldsymbol{\nu})(Q^{-1}+Q\overline{u}^2)_{\boldsymbol{\nu}} &= g(\boldsymbol{\omega}\cdot\boldsymbol{\nu}) \sum_{n=0}^{\infty} \mathcal{X}_n(\boldsymbol{\omega}\cdot\boldsymbol{\nu})(Q^{-1}+Q\overline{u}^2)_{\boldsymbol{\nu}} \\ &= g(\boldsymbol{\omega}\cdot\boldsymbol{\nu}) \sum_{n=0}^{\infty} \mathcal{X}_n(\boldsymbol{\omega}\cdot\boldsymbol{\nu})(g^{[n]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon))^{-1} g^{[n]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)(Q^{-1}+Q\overline{u}^2)_{\boldsymbol{\nu}} \\ &= g(\boldsymbol{\omega}\cdot\boldsymbol{\nu}) \sum_{n=0}^{\infty} \Big(i\boldsymbol{\omega}\cdot\boldsymbol{\nu} - \mathcal{M}^{[n-1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)\Big)\, g^{[n]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)(Q^{-1}+Q\overline{u}^2)_{\boldsymbol{\nu}} \\ &= g(\boldsymbol{\omega}\cdot\boldsymbol{\nu}) \sum_{n=0}^{\infty} \Big(i\boldsymbol{\omega}\cdot\boldsymbol{\nu} - \mathcal{M}^{[n-1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)\Big) \sum_{k=1}^{\infty} \varepsilon^k \sum_{\theta\in\overline{\Theta}^{\mathcal{R}}_{k,\boldsymbol{\nu}}(n)} \mathrm{Val}(\theta), \end{aligned} \tag{8.5}$$

where $\overline{\Theta}^{\mathcal{R}}_{k,\boldsymbol{\nu}}(n)$ differs from $\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu}}(n)$ as it contains also trees which can have one renormalized self-energy graph T with exiting line ℓ_0, if ℓ_0 denotes the root line of θ; for such trees the line entering T will be on a scale $p \geq 0$, while the renormalized self-energy graph T will have a scale $n_T = j$, with $j+1 \leq \min\{n,p\}$ (by definition of renormalized self-energy graph).

Then we obtain, by explicitly separating in (8.5) the trees containing such self-energy graphs from the others,

$$\begin{aligned} g(\boldsymbol{\omega}\cdot\boldsymbol{\nu})(Q^{-1}+Q\overline{u}^2)_{\boldsymbol{\nu}} &= g(\boldsymbol{\omega}\cdot\boldsymbol{\nu}) \sum_{n=0}^{\infty} \Big(i\boldsymbol{\omega}\cdot\boldsymbol{\nu} - \mathcal{M}^{[n-1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)\Big) \sum_{k=1}^{\infty} \varepsilon^k \sum_{\theta\in\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu}}(n)} \mathrm{Val}(\theta) \\ &\quad + g(\boldsymbol{\omega}\cdot\boldsymbol{\nu}) \sum_{n=1}^{\infty} \Big(i\boldsymbol{\omega}\cdot\boldsymbol{\nu} - \mathcal{M}^{[n-1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)\Big)\, g^{[n]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon) \\ &\quad \sum_{p=n}^{\infty} \sum_{j=0}^{n-1} M^{[j]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon) \sum_{k=1}^{\infty} \varepsilon^k \sum_{\theta\in\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu}}(p)} \mathrm{Val}(\theta) \\ &\quad + g(\boldsymbol{\omega}\cdot\boldsymbol{\nu}) \sum_{n=2}^{\infty} \Big(i\boldsymbol{\omega}\cdot\boldsymbol{\nu} - \mathcal{M}^{[n-1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)\Big)\, g^{[n]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon) \\ &\quad \sum_{p=0}^{n-1} \sum_{j=0}^{p-1} M^{[j]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon) \sum_{k=1}^{\infty} \varepsilon^k \sum_{\theta\in\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu}}(p)} \mathrm{Val}(\theta) \end{aligned} \tag{8.6}$$

which, by the definitions (8.1), can be written as

$$
\begin{aligned}
g(\boldsymbol{\omega}\cdot\boldsymbol{\nu})(Q^{-1}+Q\overline{u}^2)_{\boldsymbol{\nu}} &= g(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\Big[\sum_{n=0}^{\infty}\Big(i\boldsymbol{\omega}\cdot\boldsymbol{\nu}-\mathcal{M}^{[n-1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)\Big)\,\overline{u}_{n,\boldsymbol{\nu}} \\
&+\sum_{n=1}^{\infty}\mathcal{X}_n(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)\sum_{p=n}^{\infty}\sum_{j=0}^{n-1}M^{[j]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)\overline{u}_{p,\boldsymbol{\nu}}+\sum_{n=2}^{\infty}\mathcal{X}_n(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)\sum_{p=0}^{n-1}\sum_{j=0}^{p-1}M^{[j]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)\overline{u}_{p,\boldsymbol{\nu}}\Big].
\end{aligned}
\tag{8.7}
$$

We can write

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\mathcal{X}_n(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\sum_{p=n}^{\infty}\sum_{j=0}^{n-1}M^{[j]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)\overline{u}_{p,\boldsymbol{\nu}}+\sum_{n=2}^{\infty}\mathcal{X}_n(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\sum_{p=1}^{n-1}\sum_{j=0}^{p-1}M^{[j]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)\overline{u}_{p,\boldsymbol{\nu}}\Big) \\
&\quad=\sum_{p=1}^{\infty}\overline{u}_{p,\boldsymbol{\nu}}\Big(\sum_{j=0}^{p-1}\sum_{n=j+1}^{p}M^{[j]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)\mathcal{X}_n(\boldsymbol{\omega}\cdot\boldsymbol{\nu})+\sum_{j=0}^{p-1}\sum_{n=p+1}^{\infty}M^{[j]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)\mathcal{X}_n(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\Big) \\
&\quad=\sum_{p=1}^{\infty}\overline{u}_{p,\boldsymbol{\nu}}\sum_{j=0}^{p-1}M^{[j]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)\sum_{n=j+1}^{\infty}\mathcal{X}_n(\boldsymbol{\omega}\cdot\boldsymbol{\nu}) \\
&\quad=\sum_{n=1}^{\infty}\overline{u}_{n,\boldsymbol{\nu}}\sum_{j=0}^{n-1}M^{[j]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)\chi_0(|\boldsymbol{\omega}\cdot\boldsymbol{\nu}|)\ldots\chi_j(|i\boldsymbol{\omega}\cdot\boldsymbol{\nu}|-\mathcal{M}^{[j-1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)|) \\
&\qquad\qquad\sum_{s=j+1}^{\infty}\chi_{j+1}(|i\boldsymbol{\omega}\cdot\boldsymbol{\nu}-\mathcal{M}^{[j]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)|)\ldots\psi_s(|i\boldsymbol{\omega}\cdot\boldsymbol{\nu}-\mathcal{M}^{[s-1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)|) \\
&\quad=\sum_{n=1}^{\infty}\overline{u}_{n,\boldsymbol{\nu}}\sum_{j=0}^{n-1}M^{[j]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)\chi_0(|\boldsymbol{\omega}\cdot\boldsymbol{\nu})\ldots\chi_j(|\boldsymbol{\omega}\cdot\boldsymbol{\nu}|),
\end{aligned}
\tag{8.8}
$$

where the identity (8.4) has been used in the last line (with the correct interpretation of the term with $s=j+1$ explained after (8.4)).

By the second definition in (6.3) one has

$$
\sum_{j=0}^{n-1}M^{[j]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)\chi_0(|\boldsymbol{\omega}\cdot\boldsymbol{\nu})\ldots\chi_j(|\boldsymbol{\omega}\cdot\boldsymbol{\nu}|)=\mathcal{M}^{[n-1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}),
\tag{8.9}
$$

so that, by inserting (8.8) in (8.6), after having used (8.9) we obtain

$$
\begin{aligned}
g(\boldsymbol{\omega}\cdot\boldsymbol{\nu})(Q^{-1}+Q\overline{u}^2)_{\boldsymbol{\nu}} &= g(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\sum_{n=0}^{\infty}\Big[\Big(i\boldsymbol{\omega}\cdot\boldsymbol{\nu}-\mathcal{M}^{[n-1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)\Big)+\mathcal{M}^{[n-1]}(\boldsymbol{\omega}\cdot\boldsymbol{\nu};\varepsilon)\Big]\overline{u}_{n,\boldsymbol{\nu}} \\
&= g(\boldsymbol{\omega}\cdot\boldsymbol{\nu})\sum_{n=0}^{\infty}(i\boldsymbol{\omega}\cdot\boldsymbol{\nu})\overline{u}_{n,\boldsymbol{\nu}}=\sum_{n=0}^{\infty}\overline{u}_{n,\boldsymbol{\nu}}=\overline{u}_{\boldsymbol{\nu}},
\end{aligned}
\tag{8.10}
$$

so that (8.1) follows.

Note that at each step only absolutely converging series have been dealt with, so that the above analysis is rigorous and not only formal.

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