# Local and Global Continuity of the Integrated Density of States

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Abstract. The integrated density of states (IDS) $N(E)$ is the distribution function of a nonnegative measure $\nu$, the density of states measure (DOS). This measure usually obtained as the weak infinite-volume limit of the local eigenvalue counting function for the system restricted to a finite-volume region. For Schrödinger operators with random potentials, the eigenvalue counting function for the finite-volume system satisfies an estimate called a Wegner estimate. We present new local and global-in-energy Wegner estimates for random Schrödinger operators with Anderson-type random potentials. These estimates are strong enough to imply that the DOS measure is absolutely continuous with a density in $L^q_{loc}(\mathbb{R})$, for any $1 \leq q < \infty$. The IDS is also proved to be locally or globally Hölder continuous with Hölder exponent $1/q$, for any $q > 1$.

## 1. Introduction: Models and Main Results

This paper presents some new results on the local continuity properties of the integrated density of states (IDS), and on the local regularity of the density of states (DOS) measure, for some random operators. The proofs of these local-in-energy results rely on some recent progress on continuity properties of the IDS *at all energies* by the authors. We show how the methods of [**6**] can be localized to energy intervals that provide new results on the local properties of the IDS for some random models, especially those with constant magnetic fields. The results of this paper also apply to the random operators describing acoustic and electromagnetic waves in randomly perturbed media, and we refer the reader to [**10, 13, 14**].

We will consider the family of Schrödinger operators $H_\omega(\lambda)$ on $L^2(\mathbb{R}^d)$, constructed from a deterministic, background operator $H_0 = (-i\nabla - A_0)^2 + V_0$. We assume that this operator is self-adjoint with operator core $C_0^\infty(\mathbb{R}^d)$, and that $H_0 \geq -M_0 > -\infty$, for some finite constant $M_0$. We consider an Anderson-type random potential $V_\omega$ constructed from the single-site potential $u$ as

$$V_\omega(x) = \sum_{j \in \mathbb{Z}^d} \lambda_j(\omega) u(x-j). \tag{1.1}$$

The family of random Schrödinger operators is given by

$$H_\omega(\lambda) = H_0 + \lambda V_\omega. \tag{1.2}$$

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By re-scaling the random variables, if necessary, we can normalize $V_\omega$ so that $\|V_\omega\|_\infty \leq 1$. In this way, the parameter $\lambda > 0$ becomes a measure of the disorder. The main results are independent of the disorder (provided it is nonzero for Theorem 1.3 and in the Landau case).

We need two hypotheses throughout the paper on the single-site potential $u$, and on the distribution of the random variables. We denote the cube of side length $L > 0$ centered at $x \in \mathbb{R}^d$ by $\Lambda_L(x)$.

**(H1):** The single-site potential $u \neq 0$ is nonnegative with compact support, $u \in L^\infty(\mathbb{R}^d)$, and $\|u\|_\infty \leq 1$.

**(H2):** The random coupling constants $\{\lambda_j(\omega) \mid j \in \mathbb{Z}^d\}$, are independent and identically distributed. The distribution has a density $h_0 \in L^\infty(\mathbb{R})$ with $\operatorname{supp} h_0 \subset [0,1]$.

In this partially expository note, we will replace hypothesis (H1) with the following stronger one whenever it simplifies the discussion without further mention:

**(H1a):** The single-site potential $u \neq 0$ is nonnegative with compact support with $\operatorname{supp} u \subset \Lambda_1(0)$, $u \in L^\infty(\mathbb{R}^d)$, and $\|u\|_\infty \leq 1$.

The hypothesis (H1a) on the single-site potential is stronger than needed, but it simplifies the discussion here by eliminating contributions near the boundary of $\Lambda$ coming from the potential outside of $\Lambda$. These terms can be dealt with in a routine manner as presented in [**6**]. We note that the case of small support is the most interesting one. If the support of $u$ is large, that is, if $\Lambda_1(0) \subset \operatorname{supp} u$, then the proof of the Lipschitz continuity of the IDS is much easier, cf. [**4, 19**].

In order to discuss the IDS, we need Hamiltonians $H_\Lambda$ that describe the system constrained to a finite-volume region $\Lambda \subset \mathbb{R}^d$. For $\Lambda \subset \mathbb{R}^d$, we denote the lattice points in $\Lambda$ by $\tilde{\Lambda} = \Lambda \cap \mathbb{Z}^d$, and we let $\chi_\Lambda$ be the characteristic function for $\Lambda$. For a given bounded $\Lambda \subset \mathbb{R}^d$, we take $H_0^\Lambda$ and $H_\omega^\Lambda$ to be the restrictions of $H_0$ and $H_\omega$, respectively, to the region $\Lambda$, with self-adjoint boundary conditions on the boundary of $\Lambda$, $\partial\Lambda$. We assume, without loss of generality, that the boundary conditions have been chosen so that these operators are bounded from below by a constant, independent of $\Lambda$, for all large $\Lambda$. We denote by $E_0^\Lambda(\cdot)$ and $E_\Lambda(\cdot)$ the spectral families for $H_0^\Lambda$ and $H_\omega^\Lambda$, respectively. Under the simplifying hypothesis (H1a), the local potential $V_\Lambda$ is given by

$$\begin{aligned} V_\Lambda(x) &= V_\omega(x)\chi_\Lambda(x) \\ &= \sum_{j \in \tilde{\Lambda}} \lambda_j(\omega) u(x-j)\chi_\Lambda(x). \end{aligned} \tag{1.3}$$

When $\Lambda$ is the union of unit cubes, our hypothesis (H1a) implies that $\operatorname{supp} V_\Lambda \subset \Lambda$.

We define the IDS $N(E)$ for $H_\omega$ using the eigenvalue counting function for $H_\omega^\Lambda$. Let $N_\Lambda(E)$ be the number of eigenvalues of $H_\omega^\Lambda$, with self-adjoint boundary conditions, less than or equal to $E$. This function depends on the realization $\omega$, and on the boundary conditions. The integrated density of states is defined by

$$N(E) = \lim_{|\Lambda| \to \infty} \frac{N_\Lambda(E)}{|\Lambda|}, \tag{1.4}$$

when this limit exists. When it exists, the IDS $N(E)$ is a monotonic function. We assume that $N(E)$ has been defined to be right continuous. It has at most a countable number of discontinuities. *We will always assume that the IDS $N(E)$*

*exists almost surely for the random family of operators considered here.* For example, if the family $H_\omega$ is an ergodic family of random Schrödinger operators with a reasonable random potential, it is known that this limit exists and is independent of the realization $\omega$ almost surely (cf. [**3, 18, 21**]). Furthermore, it is known that the IDS is independent of the boundary conditions taken on the finite volume $\Lambda$, cf. [**12, 18, 20**]. We will give a common condition on $V_\omega$ in section 4 that guarantees the existence of the IDS.

The IDS $N(E)$ is the distribution function of a nonnegative measure $\nu$ called the *density of states* (DOS) measure. The DOS measure is defined via the Riesz-Markov Theorem and the following expression, whenever the limit exits: For any real function $f \in C_0(\mathbb{R})$, the density of states measure $d\nu$ satisfies

$$\begin{aligned} \int f(s) d\nu(s) &= \lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|} \, Tr(\chi_\Lambda f(H_\omega^\Lambda)\chi_\Lambda) \\ &= \lim_{|\Lambda|\to\infty} \int f(s) d\nu_\Lambda(s). \end{aligned} \tag{1.5}$$

One usually proves the vague convergence of the measures $\nu_\Lambda$ to the DOS measure $\nu$ by proving that the limit of the distribution functions on the right side of (1.4) exists at points of continuity of the limit function. Properties of the IDS follow from estimates on the expectation of the eigenvalue counting function for local Hamiltonians known as *Wegner estimates*, after Wegner who first considered them [**29**].

**1.1. Local-in-Energy Results on the IDS and the DOS.** It is interesting and useful to note that the main results of [**6**] can be localized in the energy. We give two results, Theorem 1.1 and Theorem 1.3, that are the analogs of Theorem 1.1 and Theorem 1.2 from [**6**], respectively. We also give results on the behavior of the DOS measure. Our first theorem deals with the simple case for which the background operator $H_0$ has a Hölder continuous IDS on some interval. Although the proof of this theorem is rather simple, the main limitation is that the resulting Hölder exponent is considerably smaller than expected.

THEOREM 1.1. *In addition to hypotheses (H1) and (H2), we assume that the operator $H_0$ admits an integrated density of states $N_0(E)$ that is locally Hölder continuous on an open interval $I \subset \mathbb{R}$ for some exponent $0 < q_1 \leq 1$. That is, there exists a finite, positive constant $C_0(I) > 0$ so that the IDS satisfies*

$$|N_0(E') - N_0(E)| \leq C_0(I)|E' - E|^{q_1}, \text{ for } E, E' \in I. \tag{1.6}$$

*Then, the IDS for $H_\omega(\lambda)$ is Hölder continuous on $I$ for any exponent $0 < q < q_1/(q_1 + 2)$.*

We can conclude from this theorem that the DOS measure $\nu$ restricted to the interval $I$ is absolutely continuous with respect to Lebesgue measure.

COROLLARY 1.2. *Under the assumption of Theorem 1.1, the DOS measure $\nu|I$ is absolutely continuous with respect to Lebesgue measure. The local density $f_I \in L^p_{loc}(I)$, for any $1 \leq p < 1 + \frac{q_1}{2}$.*

We can improve the Hölder exponent considerably with more work and an additional assumption that the background operator $H_0$ is periodic. We assume that both $V_0$ and $A_0$ are periodic with respect to the group $\Gamma = \mathbb{Z}^d$ because of the form of the Anderson-type potential (1.1). We note that we could work with a nondegenerate lattice $\Gamma$, by defining a corresponding Anderson-type potential, but we will explicitly treat the case $\Gamma = \mathbb{Z}^d$. Let $\Gamma^*$ denote the dual lattice, that is, $\Gamma^* = \{\gamma' \mid \gamma \cdot \gamma' \in 2\pi\mathbb{Z}, \text{ for all } \gamma \in \Gamma\}$. We let $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ be the torus, and $(\mathbb{T}^d)^* = (\mathbb{R}^d)^*/(\mathbb{Z}^d)^*$ be the dual torus. We denote by $C_0$ the unit cell for $\Gamma = \mathbb{Z}^d$, and by $C_0^*$ the unit cell for $\Gamma^* = (\mathbb{Z}^d)^*$. The Floquet decomposition of $H_0$ yields a family of operators $H_0(\theta)$, for $\theta \in (\mathbb{R}^d)^*$. Each operator $H_0(\theta)$ is self-adjoint on $L^2(\mathbb{T}^d)$, and has a compact resolvent. We denote the eigenvalues of $H_0(\theta)$ by $E_n(\theta)$. The spectrum of $H_0$ is given by

$$\sigma(H_0) = \bigcup_{\theta \in (\mathbb{T}^d)^*} \sigma(H_0(\theta)) = \bigcup_{n \in \mathbb{N}} \bigcup_{\theta \in (\mathbb{T}^d)^*} E_n(\theta). \tag{1.7}$$

For an open, relatively compact interval $\Delta \subset \mathbb{R}$, we let $E_0(\Delta, \theta)$ denote the spectral projector of $H_0(\theta)$ onto the eigenspace of $H_0(\theta)$ spanned by its eigenfunctions with eigenvalues $E_n(\theta) \in \Delta$. Because of the discreteness of the spectrum of $H_0(\theta)$, the dimension of $Ran E_0(\Delta, \theta)$ is finite and locally constant.

We consider Hamiltonian $H_\omega(\lambda)$ restricted to the cube $\Lambda$ with periodic boundary conditions (PBC). If the cube $\Lambda$ has side length $n$, denoted by $\Lambda_n$, we can consider it as a fundamental unit cell for the lattice $n\mathbb{Z}^d$ and take a Floquet decomposition relative to it. We let $H_0^n(\theta)$ and $E_j^{(n)}(\theta)$ denote the corresponding fibered operators and their eigenvalues. Note that the operator $H_0^{\Lambda_n}$, with PBC on $\partial\Lambda_n$, coincides with the fibered operator $H_0^n(\theta = 0)$. We define the IDS for $H_\omega(\lambda)$ as above using the eigenvalue counting function for $H_\omega^\Lambda$ with PBC. We define another, deterministic potential associated with a bounded region $\Lambda$ by

Theorem 1.3. *Assume hypotheses (H1) and (H2). Let $E_0 \in \mathbb{R}$ and let $\tilde{\Delta} \subset \mathbb{R}$ be an interval centered on $E_0$ and of length $16D_0^2$, where $D_0$ is defined in (**??**). Suppose that there exists a finite constant $C(\tilde{\Delta}, u) > 0$ so that, for all $\theta \in (\mathbb{T}^d)^*$, the single-site potential $u$ satisfies*

*Let us note that as $D_0 \sim \|u\|_\infty$, it is clear from the positivity condition (**??**) that the lower bound $C(\tilde{\Delta}, u) \to 0$ as $D_0 \to 0$. Since we have made no assumption on the IDS $N_0(E)$ for the background operator $H_0$, we expect that the interval $\Delta$ must also shrink as the random potential vanishes. Although the periodicity of the operator $H_0$ implies that the IDS is absolutely continuous (cf. [**22**]), we do not know anything else about the regularity of its derivative $dN_0/d\lambda$. We have the analog of Corollary 1.2.*

Corollary 1.4. *Under the assumptions of Theorem 1.3, the DOS measure $\nu|\Delta$ is absolutely continuous with respect to Lebesgue measure. The local density $f_I \in L^p_{loc}(\Delta)$, for any $1 \leq p < \infty$.*

*We give a condition for the positivity condition (**??**) to hold. We say that $H_0$ has the unique continuation property on an interval $J \subset \mathbb{R}$ if the following condition holds. For any $E \in J \subset \mathbb{R}$ and any function $\varphi$ in the domain of $H_0$,* *if $\varphi$ satisfies $(H_0 - E)\varphi = 0$, and $\varphi$ vanishes on some open set, then $\varphi$ vanishes identically. It is well-known that a large class of Schrödinger operators satisfy the unique continuation property for $J = \mathbb{R}$. For example, the operator $H_0$ has the unique continuation property if, in dimension $d \geq 3$, $V_0 \in L^{d/2}_{loc}(\mathbb{R}^d)$, the vector potential $A_0 \in L^d_{loc}(\mathbb{R}^d)$, and $\nabla A_0 \in L^{d/2}_{loc}(\mathbb{R}^d)$. We refer to* [**30**], *and references therein.*

PROPOSITION 1.5. *We assume hypotheses (H1) and (H2). In any dimension $d \geq 1$, if $H_0$ has the unique continuation property on the interval $\tilde{\Delta}$ of Theorem 1.3, and $u > 0$ on a nonempty open set, then condition (**??**) holds.*

**1.2. Global-in-Energy Results on the IDS and the DOS.** *For comparison, we state the two main results of* [**6**]. *We use the same notation as above. The first result treats the case when the IDS $N_0(E)$ for $H_0$ is known to be Hölder continuous on $\mathbb{R}$.*

THEOREM 1.6. *In addition to hypotheses (H1) and (H2), we assume that the operator $H_0$ admits a density of states $N_0(E)$ that is locally Hölder continuous on $\mathbb{R}$ for some exponent $0 < q_1 \leq 1$, i. e. that satisfies*

$$|N_0(E') - N_0(E)| \leq C_0 |E' - E|^{q_1} \tag{1.8}$$

*where the finite constant $C_0 > 0$ is locally uniform in energy. Then, the IDS for $H_\omega(\lambda)$ is Hölder continuous on $\mathbb{R}$ for any exponent $0 < q < q_1/(q_1 + 2)$.*

*Similarly, if the background operator $H_0$ is periodic, then we have the following regularity result that improves the Hölder exponent.*

THEOREM 1.7. *Assume hypotheses (H1) and (H2), and that $H_0$ is $\mathbb{Z}^d$-periodic. Let $\tilde{\Delta} \subset \mathbb{R}$ be any bounded, closed interval. Suppose that for all $\theta \in (\mathbb{T}^d)^*$ there exists a finite constant $C(\tilde{\Delta}, u) > 0$, so that the single-site potential $u$ satisfies*

$$E_0(\tilde{\Delta}, \theta) u E_0(\tilde{\Delta}, \theta) \;\geq\; C(\tilde{\Delta}, u) E_0(\tilde{\Delta}, \theta). \tag{1.9}$$

*Then, the IDS for the random family $H_\omega(\lambda) = H_0 + \lambda V_\omega$, for $\lambda \neq 0$, is Hölder continuous on $\mathbb{R}$, for any Hölder exponent $0 < q < 1$. If the dimension $d = 1$, and $H_0 = -d^2/dx^2$, then the IDS for $H_\omega(\lambda)$ is Lipschitz continuous at all energies.*

*The DOS measure is correspondingly absolutely continuous with respect to Lebesgue measure.*

COROLLARY 1.8. *Under the assumptions of either Theorem 1.6, or Theorem 1.7, the DOS measure $\nu$ is absolutely continuous with respect to Lebesgue measure. The local density $f \in L^p_{loc}(\mathbb{R})$, for any $1 \leq p_0$, where for Theorem 1.6, we have $p_0 = 1 + \frac{q_1}{2}$, and for Theorem 1.7, we have $p_0 = \infty$.*

*All of these results can be improved to local and global Lipschitz continuity for the one-dimensional case, with $H_0 = -d^2/dx^2$, using better estimates on the spectral shift function* [**23, 24**]. *Furthermore, the exponent of continuity of the DOS measure can be taken to be $q = 1$ in Corollaries 1.2, 1.4, and 1.8. The importance of the spectral shift function to these estimates is discussed further in section 5.*

*Similar results for the one-dimensional case were recently obtained by E. Giere* [**16**] *using different methods.*

**1.3. Outline of Contents.** *We will sketch the proof of Theorem 1.1 in section 2, and the proof of Theorem 1.3 in section 3. In section 4, we prove Corollaries 1.2, 1.4, and 1.8. In section 5, we discuss some related problems, including the question of Lipschitz continuity of the IDS, an application to the continuity of the IDS for Landau Hamiltonians, and regularity of the IDS.*

## 2. Preservation of Good Behavior of the IDS in an Interval

*In this section, we outline the proof Theorem 1.1. The proof follows closely the proof of Theorem 1.1 in* [**6**]. *It is natural to expect that if the IDS for the unperturbed operator $H_0$ is well-behaved on an interval $\tilde{\Delta} = [E_-, E_+] \subset \mathbb{R}$, then the IDS for $H_\omega$ will also be well-behaved on a slightly small interval. We first note that for $|\Lambda|$ sufficiently large, depending on $\tilde{\Delta}$, the hypothesis of Theorem 1.1 implies that* [**6**],

$$TrE_0^\Lambda(\tilde{\Delta}) \;\leq\; C_1(E_+, d)|\tilde{\Delta}|^{q_1}|\Lambda|, \tag{2.1}$$

*where the finite constant $C_1(E_+, d) > 0$ depends on the constant $C_0 > 0$ in (1.4), the dimension $d$, and the energy $E_+$. To prove result (2.1), let $N_{0,\Lambda}(E)$ be the number of eigenvalues of $H_0^\Lambda$ less than or equal to $E$. Let us first assume that $N_0(E_-) \neq 0$. Then, by the definition of the density of states, one has*

$$N_0(E_+)(1 - |E_+ - E_-|^{q_1}) \leq \frac{N_{0,\Lambda}(E_+)}{|\Lambda|} \leq N_0(E_+)(1 + |E_+ - E_-|^{q_1}),$$
$$N_0(E_-)(1 - |E_+ - E_-|^{q_1}) \leq \frac{N_{0,\Lambda}(E_-)}{|\Lambda|} \leq N_0(E_-)(1 + |E_+ - E_-|^{q_1}).$$

*Using (1.4), for $\Lambda$ sufficiently large, we get*

$$\begin{aligned} TrE_0^\Lambda(\tilde{\Delta}) &\leq (N_{0,\Lambda}(E_+) - N_{0,\Lambda}(E_-)) \\ &\leq |\Lambda|[N_0(E_+)(1 + |E_+ - E_-|^{q_1}) - N_0(E_-)(1 - |E_+ - E_-|^{q_1})] \\ &\leq \{C_0 + 2N_0(E_+)\}|\Lambda||E_+ - E_-|^{q_1} \\ &\leq C_1(E_+, d)|\Lambda||\tilde{\Delta}|^{q_1}. \end{aligned}$$

*We used the monotonicity of the IDS and the bound $N_0(E) \leq C(E)$, for a finite constant $C(E)$. If it happens that $N_0(E_-) = 0$, then (2) follows from the fact that $\frac{N_{0,\Lambda}(E_+)}{|\Lambda|} \leq |E_+ - E_-|^{q_1}$, for $|\Lambda|$ sufficiently large.*

Proof. 1. Suppose that the interval $I = (E_1, E_2)$, and that $E_0 \in I$. Consider a bounded interval $\Delta \equiv [\Delta_-, \Delta_+] \subset I$, with $E_0 \in \Delta$, and with $|\Delta| < 1$. Let $\alpha \in \mathbb{R}$ be an index satisfying $0 < \alpha < 1$. Let $\tilde{\Delta}$ be the interval $\tilde{\Delta} = [\Delta_- - |\Delta|^\alpha, \Delta_+ + |\Delta|^\alpha]$, so that $\Delta \subset \tilde{\Delta}$ and $d_0 = \text{dist}\,(\Delta, \tilde{\Delta}^c) = |\Delta|^\alpha$. We choose $|\Delta|$ small enough so that $(1/2)|\Delta| + |\Delta|^\alpha < \min(E_0 - E_1, E_2 - E_0)$. We decompose the spectral projector $E_\Lambda(\Delta)$ relative to $H_0^\Lambda$ as

$$TrE_\Lambda(\Delta) = TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta}) + TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta}^c). \tag{2.2}$$

We estimate the first term on the right in (2.2) using the assumption (2.1) on $H_0^\Lambda$ and the positivity of the projectors. For $\Lambda$ sufficiently large (depending on $\tilde{\Delta}$), this gives

$$TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta}) \;\leq\; C_1|\Delta|^{\alpha q_1}|\Lambda|, \tag{2.3}$$

where we write $C_1$ for the constant $C_1(E_+, d)$ appearing in (2.1).
2. To deal with the second term in (2.2), we write

$$\begin{aligned} TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta}^c) &= TrE_\Lambda(\Delta)(H_\omega^\Lambda - E_0)E_0^\Lambda(\tilde{\Delta}^c)(H_0^\Lambda - E_0)^{-1} \\ &\quad -\lambda TrE_\Lambda(\Delta)V_\Lambda E_0^\Lambda(\tilde{\Delta}^c)(H_0^\Lambda - E_0)^{-1} \\ &= (i) + (ii). \end{aligned} \tag{2.4}$$

For the first term $(i)$, we use the estimate $\|E_0^\Lambda(\tilde{\Delta}^c)(H_0^\Lambda - E_0)^{-1}\| \leq |\Delta|^{-\alpha}$ and easily obtain:

$$|(i)| \;\leq\; |\Delta|^{1-\alpha}\; TrE_\Lambda(\Delta). \tag{2.5}$$

As long as $\alpha < 1$, this term can be absorbed into the left side of (2.2). For the term $(ii)$ in (2.4), we insert another factor of $(H_0^\Lambda - E_0)(H_0^\Lambda - E_0)^{-1}$, and obtain,

$$\begin{aligned} (ii) &= -\lambda Tr(H_\omega^\Lambda - E_0)E_\Lambda(\Delta)V_\Lambda E_0^\Lambda(\tilde{\Delta}^c)(H_0^\Lambda - E_0)^{-2} \\ &\quad +\lambda^2 TrV_\Lambda E_\Lambda(\Delta)V_\Lambda E_0^\Lambda(\tilde{\Delta}^c)(H_0^\Lambda - E_0)^{-2} \\ &= (iii) + (iv). \end{aligned} \tag{2.6}$$

The term $(iii)$ is estimated just as term $(i)$ in (2.5), and we obtain

$$|(iii)| \;\leq\; \lambda|\Delta|^{1-2\alpha}\; \|\tilde{V}_\Lambda\|\; TrE_\Lambda(\Delta). \tag{2.7}$$

For the term $(iv)$ in (2.7), we obtain

$$|(iv)| \;\leq\; \lambda^2|\Delta|^{-2\alpha}\; TrV_\Lambda E_\Lambda(\Delta)V_\Lambda. \tag{2.8}$$

In order to estimate the last term, we note that hypothesis (H1) implies that $0 \leq V_\Lambda^2 \leq \tilde{V}_\Lambda$, where $\tilde{V}_\Lambda(x) = \sum_{j\in\tilde{\Lambda}} u(x-j)$. We then have

$$TrV_\Lambda E_\Lambda(\Delta)V_\Lambda \;\leq\; Tr\tilde{V}_\Lambda E_\Lambda(\Delta). \tag{2.9}$$

We now take the expectation. Using the result on the spectral shift function [**9**] discussed in section 5, we have

$$\mathbb{E}\{Tr(\lambda\tilde{V}_\Lambda E_\Lambda(\Delta))\} \;\leq\; C_2(q_2, u)|\Delta|^{q_2}|\Lambda|, \tag{2.10}$$

for any $0 < q_2 < 1$, and $|\Lambda|$ large enough, so that

$$\mathbb{E}\{|(iv)|\} \;\leq\; \lambda|\Delta|^{q_2-2\alpha}C_2(q_2, u)|\Lambda|. \tag{2.11}$$

3. Gathering the estimates from (2.3), (2.5), (2.7), and (2.11), we get

$$\begin{aligned} &\{1 - |\Delta|^{1-\alpha} - \lambda|\Delta|^{1-2\alpha}\|\tilde{V}_\Lambda\|\}\; \mathbb{E}\{TrE_\Lambda(\Delta)\} \\ &\leq\; (\lambda|\Delta|^{q_2-2\alpha}C_2(q_2, u) + C_1|\Delta|^{\alpha q_1})|\Lambda|. \end{aligned} \tag{2.12}$$

It is clear that if $\alpha < 1/2$ and $|\Delta|$ is taken sufficiently small, relative to the disorder $\lambda > 0$ (recall that $\|\tilde{V}_\Lambda\|_\infty = \mathcal{O}(1)$), then the prefactor on the left side of (2.12) is invertible. Finally, we require that $\alpha < q_2/(q_1 + 2) < 1/2$ in order that the right side of (2.12) be bounded above by $|\Delta|^q$, for $0 < q < q_1 q_2/(q_1 + 2)$. As $q_2 < 1$ is arbitrary, the bound

$$\frac{1}{|\Lambda|}\mathbb{E}\{TrE_\Lambda(\Delta)\} \leq C|\Delta|^q$$

holds for any $0 < q < q_1/(q_1+2)$, where $0 < q_1 \leq 1$ is the Hölder exponent of continuity for $N_0$, and for $\Lambda$ sufficiently large. Taking the limit $|\Lambda| \to +\infty$, we complete the proof of Theorem 1.1. □

## 3. IDS for Periodic $H_0$

*In this section, we sketch the proof of Theorem 1.3. The main assumption of Theorem 1.3 implies the following positivity condition* [**6**]*:*

$$E_0^\Lambda(\tilde{\Delta})\tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta}) \ \geq C(\tilde{\Delta},u)E_0^\Lambda(\tilde{\Delta}), \tag{3.1}$$

*where we have taken $\theta = 0$ in (1.9) corresponding to periodic boundary conditions on $\partial\Lambda$.*

Proof. 1. We begin with a decomposition of the spectral projector as in (2.2),

$$TrE_\Lambda(\Delta) = TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta}) + TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta}^c), \tag{3.2}$$

where $E_0 \in \Delta \subset \tilde{\Delta}$ as in the theorem. Let $d_0 \equiv \text{dist}\ (E_0, \tilde{\Delta}^c)$. In this case, it is easy to estimate the term involving $\tilde{\Delta}^c$. Using standard estimates, we obtain

$$TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta}^c) \leq \frac{|\Delta|}{d_0}TrE_\Lambda(\Delta) + \frac{1}{d_0}TrE_\Lambda(\Delta)(\lambda^2 V_\Lambda^2). \tag{3.3}$$

2. We use the lower bound (3.1) in order to estimate the first term on the right in (3.2). Substituting this into the trace for $E_0^\Lambda(\tilde{\Delta})$, we obtain

$$\begin{aligned} TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta}) \quad &\leq \quad \frac{1}{C(\tilde{\Delta},u)}\Big\{TrE_\Lambda(\Delta)\tilde{V}_\Lambda - TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta}^c)\tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta}) \\ &\qquad -TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta})\tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta}^c)\Big\}, \end{aligned} \tag{3.4}$$

where we dropped the positive term $TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta}^c)\tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta}^c)E_\Lambda(\Delta)$ since it occurs with a negative sign. Following the calculation in [**6**], we arrive at

3. We now combine (3.3) and (**??**) and obtain

$$\begin{aligned} TrE_\Lambda(\Delta) \quad &= \quad TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta}) + TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta}^c) \\ &\leq \quad \left\{\frac{|\Delta|}{d_0} + \frac{4D_0^2|\Delta|}{d_0 C(\tilde{\Delta},u)^2}\right\}TrE_\Lambda(\Delta) \\ &\qquad + \left\{\frac{2}{C(\tilde{\Delta},u)} + \frac{C_1(u,d)\lambda^2}{d_0}\left(1 + \frac{4D_0^2}{C(\tilde{\Delta},u)^2}\right)\right\}TrE_\Lambda(\Delta)\tilde{V}_\Lambda. \end{aligned} \tag{3.5}$$

Since $|\tilde{\Delta}| = 16D_0$, we have $d_0 = 8D_0$. The condition on $\Delta$ implies that $|\Delta|/d_0 \leq 1/4$, and the second term $4D_0^2|\Delta|/d_0C(\tilde{\Delta},u)^2 < 1/2$. So the coefficient of $TrE_\Lambda(\Delta)$ on the right side of (3.5) is less than $3/4$, so this term can be moved to the right. In order to estimate the second term, $TrE_\Lambda(\Delta)\tilde{V}_\Lambda$, we must take the expectation and use the estimate on the spectral shift function proved in [**9**]. □

There is an alternate manner to express the result of Theorem 1.3. Estimate (3.3) can also be written as

$$TrE_\Lambda(\Delta)E_0^\Lambda(\tilde{\Delta}^c) \leq |\Delta|TrE_\Lambda(\Delta) + \frac{1}{d_0^2}TrE_\Lambda(\Delta)(\lambda^2 V_\Lambda^2). \tag{3.6}$$

The effect of keeping $d_0$ with the second term is that the coefficient of $TrE_\Lambda(\Delta)$ on the right side of (3.5) can be written as

## 4. The Density of States Measure

Let us consider a general random Schrödinger operator $H_\omega = H_0 + V_\omega$, where $V_\omega$ is some random process, not necessarily of Anderson-type. It is well-known that under mild regularity assumptions on the random potential $V_\omega$ the IDS $N(E)$ exists almost surely at all points of continuity. For example (cf. [**3, 18, 21**]), if $p >$ max $(d/2, 1)$, and $r = (dp/2)(p - d/2)^{-1}$, and if $V_\omega$ satisfies

THEOREM 4.1. *Under the hypotheses (H1)–(H2), the DOS measure for $H_\omega$ exists and is locally or globally absolutely continuous with respect to Lebesgue measure. For any Borel set $A \subset \mathbb{R}$, as stated in Corollaries 1.2, 1.4, or 1.8, we have*

PROOF. 1. We first note that the proofs of Theorems 1.1 and 1.3 can be extended to include arbitrary Borel sets. We unify notation as follows. Let the set $J \subset \mathbb{R}$ denote the interval $\Delta \subset I$ in the proof of Theorem 1.1 for which the Wegner estimate holds, or let it denote the interval $\Delta$ in Theorem 1.3. For an arbitrary Borel set $A \subset \mathbb{R}$, we have

2. Let $f_J$ be the local density of the measure $\nu$. By the general theory, the density $f_J \in L^1_{loc}(\mathbb{R})$. Since the IDS $N(E)$ is locally Hölder continuous for any exponent $q > 1$, the density is better behaved. For any $\lambda \in \mathbb{R}$, let $F_\lambda$ be the subset of $\mathbb{R}$ defined by $F_\lambda \equiv \{t \in \mathbb{R} \mid f_J(t) > \lambda\}$. The $\nu$-measure of this set is bounded from below by

We remark that we first proved that if the IDS satisfies a Hölder continuity condition with exponent $0 < q < 1$, then the density is in weak-$L^p$, for $p = 1/(1-q)$. Let us note that under the best condition of Lipschitz continuity of the IDS, the density $f_J$ is given by the derivative $dN/d\lambda$, and this is in $L^\infty_{loc}$. This corresponds to $q = 1$. On the other hand, if we merely know that the DOS $\nu$ is absolutely continuous with respect to Lebesgue measure, then we are in the $q = 0$ case and the density $f_J$ is in $L^1_{loc}$.

## 5. Concluding Remarks

We make some comments about the results.

1. **Lipschitz Continuity.** It is believed that under hypotheses (H1)–(H2) the IDS is Lipschitz continuous. The above proofs provide for Lipschitz continuity provided there is an improved estimate on the spectral shift function (SSF) (for background on the SSF cf. [**1, 31**]). In [**9**], the following estimate on the SSF is proved. Suppose that $H_0$ and $H$ are two self-adjoint operators on a separable Hilbert space with $V \equiv H - H_0 \in \mathcal{I}_{1/p}$, for any $p \geq 1$. The Schatten-von Neumann ideals $\mathcal{I}_q$ are the classes of bounded operators $A$ having singular values $\mu_j(A)$ satisfying $\|A\|_q^q \equiv$

$\sum_j \mu_j(A)^q < \infty$, cf. [**25**]. Under these conditions, the SSF $\xi(\lambda; H, H_0) \in L^p(\mathbb{R})$, and satisfies the estimate

2. **Landau Hamiltonians.** A nice application of the method of proof of Theorem 1.1 is to the Hölder continuity of the IDS for the random Landau Hamiltonian outside of a discrete set. The unperturbed operator $H_L$ on $L^2(\mathbb{R}^2)$ has the form

$$H_L = (-i\nabla - A)^2, \text{ where } A(x_1, x_2) = \frac{B}{2}(-x_2, x_1), \tag{5.1}$$

where $B > 0$ is the magnetic field strength. The spectrum is pure point and consists of an increasing sequence of infinitely degenerate, isolated eigenvalues $\{E_j(B) = (2j+1)B \mid j = 0, 1, \dots, \}$. The IDS $N_0(E)$ for this model is a piecewise constant, monotone increasing function (cf. e. g. [**20**]). In particular, the IDS $N_0(E)$ is constant on any open interval of the form $I_n \equiv (E_n(B), E_{n+1}(B))$. Consequently, if $\tilde{\Delta} \subset I_n$, the local estimate (2.1) becomes

The perturbed family of Landau operators is

$$H_\omega = H_L + \lambda V_\omega, \tag{5.2}$$

where $V_\omega$ is the Anderson-type random perturbation given in (1.1). It is known that $N(E)$ is locally Lipschitz continuous in the following sense. Given an $N > 0$, there is a $B_N > 0$ so that for $B > B_N$, the IDS $N(E)$ is Lipschitz continuous on $(0, (2N+1)B) \backslash \{E_j(B) \mid j = 0, 1, \dots, N\}$ [**5, 27**]. Under some additional conditions, Wang [**28**] also proved that $N(E)$ is smooth outside of a given Landau level for sufficiently large magnetic field strength. By applying a slight extension of Theorem 1.1, possible in this case since (**??**) holds for any $q_1 \geq 0$, we are able to prove the following proposition that improves these known results.

Proposition 5.1. *The IDS for the random Landau Hamiltonian $H_\omega$, defined in (5.1)–(5.2) is Hölder continuous on each interval $I_n = (E_n(B), E_{n+1}(B))$ with any Hölder exponent $0 < q < 1$.*

There has been some discussion as to the behavior of the IDS at the Landau energies $E_j(B)$. If the single-site potential $u$ in (1.1) satisfies $u|\Lambda_1(0) > \epsilon\chi_{\Lambda_1(0)} > 0$, for some $\epsilon > 0$, then the IDS is locally Lipschitz continuous at all energies [**5**]. In [**6**], we proved a general theorem on the global Hölder continuity of the IDS under assumptions (H1)–(H2), and under the assumption that the magnetic flux through a unit square is $2\pi$ times a rational number. The proof of this result relies on the methods of the proof of Theorem 1.6. In the absence of the rational flux assumption, we have been able to prove that this IDS is continuous at all energies [**7**]. It is still an open problem to prove the Hölder continuity of the IDS at all energies with no flux condition.

3. **Regularity.** Very little is known about the regularity of the IDS for the models discussed here. Based on results for lattice models (cf. [**2, 3, 11, 21**], it is clear that the smoothness of the IDS is related to the smoothness of the probability density $h_0$. Consequently, one must strengthen hypothesis (H2) and require some smoothness on the distribution $h_0$.

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