FURTHER RIGIDITY PROPERTIES OF CONFORMAL ANOSOV SYSTEMS

R. DE LA LLAVE

Abstract. We consider systems that have some hyperbolicity behavior and which preserve conformal structures on the stable and unstable bundles. We show that two such systems that are topologically conjugate are smoothly conjugate. This is somewhat more general than a conjecture of the author in 2002. Related results have also been obtained by B. Kalinin and V. Sadovskaia.

1. Introduction

The goal of this paper is to show rigidity properties of systems with some hyperbolicity behavior which preserve conformal structures along stable and unstable foliations.

The main result is valid only for Anosov diffeomorphisms and flows, but much of the other results are valid under less restrictive assumptions on the hyperbolicity of the system. The only obstruction to obtaining the full result for partially hyperbolic systems is a technical regularity result, which we believe is true and within reach. We discuss these generalizations in Remark 3.1.

The main result of this paper is the following

Theorem 1.1. *Let f, g be C^r transitive Anosov diffeomorphisms (resp. X_t, Y_t be transitive Anosov flows) of a compact manifold M of dimension d. $r \in \mathbb{N} + [0, 1)$, $r > 1$.*

Assume that

i) $\dim(E^s), \dim(E^u) \geq 2$.

ii) *There exists a homeomorphism h such that*

$$f \circ h = h \circ g. \tag{1}$$

resp.

$$X_t \circ h = h \circ Y_t. \tag{2}$$

iii) *Each of the diffeomorphisms f, g preserve C^1 conformal structures on the stable and unstable bundles.*

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Then

$$h \in C^{r'} \tag{3}$$

where $r = \begin{cases} r - \epsilon & \text{if} \quad r \in \mathbb{N} \\ r & \text{if} \quad r \in \mathbb{N} + (0,1) \end{cases}.$

Remark 1.2. We note that the assumption iii) can be weakened in two directions:

a) In [dlLar], it is shown that if $M = \mathbb{T}^d$, – or has some triviality properties of the tangent bundle – the fact that f preserves a smooth conformal structure on the stable and unstable bundles holds when f is sufficiently C^1-close to a linear automorphism and

$$f^n(x) = x \implies Df^n(x)|_{E^s_x} = (\mu^s_x)^n \operatorname{Id}_{E^s_x}, \quad Df^n(x)|_{E^u_x} = (\mu^u_x)^n \operatorname{Id}_{E^u_x} \tag{4}$$

Note that in (4) we do not assume that the μ_x are independent of x.

One of the consequences of Theorem 1.1 is that if (4) holds and f is C^1-close to a linear automorphism of the torus, then, we have that indeed that μ^s_x are independent of x.

b) In [KdlLS02], it is shown that the preservation of a continuous conformal structure is implied by the preservation of an structure that belongs to L^p with p high enough and that f is C^1 close to a measure preserving diffeomorphism. Similarly, in [KdlLS02], one shows that the regularity of the conformal structure can be bootstrapped till it is roughly $r - 1$.

c) The papers [Sad02], [KS02] do not require that the systems are transitive. One of the first steps of those papers is to show that Anosov diffeomorphisms that preserve conformal structures are volume preserving.

If one used the arguments from these papers for the first part of the arguments, the assumption that the systems are transitive could be eliminated for diffeomorphisms. We have chosen not to follow this route to have more arguments documented in the literature.

We point, however that we do not use much transitivity. The main place is in the application of the theory of SRB measures, which includes transitivity in its assumptions. It is possible that this use of transitivity could be eliminated by considering the arguments in the basins of attraction of the different ergodic components. □

Remark 1.3. The Theorem 1.1 remains valid if Assumption i) is replaced by $\dim E^s = 1$, $\dim E^u \geq 2$, and, whenever $g^n(x) = x$, hence $f^n \circ h(x) = h(x)$ we have

$$\operatorname{Spec}(Dg^n(x)|_{E^s,x}) = \operatorname{Spec}((Df^n) \circ h(x)|_{E^s_{h(x)}})$$

We fact that $\dim E^s = 1$ implies that the Anosov system is transitive. See [New70]. Therefore, for diffeomorphisms, the assumption of transitivity becomes redundant if one modifies Assumption i) as indicated. For flows, the assumption of transitivity is not implied by the manifolds being one-dimensional, [FW80].

We develop the proof of Theorem 1.1 under the assumptions indicated in this remark in Section 3.1

□

The main technique of this paper is a sequence of arguments bootstrapping the regularity of the conjugacies. Many of the arguments have a similar flavor. We derive formulas for the increments of the derivatives along the (un)stable directions. These formulas will hold in a rather weak sense. Nevertheless, by examining them, we will discover that the derivatives – or the increments – hold in a somewhat stronger sense. Eventually, we will reach a formula so strong that can be differentiated term by term. Therefore, we establish that the conjugacy is smooth when restricted to the stable and unstable foliations.

Once we have that the conjugacy is smooth along the stable and unstable foliations, we can appeal to a regularity result (Lemma 2.5 of [dlLMM86]) that states that functions smooth along the stable and unstable manifold are actually smooth. As is usual in bootstrap arguments, the more complicated parts happen at the beginning, when we are allowed to assume less properties of the function. We have indicated the milestones that we will be accomplishing as titles of the subsections.

Results very similar to Theorem 1.1 have been obtained by more geometric methods in the remarkable papers [Sad02], [KS02]. The geometric argument in [Sad02], [KS02] has several advantages with the argument presented here. In particular, in the Anosov case, it shows that the stable and unstable foliations are smooth and that they preserve an affine connection. Hence, the papers [Sad02], [KS02] can use the results of [Kan93], [BFL92], [Yue96], [BCG95] to get, just from the preservation of the conformal structures along the stable and unstable manifolds – an assumption that looks almost local – global information about the manifold.

Besides, being a different method, the present method has the advantage that it applies for flows and, we hope, it can apply to partially hyperbolic systems provided that a regularity result which we formally conjecture is true.

2. Proof of Theorem 1.1

The proof presented here consists of a sequence of arguments that bootstrap the regularity of the conjugacy h using that it satisfies that equation (1) (or (2).) As in [dlLMM86], we will bootstrap the regularity along the stable and unstable manifolds separately and then, we will appeal to regularity results to conclude that a function which is regular restricted to the stable and unstable manifolds is regular in the usual sense.

We emphasize that the fact that we can bootstrap the regularity along the stable and unstable foliations is independent of the fact that they are transversal. Hence, this part of the argument can be carried for partially hyperbolic systems or for Anosov flows. (We use at certain stages the fact that the system admits a SRB measure).

In the case of Anosov flows, however, since the conjugacies have to be smooth along the direction of the flow too, we are in a situation where we can apply the regularity results of [dlLMM86] and obtain the desired conclusions.

We conjecture that the required regularity result – an analogue of Lemma 2.5 of [dlLMM86] – is true for partially hyperbolic systems satisfying the accessibility property. If such conjecture was true, then Theorem 1.1 will go through for partially hyperbolic systems that have accessibility and SRB measures.

We think that it would be extremely interesting to integrate the arguments presented here with the arguments of [Sad02], [KS02].

In the following, we will present the proof with the notation of Anosov diffeomorphisms, but we call attention to the fact that the bootstrap of regularity only uses that the manifolds are stable and unstable. Hence it applies just the same for stable manifolds of a flow or a partially hyperbolic system.

We will denote the stable manifold of the point $x \in M$ under the map f as $W_x^{s,(f)}$ but we will suppress the (f) if it is understood from the context and will not lead to confusion.

2.1. **h is conformal along the stable and unstable leaves.** The first step of the proof of Theorem 1.1 is:

Lemma 2.1. *$h : W_x^{s,(g)} \to W_{h(x)}^{s,(f)}$ is quasi-conformal when $W_x^{s,(g)}, W_{h(x)}^{s,(f)}$ are endowed with the conformal structures preserved by g, f respectively.*

Of course, there is an analogous result for the unstable manifolds, which can be deduced from Lemma 2.1 by considering the inverse mappings (a direct proof only requires trivial modifications).

Lemma 2.1 was already established in [dlLar] Section 3.2. We recall that main idea of the argument was to show that we can construct a map k such that $k : W_x^{s,(g)} \to W_{h(x)}^{s,(f)}$, $k|_{W_x^{s(g)}}$ has uniformly bounded first derivatives and $d_s(h(x), k(x))$ is bounded, were we denote by d_s the distance measured along $W_x^{s,(g)}$. Such map – note that we do not require that k is a local homeomorphism – is constructed by convolving with suitable kernels after taking local coordinates.

By an argument quite similar to that of [Shu69], we have:

$$h = \lim_{n\to\infty} f^n \circ k \circ g^{-n} \tag{5}$$

and the limit is reached uniformly.

Using the conformality of f, g and the fact that k is uniformly quasi-conformal in all the leaves, we obtain that the distortion of $f^n \circ k \circ g^{-n}|_{W_x^{s(g)}}$ is uniformly bounded. Since the limit h is a homeomorphism, standard results in the theory of quasi-conformal maps show that h is quasi-conformal. (See [Väi71] Theorem 37.2, Corollary 37.3 p. 125 ff.)

We emphasize that the argument to establish Lemma 2.1 does not use at all the fact that the eigenvalues at corresponding periodic points are the same. We only use that there is a conformal structure present. □

2.2. **The stable Jacobian of the conjugacy is bounded.** We will introduce the notation $J^{s,(g)}[h]$ to denote the Jacobian of h restricted to the stable leaves. Similar notations will be used for the Jacobians of other maps, for the restrictions to the unstable foliations. If the map whose foliation is understood, we will suppress the superindex in parenthesis indicating it.

So far, we have shown that $J^{s,(g)}[h]$ is well defined, but we have not yet established any control on its size.

It is not difficult to show that $J^{s,(g)}[h]$ is uniformly integrable on balls of size one of the stable leaves. It suffices to observe that

$$\int_{B_r \cap W^{s,(g)}_x} J^{s,(g)}[h](x)\, dx = |h(B_r \cap W^{s,(g)}| \tag{6}$$

where by $|h(B_r \cap W^{s,(g)}|$ we mean the Riemannian volume with respect to the metric in $W^{s,(f)}_{h(x)}$. For subsequent arguments, it will be more useful to obtain control of $\log J^s[h](x)$.

In order to obtain control on $\log J^s[h](x)$, we will use an argument similar to that in [LS72]. This argument uses the theory of SRB measures. We refer to [Sin72], [Bow75], [Rue76], [BR75] for the classical theory for Anosov or Axiom A systems, and to [PS82] for results for partially hyperbolic systems.

We point out that in this section, the fact that the systems are conformal will not play any role. The arguments apply just as well to any Anosov system such that the conjugacy is absolutely continuous along stable manifolds.

We denote by $\mu^{+,(g)}$, $\mu^{-,(g)}$, the future and past SRB measures for the map g. If the map g is understood, we will suppress it from the notation.

One of the characterizations of $\mu^{-,(g)}$ is that it has a density along the stable directions. By the fact that h is absolutely continuous on the stable manifolds and uniformly integrable on balls of size one, we obtain that map $h_*\mu^{-,(g)}$ is a measure which is invariant under f and that it has densities along the stable manifolds of f. Hence, we obtain that

$$\mu^{-,(g)} = h_*\mu^{-,(f)}. \tag{7}$$

We also recall that another characterization of SRB measure $\mu^{-,(g)}$ is that they are Gibbs measures with respect to the potential $\log J^s[g]$. Similarly for $\mu^{-,(f)}$. The measure $h_*\mu^{-,(f)}$ is invariant under g and and Gibbs with respect to the potential $\log J^s[f] \circ h$. The fact that we have (7) implies, by the theory in [Sin72] Theorem 4 – see also [Bow75] Proposition 4.5 – that, for a Hölder function Ψ and a constant K, we have:

$$\log J^s[f] \circ h - \log J^s[g] = \Psi \circ g - \Psi + K \tag{8}$$

Moreover, K is the difference between the topological pressures of the functions $\log J^s[f] \circ u$ and $\log J^s[g]$.

It is not hard to show, using the thermodynamic formalism that $K = 0$ using the fact that f, g are conjugate and, therefore have the same entropy. Nevertheless, we will present an elementary proof below. We will show that, if $K = 0$, then, $\log J^s[h]$ is bounded. The same argument will reach a contradiction with (6) under the assumption that $K \neq 0$.

We start by showing that if K in (8) is 0, then $\log J^s[h]$ is bounded. The argument will be very similar to the one we applied to use that the function h is quasi-conformal. In the case that the manifolds are one-dimensional, it was used in [dlL87].

We first note that to show that $\log J^s[h]$ is bounded it suffices to show that, with the same function k as in (5), we have

$$|\log J^{s,(g)}[f^n \circ k \circ g^{-n}](x)| \leq C \tag{9}$$

where C is independent of n, x.

Indeed, Given a ball B_r in a stable manifold of g, and $\sigma > 0$, by the uniform convergence of $h_n \equiv f^n \circ k \circ g^{-n}$, we can find n such that $B_r \subset h^{-1} \circ h_n(B_{r+\sigma})$, (where $B_{r+\sigma}$ is, a ball with the same center and radius $r+\sigma$). Then, $h(B_r) \subset h_n(B_{r+\sigma})$. By the assumption that $J^s[h_n]$ is uniformly bounded, we can bound $|h(B_r)| \leq C|B_{r+\sigma}|$, where, again, $|S|$ denotes the Riemannian volume on $W^{s,(f)}$. Taking $\sigma = 1/10r$, we can bound $|B_{r+\sigma}| \leq C|B_r|$. Hence, under (9) we have shown that $|h(B_r)| \leq C|B_r|$.

To establish (9) we expand the logarithm in the LHS

$$\begin{aligned} \log J^{s,(g)}[f^n \circ k \circ g^{-n}](x) = &\sum_{i=1}^{n} \log J^s[f] \circ f^i \circ k \circ g^{-n}(x) \\ &+ \log J^s[k] \circ g^{-n}(x) \\ &+ \sum_{i=1}^{n} \log J^s[g^{-1}] \circ g^{-i+n} \circ g^{-n}(x) \end{aligned} \tag{10}$$

We want to show that the LHS of (10) is bounded uniformly in n, x.

Since $\log J^s[k]$ is bounded and for every function Γ we have $\sup \Gamma \circ g^{-n}(x) = \sup \Gamma(x)$, to obtain that the LHS of (10) is bounded it suffices to show – relabeling the indexes in the last sum – that:

$$\sum_{i=1}^{n} \log J^s[f] \circ f^i \circ k + \sum_{i=1}^{n} \log J^s[g^{-1}] \circ g^i \tag{11}$$

is bounded.

The boundedness of (10) follows from the identity – derived by applying (8) repeatedly –

$$\sum_{i=1}^{n} \log J^s[f] \circ f^i \circ h + \sum_{i=1}^{n} \log J^s[g^{-1}] \circ g^i = \Psi \circ g^i - \Psi \tag{12}$$

which is bounded.

We also observe that $d(f^i \circ h(x), f^i \circ k(x)) \le C\lambda^i$ – where $\lambda < 1, C$ are independent of x – and that $J^s[f]$ is Hölder so that we can bound

$$|\log J^s[f] \circ f^i \circ h - \log J^s[f] \circ f^i \circ | \le C\lambda^{i\alpha}$$

Hence, the difference between (10) and (12) can be bounded by a geometric series.

To reach a contradiction with the fact that $K \neq 0$, we note that the argument presented above remains the same except that (12) is replaced by

$$\sum_{i=1}^{n} \log J^s[f] \circ f^i \circ h + \sum_{i=1}^{n} \log J^s[g^{-1}] \circ g^i = \Psi \circ g^i - \Psi + nK \tag{13}$$

. Therefore, we obtain, even in the case that $K \neq 0$ that $J^s[f^n \circ k \circ g^{-n}](x)e^{-nK}$ remains bounded. If $K > 0$, this is a contradiction with the fact that given a ball $B \subset W^{s,(g)}_x$, $f^n \circ k \circ g^{-n} B$ converges uniformly to $h(B)$, which is a bounded set. In particular, the measure of $f^n \circ k \circ g^{-n}(B)$ should remain bounded.

We can reach a similar contradiction with $K < 0$ by reversing the roles of f, g and those of h, h^{-1} in the argument above.

Remark 2.2. Once that we have that $\log J^s[h]$ is bounded, we note that $\log J^s[h]$ satisfies the cohomology equations (8).

It is well known [PY99], [dlL01], that the bounded measurable solutions of (8) for systems that have ergodic SRB measures are unique up to an additive constant and, furthermore are Hölder. This shows that the $\log J^s[h] = \Psi + C$. Hence, $\log J^s[h]$ is Hölder.

Of course, in the one-dimensional case, where the Jacobian can be identified with the derivative, this shows that h is $C^{1+\alpha}$.

Later we will do better exploiting the quasi-conformal structure. □

Remark 2.3. Another corollary of the arguments of this section is that if the Jacobians along stable directions match and the system g is Anosov transitive, the conjugacy is absolutely continuous on stable leaves.

To establish this result, we only need to point out that we can use Livsic Theorem to show that there exists a Ψ satisfying (8) with $K = 0$. From them on, we can use the same argument as that presented here.

□

2.3. **The derivative of the conjugacy is bounded.** One important reason for us is why the estimates of the Jacobians are so useful for us is that, for conformal mappings, estimates on the Jacobian imply estimates on the derivatives.

Let M, $\tilde{M}$ be two manifolds, E, $\tilde{E}$ be subbundles of TM, $T\tilde{M}$ respectively.

Let σ, $\tilde{\sigma}$ be Riemannian metrics on E, $\tilde{E}$ continuous over the corresponding manifolds.

Given a mapping $F : M \to \tilde{M}$ such that $DF(x) : E_x \to \tilde{E}_{F(x)}$ we define

$$K_{E,\tilde{E},\sigma,\tilde{\sigma}}(x) = \max_{\substack{v\in E_x\\ \|v\|_\sigma=1}} \|DF(x)v\|_{\tilde{\sigma}} \Big/ \min_{\substack{v\in E_x\\ \|v\|_\sigma=1}} \|DF(x)v\|_{\tilde{\sigma}}$$

We say that F is A-quasi-conformal when $K_{E,\tilde{E},\sigma,\tilde{\sigma}}(x) \le A$.

Proposition 2.4. *Let $F : M \to \tilde{M}$ be E, $\tilde{E}$, σ, $\tilde{\sigma}$, A-quasi-conformal then, for a constant C which depends only on σ, $\tilde{\sigma}$ and the background metric we have*

$$\|DF(x)\| \le CA^{\frac{\nu-1}{\nu}} J_F^{1/\nu}$$

The proof of the proposition is very easy. Using the singular value decomposition of linear algebra, we know that there are matrices U, V such that U is $\sigma'_{F(x)}$ unitary, V is σ_x unitary and $UDF(x)V \equiv D$ is diagonal with pointwise entries $d_1 \le d_2 \cdots \le d_\nu$.

Then we have:

$$\begin{aligned} \|DF(x)\|_{\sigma,\sigma'} &= d_\nu \\ K\big(DF(x)\big) &= \frac{d_\nu}{d_1} \\ J_{\sigma,\sigma'}\big(DF(x)\big) &= d_1 \cdots d_\nu \end{aligned}$$

Because of the continuity of the metric, the norm and the Jacobian with respect to the background metric are equivalent to those with respect to σ, $\tilde{\sigma}$.

Therefore, we have

$$A^{-(\nu-1)}\|DF(x)\|^\nu_{\sigma,\tilde{\sigma}} \le J_{\sigma,\tilde{\sigma}}\big(DF(x)\big) \le \|DF(x)\|^\nu_{\sigma,\tilde{\sigma}}$$

□

As an immediate consequence of Proposition 2.4 we obtain

$$\|D_{s,(g)}h\| \in L^\infty \ . \tag{14}$$

where by $D_{s,(g)}$ we mean, following [dlLMM86], the derivative along the leaves of the stable foliation for g. When there is no possibility of confusion we will omit the map g from the notation.

Of course, a similar argument shows that $D_{u,(g)}h \in L^\infty$.

The result (14) implies that h is Lipschitz restricted to the stable and unstable manifold.

We also note that a similar argument with f, g exchanged shows that $D_s(h^{-1})$ is uniformly bounded. Hence $\|(D_sh)^{-1}\|$ is also uniformly bounded.

Remark 2.5. For Anosov systems, because of the local product structure, we have shown that h is Lipschitz. In the case that the system is partially hyperbolic, we would have established some Hölder exponents which are depending only on the accessibility properties. In particular, they are independent of h. This will not be a difficulty for the subsequent argument since we will continue bootstrapping the regularity along stable and unstable manifolds separately. □

2.4. **Further regularity along stable and unstable directions.** The next part of the argument will be based on the observation that, taking derivatives along the stable directions of g in (1) we have:

$$D_s h(x) = Df^{-1} \circ h \circ g(x) D_s h \circ g(x) D_s g(x) \ . \tag{15}$$

(Note that we have omitted the indexes that indicate which maps give rise to the stable foliations that originate the operators D_s. All of them in this case are with respect to the foliations invariant under g.)

Equation (15) can be considered as saying that $D_s h$ is a coboundary between $D_s f^{-1} \circ h \circ g(x)$ and $D_s g(x)$.

By the results in the previous section, we already have that $D_s h$ is an L^∞ coboundary. There are already several papers in the literature which show that coboundaries which are measurable are automatically smooth (see, e.g. [NT98] for the case of continuous functions). In this paper we will adapt the method of [dlL02].

We denote by $\mathcal{L}(X, Y)$ the space of Linear operators from X to Y endowed with the supremum norm.

We note that $D_s h(x) \in \mathcal{L}(E^{s,(g)}_x, E^{s,(f)}_{h(x)})$. Similarly

$$Df^{-1} \in \mathcal{L}(E^{s,(f)}_x, E^{s,(f)}_{f^{-1}(x)})$$
$$Dg(x) \in \mathcal{L}(E^{s,(f)}_x, E^{s,f}_{f^{-1}(x)})$$

We introduce the operator $\eta(x)$

$$\eta(x) : \mathcal{L}(E^{s,(g)}_{g(x)}, E^{s,(f)}_{h \circ g(x)}) \to \mathcal{L}(E^{s,(g)}_x, E^{s,f}_{h(x)}$$

by

$$\eta(x)\Gamma_{g(x)} = D_s f \circ h \circ g(x) \Gamma_{g(x)} D_s g(x)$$

We also introduce the notation

$$\Psi(x) = D_s h(x)$$

Hence, the equation (15) can be written more compactly as:

$$\Psi(x) = \eta(x) \Psi \circ g(x) \ . \tag{16}$$

Iterating (15) we have

$$\Psi(x) = \eta(x) \cdots \eta \circ g^n(x) \Psi \circ g^{n+1}(x) \tag{17}$$

The subsequent analysis will use heavily expressions such as (17) to bootstrap the regularity of Ψ.

The analysis will be somewhat similar to that of [dlL02].

For the moment, we will just estimate $\eta(x) \cdots \eta \circ g^n(x)$.

The following bounds will be crucial for subsequent analysis:

Lemma 2.6. *With the notation above, we have*

$$\|\eta(x)\cdots\eta\circ g^n(x)\| \le C$$

where C depends only on the properties of the metrics. In particular, it is independent of n.

Proof. Clearly we have

$$\|\eta(x)\cdots\eta\circ g^n(x)\| \le \|Df^{-n}\circ h\circ g(x)\|\|Dg^n(x)\| \tag{18}$$

By (18) and Proposition 2.4, to prove Lemma 2.6, it suffices to bound uniformly in n, x.

$$E(n,x) = (J^s[f^{-n}])\circ h\circ g^n(x)J^s[g^n](x)$$

We note that because of (1) we have

$$E(n,x) = J^s[h](x)\big/J^s[h]\circ g^n(x) \tag{19}$$

and the R.H.S. is bounded uniformly in n, x because of the fact that the stable Jacobian is Hölder, and bounded above and below away from zero. □

Remark 2.7. Even if we will not use it here, we note that the fact that the maps are conformal, show that the rates of growth in along all the directions in an orbit are the same. The proof of stable foliations in [HK90] allow us to conclude that the stable and unstable foliations are C^1.

Actually, following [HP70], once we know that the foliations are C^1, we can use the fact that the derivatives of the jet satisfy an equation that can be analyzed with the fiber contraction theorem, to obtain that the foliations are $C^{1+\gamma}$.

Even if these considerations will not play any role in our discussions, the regularity of the foliations is a very important part of the argument in [Sad02] [KS02] and it allows them to reconstruct global properties of the manifold. □

2.5. D_sh **is Hölder.** The following will be the main result of this section.

Lemma 2.8. *Assume that D_sg, $D_sf\circ h$ are $C^{\alpha,(s)}$, $\alpha>0$ (resp. $C^{\alpha,(u)}$, $\alpha>0$). Then*

$$D_sh\in C^{\alpha,(s)} \quad (\textit{resp. } C^{\alpha,(u)}).$$

Note that we have assumed $f\in C^r$ with $r>1$. We have also shown that h is Lipschitz restricted to the and unstable stable foliation Hence, using that the map is Anosov, we can take

$$\alpha = \min(r-1,\mathrm{Lip}).$$

(In the case that the map is partially hyperbolic, we will need to rearrange the argument slightly.)

In the case that $\alpha=\mathrm{Lip}$ — which is the case relevant for higher regularity — we obtain that D_sh is differentiable almost everywhere along the stable and unstable leaves.

We also call attention that in the course of the proof of Lemma 2.8, we will introduce a remarkable formula (25) giving the "increments" of $D_s h$ along the stable manifolds. This formula will be the basis for subsequent bootstraps.

We will need to introduce some notation which is very similar to that of [dlL02], [KdlLS02].

Given a vector field u in M we denote by $\delta_u(x) = \exp_x(u(x))$ where exp denotes the Riemannian geometry exponential.

In subsequent arguments we will need that the vector fields u are such that δ_u preserves the (un)stable foliation of g and that they are absolutely continuous.

The reason to require that δ is absolutely continuous is that we are going to consider $\Psi \circ \delta$ where $\Psi \in L^\infty$ and evaluate identities that hold for general points x at $\delta(x)$.

We note that maps satisfying the previous requirements are plentiful because of the absolute continuity of the foliation. That is

Proposition 2.9. *If $d(x, y) \le \varepsilon_0$, $y \in W_x^{s,(g)}$ then there exists δ as above, such that*

$$\delta(x) = y \ , \quad \|\delta - \mathrm{Id}\|_{L^\infty} \le C d(x, y) \ , \quad \|J[\delta]\|_{L^\infty} \le C \ .$$

The proof of the proposition is obvious if we consider a coordinate patch such as those introduced in Theorem 9 in [Ano69]. There, it is shown how it is possible to make changes of variables that reduce the stable foliations to a product foliation in $\mathbb{R}^d$. This changes of variables are, moreover absolutely continuous. Hence, the problem considered in Proposition 2.9 can be refered just to the a trivial foliation.

We will also need to compare tangent spaces at nearby points. We just recall the device of *"connectors"* introduced in [HPPS70].

Let x, y be points which are close enough that we can find a unique shortest geodesic.

Define by $S_x^y : T_x M \to T_y M$ to be the map obtained by transporting the vectors from $T_x M$ to $T_y M$ along the shortest geodesic.

Note that since we are assuming that the background metric is analytic, we obtain that S_x^y depends analytically on x, y provided that $d(x, y) \le \varepsilon_0$ for some $\varepsilon_0 > 0$.

Note that, if both sides can be defined, we have

$$S_x^z = S_y^z \circ S_x^y$$

In particular

$$\mathrm{Id} = S_y^x \circ S_x^y \ . \tag{20}$$

Using the notations of $\Psi = D_s h$ and η introduced in section 2.3 we will show that

$$\|\Psi(x) - S^x_{\delta_v(x)} \Psi \circ \delta_v(x)\| \le C \|v\|^\alpha_{C^0}$$

provided that δ_v preserves the stable foliation.

Later we will provide with a different argument to show a similar result when δ_v preserves the unstable foliation.

Assuming that δ preserves the stable foliation of g and that $\delta(x) - x$ is small enough so that we can define the transport, then we see that $g^n \circ \delta(x)$, $g^n(x)$ are also close and we define the $S^{g^n(x)}_{g^n\circ\delta(x)}$.

Evaluating (17) at $\delta(x)$ and inserting factors $S^{g^i\circ\delta(x)}_{g^i(x)} S^{g^i(x)}_{g^i\circ\delta(x)}$, which are just the identity, we obtain:

$$
\begin{aligned}
S^x_{\delta(x)}\Psi\circ\delta(x) =& S^x_{\delta(x)}\eta\circ\delta(x) S^{g(x)}_{g\circ\delta(x)} S^{g\circ\delta(x)}_{g(x)}\eta\circ g\circ\delta(x) S^{g^2(x)}_{g^2\circ\delta(x)} S^{g^2\circ\delta(x)}_{g^2(x)} \\
&\cdots \\
&\cdot S^{g^n(x)}_{g^n\circ\delta(x)} S^{g^n\circ\delta(x)}_{g^n(x)} \eta\circ g^n\circ\delta(x) \\
&\cdot S^{g^{n+1}(x)}_{g^{n+1}\circ\delta(x)} S^{g^{n+1}\circ\delta(x)}_{g^{n+1}(x)} \Psi\circ g^{n+1}\circ\delta(x)
\end{aligned}
\tag{21}
$$

Introducing the notation

$$
\begin{aligned}
\tilde\eta\circ\delta(x) &= S^x_{\delta(x)}\eta\circ\delta(x) S^{g\circ\delta(x)}_{g(x)} \\
\tilde\Psi\circ\delta(x) &= S^x_{\delta(x)}\Psi\circ\delta(x)
\end{aligned}
\tag{22}
$$

We see that (21) can be written

$$
\tilde\Psi\circ\delta(x) = \tilde\eta\circ\delta(x)\tilde\eta\circ g\circ\delta(x)\cdots\tilde\eta\circ g^n\circ\delta(x)\tilde\Psi\circ g^{n+1}\circ\delta(x)
\tag{23}
$$

Remark 2.10. There is a nice geometric intuition behind the calculation leading to (21).

Namely, we are trying to analyze the objects along the orbit $g^i\circ\delta(x)$ by referring them to objects in the orbit $g^i(x)$.

This is accomplished, of course by conjugating with the identifications S. This will allow us to compare operators along two neighboring orbits by referring them to common points. □

Adding and subtracting in (17) and (21), we obtain:

$$
\begin{aligned}
\Psi(x) - \tilde\Psi\circ\delta(x) =& \big[\eta(x) - \tilde\eta\circ\delta(x)\big]\tilde\eta\circ g\circ\delta(x)\cdots\tilde\eta\circ g^n\circ\delta(x)\tilde\Psi\circ g^{n+1}\circ\delta(x) \\
&+ \quad \cdots \quad + \\
&\eta(x)\cdots\eta\circ g^{i-1}(x)\big[\eta\circ g^i(x) - \tilde\eta\circ g^i\circ\delta(x)\big]\tilde\eta\circ g^n\circ\delta(x)\tilde\Psi\circ g^{n+1}\delta(x) \\
&+ \quad \cdots \quad + \\
&+ \eta(x)\cdots\eta\circ g^{n-1}(x)\big[\eta\circ g^n(x) - \tilde\eta\circ g^n\circ\delta(x)\big]\tilde\Psi\circ g^{n+1}\delta(x) \\
&+ \eta(x)\cdots\eta\circ g^n(x)\big[\Psi\circ g^{n+1}(x) - \tilde\Psi\circ g^{n+1}\circ\delta(x)\big]
\end{aligned}
\tag{24}
$$

We denote by $T_i(x)$ the general term of (24), namely:

$$
T_i(x) = \eta(x)\cdots\eta\circ g^{i-1}(x)[\eta\circ g^i(x) - \tilde\eta\circ g^i\circ\delta(x)]\Psi\circ g^{i+1}\circ\delta(x)
$$

We want to show that

$$\Psi(x) - \tilde{\Psi} \circ \delta(x) = \sum_{i=1}^{\infty} T_i(x) \tag{25}$$

almost everywhere x.

First of all we want to show that $\sum_{i=1}^{\infty} T_i(x)$ converges almost everywhere as well as in L^∞ sense. This follows from

$$\|T_i\|_{L^\infty} \le \lambda_c^{i\alpha} C \|\delta - \mathrm{Id}\|_{C^0}^{\alpha} \tag{26}$$

To prove (26) we use that, by Lemma 2.6 the factors $\eta(x) \cdots \eta \circ g^{i-1}(x)$ are uniformly bounded independently of i, x.

Also $\tilde{\Psi} \circ g^{n+1} \circ \delta(x)$ is uniformly bounded in L^∞.

Because η is C^α and δ is adapted to the stable foliation we have

$$\|\eta \circ g^i - \tilde{\eta} \circ g^i \circ \delta\|_{L^\infty} \le \lambda^{i\alpha} C \|\delta - \mathrm{Id}\|_{L^\infty}^{\alpha}$$

This proves (26).

To estimate the last term in (24), we follow an argument in [NP99].

We note that, by Lusin theorem, Ψ is continuous in as set of measure at least $1 - \varepsilon$.

We obtain that, because of the ergodicity of g, $\Psi \circ g^{n+1}(x) - \tilde{\Psi} \circ g^{n+1} \circ \delta(x)$ converges to zero along a subsequence.

Hence, we obtain that (25) holds almost everywhere.

This shows that $D_s h \in C^{\alpha,s}$.

We note that if Ψ satisfies (15) then we also have

$$\Psi(x) = \eta^{-1} \circ g^{-1}(x) \Psi \circ g^{-1}(x) \ .$$

Hence, applying the result we have for the regularity along the stable directions, we obtain the claim for the unstable directions. □

One consequence that we will use later is that the formula (25) holds not only almost everywhere but everywhere. Hence, we do not need to use the device of studying increments by comparing with absolutely continuous vector fields.

We have when $y \in W_x^{s(g)}$

$$\begin{aligned} \Delta(y) &\equiv \Psi(x) - S_y^x \Psi(y) \\ &= \sum_{i-1}^{\infty} \eta(x) \cdots \eta \circ g^i(x) [\eta \circ g^i(x) - \tilde{\eta} \circ g^i(y)] \Psi \circ g^{i+1}(y) \end{aligned} \tag{27}$$

2.6. $D_s h$ is differentiable.

Lemma 2.11. *If $D_s f^{-1} \circ h$, $Dg \in C^{1+\alpha,(s)}$, $\alpha > 0$ (resp. $\in C^{1+\alpha,(u)}$, $\alpha > 0$) then $D_s h \in C^{1+\alpha,(s)}$. (resp. $C^{1+\alpha,(u)}$.)*

The proof of Lemma 2.11 is very simple. Given (27) it is natural to guess what the formula for the derivative should be (see (28)). Then we will prove that this formula is indeed a derivative by estimating the remainder. We call attention to the fact that the formula (28) will be the basis for subsequent bootstraps.

We guess

$$G(x) = \sum_{i=1}^{\infty} \eta(x) \cdots \eta \circ g^i(x)(D_s\tilde{\eta} \circ g^i)(x)\Psi \circ g^{i+1}(x)$$
$$= \sum_{i=1}^{\infty} \Psi(x)\Psi^{-1} \circ g^{n-1}(x)(D_s\tilde{\eta} \circ g^i)(x)\Psi \circ g^{i+1}(x) \tag{28}$$

First of all, we observe that the formula in (28) converges uniformly.

The reason is that, because of the chain rule we have for a vector field X tangent to the stable direction:

$$D_s(\tilde{\eta} \circ g^i)(x)\, X(\tilde{\eta} \circ g^i)(x) = (D_s\eta) \circ g^i Dg^i(x)X$$

Because the vector field X is tangent to the stable direction we see that $\|SDg^i(x)X\| \le \lambda^i\|X\|$. Hence, $\|D_s\tilde{\eta} \circ g^i(x)\| \le C\lambda^i$ and, therefore, the general term of (28) can be bounded by a decreasing exponential.

Once we know that the formula (28) makes sense, to establish that it is a true derivative, we just need to estimate the error in the linear approximations. This will be easy because the formula for the increments is given in (25). When we substract the linear approximation given by (28) from the increment formula (25), we obtain:

(29)
$$\Psi(x) - S_y^x\Psi(y) - G(x)[y-x]$$
$$= \sum_i \eta(x) \cdots \eta \circ g^i(x)[\eta g^i(x) - \tilde{\eta} \circ g^i(y) - D_y\tilde{\eta} \circ g^i(y)]\Psi \circ g^{i+1}(x)$$
$$+ \sum_i \eta(x) \cdots \eta \circ g^i(x)[\eta \circ g^i(x) - \tilde{\eta} \circ g^i(y)][\Psi \circ g^{i+1}(x) - \tilde{\Psi} \circ g^{i+1}(y)]$$

The desired result will be established when we estimate the (29) and show that it can be bounded by something that is $o(d_s(x,y))$. We will start by bounding the general term of each of the sums in (29) and then add up all the estimates obtained for each of the terms.

We again note that the factor $\eta(x) \cdots \eta \circ g^i(x)$ are bounded uniformly. So is $\Psi \circ g^{i+1}(x)$.

Because $\eta \in C^{1+\alpha,(s)}$, we have that

$$|[\eta \circ g^i(x) - \tilde{\eta} \circ g^i(y) - [D_s\tilde{\eta}g^i](x)[x-y]]| \le \lambda^{(1+\alpha)i} d_s(x-y)^{1+\alpha}$$

This shows that the first sum in (29) can be estimated by $Cd(x,y)^{1+\alpha}$.

Now we estimate the second sum in (29).

Because Ψ is $C^{\mathrm{Lip},(s)}$ we have:

$$|\Psi \circ g^i(x) - \tilde{\Psi} \circ g^i(y)| \le C\lambda^i d_s(x,y)\ .$$

Since η is Lipschitz we can bound:

$$\|\eta \circ g^i(x) - \tilde{\eta} \circ g^i(y)\| \leq C\lambda^i d_s(x, y)$$

Hence the second sum can be bounded by $Cd_s(x, y)^2$

This finished the proof that (28) is indeed the derivative along the stable direction.

This finished the proof of Lemma 2.11.

2.7. **Higher derivatives along the stable directions.** Note that we have also established (28) a formula for the derivative along the stable direction.

Establishing the existence of higher order derivatives is now very easy because we can take derivatives of (28) term by term. The proof is from now on, quite similar to the proofs in [dlLMM86] and [BdlLW96].

We just need to show that the expression for the derivative given in (28) can be differentiated term by term. These formal derivatives will converge uniformly. Hence, it is standard to conclude that these sums are indeed the true derivatives. The estimates needed are a particular case of Lemma 4 of [BdlLW96] – which are very similar to those of [dlLMM86] p. 587.

Following [dlLMM86], for $k \in \mathbb{N}$, we say that a function φ is in $C_s^{k,(g)}$ when the restriction to the stable leafs of g are uniformly C^k and the k jets of these restrictions are continuous over the manifold.

We say that $\varphi \in C_s^{k+\alpha,(g)}$ when $\varphi \in C_s^{k,(g)}$ and the k jets are uniformly Hölder when restricted to the stable leaves of the stable leaves. The space $C_s^{k,(g)}$ is a Banach space when endowed with the norm of the sup of the k-jets. Similarly, $C_s^{k+\alpha,(g)}$ is a Banach space when endowed with the norm obtained taking the maximum of the $C^{k,(g)}$ norm and the seminorm $H^{\alpha}_{s,(g)}$ for the k-jet.

Clearly, we have

$$\varphi \in C_s^{k,(g)} \iff D_{s,(g)}\varphi \in C_s^{k-1,(g)}$$

and the same with fractional exponents.

Lemma 2.12. *Assume that $\Psi, D_s\eta \in C_s^{t,(g)}$ $t \in \mathbb{N}+[0, \mathrm{Lip}]$ and that $g \in C^r$, $r \geq t$, $t > 0$. Then, the sum in (28) converges uniformly in $C_s^{t,(g)}$.*

Proof. This lemma is very similar to the estimates in [dlLMM86] p. 587 and [BdlLW96] Lemma 4. The estimates in the above papers give

$$\begin{aligned} &\|D_s^j \Psi^{-1} \circ g^{n-1}\|_{L^\infty} \leq C_j \|\Psi^{-1}\|_{C_s^j} \lambda^{j(n+1)} j^{n+1} \\ &\|D_s \tilde{\eta} \circ g^i\|_{L^\infty} \leq C_j \|D_s\tilde{\eta}\|_{C_s^j} \lambda^{ji} j^i \\ &\|\Psi \circ g^{i+1}\|_{L^\infty} \leq C_j \|\Psi\|_{C_s^j} \lambda^{j(i+1)} j^{i+1} \end{aligned} \tag{30}$$

where C_j are numbers that depend on j and on $\|g\|_{C^j}$ but which are independent of i, n.

Using Leibniz rules for the derivatives, we obtain that, as claimed the $C^{r,(g)}_s$ norm of the general term in (28) is bounded by a decreasing exponential.

When we need to consider fractional derivatives, the arguments we We denote by $d_s(x, y)$ the distance of two points in the same leaf of the stable foliation measured along the leaves. We also denote

$$H^\alpha_s(\eta) = \sup_{0\le d_s(x,y)\le\gamma} (\eta(x) - \eta(y))/d_s(x, y)^\alpha$$

We will assume without loss of generality that the metric is adapted. That is, $d_s(f(x), f(y)) \le \lambda d_s(x, y)$ for some $\lambda < 1$. This immediately leads to

$$H^\alpha_s(\eta \circ f) \le H^\alpha_s(\eta)\lambda^\alpha. \tag{31}$$

The bounds for the H^α_s of the high derivatives follow by estimating the same formulas that we use for the integer derivatives.

□

Applying repeatedly, Lemma 2.12, we obtain the desired result. Note that, if we assume that $\Psi, \eta \in C^{j,(g)}_s$, then Lemma 2.12 shows that the sum in (28) converges in $C^{j,(g)}_s$, therefore, $G = D_s h$ is in $C^{j,(g)}_s$ and, therefore $h \in C^{j+1,(g)}_s$ and therefore $\Psi, \eta \in C^{j+1,(g)}_s$. The induction stops when we reach the regularity assumed in g, f.

Remark 2.13. Note that the argument presented to obtain the higher derivatives is very reminiscent of the argument presented for the bootstrap of regularity of conjugacies in [dlL92]. In Theorem 6.1 of [dlL92], it is established that once a conjugacy is smoother than an critical value, determined out of the bunching of the spectrum of expansions and contraction, then the conjugacy is as smooth as the map. (See also the last parts of [NT98].)

For maps preserving a conformal metric, the expansions of all the vectors along the stable manifold is roughly the same. Hence, the critical value for Theorem 6.1 in [dlL92] – expressed in terms of bunching properties of the expansion properties – is $1 + \epsilon$. Hence, once that we have that $h \in C^{1+\alpha}$, it is also possible to just adapt the arguments in [dlL92]. □

3. Regularity in the usual classes

We have so far shown that the conjugacy $h \in C^{r,(g)}_s$. A similar argument shows that $h \in C^{r,(g)}_u$.

For the case of difeomorphisms, we are done by invoking a regularity lemma that says that functions with the above properties are $C^{r'}$. In [dlLMM86] one can find a prove of this result using elliptic regularity theory. Other proofs are available in [Jou88]. (Even other versions less technically optimal appear in [HK90], [dlL92], [dlL97]).

For the case of flows, we point out that exactly the same argument applies for the stable and unstable manifolds of the flow. It is also trivial to observe that if (2) holds, then the conjugacy is smooth along the direction of the flow. Then, we can apply the regularity results to obtain that the conjugacy is $C_{cs}^{r',X}$ – where the subindex cs refers to the center stable direction. Then, a further application of the regularity result considering now the cs foliation and the unstable one finishes the proof of Theorem 1.1.

Remark 3.1. It is interesting to note that, except for Section 3, all the previous sections work for partially hyperbolic maps that have SRB measures with ergodic properties.

It is reasonable to conjecture that there should be an analogue of the regularity result for foliations of Anosov systems for foliations of partially hyperbolic systems that have a certain accessibility property. Following the scheme of proof in [dlLMM86], it seems that it would be enough to use some version of the theory of hypoelliptic equations rather than the elliptic regularity theory used in [dlLMM86].

One would expect that the regularity recovered, would be a fraction of the regularity of the derivatives obtained.

Once such result is on hand, then it will automatically follow a version of Theorem 1.1 for partially hyperbolic systems. We hope to come to this problem in a the future. □

3.1. **Results for systems with one-dimensional foliations.** We will present two arguments that are very similar to the arguments in [dlL87], [dlLM88] and in [Pol88], [dlL92].

In the first argument, we note that, because of the condition on the eigenvalues, the fact that the system is transitive and the result of [Liv72], we can find a function Ψ satisfying (8) (with $K = 0$). Then, we can repeat the argument as in Section2.2. We obtain that the Jacobian of the derivative is bounded. Of course, in the one-dimensional case, the Jacobian of the derivative is the same as the derivative. Hence, showing that the derivative is bounded is obvious.

An alternative argument is based on the fact that the SRB measure is also characterized as the weak limit of measures based on periodic orbits with weights which are proportional to the stable Jacobians. Given the equality of the stable derivatives at corresponding periodic points, we obtain that the SRB measure is transported by the conjugacy.

¿From then, one can use that the densities are transported and use that they are smooth to get that the derivatives are bounded. The argument presented before applies also to the one-dimensional case.

Using the commutativity of the derivatives one can use simpler arguments. Several different arguments to bootstrap regularity taking advantage of the fact that the stable manifold is one-dimensional can be found in [dlL87], [dlL92], [dlL97]. We refer to [dlL97] for a comparison between these one-dimensional arguments.

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Dept. of Mathematics, Univ. of Texas, Austin TX 78712
E-mail address, R. de la Llave: `llave@math.utexas.edu`