RELAXATION TIME OF ANISOTROPIC SIMPLE EXCLUSION PROCESSES AND QUANTUM HEISENBERG MODELS

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Abstract. Motivated by an exact mapping between anisotropic half integer spin quantum Heisenberg models and asymmetric diffusions on the lattice, we consider an anisotropic simple exclusion process with N particles in a rectangle of $\mathbb{Z}^2$. Every particle at row h tries to jump to an arbitrary empty site at row $h \pm 1$ with rate $q^{\pm 1}$, where $q \in (0,1)$ is a measure of the drift driving the particles towards the bottom of the rectangle. We prove that the spectral gap of the generator is uniformly positive in N and in the size of the rectangle. The proof is inspired by a recent interesting technique envisaged by E. Carlen, M.C. Carvalho and M. Loss to analyze the Kac model for the non linear Boltzmann equation. We then apply the result to prove precise upper and lower bounds on the energy gap for the spin–S, $\mathrm{S} \in \frac{1}{2}\mathbb{N}$, XXZ chain and for the 111 interface of the spin–S XXZ Heisenberg model, thus generalizing previous results valid only for spin $\frac{1}{2}$.

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1. Introduction

Some years ago it was discovered by Alcaraz [2] that a class of asymmetric reversible simple exclusion processes on $\mathbb{Z}^d$ related to models of diffusion limited chemical reactions, are unitarily equivalent to certain anisotropic quantum Heisenberg Hamiltonian, known as XXZ models, that have received in recent years increasing attention in connection with the analysis of quantum domain walls (see [1], [11], [14] and references therein). Such an equivalence implies that the spectrum of (minus) the Markov generator of the process coincides with the spectrum of the quantum Hamiltonian. In particular the energy gap above the quantum ground state, a key quantity in the theory of quantum spin systems, becomes identical to the spectral gap of the process and a variety of probabilistic techniques come into play in order to obtain meaningful estimates. Such an observation was exploited recently in [7] to prove sharp bounds on the energy of low lying excitations above a ground state describing a 111 interface for a spin $\frac{1}{2}$ XXZ model. The extension of such results to higher half integer spin $\mathrm{S} \in \frac{1}{2}\mathbb{N}$ requires additional finer analysis and led us to consider the following model.

1.1. Setup.

Given two natural numbers L, H we consider the rectangle

$$\Lambda = \{(i,h) \in \mathbb{Z}^2 : \; i = 1, \dots, L \text{ and } h = 1, \dots, H\} \tag{1.1}$$

For each i, Λ_i stands for the *stick* at i given by $\Lambda_i = \{(i,h) \; : \; h = 1, \dots, H\}$. At each $x \in \Lambda$ we have a variable $\alpha_x \in \{0,1\}$: we say that site x is occupied (by a particle) if $\alpha_x = 1$ and empty otherwise. The set of configurations $\{0,1\}^\Lambda$ is denoted by Ω and it is

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naturally decomposed in single stick configurations: $\alpha \in \Omega$ will be written often in the form $\alpha = (\eta_1, \dots, \eta_L)$ with $\eta_i \in \{0,1\}^H$ denoting the restriction of α to the stick Λ_i.

Given a parameter $q \in (0,1)$ we define the product probability measure μ on Ω.

$$\mu(f) = \sum_{\alpha \in \Omega} \mu(\alpha) f(\alpha)\,, \quad \mu(\alpha) = \prod_{i=1}^{L} \prod_{h=1}^{H} \frac{q^{2h\alpha_{(i,h)}}}{1+q^{2h}} \tag{1.2}$$

where f is a generic function $f : \Omega \to \mathbb{R}$. According to μ particles prefer to live on the region of small h, i.e. the bottom of the box Λ if we interpret h as a vertical coordinate. We define n_i as the number of particles in the stick Λ_i: $n_i(\alpha) = n_i(\eta_i) = \sum_{h=1}^{H} \alpha_{(i,h)}$ and consider the conditional probability measure

$$\nu = \nu_N = \mu\Big(\ \cdot \mid \sum_{i=1}^{L} n_i = N\Big) \tag{1.3}$$

The variance of a function f w.r.t. ν will be written as usual in one of the following ways

$$\mathrm{Var}(f) = \nu(f,f) = \nu\big((f-\nu(f))^2\big)\,.$$

1.2. **The process and main result.** The asymmetric diffusion that will be analyzed in the sequel can be described as follows. Every particle at row h tries to jump to an arbitrary empty site at row $h+1$ with rate q and to an empty site at row $h-1$ with rate $1/q$. The Markov generator is defined by the operator

$$\mathcal{L}f(\alpha) = \frac{1}{L} \sum_{i=1}^{L} \sum_{j=1}^{L} \sum_{h=1}^{H-1} c_{(i,h);(j,h+1)}(\alpha) \nabla_{(i,h);(j,h+1)} f(\alpha) \tag{1.4}$$

where we use the notation

$$\nabla_{(i,h);(j,h+1)} f(\alpha) = f(\alpha^{(i,h);(j,h+1)}) - f(\alpha)\,, \tag{1.5}$$

$\alpha^{(i,h);(j,h+1)}$ denoting the configuration in which the values of α at (i,h) and $(j,h+1)$ have been interchanged while the rest is kept unchanged. The rates $c_{(i,h);(j,h+1)}$ are given by

$$c_{(i,h);(j,h+1)}(\alpha) = q^{\alpha_{(i,h)} - \alpha_{(j,h+1)}}\,. \tag{1.6}$$

Simple computations show that $\mathcal{L}$ is self adjoint in $L^2(\nu)$, the associated Dirichlet–form being

$$\mathcal{D}(f,f) = \nu(f(-\mathcal{L})f) = \frac{1}{L} \sum_{i=1}^{L} \sum_{j=1}^{L} D_{ij}(f) \tag{1.7}$$

$$D_{ij}(f) := \frac{1}{2} \sum_{h=1}^{H-1} \nu\big[c_{(i,h);(j,h+1)} \big(\nabla_{(i,h);(j,h+1)} f\big)^2\big]$$

Our main result then says that decay to equilibrium for the dynamics defined by (1.4) occurs exponentially fast in the $L^2(\nu)$ norm, uniformly in L, H and N. As a corollary we shall obtain an interesting estimate on the energy gap for a class of quantum XXZ Hamiltonian, see Theorems 4.1 and 5.1 below, that extends in particular previous results in [7] and [10].
More precisely let

$$\gamma(L,H) = \sup_{N} \sup_{f \in L^2(\nu)} \frac{\mathrm{Var}(f)}{\mathcal{D}(f,f)} \tag{1.8}$$

where the number of particles N in $\sup_N$, using the particle–hole symmetry, is assumed to range from 1 to $\frac{LH}{2}$.

Theorem 1.1. *For every $q \in (0,1)$ there exists $C < \infty$ such that*

$$\sup_{L,H} \gamma(L,H) \leqslant C\,.$$

Remark 1.2. Recently we learned [3] that for $L = 1$ and $N = \frac{H}{2}$ the mixing time (namely the smallest time such that $\sup_{\alpha,\tilde{\alpha}} \nu\left(|e^{t\mathcal{L}}(\alpha,\cdot) - e^{t\mathcal{L}}(\tilde{\alpha},\cdot)|\right) \leq \frac{1}{2}$) grows like H. Remarkably in the same setting the logarithmic Sobolev constant grows at least like H^2.

1.3. **Applications.** Some of the applications of Theorem 1.1 we have in mind, particularly those to quantum Heisenberg models, are linked to the analysis of the restriction of the above defined process to the horizontal sums of the basic variables $\alpha_{i,h}$ given by

$$\omega_h = \sum_{i=1}^{L} \alpha_{(i,h)}\,, \qquad h = 1,\dots,H$$

We will show below that the evolution of the new variables $\{\omega_h\}_{h=1}^{H}$ is still Markovian and that it can be interpreted as describing the fluctuations of a non–negative profile subject to a fixed area constraint.

Let $\mathcal{P}_L$ denote the set of permutations of $\{1,\dots,L\}$. Given $\pi \in \mathcal{P}_L$ we write $\alpha^{\pi,h}$ for the configuration

$$\alpha^{\pi,h}_{(i,h')} = \begin{cases} \alpha_{(i,h')} & h' \neq h \\ \alpha_{(\pi(i),h)} & h' = h \end{cases}, \qquad i = 1,\dots,L \tag{1.9}$$

The subspace $\mathcal{S}$ of horizontally symmetric functions is defined by

$$\mathcal{S} = \{f \in L^2(\nu) :\ f(\alpha) = f(\alpha^{\pi,h})\,,\ \forall \pi \in \mathcal{P}_L,\ \forall h = 1,\dots,H\}\,. \tag{1.10}$$

and it clearly consists of functions which only depend on the horizontal sums.

Given $f \in \mathcal{S}$ we write $\hat{f}(\omega) = f(\alpha)$. In this way we identify $\mathcal{S}$ with the space $L^2(\hat{\Omega},\hat{\nu})$, $\hat{\Omega} = \{0,1,\dots,L+1\}^H$ and $\hat{\nu}$ the marginal of ν on horizontal sums $\omega = \{\omega_h\}$. The probability $\hat{\nu}(\omega)$ of a single $\omega \in \hat{\Omega}$ compatible with the global constraint $\sum_h \omega_h = N$ is easily computed to be

$$\hat{\nu}(\omega) = \frac{1}{Z} \prod_{h=1}^{H} \binom{L}{\omega_h} q^{2h\omega_h} \quad Z = \sum_{\substack{\omega \in \hat{\Omega}: \\ \sum_h \omega_h = N}} \prod_{h=1}^{H} \binom{L}{\omega_h} q^{2h\omega_h} \tag{1.11}$$

Observe that $\mathcal{S}$ is an invariant subspace for the generator $\mathcal{L}$, i.e. $\mathcal{L}\mathcal{S} \subset \mathcal{S}$. In fact a simple computation shows that for every $f \in \mathcal{S}$, $\mathcal{L}f(\alpha)$ can be written as

$$\widehat{\mathcal{L}}\hat{f}(\omega) = \frac{1}{L} \sum_{h=1}^{H-1} \Big\{ w_{+,h}(\omega)\left[\hat{f}(\omega^{+,h}) - \hat{f}(\omega)\right] + w_{-,h}(\omega)\left[\hat{f}(\omega^{-,h}) - \hat{f}(\omega)\right] \Big\} \tag{1.12}$$

$$w_{+,h} := q^{-1}\,(L - \omega_h)\omega_{h+1}\,, \quad w_{-,h} := q\,(L - \omega_{h+1})\omega_h\,,$$

$$\omega^{\pm,h}_{h'} := \begin{cases} \omega_{h'} & h' \neq h, h+1 \\ \omega_h \pm 1 & h' = h \\ \omega_{h+1} \mp 1 & h' = h+1 \end{cases}$$

This defines a Markov generator $\widehat{\mathcal{L}}$ which is symmetric in $L^2(\hat{\Omega}, \hat{\nu})$. The corresponding process (the restriction to $\{\omega_h\}$ of the original anisotropic exclusion dynamics) can be interpreted as describing fluctuations of a non-negative profile $\omega := \{\omega_h\}_{h=1}^H$ subject to a fixed area constraint ($\sum_h \omega_h =$ constant). In view of the anisotropy the profile is rather sharply localized: letting $\rho = N/L$ we see that $\omega_h \approx L$ for heights h below ρ and $\omega_h \approx 0$ above ρ with high probability. By Theorem 1.1 relaxation to equilibrium in $L^2(\hat{\Omega}, \hat{\nu})$ is exponentially fast uniformly in ρ.
In the case $L = 2$ the previous analysis admits another interesting interpretation as a model for diffusion limited chemical reactions, see [2, 5, 6] and references therein. Namely describe the state $\omega_h = 2$ as the presence at h of a particle of type A, $\omega_h = 0$ as a particle of type B and $\omega_h = 1$ as the absence of particles (*inert*). If n_A, n_B denote the size of the two populations we see that the difference $n_A - n_B$ is conserved and this system can be studied as a model for asymmetric diffusion with creation and annihilation of the two species. Particles of type A have a constant drift towards the bottom ("small h" region) while particles of type B have the same drift towards the top ("large h" region). They perform asymmetric simple exclusion with respect to the *inert* sites but when they meet (*i.e.* when they become nearest neighbors) they can produce the annihilation reaction $A + B \to$ *inert*. The reverse reaction *inert* $\to A + B$ restores steady state fluctuations given by the canonical measure.
While Theorem 1.1 implies immediately L^2-exponential ergodicity for the above process, a direct proof of the result for the two-particle model seemed difficult to us.

1.4. **Some ideas for the proof of Theorem 1.1.** We conclude this introductory section with some comments on the main ideas behind the proof of Theorem 1.1. Our main source of inspiration has been a recent work by E. Carlen, M.C. Carvalho and M. Loss on the rate of approach to equilibrium for the Kac model of the non linear Boltzmann equation [8]. Like in other approaches to bound the spectral gap for large reversible Markov chains, the first idea is to recursively bound $\gamma(L, H)$ in terms $\gamma(1, H)$, the latter being finite uniformly in H because of Theorem 4.3 in [7]. The starting point, as e.g. in the martingale approach of H.-T. Yau [15], is a decomposition of the variance of an arbitrary function f as

$$\mathrm{Var}(f) = \frac{1}{L}\sum_{k=1}^{L} \nu\big(\mathrm{Var}(f \mid \mathcal{F}_k)\big) + \frac{1}{L}\sum_{k=1}^{L} \mathrm{Var}\big(\nu(f \mid \mathcal{F}_k)\big)\,.$$

where $\mathcal{F}_k$ denote the $\sigma-$algebra generated by the stick-variables η_k, $k = 1, \ldots, , L$. It is easy to check (see section 3 below) that the first term can be bounded in terms of $\gamma(L-1, H) \times \mathcal{D}(f, f)$. The main new idea comes in the analysis of the second term and consists in introducing the stochastic symmetric operator

$$Pf = \frac{1}{L}\sum_{k=1}^{L} \nu(f \mid \mathcal{F}_k)\,,$$

and observing that for any mean zero function f the following identity holds true:

$$\frac{1}{L}\sum_{k=1}^{L} \mathrm{Var}\big(\nu(f \mid \mathcal{F}_k)\big) = \nu\big(fPf\big)\,.$$

Thus

$$\mathrm{Var}(f) - \frac{1}{L}\sum_{k=1}^{L} \mathrm{Var}\big(\nu(f \mid \mathcal{F}_k)\big) = \mathrm{Var}(f) - \nu\big(fPf\big) = \nu\big(f(\mathbb{1} - P)f\big)\,.$$

so that one is left with the problem of establishing an estimate from below on the spectral gap of P which is sharp enough to allow a successful iteration in L for $\gamma(L, H)$. The key point now is that, because of the particular form of P and of the symmetry of the measure ν, the estimate of the spectral gap of P boils down to the estimate from below of the spectral gap of a particular one dimensional random walk that can be described as follows. Let $n_\pm$ be the minimum and maximum number of particles allowed in a single stick, say the first one. Then the state space for the random walk is the interval $[n_-, n_- + 1, \dots, n_+]$ and the transition kernel $q(n \to m)$ is given by $\nu\big(n_1 = m \,|\, n_2 = n\big)$. It is easy to check that such a process is ergodic iff $L \geq 3$. The study of its relaxation time represents in some sense the technical core of the paper and it requires a rather fine analysis based on equivalence of ensembles type of results.
The rest of the paper is organized as follows.

(i) In the next section we define and analyze the one dimensional random walk mentioned above.
(ii) In section 3 we prove Theorem 1.1.
(iii) In sections 4,5 we discuss the main applications of Theorem 1.1 to quantum Heisenberg XXZ models.

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2. Spectral gap of the long–jump random walk in a single stick

In this section we are going to study a one-dimensional process which plays a key role in the recursive proof of Theorem 1.1.

Let ν_0 be the marginal of ν, the canonical measure defined in (1.3), on a single stick configuration. If $\pi_i : \{0,1\}^\Lambda \to \{0,1\}^{\Lambda_i}$ denotes the canonical projection onto single stick configurations ($\pi_i\alpha = \eta_i$) we may write $\nu_0 = \nu \circ \pi_1^{-1}$. Let $\mathbb{H}$ denote the space $L^2(\{0,1\}^{\Lambda_1}, \nu_0)$ and denote by $\langle \cdot, \cdot \rangle$ the corresponding scalar product:

$$\langle \varphi, \psi \rangle = \nu\big((\varphi \circ \pi_1)(\psi \circ \pi_1)\big)\,, \qquad \varphi, \psi \in \mathbb{H}.$$

The σ-algebra generated by the single stick variable η_i is denoted by $\mathcal{F}_i$. The operator $K : \mathbb{H} \to \mathbb{H}$ is defined by

$$K\varphi = \nu\big(\varphi \circ \pi_2 \,|\, \mathcal{F}_1\big)\,,$$

or equivalently by the bilinear form

$$\langle \varphi, K\psi \rangle = \nu\big((\varphi \circ \pi_1)(\psi \circ \pi_2)\big)\,.$$

K is a stochastic, symmetric linear operator on $\mathbb{H}$. The number of particles in a stick is denoted by n and we call $\rho = N/L$ its average value. The centered variable $n - \rho$ is denoted by $\bar{n}$. Observe that $\bar{n}$ is an eigenfunction of K with eigenvalue $-1/(L-1)$:

$$K\bar{n} = -\frac{1}{L-1}\,\bar{n}\,. \tag{2.1}$$

Our main result in this section states that apart from the values 1 and $-1/(L-1)$ the spectrum of K is concentrated around 0 within an interval of radius $O(L^{-1-\delta})$ for some $\delta > 0$, uniformly in N, H.

Theorem 2.1. *There exist constants $\delta > 0$, $k < \infty$ and $L_0 < \infty$ independent of N, H such that if $L \geqslant L_0$*

$$\big|\,\langle \varphi, K\varphi \rangle\,\big| \leqslant k\, L^{-1-\delta}\, \langle \varphi, \varphi \rangle \tag{2.2}$$

for all $\varphi \in \mathbb{H}$ with $\nu_0(\varphi) = 0$ and $\langle \varphi, \bar{n} \rangle = 0$.

Proof. Let $\mathbb{H}_0$ denote the subspace of $\mathbb{H}$ of functions which only depend on the number of particles. Let also E denote the orthogonal projection onto $\mathbb{H}_0$, i.e. if $\mathcal{F}_0$ denotes the σ-algebra generated by the random variable n we have

$$E\varphi = \nu_0\big(\varphi \,|\, \mathcal{F}_0\big)\,.$$

A simple computation now shows that K commutes with E and

$$K\varphi = KE\varphi\,, \qquad \varphi \in \mathbb{H}.$$

These observations prove that $\langle \varphi, K\varphi \rangle = \langle E\varphi, KE\varphi \rangle$ and in order to prove the claim (2.2) we can restrict to $\varphi \in \mathbb{H}_0$.

Let us introduce some handy notations. We write $\nu(n)$ for the probability that in one given stick there are n particles, $\nu(n \cap m)$ for the probability that in two given sticks there are n and m particles respectively, and $\nu(n\,|\,m)$ for the probability that in one given stick there are n particles conditioned to the event that in a different given stick there are m particles. With these symbols we have

$$K\varphi(n) = \sum_m \nu(m\,|\,n)\varphi(m) \qquad \varphi \in \mathbb{H}_0\,.$$

The operator K then describes a random walk on the integers with transition probabilities Prob $(n \to m) = \nu(m|n)$. The desired estimate (2.2) can be written as

$$\sum_{n,m} \nu(m)\nu(n)\varphi(m)Q(m,n)\varphi(n) \leqslant k\, L^{-1-\delta} \langle \varphi, \varphi \rangle \ , \tag{2.3}$$

$$Q(m,n) := \frac{\nu(n\,|\,m)}{\nu(n)} - 1 \tag{2.4}$$

for all $\varphi \in \mathbb{H}_0$ such that $\nu_0(\varphi) = 0$ and $\langle \varphi, \bar{n} \rangle = 0$. The rest of this section is concerned with the proof of (2.3).

The idea is to split the sum in (2.3) in a region of typical values of m, n where things are controlled by a careful expansion and a region of atypical values whose contribution is shown to be negligible by tail estimates. Unfortunately the definition of typical and atypical values of n, m strongly depends on the value of the particle density ρ and we will be forced to distinguish between two cases, conventionally denoted `large density` and `small density` case, depending whether $\rho \geq L^{-\frac{3}{4}}$ or $\rho < L^{-\frac{3}{4}}$. Before entering into the details of the proof we first need to establish some preliminary useful bounds.

2.1. **Technical bounds.** The grand canonical measure $\mu = \mu^{\lambda(\rho)}$ with density $\rho = N/L$ is the product measure on Λ (see (1.2)) with

$$\mu^{\lambda(\rho)}(\alpha) = \prod_{i=1}^{L} \prod_{h=1}^{H} \frac{q^{2(h-\lambda(\rho))\alpha_{(i,h)}}}{1 + q^{2(h-\lambda(\rho))}} \tag{2.5}$$

Here the parameter $\lambda(\rho) \in \mathbb{R}$, often called the chemical potential, is such that the average number of particles in any given stick is equal to ρ. The variance $\mu(\bar{n}^2)$ of the number of particles in a stick is denoted by σ^2. Simple computations - as in [7], Lemma 3.2 - show that for every $q \in (0,1)$, there exists $k < \infty$ such that

$$k^{-1}(\rho \wedge 1) \leqslant \sigma^2 \leqslant k\,(\rho \wedge 1)\,.$$

Similarly $\mu(\bar{n}^4) \leqslant k(\rho \wedge 1)$. As a rule, here and throughout the rest of this section the letter k will be used to denote a finite constant whose value may change from line to line. What is essential is that it only depends on q and is uniform in all other parameters: L,H, and N.

We introduce the characteristic function

$$F(t) = \mu\big(e^{i\frac{t}{\sigma\sqrt{L}}\bar{n}}\big)\,.$$

We shall rely on the following simple estimate.

Lemma 2.2. *For all $t \in [-\pi\sigma\sqrt{L}, \pi\sigma\sqrt{L}]$*

$$|F(t)| \leqslant e^{-k\frac{t^2}{L}} \tag{2.6}$$

Proof. Writing

$$n = \sum_h \alpha_h\,,$$

where we drop the horizontal stick label, we have

$$|F(t)| = \prod_h g_h(t)\,, \qquad g_h(t) := \big|\mu\big(e^{i\frac{t}{\sigma\sqrt{L}}\alpha_h}\big)\big|\,.$$

Using the inequality $x \leqslant e^{(x^2-1)/2}$, $x \in (0,1)$, we have

$$\begin{aligned} g_h(t) &\leqslant \exp\big(\frac{1}{2}(g_h(t)^2-1)\big) \\ &= \exp\big\{-\frac{1}{2}\mathrm{Var}_\mu\big[\cos(t\alpha_h/\sigma\sqrt{L})\big] - \frac{1}{2}\mathrm{Var}_\mu\big[\sin(t\alpha_h/\sigma\sqrt{L})\big]\big\} \end{aligned}$$

Define $\sigma_h^2 = \mathrm{Var}_\mu(\alpha_h)$ and compute

$$\begin{aligned} \mathrm{Var}_\mu\big[\cos(t\alpha_h/\sigma\sqrt{L})\big] &= \sigma_h^2\big(\cos(t/\sigma\sqrt{L})-1\big)^2 \\ \mathrm{Var}_\mu\big[\sin(t\alpha_h/\sigma\sqrt{L})\big] &= \sigma_h^2\big(\sin(t/\sigma\sqrt{L})\big)^2 . \end{aligned}$$

It follows

$$g_h(t) \leqslant \exp\big\{-\sigma_h^2(1-\cos(t/\sigma\sqrt{L}))\big\} \leqslant \exp\big(-kt^2\sigma_h^2/\sigma^2 L\big),$$

where we use the inequality $1-\cos s \geqslant ks^2$, for some $k>0$ and all $s \in [-\pi,\pi]$. The lemma now follows from $\sigma^2 = \sum_h \sigma_h^2$. ☐

It is useful to compare the canonical measure ν, given again by (1.3), with the grand canonical probability μ. In particular, it follows from Lemma 2.2 - as shown in [7], Proposition 3.8 - that there exists $k<\infty$ independent of the density ρ such that

$$\nu(f) \leqslant k\,\mu(f) \tag{2.7}$$

for any function $f \geq 0$ depending at most on $\frac{L}{2}$ variables $\{\eta_i\}$. We shall need the following estimates on the tails of our distributions.

Lemma 2.3. *There exist constants $a>0$ and $k<\infty$ depending only on q such that*

$$\nu(n\,|\,m)\nu(m\,|\,n) \leq k\,e^{-a\,[(n-\rho)^2+(m-\rho)^2]} \tag{2.8}$$

$$\nu(n\,|\,m)\nu(m\,|\,n) \leq k\,\rho^{n+m} \tag{2.9}$$

Proof. We start with the proof of (2.8). Given $n \in \mathbb{N}$ we denote by $\lambda(n)$ the chemical potential such $\mu^{\lambda(n)}(n_1) = n$. We write simply λ for the chemical potential $\lambda(\rho)$. We have

$$\nu(n) \leq k\,\mu^\lambda(n) = k\,e^{-V_\lambda(n)}\mu^{\lambda(n)}(n) \leq k\,e^{-V_\lambda(n)} \tag{2.10}$$

where

$$V_\lambda(n) = c\,(\lambda(n)-\lambda)n - \log\left(\frac{Z^{\lambda(n)}}{Z^\lambda}\right), \quad c := -\log q\,,$$

and Z^λ denotes the partition function

$$\prod_{h=1}^{H}\left(1+q^{2(h-\lambda)}\right).$$

Now $V_\lambda(\rho)=0$ and we write

$$V_\lambda(n) = \int_\rho^n ds\,\frac{d}{ds}V_\lambda(s)\,.$$

A standard computation gives that $\frac{d}{ds}V_\lambda(s) = c\,(\lambda(s)-\lambda)$ and

$$\frac{d^2}{ds^2}V_\lambda(s) = c\,\frac{d}{ds}\lambda(s) = \frac{1}{\mu^{\lambda(s)}(\bar{n}^2)} \geqslant a$$

for some $a > 0$ independent of s. Therefore

$$V_\lambda(n) \geqslant a\,(n-\rho)^2\,.$$

By (2.10) we have the tail estimate

$$\nu(n) \leqslant k e^{-a\,(n-\rho)^2}\,. \tag{2.11}$$

We now apply the above bound to each term $\nu(n\,|\,m)$. In this case the starting chemical potential λ corresponds to a density $\rho' = \frac{N-m}{L-1}$ so that (2.11) yields

$$\nu(n\,|\,m) \leq k\,e^{-a\,\left(n-\frac{N-m}{L-1}\right)^2}$$

Our original claim (2.8) now follows from the uniform (in N, L) convexity of the function

$$G(x,y) := \left(x - \frac{N-y}{L-1}\right)^2 + \left(y - \frac{N-x}{L-1}\right)^2$$

around the unique minimum $x = \rho$, $y = \rho$.

To prove (2.9) we use again the bound (2.7) and estimate

$$\nu(n\,|\,m) \leq k\mu^{\lambda(\frac{N-m}{L-1})}(n) \leq k\left(\frac{N-m}{L-1}\right)^n \leq k\rho^n\,.$$

□

We are in a position to complete the proof of Theorem 2.1. As anticipated we will need to distinguish between "large" and "small" values of the density ρ.

2.2. **The large density case.** Here we assume $\rho \geqslant L^{-\frac{3}{4}}$. Define the set

$$\mathcal{B} = \{(n,m) :\ |\bar{n}| + |\bar{m}| \leqslant B\log L\}\,, \tag{2.12}$$

with $B < \infty$ a constant to be fixed later. We rewrite the sum in (2.3) as

$$\sum_{(n,m)\in\mathcal{B}} \nu(n)\nu(m)\varphi(n)Q(m,n)\varphi(m) + \sum_{(n,m)\notin\mathcal{B}} \nu(n)\nu(m)\varphi(n)Q(m,n)\varphi(m)\,. \tag{2.13}$$

The second term in (2.13) is estimated with the help of Lemma 2.3. To see this use Schwarz' inequality to write

$$\begin{aligned}
&\sum_{(n,m)\notin\mathcal{B}} \nu(m)\nu(n)\varphi(m)Q(m,n)\varphi(n)\\
&\leq \langle\varphi,\varphi\rangle\Big[\sum_{(n,m)\notin\mathcal{B}} \nu(m)\nu(n)Q(m,n)^2\Big]^{\frac{1}{2}}\\
&\leq \langle\varphi,\varphi\rangle\Big[\sum_{(n,m)\notin\mathcal{B}} \nu(n)\nu(m)2\,[1+\frac{\nu(n\,|\,m)^2}{\nu(n)^2}]\Big]^{\frac{1}{2}}\,.
\end{aligned}$$

Since

$$\nu(n)\nu(m)\frac{\nu(n\,|\,m)^2}{\nu(n)^2} = \nu(n\,|\,m)\nu(m\,|\,n)\,,$$

an application of (2.8) and (2.11) gives

$$\sum_{n,m\notin\mathcal{B}} \nu(n)\nu(m)\varphi(n)Q(m,n)\varphi(m) \leqslant k\,L^{-2}\,\langle\varphi,\varphi\rangle\,, \tag{2.14}$$

provided B is sufficiently large.

The key step in the proof of Theorem 2.1 in the case of large density will be the following expansion.

Lemma 2.4. *For all $q \in (0,1)$, $B < \infty$, there exist constants $k < \infty$, $\zeta > 0$ such that if $\rho \geqslant L^{-\frac{3}{4}}$ then*

$$Q(m,n) = -\frac{\bar{n}\bar{m}}{\sigma^2 L} + \mathcal{R}(n,m) \tag{2.15}$$

with the remainder $\mathcal{R}$ satisfying

$$\Big[\sum_{\substack{n,m \\ |\bar{n}|+|\bar{m}| \,\leqslant\, B\log L}} \nu(n)\nu(m)|\mathcal{R}(n,m)|^2 \Big]^{1/2} \le kL^{-1-\zeta}\,. \tag{2.16}$$

Proof. We write

$$\begin{aligned} \nu(n) &= \frac{\mu(n)}{2\pi\sigma\sqrt{L}\mu(N)} \int dt\, F(t)^{L-1} e^{i\frac{t}{\sigma\sqrt{L}}\bar{n}} \\ \nu(n\cap m) &= \frac{\mu(n)\mu(m)}{2\pi\sigma\sqrt{L}\mu(N)} \int dt\, F(t)^{L-2} e^{i\frac{t}{\sigma\sqrt{L}}[\bar{n}+\bar{m}]} \end{aligned}$$

where all the integrals are over the interval $[-\pi\sigma\sqrt{L}, \pi\sigma\sqrt{L}]$. Since

$$2\pi\sigma\sqrt{L}\mu(N) = \int dt\, F(t)^L$$

we can write

$$Q(m,n) = \frac{\nu(n\cap m) - \nu(n)\nu(m)}{\nu(n)\nu(m)} = \frac{\mathrm{NUM}}{\mathrm{DEN}}$$

with

$$\begin{aligned} \mathrm{NUM} := &\int dt\, F(t)^{L-2} e^{i\frac{t}{\sigma\sqrt{L}}[\bar{n}+\bar{m}]} \int dt'\, F(t')^L \\ &- \int dt\, F(t)^{L-1} e^{i\frac{t}{\sigma\sqrt{L}}\bar{n}} \int dt'\, F(t')^{L-1} e^{i\frac{t'}{\sigma\sqrt{L}}\bar{m}} \end{aligned}$$

and

$$\mathrm{DEN} := \int dt\, F(t)^{L-1} e^{i\frac{t}{\sigma\sqrt{L}}\bar{n}} \int dt'\, F(t')^{L-1} e^{i\frac{t'}{\sigma\sqrt{L}}\bar{m}}$$

Notice that, because of the Gaussian upper bound of Lemma 2.2 we have $|F(t)|^L \leqslant e^{-at^2}$ and only the region $|t| \le k\log L$ (for some large but fixed k) will have to be taken care of. In order to be precise about the nature of the various error terms, in what follows we will denote by $\varepsilon(L)$ a generic term which, upon multiplication by L, still goes to zero (as $L \to \infty$) as an inverse power of L uniformly in the range of $|t| \leqslant k\log L$. We first observe that $F(t) = 1 - \frac{t^2}{2L} + \varepsilon(L)$. Indeed by expanding F around $t = 0$ the third order error term is bounded from above by

$$k\frac{|t|^3}{(\sigma^2 L)^{3/2}}\mu(|\bar{n}|^3) \le k\frac{|t|^3}{\sigma L^{3/2}} \le \varepsilon(L)$$

where we use the bound

$$\mu(|\bar{n}|^3) \leqslant k\,\sigma\,\mu(\bar{n}^4)^{\frac{1}{2}} \leqslant k\,\sigma^2$$

together with $\sigma^2 \geqslant k(\rho \wedge 1)$ and $\rho \geqslant L^{-\frac{3}{4}}$. This implies $F(t)^{-1} = 1 + \frac{t^2}{2L} + \varepsilon(L)$ and $F(t)^{-2} = 1 + \frac{t^2}{L} + \varepsilon(L)$. Then if we write $e^{i\frac{t}{\sigma\sqrt{L}}\bar{n}} = 1 + \delta_n(t)$ we have

$$\mathrm{NUM} = \int dt F(t)^L \delta_n(t)\delta_m(t) \int dt' F(t')^L - \int dt F(t)^L \delta_n(t) \int dt' F(t')^L \delta_m(t') + \varepsilon(L)\,.$$

If we modify further and define

$$\hat{\delta}_n = \delta_n - i\frac{t}{\sigma\sqrt{L}}\bar{n}\,, \qquad I(L) = \int dt F(t)^L\, t$$

we then have,

$$\begin{aligned}
\mathrm{NUM} = &-\frac{\bar{n}\bar{m}}{\sigma^2 L}\Big(\int dt\, t^2\, F(t)^L \int dt'\, F(t')^L - I(L)^2 \Big) \\
&+ \int dt\, F(t)^L \hat{\delta}_n(t)\, \hat{\delta}_m(t) \int dt'\, F(t')^L - \int dt\, F(t)^L \hat{\delta}_n(t) \int dt'\, F(t')^L \hat{\delta}_m(t') \\
&+ i\,\frac{\bar{m}}{\sigma\sqrt{L}} \int dt\, F(t)^L\, t\, \hat{\delta}_n(t) \int dt'\, F(t')^L + i\,\frac{\bar{n}}{\sigma\sqrt{L}} \int dt\, F(t)^L\, t\, \hat{\delta}_m(t) \int dt'\, F(t')^L \\
&- i\,\frac{\bar{m}}{\sigma\sqrt{L}} \int dt\, F(t)^L \hat{\delta}_n(t) \int dt'\, F(t')^L\, t' - i\,\frac{\bar{n}}{\sigma\sqrt{L}} \int dt\, F(t)^L \hat{\delta}_m(t) \int dt'\, F(t')^L\, t' + \varepsilon(L)\,.
\end{aligned}$$

Now observe that $\frac{1}{L}I(L) = \varepsilon(L)$ and

$$\frac{1}{L} \int dt\, t^2\, F(t)^L \int dt'\, F(t')^L = \frac{2\pi}{L} + \varepsilon(L)\,.$$

Using also $|\hat{\delta}_n(t)| \leq k\frac{\bar{n}^2 t^2}{\sigma^2 L}$ it follows that

$$\mathrm{NUM} = -2\pi\,\frac{\bar{n}\bar{m}}{\sigma^2 L} + R(n,m) + \varepsilon(L)$$

with

$$|R(n,m)| \leq k\Big[\, \frac{|\bar{n}|\,|\bar{m}|}{\sigma^2}\,\varepsilon(L) + \frac{\bar{n}^2\bar{m}^2}{(\sigma^2 L)^2} + \frac{|\bar{n}|\,\bar{m}^2 + |\bar{m}|\,\bar{n}^2}{(\sigma^2 L)^{3/2}}\,\Big]$$

What is crucial for us is that

$$\Big[\sum_{\substack{n,m \\ |\bar{n}|+\bar{m}| \,\leqslant\, B\log L}} \nu(n)\nu(m)|R(n,m)|^2\Big]^{1/2} \leq \varepsilon(L) \tag{2.17}$$

provided that $\rho \geq L^{-\frac{3}{4}}$. The above estimate actually holds without the restriction $|\bar{n}| + |\bar{m}| \leqslant B\log L$ as it is easily seen using $\nu(\bar{n}^4) \leqslant k\mu(\bar{n}^4) \leqslant k\sigma^2$ and the bound $(\rho \wedge 1) \leqslant k\sigma^2$. On the other hand if we repeat the reasoning for the denominator DEN we obtain

$$\mathrm{DEN} = 2\pi + \hat{R}(n,m)$$

with a remainder $\hat{R}(n,m)$ satisfying

$$\sup_{\substack{n,m: \\ |\bar{n}|+|\bar{m}| \,\leqslant\, B\log L}} |\hat{R}(n,m)| \leqslant kL^{-\zeta} \tag{2.18}$$

for some $\zeta > 0$ whenever $\rho \geqslant L^{-\frac{3}{4}}$. In conclusion $Q(m,n)$ has been written as in (2.15) and (2.17)–(2.18) imply (2.16). ☐

We are now able to finish the proof of Theorem 2.1 in the case of large densities. Observe that

$$\sum_{(n,m)\in\mathcal{B}} \nu(n)\nu(m)\varphi(n)\varphi(m)\bar{n}\bar{m} = \langle\varphi,\bar{n}\rangle^2 - \sum_{(n,m)\notin\mathcal{B}} \nu(n)\nu(m)\varphi(n)\varphi(m)\bar{n}\bar{m}\,.$$

By assumption $\langle\varphi,\bar{n}\rangle = 0$. Then

$$\Big|\sum_{(n,m)\in\mathcal{B}} \nu(n)\nu(m)\varphi(n)\varphi(m)\bar{n}\bar{m}\Big|$$
$$\leqslant \Big[\sum_{(n,m)\notin\mathcal{B}} \nu(n)\nu(m)\bar{n}^2\bar{m}^2\Big]^{\frac{1}{2}}\,\langle\varphi,\varphi\rangle\,\leqslant kL^{-2}\,\langle\varphi,\varphi\rangle$$

where the last estimate can be easily obtained from (2.11). By Lemma 2.4 we then have

$$\Big|\sum_{(n,m)\in\mathcal{B}} \nu(n)\nu(m)\varphi(n)Q(n,m)\varphi(m)\Big|$$
$$\leqslant \langle\varphi,\varphi\rangle\left\{k\,L^{-2} + \Big[\sum_{(n,m)\in\mathcal{B}} \nu(n)\nu(m)|\mathcal{R}(n,m)|^2\Big]^{1/2}\right\} \leqslant k\,L^{-1-\zeta}\,\langle\varphi,\varphi\rangle$$

Together with (2.14) these bounds imply Theorem 2.1 when $\rho \geqslant L^{-\frac{3}{4}}$.

2.3. **The case of small density.** When $\rho \leq L^{-\frac{3}{4}}$ the strategy for the proof of (2.3) has to be slightly modified since various previous technical estimates are no longer valid. We can however take advantage of the thinner tails of the distribution of the number of particles. In this respect we observe that

$$\Big[\sum_{\substack{n,m \\ n+m\,\geqslant\,2\,,\,nm\neq 1}} \nu(n)\nu(m)\,Q(n,m)^2\Big]^{\frac{1}{2}} \leq kL^{-1-\zeta} \tag{2.19}$$

with $\zeta = 1/8$. Indeed, thanks to (2.9)

$$\Big[\sum_{\substack{n,m \\ n+m\geq 3}} \nu(n)\nu(m)\,Q(n,m)^2\Big]^{\frac{1}{2}} \leq k\,\rho^{\frac{3}{2}} \leqslant k\,L^{-1-\zeta}\,.$$

On the other hand we can examine explicitly the case $n = 2, m = 0$. We have

$$Q(2,0) = \frac{1}{\nu(0)} - 1 - \frac{\nu(\{n=2\}\cap\{m\geq 1\})}{\nu(0)\nu(2)}\,.$$

Using the bounds $\nu(0) \geqslant 1 - k\rho$, $\nu(n) \leqslant k\rho^n$ and $\nu(\{n=2\}\cap\{m\geq 1\}) \leqslant k\rho^3$ one has

$$\nu(2)\nu(0)Q(2,0)^2 \leqslant k\rho^4\,.$$

We have proved (2.19).

At this point we may proceed as in (2.13) with the choice

$$\mathcal{B} := \{n,m:\; n\leq 1,\, m\leq 1\}\,.$$

The second term in (2.13) is controlled by (2.19). The first term is given by

$$\sum_{n\leq 1}\sum_{m\leq 1} \varphi(n)\varphi(m)\big[\,\nu(n\cap m) - \nu(n)\nu(m)\,\big]\,.$$

We are going to study the eigenvalues of the symmetric 2×2 matrix

$$\mathcal{M}(n,m) := \big[\, \nu(n \cap m) - \nu(n)\nu(m) \,\big]\,, \quad n,m \in \{0,1\}\,.$$

For $n = 0, 1 \dots N$, let M_n be the number of sticks with exactly n particles. The identities $L = \sum_n M_n$ and $N = \sum_n nM_n$ imply

$$M_0 = L - N + \sum_{n \geq 2} (n-1) M_n\,, \qquad M_1 = N - \sum_{n \geq 2} nM_n\,.$$

Denoting by n_i the number of particles in the stick Λ_i, we have the estimates

$$\begin{aligned} \mathrm{Var}_\nu(M_0) &\leq k\mathrm{Var}_\mu \left(\sum_i (n_i - 1) \mathbb{1}_{\{n_i \,\geqslant\, 2\}} \right) \leq kL\rho^2 \\ \mathrm{Var}_\nu(M_1) &\leq k\mathrm{Var}_\mu \left(\sum_i n_i \mathbb{1}_{\{n_i \,\geqslant\, 2\}} \right) \leq kL\rho^2\,. \end{aligned}$$

We compute

$$\begin{aligned} \nu(\{n_1 = 0\} \cap \{n_2 = 0\}) &= \frac{1}{L-1} \sum_{i=2}^{L} \nu(\{n_1 = 0\} \cap \{n_i = 0\}) \\ &= \frac{1}{L-1}\, \nu(\mathbb{1}_{\{n_1 = 0\}}(M_0 - 1)) = \frac{1}{L(L-1)}\, \nu(M_0(M_0-1)) \\ &= \frac{1}{L(L-1)}\, \mathrm{Var}(M_0) + \frac{1}{L(L-1)}\, \nu(M_0)^2 - \frac{1}{L(L-1)}\, \nu(M_0)\,. \end{aligned}$$

It follows that $\mathcal{M}(0,0) = \nu(\{n_1 = 0\} \cap \{n_2 = 0\}) - \nu(1)\nu(0)$ can be written

$$\mathcal{M}(0,0) = -\frac{\nu(0)(1-\nu(0))}{L-1} + \frac{1}{L(L-1)} \mathrm{Var}(M_0)\,.$$

Similarly

$$\begin{aligned} \mathcal{M}(1,1) &= -\frac{\nu(1)(1-\nu(1))}{L-1} + \frac{1}{L(L-1)} \mathrm{Var}(M_1) \\ \mathcal{M}(0,1) &= \frac{\nu(0)\nu(1)}{L-1} + \frac{1}{L(L-1)}\, \mathrm{Cov}_\nu(M_0, M_1)\,. \end{aligned}$$

Introducing the matrix

$$A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\,,$$

the above computations show that

$$\mathcal{M} = \frac{\rho}{L-1}\, A + \widetilde{\mathcal{M}}$$

with a symmetric matrix $\widetilde{\mathcal{M}}$ such that $|\widetilde{\mathcal{M}}(n,m)| \leqslant O(\rho^2/L)$, $n,m \in \{0,1\}$. Then we have

$$\Big| \sum_{\substack{n \,\leqslant\, 1 \\ m \,\leqslant\, 1}} \varphi(n)\varphi(m)\mathcal{M}(n,m) \Big| \leqslant \frac{\rho}{L-1}\, [\varphi(0) - \varphi(1)]^2 + k\frac{\rho^2}{L}\, [\varphi(1)^2 + \varphi(0)^2] \tag{2.20}$$

Observe that by the orthogonality $\langle \varphi, \bar{n} \rangle = 0$ and Schwarz inequality

$$\Big| \sum_{\substack{n \leqslant 1 \\ m \leqslant 1}} \nu(n)\nu(m)\varphi(n)\varphi(m)\bar{n}\bar{m} \Big|$$

$$\leqslant \langle \varphi, \varphi \rangle \Big[\sum_{\substack{n+m \geqslant 2 \\ nm \neq 1}} \nu(n)\nu(m)\bar{n}^2\bar{m}^2 \Big]^{\frac{1}{2}} \leqslant k\,\rho^{\frac{3}{2}} \langle \varphi, \varphi \rangle$$

On the other hand

$$\sum_{\substack{n \leqslant 1 \\ m \leqslant 1}} \nu(n)\nu(m)\varphi(n)\varphi(m)\bar{n}\bar{m} = \rho^2[\varphi(0) - \varphi(1)]^2 + O(\rho^3)[\varphi(1)^2 + \varphi(0)^2]\,.$$

Collecting these estimates we arrive at

$$[\varphi(0) - \varphi(1)]^2 \leqslant k\,\rho^{-\frac{1}{2}} \langle \varphi, \varphi \rangle + k\,\rho\,[\varphi(1)^2 + \varphi(0)^2]$$

Using also the bound

$$[\varphi(1)^2 + \varphi(0)^2] \leqslant k\rho^{-1} \langle \varphi, \varphi \rangle$$

and going back to (2.20) we finally obtain

$$\Big| \sum_{\substack{n \leqslant 1 \\ m \leqslant 1}} \varphi(n)\varphi(m)\mathcal{M}(n,m) \Big| \leqslant k\frac{\rho^{\frac{1}{2}}}{L} \langle \varphi, \varphi \rangle \leqslant kL^{-1-\zeta} \langle \varphi, \varphi \rangle\ .$$

This ends the proof of Theorem 2.1

3. Proof of Theorem 1.1

Recall the definition of the stochastic operator K introduced at the beginning of the previous section. We set

$$w(L,H) = \sup_N \big[\,\mathrm{gap}(\mathbb{1}-K)\big]^{-1} \tag{3.1}$$

where

$$\mathrm{gap}(\mathbb{1}-K) = \inf_{\substack{\varphi\in\mathbb{H}:\\ \nu_0(\varphi)=0}} \frac{\langle\varphi,(\mathbb{1}-K)\varphi\rangle}{\langle\varphi,\varphi\rangle}\,. \tag{3.2}$$

Define also

$$\tilde\gamma(L,H) = \sup_N \sup_{f\in L^2(\nu)} \frac{\mathrm{Var}(f)}{\tilde{\mathcal{D}}(f,f)} \tag{3.3}$$

where we introduce the modified Dirichlet form

$$\tilde{\mathcal{D}}(f,f) = \frac{1}{L}\sum_i\sum_{j\neq i} D_{ij}(f)\,. \tag{3.4}$$

Clearly $\mathcal{D}(f,f)\geqslant\tilde{\mathcal{D}}(f,f)$ for all f and $\gamma(L,H)\leqslant\tilde\gamma(L,H)$. Note also that $\tilde{\mathcal{D}}(f,f)$ is still ergodic: $\tilde{\mathcal{D}}(f,f)=0$ implies $f=\text{const.}$ for all $L\geqslant 3$. More precisely, using e.g. the method of Lemma 2.6 in [7] one can easily prove that there exists $k=k(q)<\infty$ such that for every $L\geqslant 3$ one has $\mathcal{D}(f,f)\leqslant k\tilde{\mathcal{D}}(f,f)$ for all $f\in L^2(\nu)$. The key step in the proof of Theorem 1.1 is represented by the following proposition whose proof is postponed to the end of the section.

Proposition 3.1. *There exists $L_0<\infty$ such that for any $L\geqslant L_0$ and any $H\geqslant 1$*

$$\tilde\gamma(L,H)\leqslant\big[1\vee w(L,H)\big]\,\tilde\gamma(L-1,H) \tag{3.5}$$

To complete the proof of Theorem 1.1 we need an estimate on $w(L,H)$. In view of Theorem 2.1 we know that for every $\varphi\in\mathbb{H}$, such that $\nu_0(\varphi)=0$ we have

$$\langle\varphi,(\mathbb{1}-K)\varphi\rangle\;\geqslant\big(1-k\,L^{-1-\delta}\big)\,\langle\varphi,\varphi\rangle \tag{3.6}$$

for $L\geqslant L_0$ with uniform constants $\delta>0$ and $L_0,k<\infty$. It is then immediate to deduce $w(L,H)\leqslant 1+kL^{-1-\delta}$ and therefore

$$\prod_{L=L_0}^{\infty} w(L,H)\leqslant C\,, \tag{3.7}$$

with some uniform constant $C<\infty$. Then Theorem 1.1 follows from Proposition 3.1 and the bound

$$\sup_H\tilde\gamma(L_0,H)<\infty\,,\qquad L_0\geqslant 3\,.$$

The latter is easily deduced e.g. from results in [7].

3.1. **Proof of Proposition 3.1.** Let $\mathcal{F}_k$ denote the $\sigma-$algebra generated by the stick-variables η_k, $k=1,\dots,,L$. For any f one has the decomposition

$$\mathrm{Var}(f)=\frac{1}{L}\sum_{k=1}^{L}\Big\{\nu\big(\mathrm{Var}(f\,|\,\mathcal{F}_k)\big)+\mathrm{Var}\big(\nu(f\,|\,\mathcal{F}_k)\big)\Big\}\,. \tag{3.8}$$

We first establish the estimate

$$\frac{1}{L}\sum_{k=1}^{L}\nu\big(\mathrm{Var}(f\,|\,\mathcal{F}_k)\big)\leqslant\frac{L-2}{L-1}\,\tilde\gamma(L-1,H)\tilde{\mathcal{D}}(f,f)\,. \tag{3.9}$$

By definition of $\tilde\gamma$ for every k one has

$$\nu\big(\mathrm{Var}(f\,|\,\mathcal{F}_k)\big)\leqslant\tilde\gamma(L-1,H)\,\frac{1}{L-1}\sum_{i\neq k}\sum_{j\neq i,k}D_{ij}(f)\,.$$

Summing over k and using

$$\sum_k\sum_{i\neq k}\sum_{j\neq i,k}D_{ij}(f)=(L-2)\sum_i\sum_{j\neq i}D_{ij}(f)=L(L-2)\tilde{\mathcal{D}}(f,f)$$

we obtain (3.9).

We turn to an estimate on the second term in (3.8). Consider the non-negative stochastic operator $P:L^2(\nu)\to L^2(\nu)$ defined by

$$Pf=\frac{1}{L}\sum_{k=1}^{L}\nu(f\,|\,\mathcal{F}_k)\,. \tag{3.10}$$

We may assume without loss that $\nu(f)=0$. Then we have the identity

$$\frac{1}{L}\sum_{k=1}^{L}\mathrm{Var}\big(\nu(f\,|\,\mathcal{F}_k)\big)=\nu(fPf)\,. \tag{3.11}$$

From (3.8), (3.9) and (3.11) we obtain

$$\nu(f(\mathbb{1}-P)f)\leqslant\frac{L-2}{L-1}\,\tilde\gamma(L-1,H)\tilde{\mathcal{D}}(f,f)\,. \tag{3.12}$$

In order to estimate from below the left hand side in (3.12) we are going to prove the bound

$$\mathrm{gap}(\mathbb{1}-P)\geqslant\frac{L-2}{L-1}\,\big[1\wedge\mathrm{gap}(\mathbb{1}-K)\big]\,. \tag{3.13}$$

Here $\mathrm{gap}(\mathbb{1}-P)$ stands for the smallest nonzero eigenvalue of $\mathbb{1}-P$. Note that (3.13) and (3.12) immediately imply the proposition.

Take $f\in L^2(\nu)$ such that $\nu(f)=0$ and

$$Pf=\lambda f\,,\qquad\lambda>0\,.$$

Then f is of the form

$$f(\eta)=\sum_\ell\varphi_\ell(\eta_\ell)\,, \tag{3.14}$$

with $\varphi_\ell\in\mathbb{H}$ and we have the identity

$$\lambda\sum_\ell\varphi_\ell(\eta_\ell)=\frac{1}{L}\sum_\ell\sum_k\nu(\varphi_k(\eta_k)\,|\,\eta_\ell)\,. \tag{3.15}$$

Define now the function $\Phi_f \in \mathbb{H}$:

$$\Phi_f = \sum_\ell \varphi_\ell \,.$$

Taking conditional expectation with respect to $\mathcal{F}_j$ in (3.15) a simple computation yields

$$\lambda K \Phi_f + \lambda(\mathbb{I} - K)\varphi_j = \frac{L-1}{L} K^2 \Phi_f + \frac{1}{L} K(2-K)\Phi_f + \frac{1}{L}(\mathbb{I} - K)^2 \varphi_j \,.$$

Summing over j and factorizing we obtain

$$\left[K - \frac{\lambda L - 1}{L-1}\right]\left[K + \frac{1}{L-1}\right]\Phi_f = 0 \tag{3.16}$$

The above identity says that if $(K + \frac{1}{L-1})\Phi_f \neq 0$ then $\mu := \frac{\lambda L-1}{L-1}$ is in the spectrum of K. In this case then

$$1 - \lambda = \frac{L-1}{L}(1-\mu) \geqslant \frac{L-1}{L}\,\mathrm{gap}(\mathbb{I} - K) \geqslant \frac{L-2}{L-1}\,\mathrm{gap}(\mathbb{I} - K) \,.$$

We have to study the eigenfunctions f of P such that the corresponding Φ_f satisfies

$$\left[K + \frac{1}{L-1}\right]\Phi_f = 0 \,. \tag{3.17}$$

Then Theorem 2.1 implies that $\Phi_f = A\bar{n}$ for some $A \in \mathbb{R}$. On the other hand every φ_ℓ can be decomposed as

$$\varphi_\ell = a_\ell \bar{n} + \hat{\varphi}_\ell$$

with $a_\ell = \langle \varphi_\ell, \bar{n}\rangle$ and $\langle \hat{\varphi}_\ell, \bar{n}\rangle = 0$. Note that in view of (3.14) and the conservation law $\sum_\ell \bar{n}(\eta_\ell) = 0$, there is no restriction in assuming $\sum_\ell a_\ell = 0$. Therefore

$$A\,\langle \bar{n}, \bar{n}\rangle = \langle \Phi_f, \bar{n}\rangle = \langle \bar{n}, \bar{n}\rangle \sum_\ell a_\ell = 0$$

Thus if Φ_f solves (3.17) then necessarily $\Phi_f = 0$. It remains to study the latter case in detail.

Typical examples for which $\Phi_f = 0$ are obtained by choosing each φ_ℓ proportional to $\bar{n}$. Call $\mathcal{A}$ the class of all $f \in L^2(\nu)$ of the form (3.14) with $\varphi_\ell = a_\ell\,\bar{n}$ with arbitrary $\underline{a} := \{a_1, \dots, a_L\} \in \mathbb{C}^L$. A simple computation gives

$$(\mathbb{I} - P)f = \frac{L-2}{L-1} f \,, \qquad f \in \mathcal{A} \tag{3.18}$$

By (3.18) we may then restrict to the orthogonal complement $\mathcal{A}^\perp$ to prove our claim (3.13). We are going to prove that there exists a finite $L_0 = L_0(q)$ such that if $f \in \mathcal{A}^\perp$ and $\Phi_f = 0$

$$(1 - \lambda) \geqslant \frac{L-2}{L-1} \,, \qquad \forall L \geq L_0. \tag{3.19}$$

Let us first check that if $f \in \mathcal{A}^\perp$ then the corresponding φ_ℓ are all orthogonal (in $\mathbb{H}$) to the number of particles:

$$\langle \varphi_\ell, \bar{n}\rangle = 0 \,, \qquad \ell = 1, \dots, L \tag{3.20}$$

Indeed $f \in \mathcal{A}^\perp$ means

$$
\begin{aligned}
0 &= \sum_{k,\ell} a_k \nu(\varphi_\ell(\eta_\ell)\bar{n}_k) = \sum_k a_k \Big[\sum_{\ell \neq k} \langle \varphi_\ell, K\bar{n}\rangle + \langle \varphi_k, \bar{n}\rangle\Big] \\
&= \frac{1}{L-1}\sum_k a_k \Big[-\langle \Phi_f, \bar{n}\rangle + L\,\langle \varphi_k, \bar{n}\rangle\Big] = \frac{L}{L-1}\sum_k a_k\,\langle \varphi_k, \bar{n}\rangle
\end{aligned}
$$

Here we are using (2.1) and $\Phi_f = 0$. The above identity implies (3.20) in view of the arbitrariness of $\underline{a}$. Using again $\Phi_f = 0$ we compute

$$
\begin{aligned}
\nu(f^2) &= \sum_{k,\ell} \nu(\varphi_k(\eta_k)\varphi_\ell(\eta_\ell)) = \sum_k \Big[\sum_{\ell\neq k} \langle \varphi_k, K\varphi_\ell\rangle + \langle \varphi_k, \varphi_k\rangle\Big] \\
&= \sum_k \Big[\langle \varphi_k, K\Phi_f\rangle + \langle \varphi_k, (\mathbb{1} - K)\varphi_k\rangle\Big] = \sum_k \langle \varphi_k, (\mathbb{1}-K)\varphi_k\rangle
\end{aligned}
$$

A similar computation yields

$$
\nu(f(\mathbb{1}-P)f) = \frac{1}{L}\sum_k \langle \varphi_k, (\mathbb{1}-K)((L-1)\mathbb{1}+K)\varphi_k\rangle
$$

Writing $\tilde{\varphi}_\ell = (\mathbb{1}-K)^{1/2}\varphi_\ell$ one has

$$
(1-\lambda)\sum_\ell \langle \tilde{\varphi}_\ell, \tilde{\varphi}_\ell\rangle = \frac{1}{L}\sum_\ell \langle \tilde{\varphi}_\ell, ((L-1)\mathbb{1}+K)\tilde{\varphi}_\ell\rangle\ .
$$

Now observe that $\langle \tilde{\varphi}_\ell, \bar{n}\rangle = 0$. This follows from (3.20), (2.1) and the self adjointness of K. By Theorem 2.1 we then infer

$$
\big|\langle \tilde{\varphi}_\ell, K\tilde{\varphi}_\ell\rangle\big| \leqslant k\,L^{-1-\delta}\ \langle \tilde{\varphi}_\ell, \tilde{\varphi}_\ell\rangle
$$

for some uniform constants $\delta > 0$, $k < \infty$. We conclude that

$$
(1-\lambda) \geqslant \frac{L-1}{L} - \frac{k}{L^{2+\delta}} \geqslant \frac{L-2}{L-1}
$$

for L large. This finishes the proof of (3.13). ☐

3.2. **A remark on Bernoulli–Laplace model of diffusion.** We observe that the strategy of the above proof may be used to compute in a simple way the spectral gap for the so–called Bernoulli–Laplace process, see [13], [12] and references therein. The latter can be seen as an exclusion process on a complete graph: there are L sites with exchanges allowed between any couple of sites (i,j), $i,j = 1,\dots,L$ and with uniform rates. The configuration space is $\Omega_0 := \{0,1\}^L$ and the measure ν is the product Bernoulli measure on Ω_0 conditioned to the event $\sum_i \alpha_i = N$. The Dirichlet form is then defined by (3.4) with D_{ij} replaced by

$$
E_{ij}(f) = \frac{1}{2}\nu\big[(\nabla_{ij}f)^2\big] \tag{3.21}
$$

$$
\nabla_{ij}f(\alpha) = f(\alpha^{ij}) - f(\alpha)\,, \quad (\alpha^{ij})_k = \begin{cases} \alpha_k & k\neq i,j \\ \alpha_i & k = j \\ \alpha_j & k = i \end{cases} \tag{3.22}
$$

We are going to show that $\tilde{\gamma}(L) = \frac{1}{2}$, where

$$\tilde{\gamma}(L) = \sup_{1 \leqslant N \leqslant L-1} \sup_{f \in L^2(\nu)} \frac{\mathrm{Var}(f)}{\tilde{\mathcal{D}}(f,f)}\,. \tag{3.23}$$

As in the proof of Proposition 3.1 – see (3.12) – we obtain

$$\nu(f(\mathbb{1} - P)f) \leqslant \frac{L-2}{L-1}\,\tilde{\gamma}(L-1)\tilde{\mathcal{D}}(f,f)\,. \tag{3.24}$$

where again P is defined by (3.10) with $\mathcal{F}_k$ the σ-algebra generated by $\alpha_k \in \{0,1\}$. The analysis of the spectrum of P is much simpler now. Indeed any f of the form (3.14), with $\nu(f) = 0$, here must be of type

$$f(\alpha) = \sum_{\ell} a_\ell \bar{\alpha}_\ell\,, \qquad \bar{\alpha}_\ell := \alpha_\ell - \frac{N}{L}\,. \tag{3.25}$$

As in (3.18) a simple computation shows that if f is given by (3.25) then $(\mathbb{1} - P)f = \frac{L-2}{L-1}f$. It follows that any $f \in L^2(\nu)$ such that $\nu(f) = 0$ satisfies

$$\nu(f(\mathbb{1} - P)f) \geqslant \frac{L-2}{L-1}\nu(f^2) \tag{3.26}$$

By (3.24) we then obtain

$$\tilde{\gamma}(L) \leqslant \tilde{\gamma}(L-1) \leqslant \cdots \leqslant \tilde{\gamma}(2) \qquad L \geqslant 3\,. \tag{3.27}$$

On the other hand simple computations show that whenever $N = 1$ one has

$$\tilde{D}(f,f) = 2\mathrm{Var}_\nu(f)$$

for any $f \in L^2(\nu)$. In particular $\tilde{\gamma}(2) = 1/2$ and $\tilde{\gamma}(L) \geqslant 1/2$ for all $L \geqslant 3$. Together with (3.27) this shows that $\tilde{\gamma}(L) = 1/2$. A more detailed spectral analysis of this model including all the eigenvalues was obtained in [13] by a different technique.

4. Quantum XXZ Hamiltonian

Given $S \in \frac{1}{2}\mathbb{N}$, $H \in \mathbb{N}$, consider the Hilbert space $\mathfrak{H} = \otimes_{h=1}^{H}\mathbb{C}^{2S+1}$. The spin-S XXZ chain on $[1,H]\cap\mathbb{Z}$ with *kink* boundary conditions is defined by the operator

$$\mathcal{H}^{(S)} = \sum_{h=1}^{H-1} \mathcal{H}^{(S)}_{h,h+1}\,, \tag{4.1}$$

$$\mathcal{H}^{(S)}_{h,h+1} = S^2 - \Delta^{-1}\left(S^1_h S^1_{h+1} + S^2_h S^2_{h+1}\right) - S^3_h S^3_{h+1} + S\sqrt{1-\Delta^{-2}}\left(S^3_{h+1} - S^3_h\right).$$

Here S^i_h, $i = 1,2,3$, are the spin–S operators (the $2S+1$–dimensional irreducible representation of $SU(2)$) at every h, and the constant S^2 has been added in order to have zero ground state energy. The parameter $\Delta \in (1,\infty)$ measures the anisotropy along the third axis. The kink boundary condition is obtained through the telescopic sum $S^3_H - S^3_1 = \sum_{h=1}^{H-1}\left(S^3_{h+1} - S^3_h\right)$ and the pre-factor $S\sqrt{1-\Delta^{-2}}$ is chosen in order to obtain non-trivial ground states describing quantum domain walls (see [1], [14] and references therein).

We choose the basis of $\mathfrak{H}$ labelled by the $2S+1$ states of the third component of the spin at each site and we write it in terms of configurations

$$m = (m_1,\dots,m_H) \in \{-S,-S+1,\dots,S-1,S\}^H =: \mathcal{Q}_S$$

so that $|m\rangle = \otimes_{h=1}^{H}|m_h\rangle$ stands for the generic basis vector in $\mathfrak{H}$. With these notations, and introducing the stair-operators $S^{\pm} = S^1 \pm iS^2$, the action of S^i, $i = 1,2,3$, is given by

$$S^3_h\,|m_h\rangle = m_h\,|m_h\rangle\,, \quad S^{\pm}_h\,|m_h\rangle = c_{\pm}(S,m_h)\,|m_h \pm 1\rangle\,. \tag{4.2}$$

$$c_{\pm}(S,m_h) := \sqrt{(S\mp m_h)(S\pm m_h+1)}$$

The action of $\mathcal{H}^{(S)}$ is explicated by rewriting the pair-interaction terms as

$$\mathcal{H}^{(S)}_{h,h+1} = S^2 - (2\Delta)^{-1}\left(S^+_h S^-_{h+1} + S^-_h S^+_{h+1}\right) - S^3_h S^3_{h+1} + S\sqrt{1-\Delta^{-2}}\left(S^3_{h+1} - S^3_h\right) \tag{4.3}$$

4.1. **The spectral gap.**

The Hamiltonian $\mathcal{H}^{(S)}$ commutes with the total third component of the spin

$$S^3_{\rm tot} = \sum_{h=1}^{H} S^3_h\,.$$

We shall divide the space $\mathfrak{H}$ into sectors $\mathfrak{H}_n$, $n \in \{-SH,-SH+1,\dots,SH-1,SH\}$, given by the eigenspaces corresponding to the eigenvalue n of $S^3_{\rm tot}$. It is known [1] that for each n there is a unique (up to multiplicative constants) vector $\psi_n \in \mathfrak{H}_n$ such that $\mathcal{H}^{(S)}\psi_n = 0$, which is given by

$$\psi_n = \sum_{\substack{m\in\mathcal{Q}_S:\\ \sum_h m_h = n}} \psi(m)\,|m\rangle$$

$$\psi(m) = \prod_h q^{hm_h}\sqrt{\binom{2S}{S+m_h}}\,. \tag{4.4}$$

Here $q \in (0,1)$ is the anisotropy parameter linked to Δ by the equation

$$\Delta = \frac{1}{2}(q+q^{-1})\,. \tag{4.5}$$

The ground states ψ_n are interpreted as describing an interface profile, [1, 11]. A fundamental question associated to the stability of such "quantum interfaces" is the positivity of the spectral gap [9, 10]. The latter, denoted $\mathrm{gap}(\mathcal{H}^{(S)})$, is defined as the energy of the first excited state, i.e. the first non-zero eigenvalue of the non-negative operator $\mathcal{H}^{(S)}$. Recently this question was studied in great detail in the paper [10] by both analytical and numerical means. One of the main results of [10] is a proof of the fact that for every $S \in \frac{1}{2}\mathbb{N}$, $\mathrm{gap}(\mathcal{H}^{(S)})$ is positive uniformly in H. Furthermore it was conjectured on the basis of numerical analysis that $\mathrm{gap}(\mathcal{H}^{(S)})$ should grow linearly with S. We prove the following bounds.

Theorem 4.1. *For every $\Delta \in (1,\infty)$, there exists $\delta > 0$ such that*

$$\delta\, S \leqslant \mathrm{gap}(\mathcal{H}^{(S)}) \leqslant \delta^{-1} S$$

for all $S \in \frac{1}{2}\mathbb{N}$ and all $H \geqslant 2$.

In order to prove Theorem 4.1 we shall establish the following unitary equivalence. Let $L = 2S$ and $N = SH + n$, recall the definition of $L^2(\hat{\Omega}, \hat{\nu})$, of the subspace of horizontally symmetric functions and of the generator $\widehat{\mathcal{L}}$ introduced in (1.12). The measure $\hat{\nu}$ in (1.11) can be written, using (4.4) with $m = \omega - S$, as:

$$\hat{\nu}(\omega) = \frac{1}{\tilde{Z}}\left[\psi(\omega - S)\right]^2 \qquad \tilde{Z} = \sum_{\substack{\omega\in\hat{\Omega}:\\ \sum_h \omega_h = SH+n}} \left[\psi(\omega - S)\right]^2 . \tag{4.6}$$

For any $\varphi \in \mathfrak{H}_n$ we also write

$$\varphi = \sum_{\substack{m\in\mathcal{Q}_S:\\ \sum_h m_h = n}} \varphi(m)\,|m\rangle\ .$$

Finally we set $\tilde{\varphi}(\omega) = \varphi(\omega - S)$. Then the transformation

$$\varphi(m)\ \to\ \frac{1}{\sqrt{\hat{\nu}(\omega)}}\,\tilde{\varphi}(\omega)\ =:\ \big[U_n\varphi\big](\omega)\,, \qquad \omega = m + S$$

maps unitarily $\mathfrak{H}_n$ into $L^2(\hat{\Omega}, \hat{\nu})$.

Lemma 4.2. *For every $n \in \{-SH, -SH+1, \dots, SH-1, SH\}$*

$$U_n\,\mathcal{H}^{(S)}\,\varphi = -\frac{S}{\Delta}\,\widehat{\mathcal{L}}\,U_n\,\varphi\,, \quad \varphi \in \mathfrak{H}_n \tag{4.7}$$

Proof. From (4.3) we compute

$$\begin{aligned}\mathcal{H}^{(S)}_{(h,h+1)}\,|m\rangle &= \big(S^2 - m_h m_{h+1} + S\sqrt{1-\Delta^{-2}}(m_{h+1} - m_h)\big)\,|m\rangle \\ &\quad - (2\Delta)^{-1} c_+(S, m_h)c_-(S, m_{h+1})|m^{+,h}\rangle - (2\Delta)^{-1} c_-(S, m_h)c_+(S, m_{h+1})|m^{-,h}\rangle\,.\end{aligned}$$

Here we are using the notation

$$m^{\pm,h}_{h'} = \begin{cases} m_{h'} & h' \neq h, h+1 \\ m_h \pm 1 & h' = h \\ m_{h+1} \mp 1 & h' = h+1 \end{cases}$$

Therefore

$$(2\Delta)\big[\mathcal{H}^{(S)}_{(h,h+1)}\varphi\big](m) = \Gamma(m_h,m_{h+1})\,\varphi(m)$$
$$- c_+(S,m_h)c_-(S,m_{h+1})\varphi(m^{+,h}) - c_-(S,m_h)c_+(S,m_{h+1})\varphi(m^{-,h})$$

with

$$\Gamma(m_h,m_{h+1}) = (2\Delta)\big[S^2 - m_h m_{h+1} + S\sqrt{1-\Delta^{-2}}(m_{h+1}-m_h)\big]\,.$$

Now a computation shows that

$$\Gamma(m_h,m_{h+1}) = w_{+,h}(\omega) + w_{-,h}(\omega)\,, \qquad \omega = S+m\,,$$

with $w_{\pm,h}$ the rates defined in (1.12). Another computation shows that

$$c_+(S,\omega_h - S)c_-(S,\omega_{h+1}-S)\sqrt{\frac{\hat\nu(\omega^{+,h})}{\hat\nu(\omega)}} = w_{+,h}(\omega)$$

and similarly

$$c_-(S,\omega_h - S)c_+(S,\omega_{h+1}-S)\sqrt{\frac{\hat\nu(\omega^{-,h})}{\hat\nu(\omega)}} = w_{-,h}(\omega)\,.$$

We have then obtained

$$(2\Delta)\big[U_n\,\mathcal{H}^{(S)}_{(h,h+1)}\varphi\big](\omega) = \big(w_{+,h}(\omega)+w_{-,h}(\omega)\big)\big[U_n\varphi\big](\omega)$$
$$- w_{+,h}(\omega)\big[U_n\varphi\big](\omega^{+,h}) - w_{-,h}(\omega)\big[U_n\varphi\big](\omega^{-,h})$$

and the lemma follows. □

We are now able to finish the proof of Theorem 4.1. Recall (1.7). Since here $L = 2S$, we readily infer from Lemma 4.2 the estimate

$$\mathrm{gap}(\mathcal{H}^{(S)}) \geqslant \frac{S}{\Delta\gamma(2S,H)}\,, \tag{4.8}$$

where $\gamma(2S,H)$ is defined by (1.8). The bound $\gamma \leqslant \delta^{-1}$ is the content of Theorem 1.1. On the other hand to prove $\mathrm{gap}(\mathcal{H}^{(S)}) \leqslant \delta^{-1}\,S$ we may use the following simple argument which says that in each sector $\mathfrak{H}_n$ there are excited states with energy bounded by $\delta^{-1}\,S$. Choose $f(\omega) = \omega_{h_0}$ with $h_0 = [\rho]$, $\rho = N/(2S)$. A simple estimate shows that

$$\hat\nu\big(f(-\widehat{\mathcal{L}})f\big) \leqslant q^{-1}\,\hat\nu\big(\omega_{h_0}+\omega_{h_0+1}\big) + q\,\hat\nu\big(\omega_{h_0}+\omega_{h_0-1}\big) \leqslant k\,S\,(1\wedge\rho)$$

for some finite $k = k(q) < \infty$. On the other hand using the estimates in [7] it is possible to check that $\mathrm{Var}_{\hat\nu}(f) \geqslant \delta\,S\,(1\wedge\rho)$ for some $\delta = \delta(q) > 0$. By the variational principle

$$\mathrm{gap}(\mathcal{H}^{(S)}) \leqslant \frac{S}{\Delta}\,\frac{\hat\nu\big(f(-\widehat{\mathcal{L}})f\big)}{\mathrm{Var}_{\hat\nu}(f)} \leqslant k\,S\,.$$

5. Energy gap above the diagonal interfaces of the XXZ model

In this section we study a higher dimensional quantum XXZ Hamiltonian which is sometimes used to model a tilted interface ([11, 4, 7]). In order to avoid complicated notation we shall work in a two-dimensional setting. We later observe that our results actually hold without modifications in any dimension.

We begin by defining a cylindrical region (in two dimensions simply a rectangle) with axis along the diagonal. Given two integers R and H we define

$$\Gamma = \Gamma_{R,H} = \left\{x \in \mathbb{Z}^2 :\ -R \leqslant x_1 - x_2 \leqslant R\,,\ 1 \leqslant x_1 + x_2 \leqslant H\right\} \tag{5.1}$$

We write $\ell_x = x_1 + x_2$ for the distance of a site x form the line $x_1 = -x_2$. A bond is an oriented pair $b = (x, y)$, with $x, y \in \mathbb{Z}^2$ such that $|x_1 - y_1| + |x_2 - y_2| = 1$ (in particular $\ell_y = \ell_x \pm 1$). We call $\mathcal{B} = \mathcal{B}_{R,H}$ the set of bonds $b = (x, y)$ with $x, y \in \Gamma$ and $\ell_y = \ell_x + 1$. For any $S \in \frac{1}{2}\mathbb{N}$ the anisotropic spin–S Hamiltonian in the region Γ with kink boundary conditions is defined by

$$\mathcal{H}_R^{(S)} = \sum_{b\in\mathcal{B}} \mathcal{H}_b^{(S)}\,, \tag{5.2}$$

$$\mathcal{H}_b^{(S)} = S^2 - \Delta^{-1}\left(S_x^1 S_y^1 + S_x^2 S_y^2\right) - S_x^3 S_y^3 + S\sqrt{1-\Delta^{-2}}\left(S_y^3 - S_x^3\right)\,, \qquad b = (x, y)$$

with S_x^k, $k = 1, 2, 3$, the spin–S operators at site x. This is the higher–spin analog of the spin–$\frac{1}{2}$ cylindrical models considered in [4, 7]. As usual we consider the Hilbert space $\mathfrak{H}_\Gamma = \otimes_{x\in\Gamma}\mathbb{C}^{2S+1}$ with the basis labelled by configurations

$$|m\rangle = \otimes_{x\in\Gamma}|m_x\rangle \qquad m \in \{-S, \ldots, S\}^\Gamma =: \mathcal{Q}_{\Gamma,S}\,.$$

Clearly the total third component $S_{\rm tot}^3 = \sum_{x\in\Gamma} S_x^3$ is conserved and we may divide $\mathfrak{H}_\Gamma$ into sectors $\mathfrak{H}_{\Gamma,n}$, $n \in \{-S|\Gamma|, \ldots, S|\Gamma|\}$ according to the eigenvalues of $S_{\rm tot}^3$. Following [1] we know that in each such sector there is a unique ground state $\psi_{\Gamma,n}$ given by

$$\psi_{\Gamma,n} = \sum_{\substack{m\in\mathcal{Q}_{\Gamma,S}:\\ \sum_x m_x = n}} \psi_\Gamma(m)\,|m\rangle$$

$$\psi_\Gamma(m) = \prod_{x\in\Gamma} q^{\ell_x m_x}\sqrt{\binom{2S}{S+m_x}}\,. \tag{5.3}$$

As before $q = q(\Delta)$ is defined by (4.5). The above ground states have zero energy and we call $\mathrm{gap}(\mathcal{H}_R^{(S)})$ the first nonzero eigenvalue of $\mathcal{H}_R^{(S)}$.

Our main result here is a generalization to higher–spin models of a theorem we proved in the case $S = \frac{1}{2}$, [7].

Theorem 5.1. *For every $\Delta \in (1, \infty)$, there exists $\delta > 0$ such that for all $S \in \frac{1}{2}\mathbb{N}$, all $R, H \in \mathbb{N}$*

$$\delta\, S\, R^{-2} \leqslant \mathrm{gap}(\mathcal{H}_R^{(S)}) \leqslant \delta^{-1}\, S\, R^{-2}\,. \tag{5.4}$$

Remark 5.2. It will be clear from the proof that exactly the same estimates hold (except that now the constant δ in (5.4) may depend on the dimension) in the case where Γ is replaced by a cylinder in $d+1$ dimensions with axis along the $11\cdots1$ direction and basis given by a d-dimensional hypercube of side R.

Proof. As a first step we establish a unitary equivalence in the spirit of Lemma 4.2. Namely let $\tilde{\Omega}_\Gamma = (2S+1)^\Gamma$ denote the set of configurations $\omega_x \in \{0,\dots,2S\}$, $x \in \Gamma$. Following our previous analysis we may interpret ω_x as the number of particles at site x. Given $N \in \{0,1,\dots,2S|\Gamma|\}$ define

$$\tilde{\Omega}_{\Gamma,N} = \big\{\omega \in \tilde{\Omega}_\Gamma : \sum_{x\in\Gamma} \omega_x = N\big\}$$

Consider the probability measure on $\tilde{\Omega}_{\Gamma,N}$ given by

$$\hat{\nu}_{\Gamma,N}(\omega) = \frac{1}{Z_{\Gamma,N}}\big[\Psi_\Gamma(\omega - S)\big]^2, \quad Z_{\Gamma,N} = \sum_{\omega\in\tilde{\Omega}_{\Gamma,N}} \big[\Psi_\Gamma(\omega - S)\big]^2. \tag{5.5}$$

Then as in Lemma 4.2 we obtain that

$$\varphi(m) \to \frac{1}{\sqrt{\hat{\nu}_{\Gamma,N}(\omega)}}\,\varphi(\omega - S) =: \big[U_{\Gamma,n}\varphi\big](\omega), \qquad \omega = m + S$$

maps unitarily each sector $\mathfrak{H}_{\Gamma,n}$ into $L^2(\tilde{\Omega}_{\Gamma,N}, \hat{\nu}_{\Gamma,N})$, $N = S|\Gamma| + n$. Repeating the computation leading to (4.7), for every bond $b \in \mathcal{B}$ we have

$$U_{\Gamma,n}\,\mathcal{H}_b^{(S)}\,U_{\Gamma,n}^{-1} f = -\frac{1}{2\Delta}\big\{w_b^+\nabla_b^+ f + w_b^-\nabla_b^- f\big\}, \quad f \in L^2(\tilde{\Omega}_{\Gamma,N}, \hat{\nu}_{\Gamma,N}) \tag{5.6}$$

with the notation

$$w_b^+(\omega) = q^{-1}\,\omega_y(2S - \omega_x)\,,\ w_b^-(\omega) = q\,\omega_x(2S - \omega_y)\,, \quad b = (x,y)\,,$$

$$\nabla_b^\pm f(\omega) = f(\omega^{b,\pm}) - f(\omega)\,, \quad \omega_z^{b,\pm} := \begin{cases} \omega_z & z \neq x,y \\ \omega_x \pm 1 & z = x \\ \omega_y \mp 1 & z = y \end{cases}$$

Let us define the Markov generator $\widehat{\mathcal{G}}_{R,S}$ by

$$\widehat{\mathcal{G}}_{R,S} f(\omega) = \sum_{b\in\mathcal{B}} \big\{w_b^+(\omega)\nabla_b^+ f(\omega) + w_b^-(\omega)\nabla_b^- f(\omega)\big\}. \tag{5.7}$$

Then $\widehat{\mathcal{G}}_{R,S}$ is symmetric in $L^2(\tilde{\Omega}_{\Gamma,N}, \hat{\nu}_{\Gamma,N})$ and by (5.6) we have the unitary equivalence

$$\mathcal{H}_R^{(S)} \simeq -\frac{1}{2\Delta}\,\widehat{\mathcal{G}}_{R,S} \tag{5.8}$$

Set $\Omega_{\Gamma^{(S)}} = \{0,1\}^{\Gamma^{(S)}}$, where $\Gamma^{(S)}$ is the 3–dimensional region

$$\Gamma^{(S)} = \big\{(i,x)\ :\quad i = 1,\dots,2S\,;\ x \in \Gamma\big\}.$$

Define then $\Omega_{\Gamma^{(S)},N}$ as the set of $\alpha \in \Omega_{\Gamma^{(S)}}$ such that $\sum_{(i,x)\in\Gamma^{(S)}} \alpha_{(i,x)} = N$. Consider the probability measure ν_N on $\Omega_{\Gamma^{(S)},N}$ defined by

$$\nu_N(\alpha) = \frac{1}{Z_N}\prod_{(i,x)\in\Gamma^{(S)}} q^{2\ell_x\alpha_{(i,x)}}\,, \quad Z_N = \sum_{\alpha\in\Omega_{\Gamma^{(S)},N}}\ \prod_{(j,y)\in\Gamma^{(S)}} q^{2\ell_y\alpha_{(j,y)}} \tag{5.9}$$

Consider the subspace $\mathcal{S}_\Gamma \subset L^2(\Omega_{\Gamma^{(S)},N}, \nu_N)$ of symmetric functions defined by: $f(\alpha) = f(\alpha^{\pi,x})$ for all α, all $x \in \Gamma$ and all permutations $\pi \in \mathcal{P}_{2S}$, with $\alpha^{\pi,x}$ defined as in (1.9) replacing h by x. As in section 1, $\hat{\nu}_{\Gamma,N}$ can be looked at as the marginal of ν_N on the sums

$$\omega_x = \sum_{i=1}^{2S} \alpha_{(i,x)}\,, \qquad \alpha \in \Omega_{\Gamma^{(S)},N}$$

and $\mathcal{S}_\Gamma$ is identified with $L^2(\tilde{\Omega}_{\Gamma,N}, \hat{\nu}_{\Gamma,N})$. Then $\widehat{\mathcal{G}}_{R,S}$ may be identified with the restriction to $\mathcal{S}_\Gamma$ of $(2S)\,\mathcal{G}_{R,S}$, the Markov generator defined by

$$\begin{gathered}\mathcal{G}_{R,S} f(\alpha) = \frac{1}{2S} \sum_{i=1}^{2S} \sum_{j=1}^{2S} \sum_{(x,y)\in\mathcal{B}} c_{(i,x);(j,y)}(\alpha)\, \nabla_{(i,x);(j,y)} f(\alpha) \\ c_{(i,x);(j,y)}(\alpha) = q^{\alpha_{(i,x)} - \alpha_{(j,y)}}\,, \quad \nabla_{(i,x);(j,y)} f(\alpha) = f(\alpha^{(i,x);(j,y)}) - f(\alpha)\,.\end{gathered} \tag{5.10}$$

Here as usual $\alpha^{(i,x);(j,y)}$ denotes the configuration after the exchange between (i,x) and (j,y). At this point we have obtained a unitary equivalence

$$\mathcal{H}_R^{(S)} \simeq -\frac{S}{\Delta}\,\mathcal{G}_{R,S} \tag{5.11}$$

when the right hand side above is restricted to $\mathcal{S}_\Gamma$. Notice the analogy of (5.10) with the process introduced in section 1, see (1.4). However there is an important difference (which will be seen in a moment to be responsible for the R^{-2} factors in (5.4)): while particles diffuse asymmetrically in the 11 direction just as it happens for (1.4) along the vertical direction, we have in (5.10) in addition an essentially symmetric diffusion along the orthogonal direction (given by the lines $\ell_x =$ constant). We are now going to take care of these facts.

Observe that $\mathcal{G}_{R,S}$ is symmetric in $L^2(\Omega_{\Gamma(S)}, \nu_N)$ with Dirichlet form

$$\begin{aligned}\mathcal{E}_{R,S}(f) = \nu_N\big(f(-\mathcal{G}_{R,S})f\big) &= \frac{1}{2S} \sum_{i=1}^{2S} \sum_{j=1}^{2S} E_{(i,j)}(f) \\ E_{(i,j)}(f) := \frac{1}{2} \sum_{(x,y)\in\mathcal{B}} \nu_N &\left[c_{(i,x);(j,y)}\big(\nabla_{(i,x);(j,y)} f\big)^2\right]\end{aligned} \tag{5.12}$$

For any $x \in \Gamma$ let $t_x = x_1 - x_2$, so that x is uniquely determined by a couple (t,ℓ), $-R \leqslant t \leqslant R$, $1 \leqslant \ell \leqslant H$. We may then write

$$E_{(i,j)}(f) = \frac{1}{2} \sum_{\substack{t,s\in[-R,\dots,R]:\\ |t-s|=1}} \sum_{\ell=1}^{H-1} \chi(t,\ell)\, \nu_N \left[c_{(i,t,\ell);(j,s,\ell+1)}\big(\nabla_{(i,t,\ell);(j,s,\ell+1)} f\big)^2\right] \tag{5.13}$$

where $\chi(t,\ell) = 1$ if there exists $x \in \Gamma$ such that $x_1 - x_2 = t$ and $x_1 + x_2 = \ell$ and $\chi(t,\ell) = 0$ otherwise (notice that if $\chi(t,\ell) = 1$ then $\chi(s,\ell+1) = 1$ for $s \in [-R,\dots,R]$ with $|s-t| = 1$). Using the bounds $q \leqslant c \leqslant q^{-1}$ on the rates and the properties of the measure ν_N it is not difficult (see Lemma 2.6 in [7] for similar computations) to show that there exists $k = k(q) < \infty$ such that given arbitrary $t,s \in [-R,\dots,R]$ with $t \leqslant s-1$ and $\ell \in [1,H]$ one has

$$\begin{aligned}&\chi(t,\ell)\chi(s,\ell+1)\, \nu_N \left[c_{(i,t,\ell);(j,s,\ell+1)}\big(\nabla_{(i,t,\ell);(j,s,\ell+1)} f\big)^2\right] \\ &\qquad \leqslant k\, R \sum_{r=t}^{s-1} \chi(r,\ell)\, \nu_N \left[c_{(i,r,\ell);(j,r+1,\ell+1)}\big(\nabla_{(i,r,\ell);(j,r+1,\ell+1)} f\big)^2\right]\end{aligned} \tag{5.14}$$

From (5.13) and (5.14) we easily obtain

$$\sum_{t,s\in[-R,\dots,R]}\sum_{\ell=1}^{H-1}\chi(t,\ell)\chi(s,\ell+1)\,\nu_N\left[c_{(i,t,\ell);(j,s,\ell+1)}\big(\nabla_{(i,t,\ell);(j,s,\ell+1)}f\big)^2\right]$$
$$\leqslant\; k\,R^3\,E_{(i,j)}(f) \tag{5.15}$$

Recalling (5.12) we may summarize the above estimate with the following statement

$$\mathcal{E}_{R,S}(f)\geqslant\;\delta\,R^{-2}\mathcal{D}_{R,S}(f)\,, \tag{5.16}$$

$$\mathcal{D}_{R,S}(f):=\frac{1}{2SR}\sum_{i,j\in[1,\dots,2S]}\sum_{t,s\in[-R,\dots,R]}D_{(i,t);(j,s)}(f)\,,$$

$$D_{(i,t);(j,s)}(f):=\sum_{\ell=1}^{H-1}\chi(t,\ell)\chi(s,\ell+1)\,\nu_N\left[c_{(i,t,\ell);(j,s,\ell+1)}\big(\nabla_{(i,t,\ell);(j,s,\ell+1)}f\big)^2\right]$$

We are now able to conclude thanks to Theorem 1.1. Indeed, the Dirichlet form $\mathcal{D}_{R,S}$ defined above is a special case of the one appearing in the Theorem, when we set $L=2SR$ if R is even and $L=2S(R+1)$ if R is odd. We then have that

$$\mathcal{D}_{R,S}(f)\geqslant\;\delta\,\mathrm{Var}_{\nu_N}(f)$$

for some uniform constant δ. From (5.16) and (5.11) we see that the energy of excited states in each sector lies above $\delta\,S\,R^{-2}$ for some uniform constant $\delta>0$. The desired lower bound on $\mathrm{gap}(\mathcal{H}_R^{(S)})$ follows at once. The reverse estimate is much easier and it can be obtained again from (5.11) by modifying slightly the reasoning in Proposition 6.1 of [7].

□

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