Scattering from Subspace Potentials for Schrödinger Operators on Graphs

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Abstract

Let $\mathcal{G}$ be a simple countable connected graph and let H_0 be the discrete Laplacian on $l^2(\mathcal{G})$. Let $\Gamma \subset \mathcal{G}$ and let $V = \sum_{n\in\Gamma} V(n)(\delta_n|\cdot)\delta_n$ be a potential supported on Γ. We study scattering properties of the operators $H = H_0 + V$. Assuming that the wave operators $W^{\pm}(H, H_0)$ exist, we find sufficient and necessary conditions for their completeness in terms of a suitable criterion of localization along the subspace $l^2(\Gamma)$. We discuss the case of random subspace potentials, for which these conditions are particularly natural and effective. As an application, we discuss scattering theory of the discrete Laplacian on the half-space $\mathcal{G} = \mathbb{Z}^d \times \mathbb{Z}_+$ perturbed by a potential supported on the boundary $\Gamma = \mathbb{Z}^d \times \{0\}$.

1 Introduction

Let $\mathcal{G}$ be the set of points (a.k.a. vertices) of a simple countable connected graph and let $\rho : \mathcal{G} \times \mathcal{G} \mapsto \{0, 1, 2, ...\}$ be the distance on this graph. Note that since the metric ρ fully determines the edges of the graph (which can be identified with unordered pairs $(n, m) \in \mathcal{G} \times \mathcal{G}$ having $\rho(n, m) = 1$), we can think of the graph as being the metric space $(\mathcal{G}, \rho)$. We assume here that the degree of points in the graph is bounded, namely,

$$\gamma := \sup_n \#\{m : \rho(m, n) = 1\} < \infty.$$

Let H_0 be the discrete Laplacian on $\mathcal{H} := l^2(\mathcal{G})$, defined by,

$$(H_0\psi)(n) = \sum_{\rho(m,n)=1} \psi(m).$$

H_0 is a bounded self-adjoint operator and $\|H_0\| \leq \gamma$. Let $\Gamma \subset \mathcal{G}$ and let

$$V = \sum_{n\in\Gamma} V(n)(\delta_n|\cdot)\delta_n,$$

where the $V(n)$ are real numbers, $\delta_n(m) = \delta_{nm}$, and $(\cdot|\cdot)$ denotes the scalar product in $\mathcal{H}$. V is a self-adjoint operator which acts non-trivialy only along the subspace $l^2(\Gamma)$. (Explicitly, $(V\psi)(n) = V(n)\psi(n)$ for $n \in \Gamma$ and $(V\psi)(n) = 0$ for $n \notin \Gamma$.) We call such V a *subspace potential.*

In this paper we study scattering properties of the operators

$$H = H_0 + V. \tag{1.1}$$

The abstract model (1.1) is a natural and technically convenient generalization of many different specific models discussed in recent literature [BBP, CS, JL1, JL3, JM3, MV1, MV2].

Let us recall some well-known facts. If A is a self-adjoint operator on a Hilbert space $\mathfrak{H}$ and $\phi, \psi \in \mathfrak{H}$, then for Lebesgue a.e. $E \in \mathbb{R}$, the limits

$$(\phi|(A - E - \mathrm{i}0)^{-1}\psi) := \lim_{\epsilon\downarrow 0}(\phi|(A - E - \mathrm{i}\epsilon)^{-1}\psi)$$

exist and are finite and non-zero. We denote by $\mathbf{1}_\Theta(A)$ the spectral projection of A onto a Borel set Θ. A bounded operator B is called A-smooth on Θ if there is a constant C such for all $\phi \in \mathrm{Ran}\,\mathbf{1}_\Theta(A)$,

$$\int_{\mathbb{R}} \|B\mathrm{e}^{-\mathrm{i}tA}\phi\|^2 \mathrm{d}t \leq C\|\phi\|^2.$$

If $\Theta = \mathbb{R}$, we simply say that B is A-smooth.

Let A and B be self-adjoint operators and assume that the wave operators

$$U^{\pm} := \mathrm{s} - \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tB}\mathrm{e}^{-\mathrm{i}tA}\mathbf{1}_{\Theta}(A),$$

exist. One easily shows that $\mathrm{Ran}\, U^{\pm} \subset \mathbf{1}_{\Theta}(A)$. The wave operators $U^{\pm}$ are called complete on Θ if $\mathrm{Ran}\, U^{\pm} = \mathbf{1}_{\Theta}(B)$. The wave operators $U^{\pm}$ are complete on Θ iff the wave operators

$$\mathrm{s} - \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tA}\mathrm{e}^{-\mathrm{i}tB}\mathbf{1}_{\Theta}(A)$$

exist.

Let $\mathcal{H}_n$ be the cyclic space spanned by H and δ_n, $n \in \Gamma$, and let $\tilde{\mathcal{H}}$ be the closure of the linear span of the subspaces $\mathcal{H}_n$. If $\tilde{\mathcal{H}} = \mathcal{H}$, we say that $\{\delta_n\}_{n\in\Gamma}$ is a cyclic family for H. It is not difficult to show (see the proof of Proposition 3.1 in [JL1]) that $\tilde{\mathcal{H}}$ does not depend on the choice of V. Thus, assuming that $\{\delta_n\}_{n\in\Gamma}$ is a cyclic family for H_0 also implies that it is a cyclic family for H. From hereon we indeed assume that $\{\delta_n\}_{n\in\Gamma}$ is a cyclic family for H_0 and thus for H.

Let $R \geq 0$ be a positive integer and

$$\Gamma_R = \{n \in \mathcal{G} : \rho(n, \Gamma) \leq R\}.$$

(Note that $\Gamma_0 = \Gamma$.) We denote by $\mathbf{1}_R$ the orthogonal projection on $l^2(\Gamma_R)$.

Our main result is:

Theorem 1.1 *Let $\Theta \subset \mathbb{R}$ be an open set. Consider the following assumptions:*
(a) *The operator $H\mathbf{1}_{\Theta}(H)$ has purely absolutely continuous spectrum.*
(b) *$\mathbf{1}_1$ is H_0-smooth on Θ.*
(c) *The wave operators*

$$W^{\pm} = \mathrm{s} - \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}tH_0}\mathbf{1}_{\Theta}(H_0),$$

exist.
(d) *There is a set $\mathcal{D}$ dense in $\mathrm{Ran}\, \mathbf{1}_{\Theta}(H_0)$ such that for $\phi \in \mathcal{D}$, $\|\mathbf{1}_1\mathrm{e}^{-\mathrm{i}tH_0}\phi\| = O(|t|^{-2})$.*
(e) *For Lebesgue a.e. $E \in \Theta$ and all $n \in \Gamma$,*

$$\mathrm{Im}\,(\delta_n|(H - E - \mathrm{i}0)^{-1}\delta_n) > 0.$$

Consider the following statements:
(1) *For Lebesgue a.e. $E \in \Theta$ and all $n \in \Gamma$,*

$$\sum_{m\in\Gamma_1} |\mathrm{Im}\,(\delta_n|(H - E - \mathrm{i}0)^{-1}\delta_m)|^2 < \infty.$$

(2) *For a dense set of $\phi \in \mathrm{Ran}\, \mathbf{1}_{\Theta}(H)$,*

$$\int_{\mathbb{R}} \|\mathbf{1}_1\mathrm{e}^{-\mathrm{i}tH}\phi\|^2\mathrm{d}t < \infty.$$

(3) *The wave operators* $W^{\pm}$ *are complete on* Θ.

If (a) *holds, then* (1) $\Rightarrow$ (2). *If* (e) *holds, then* (2) $\Rightarrow$ (1). *If* (b) *holds, then* (2) $\Rightarrow$ (3). *If* (b), (c) *and* (d) *hold, then* (3) $\Rightarrow$ (2). *Hence, if* (a)–(e) *hold, then* (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

Remark 1. The same result holds if H_0 is replaced by $H_0 + U_0$, where

$$U_0 = \sum_{n\in\mathcal{G}} U_0(n)(\delta_n|\cdot)\delta_n$$

is an arbitary background potential.
Remark 2. Since $\{\delta_n\}_{n\in\Gamma}$ is a cyclic family for H, (2) $\Rightarrow$ (a). Similarly, if either (b) or (d) holds, then $H_0\mathbf{1}_\Theta(H_0)$ has purely absolutely continuous spectrum.
Remark 3. The assumption that Θ is an open set is used only in the proof of implication (2) $\Rightarrow$ (3), all the other results hold for any Borel set Θ of positive Lebesgue measure. If in (b) we assume that $\mathbf{1}_1$ is H_0-smooth, then (b) and (2) imply (3) for any Borel set Θ of positive Lebesgue measure.

Theorem 1.1 is based on a simple physical principle already used in some special cases in [JL3]. If the spectrum of H in Θ is purely absolutely continuous, then wave packets with energies in Θ must propagate. If propagation along the subspace $l^2(\Gamma)$ is supressed, then the wave packets must propagate into $l^2(\Gamma)^\perp$. Theorem 2.1 quantifies this heuristic principle and further asserts that under fairly general assumptions the "localization within the subspace" is the only physical mechanism relevent to the completeness of the wave operators.

The assumptions (a) and (e) of Theorem 1.1 concern the interacting Hamiltonian H and could be difficult to check in practice. However, due to results in [JL2], for random subspace potentials (a) and (e) can be reduced to assumptions on H_0 which can be easily verified in concrete models. Let us describe the random model and this result in detail.

Let Ω be the set of all boundary potentials,

$$\Omega = \mathbb{R}^\Gamma = \bigtimes_\Gamma \mathbb{R},$$

and let $\mathfrak{B}$ the Borel σ-algebra in Ω. The model is specified by a choice of a probability measure P on $(\Omega, \mathcal{B})$. For simplicity, we will consider only the product measures

$$P = \bigtimes_\Gamma \mu_n,$$

where each μ_n is a probability measure on $\mathbb{R}$. Note that μ_n is the probability distribution of the random variable $\Omega \ni V \to V(n)$. We say that the random variable $V(n)$ has density if the measure μ_n is absolutely continuous w.r.t. Lebesgue measure. By construction, the random variables $\{V(n)\}_{n\in\Gamma}$ are independent.

The following result is an easy consequence of the main theorem in [JL2] (we will outline its proof in Section 3.1).

Proposition 1.2 *Assume that the random variables* $\{V(n)\}_{n\in\Gamma}$ *have densities and let* $\Theta \subset \mathbb{R}$ *be a Borel set of positive Lebesgue measure. Consider the assumption:*
(g) *The operator* $H_0 \mathbf{1}_\Theta(H_0)$ *has purely absolutely continuous spectrum and for Lebesgue a.e.* $E \in \Theta$,

$$\sum_{n\in\Gamma} \mathrm{Im}\,(\delta_n|(H_0 - E - \mathrm{i}0)^{-1}\delta_n) > 0. \tag{1.2}$$

If (g) *holds and the assumption* (c) *of Theorem 1.1 holds with probability one, then the assumptions* (a) *and* (e) *hold with probability one.*

On a technical level, Theorem 1.1 is a variant of Kato's theory of smooth perturbations. Its main interest lies in applications to random subspace potentials. The scattering theory of random Schrödinger operators has received considerable attention in recent literature. Models that have been studied include slowly decaying random potentials [B, CK, Kr, RoSh], sparse random potentials [HK, MV1, MV2], and surface random potentials [JL1, JL3]. Theorem 1.1 can be effectively applied to Anderson models with surface and sparse random potentials. We will discuss the surface model in the next section. The analysis of sparse random potential models is more technical and is based on a fusion of techniques developed in this paper and in [JL1, MV1, MV2]. The scattering theory of sparse random potentials will be discussed in a continuation of this paper.

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2 Surface random potentials

We consider the same model as in [JL1, JL3]: Let $d \geq 1$ be given and let $\mathcal{G} = \mathbb{Z}^d \times \mathbb{Z}_+$, where $\mathbb{Z}_+ = \{0, 1, \cdots\}$. We denote points in $\mathcal{G}$ by $n = (n_1, \ldots, n_{d+1})$. We consider the usual metric on $\mathcal{G}$, $\rho(n,m) = |n-m|_+$, where $|n|_+ = \max_j |n_j|$. The spectrum of the corresponding discrete Laplacian H_0 is purely absolutely continuous and $\sigma(H_0) = [-2(d+1), 2(d+1)]$.

Let $\Gamma = \{n \in \mathcal{G} : n_{d+1} = 0\} = \partial\mathcal{G}$, let V be a potential supported on Γ, and

$$H = H_0 + V. \tag{2.3}$$

This particular model is motivated by the physics of disordered surfaces (see [JMP, KP]). It is obviously an example of the abstract subspace model discussed in the previous section.

We briefly review what is known about the scattering theory of the model (2.3), refering the reader for details and additional information to the original literature. In [CS, JL1] it was proven that for all V the wave operators

$$W^\pm = \mathrm{s} - \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}tH_0}$$

exist. This implies that $\sigma(H_0) \subset \sigma_{\rm ac}(H)$. The question of completeness of the wave operators $W^\pm$ has been studied in [JL1]. In this work the notion of resonant spectrum $\mathfrak{R}(H)$ has been introduced, and it was shown that the wave operators are complete on the set $\sigma(H_0) \setminus \mathfrak{R}(H)$. In [JL1] one can also find various estimates on the location of $\mathfrak{R}(H)$ (for example, if $\|V\| < 1$, then $\mathfrak{R}(H) = \emptyset$).

The resonant spectrum is characterized by the property that the projection $\mathbf{1}_R$ is H-smooth on any compact subset of $\sigma(H) \setminus \mathfrak{R}(H)$. This is a restrictive condition and in many interesting situations $\sigma(H_0) \subset \mathfrak{R}(H)$. It is also known that in general the wave operators may not be complete on $\sigma(H_0) \cap \mathfrak{R}(H)$ [JL1, MV1]. The current paper was partly motivated by the question under what conditions one may expect the completeness of the wave operators on $\sigma(H_0) \cap \mathfrak{R}(H)$.

For the model (2.3), it was shown in [JL1] that $\{\delta_n\}_{n\in\Gamma}$ is a cyclic family for H_0 (and hence for H) and that the conditions (b), (c) and (d) of Theorem 1.1 hold. Hence, Theorem 1.1 and Remark 3 after it yield:

Theorem 2.1 *Let $\Theta \subset \sigma(H_0)$ be a Borel set of positive Lebesgue measure. Consider the assumptions:*
(a) *The operator $H\mathbf{1}_\Theta(H)$ has purely absolutely continuous spectrum.*
(b) *For Lebesgue a.e. $E \in \Theta$ and all $n \in \Gamma$, $\mathrm{Im}\,(\delta_n|(H-E-\mathrm{i}0)^{-1}\delta_n) > 0$.*
Consider the statements:
(1) *For Lebesgue a.e. $E \in \Theta$ and all $n \in \Gamma$*

$$\sum_{m\in\Gamma_1} |\mathrm{Im}\,(\delta_n|(H-E-\mathrm{i}0)^{-1}\delta_m|^2 < \infty.$$

(2) *For a dense set of $\phi \in \mathrm{Ran}\,\mathbf{1}_\Theta(H)$,*

$$\int_{\mathbb{R}} \|\mathbf{1}_1 \mathrm{e}^{-\mathrm{i}tH}\phi\|^2 \mathrm{d}t < \infty.$$

(3) *The wave operators $W^\pm$ are complete on Θ.*

Then (2) $\Rightarrow$ (3). *If* (b) *holds,* (2) $\Rightarrow$ (1). *If* (a) *holds, then* (1) $\Rightarrow$ (2). *If* (a) *and* (b) *hold, then* (3) $\Rightarrow$ (2). *Hence, if* (a) *and* (b) *hold, then* (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

We now assume that V is a random subspace potential. An explicit computation (see [JL2]) shows that for all $n \in \Gamma$ and $E \in \mathrm{int}\,\sigma(H_0)$,

$$\mathrm{Im}\,(\delta_n|(H_0-E-\mathrm{i}0)^{-1}\delta_n) > 0.$$

Hence, Proposition 1.2 and Theorem 1.1 yield:

Theorem 2.2 *Assume that the random variables $\{V(n)\}_{n\in\Gamma}$ have densities and let $\Theta \subset \sigma(H_0)$ be a Borel set of positive Lebesgue measure. Then the following statements are equivalent:*
(1) *For $\mathrm{d}P \otimes \mathrm{d}E$-a.e. $(V, E) \in \Omega \times \Theta$ and for all $n \in \Gamma$,*

$$\sum_{m\in\Gamma_1} |\mathrm{Im}\,(\delta_n|(H - E - \mathrm{i}0)^{-1}\delta_m)|^2 < \infty.$$

(2) *For P-a.e. V there is a dense set of $\phi \in \mathrm{Ran}\,\mathbf{1}_\Theta(H)$ such that*

$$\int_{\mathbb{R}} \|\mathbf{1}_1 \mathrm{e}^{-\mathrm{i}tH}\phi\|^2 \mathrm{d}t < \infty.$$

(3) *The wave operators $W^\pm$ are P-a.s. complete on Θ.*

The following corollary follows easily from Theorem 2.2 and Proposition 3.1 in [JL1].

Corollary 2.3 *Assume that the random variables $\{V(n)\}_{n\in\Gamma}$ have densities and let $\Theta \subset \sigma(H_0)$ be a Borel set of positive Lebesgue measure. Assume that for $\mathrm{d}P\otimes\mathrm{d}E$-a.e. $(V, E) \in \Omega \times \Theta$ and for all $n \in \Gamma$,*

$$\liminf_{\epsilon\downarrow 0} \sum_{m\in\Gamma} |(\delta_n|(H - E - \mathrm{i}\epsilon)^{-1}\delta_m)|^2 < \infty. \tag{2.4}$$

Then the wave operators $W^\pm$ are P-a.s. complete on Θ.

The condition (2.4) should be compared with the well-known Simon-Wolff localization criterion [SW]. For comparison, we also recall the following result proven in [JM1, JM2]: if $\{V(n)\}_{n\in\Gamma}$ have densities, $\Theta \subset \mathbb{R} \setminus \sigma(H_0)$, and for $\mathrm{d}P \otimes \mathrm{d}E$-a.e. $(V, E) \in \Omega \times \Theta$ and all $n \in \Gamma$,

$$\liminf_{\epsilon\downarrow 0} \sum_{m\in\Gamma} |(\delta_n|(H - E - \mathrm{i}\epsilon)^{-1}\delta_m)|^2 < \infty, \tag{2.5}$$

then the spectrum of H in Θ is P-a.s. pure point. If $\mathrm{supp}\mu_n = \mathbb{R}$ for at least one n, then the condition (2.5) is also necessary for H to have P-a.s. pure point spectrum in Θ.

We now discuss an application of Theorem 2.2. For simplicity, we assume that all the measures μ_n are the same and equal to μ, and that $\mathrm{d}\mu = p(x)\mathrm{d}x$.

Theorem 2.4 *Assume that $d + 1 = 2$. Let U_{per} be a periodic potential supported on Γ and $H = H_0 + U_{\mathrm{per}} + \lambda V$, $V \in \Omega$, where λ is a real constant. Assume that $\langle x\rangle^\alpha p(x) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ for some $\alpha > 2/3$. Then there is a constant $\Lambda > 0$ such that for $|\lambda| > \Lambda$, the wave operators $W^\pm$ are P-a.s. complete.*

Theorem 2.5 *Assume that $d+1=2$ and let $H=H_0+\lambda V$, $V\in\Omega$. Assume that $\langle x\rangle^\alpha p(x)$ is in $L^1(\mathbb{R})$ for some $\alpha>2/3$ and in $L^\infty(\mathbb{R})$ for some $\alpha>5/3$. Then there is a constant $\Lambda>0$ such that for $|\lambda|<\Lambda$ the wave operators $W^\pm$ are P-a.s. complete.*

In [JM1] it was shown that under the conditions of these theorems the spectrum of H outside $\sigma(H_0)$ is P-a.s. dense pure point with exponentially decaying eigenfunction (for related results see [AM, G, JM2]).

Assume now that $U_0=0$ and let $\mathrm{supp}\mu$ be the support of the probability measure μ. Then, for P-a.e. V,

$$\sigma(H)=\sigma(H_0)\cup\{[2d,2d]+x+x^{-1} : x\in\mathrm{supp}\mu,\ |x|\geq 1\},$$

see [JL1]. For example, if μ is Gaussian, then $\sigma(H)=\mathbb{R}$ P-a.s. (In this case, the resonant spectrum of H is also equal to $\mathbb{R}$ P-a.s.) Thus, Theorems 2.4 and 2.5 provide (to the best of our knowledge) the first non-trivial examples of Anderson type Hamiltonians $H=H_0+V$ which have P-a.s. dense point spectrum outside $\sigma(H_0)$, purely a.c. spectrum in $\sigma(H_0)$, and the scattering between H and H_0 is complete.

Theorems 2.4 and 2.5 are closely related to the results of [JL3]. There it was shown that under the same conditions, for all $\phi\in\mathcal{H}$ and $R\geq 0$,

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\mathbb{E}\left(\|\mathbf{1}_R\mathrm{e}^{-\mathrm{i}tH}\mathbf{1}_{\sigma(H_0)}\phi\|^2\right)\mathrm{d}t=0. \tag{2.6}$$

($\mathbb{E}$ stands for the expectation). A consequence of (2.6) is that the operators H have P-a.s. no surface spectrum in $\sigma(H_0)$, see [KP, JL3, JMP].

The proof of (2.6) is based on the following estimate proven in [JL3]: under the conditions of either Theorem 2.4 or Theorem 2.5, for $n\in\Gamma$, $R\geq 0$ and $2/3<s<1$,

$$\sup_{E\in\mathbb{R},\epsilon\neq 0}\mathbb{E}\left(\sum_{m\in\Gamma_R}|(\delta_m|(H-E-\mathrm{i}\epsilon)^{-1}\delta_n)|^s\right)<\infty.$$

This estimate and Fatou's lemma yield that for $\mathrm{d}P\otimes\mathrm{d}E$ a.e. $(V,E)\in\Omega\times\mathbb{R}$,

$$\left(\sum_{m\in\Gamma_R}|(\delta_m|(H-E-\mathrm{i}0)^{-1}\delta_n)|^2\right)^{s/2}$$

$$\leq\sum_{m\in\Gamma_R}|(\delta_m|(H-E-\mathrm{i}0)^{-1}\delta_n)|^s<\infty.$$

By Theorem 2.2, the last estimate implies that the wave operators $W^\pm$ are P-a.s. complete, and Theorems 2.4 and 2.5 follow.

3 Proofs

3.1 Preliminaries

Proof of Proposition 1.2. By the remark before Theorem 1.1, we may assume that $\mathcal{H}$ is spanned by H and $\{\delta_n\}_{n\in\Gamma}$ for all V. One also easily verifies that the subspaces $\mathcal{H}_n$ and $\mathcal{H}_m$ are not orthogonal for all V and $n, m \in \mathcal{G}$. The condition (1.2) ensures that Θ is contained in an essential support of the absolutely continuous spectrum of H_0. The existence of wave operators implies that the operators $H \restriction \operatorname{Ran} W^{\pm}$ and $H_0 \restriction \operatorname{Ran} \mathbf{1}_{\Theta}(H_0)$ are unitarily equivalent. Hence, with probability one, Θ is contained in an essential support of the absolutely continuous spectrum of H and the proposition follows from Corollaries 1.1.1 and 1.1.3 in [JL2]. □

The next lemma holds for an arbitrary subspace potential V.

Lemma 3.1 *For any $m, n \in \Gamma$, the spectral measure ν_{δ_m,δ_n} for H and δ_m,δ_n, is real-valued.*

Proof. Let $\mathcal{C}(\mathbb{R})$ be the set of all real-valued, bounded, continuous functions on $\mathbb{R}$. The measure ν_{δ_m,δ_n} is real-valued iff for all $f \in \mathcal{C}(\mathbb{R})$, $(\delta_m|f(H)\delta_n)$ is a real number. Note first that for any positive integer k, $(\delta_m|(H_0+V)^k\delta_n)$ is a real number. It follows that for any polynomial p with real coefficents, $(\delta_m|p(H)\delta_n)$ is a real number. Assume now that the potential V is bounded. Then $\sigma(H)$ is a compact set and, by an approximation argument, for all $f \in \mathcal{C}(\mathbb{R})$, $(\delta_m|f(H)\delta_n)$ is a real number.

If V is unbounded, let $V_\ell(j) = V(j)$ if $|j| \le \ell$, otherwise $V_\ell(j) = 0$. Set $H_\ell = H_0 + V_\ell$. Then $H_\ell \to H$ in the strong resolvent sense and this implies that for any $f \in \mathcal{C}(\mathbb{R})$, $f(H_\ell) \to f(H)$ strongly. Hence,

$$(\delta_m|f(H)\delta_n) = \lim_{\ell\to\infty}(\delta_m|f(H_\ell)\delta_n)$$

is a real number. □

One consequence of this lemma is the identity

$$\operatorname{Im}(\delta_m|(H-E-\mathrm{i}0)^{-1}\delta_n) = \operatorname{Im}(\delta_n|(H-E-\mathrm{i}0)^{-1}\delta_m),$$

which we will often use in the sequel.

We also recall the following well-known result (see, e.g., [S]).

Lemma 3.2 *Let μ be a finite regular complex measure and $\mathrm{d}\mu = f\mathrm{d}E + \mathrm{d}\mu_{\mathrm{sing}}$ its Lebesgue decomposition. Then for Lebesgue a.e. $E \in \mathbb{R}$,*

$$f(E) = \lim_{\epsilon\downarrow 0}\pi^{-1}\int_{\mathbb{R}}\frac{\epsilon\mathrm{d}\mu(x)}{(x-E)^2+\epsilon^2}.$$

Combining the last two lemmas we derive

Lemma 3.3 *The absolutely continouous part of the spectral measure* ν_{δ_m,δ_n} *is equal to* $\pi^{-1}\mathrm{Im}\,(\delta_m|(H-E-\mathrm{i}0)^{-1}\delta_n)\mathrm{d}E$.

3.2 Proof of Theorem 1.1

Theorem 1.1 follows from Propositions 3.4 and 3.5 below.

In Proposition 3.4 we use the same notation as in Theorem 1.1. In particular, (b)-(d) refer to the assumptions of Theorem 1.1. We assume that $\mathcal{H}$ is spanned by H and $\{\delta_n\}_{n\in\Gamma}$.

Proposition 3.4 *Let* $\Theta\subset\mathbb{R}$ *be an open set. Consider the following statements:*
(1) *For a dense set of* $\phi\in\mathrm{Ran}\,\mathbf{1}_\Theta(H)$,

$$\int_{\mathbb{R}}\|\mathbf{1}_1\mathrm{e}^{-\mathrm{i}tH}\phi\|^2\mathrm{d}t<\infty. \tag{3.7}$$

(2) *The wave operators*

$$\tilde{W}^\pm=\mathrm{s}-\lim_{t\to\pm\infty}\mathrm{e}^{\mathrm{i}tH_0}\mathrm{e}^{-\mathrm{i}tH}\mathbf{1}_\Theta(H)$$

exist.
If (b) *holds, then* (1) $\Rightarrow$ (2). *If* (b), (c) *and* (d) *hold, then* (2) $\Rightarrow$ (1).

Proof. The proof is a based on the arguments used in [JL1, JL3, JM3] in the analysis of some specific examples of the abstract model (1.1). These arguments have their roots in Kato's theory of smooth perturbations.

Assume first that (b) holds. To prove that (1) $\Rightarrow$ (2) it suffices to show that for any $\phi\in\mathrm{Ran}\,\mathbf{1}_\Theta(H)$ for which (3.7) holds, the limits

$$\lim_{t\to\pm\infty}\mathrm{e}^{\mathrm{i}tH_0}\mathrm{e}^{-\mathrm{i}tH}\phi \tag{3.8}$$

exist. In what follows we fix ϕ.

Note first that

$$\lim_{|t|\to\infty}\mathbf{1}_0\mathrm{e}^{-\mathrm{i}tH}\phi=0. \tag{3.9}$$

To prove this relation, let

$$w(t):=\mathrm{e}^{\mathrm{i}tH}\mathbf{1}_0\mathrm{e}^{-\mathrm{i}tH}\phi.$$

Then, by (3.7), $\int\|w(t)\|^2\mathrm{d}t<\infty$. Since $\|w'(t)\|\leq 2\|H_0\|$, it follows from Exercise 62 in [RS] that $\lim_{|t|\to\infty}w(t)=0$.

We adopt the shorthand $\mathbf{1}_{\overline{0}}:=\mathbf{1}-\mathbf{1}_0$. Let T be a linear operator defined by

$$T\delta_n=-\sum_{m\notin\Gamma,\rho(m,n)=1}\delta_m,\qquad \text{if } n\in\Gamma,$$

$$T\delta_n = \sum_{m\in\Gamma,\rho(m,n)=1} \delta_m, \qquad \text{if } \ n \in \Gamma_1 \setminus \Gamma,$$

and $T\delta_n = 0$ if $n \notin \Gamma_1$. A simple calculation yields

$$H_0 \mathbf{1}_{\overline{0}} - \mathbf{1}_{\overline{0}} H = [H_0, \mathbf{1}_{\overline{0}}] = T. \tag{3.10}$$

Obviously, $\|T\| \leq 2\|H_0\|$ and $T\mathbf{1}_1 = \mathbf{1}_1 T$.

Let $I = [a, b] \subset \Theta$ and let γ be a simple closed curve in the complex plane that separates $[a, b]$ and $\mathbb{R} \setminus \Theta$ and encloses I. Then, for any $\psi \in \mathcal{H}$,

$$\begin{aligned}&\mathbf{1}_{\mathbb{R}\setminus\Theta}(H_0)\mathrm{e}^{\mathrm{i}tH_0}\mathbf{1}_{\overline{0}}\mathrm{e}^{-\mathrm{i}tH}\mathbf{1}_I(H)\psi = \\ &-(2\pi\mathrm{i})^{-1}\oint_\gamma \mathbf{1}_{\mathbb{R}\setminus\Theta}(H_0)\mathrm{e}^{\mathrm{i}tH_0}(H_0 - z)^{-1}(H_0\mathbf{1}_{\overline{0}} - \mathbf{1}_{\overline{0}}H)(H - z)^{-1}\mathrm{e}^{-\mathrm{i}tH}\mathbf{1}_I(H)\psi \,\mathrm{d}z,\end{aligned}$$

see the proof of Theorem XIII.31 in [RS]. Hence, for some constant C,

$$\|\mathbf{1}_{\mathbb{R}\setminus\Theta}(H_0)\mathrm{e}^{\mathrm{i}tH_0}\mathbf{1}_{\overline{0}}\mathrm{e}^{-\mathrm{i}tH}\mathbf{1}_I(H)\psi\| \leq C\oint_\gamma \|\mathbf{1}_1(H - z)^{-1}\mathrm{e}^{-\mathrm{i}tH}\mathbf{1}_I(H)\psi\| \,\mathrm{d}z.$$

Set

$$\ell(z, t) := \mathbf{1}_1(H - z)^{-1}\mathrm{e}^{-\mathrm{i}tH}\mathbf{1}_I(H)\psi.$$

The vector-valued function $\ell(z, t)$ is uniformly bounded on $\gamma \times \mathbb{R}$ and has a uniformly bounded derivative in t. Moreover, for all $z \in \gamma$, $\ell(z, t)$ is square-integrable in t. It follows that $\lim_{|t|\to\infty} \ell(z, t) = 0$ and so

$$\mathrm{s} - \lim_{t\to\pm\infty} \mathbf{1}_{\mathbb{R}\setminus\Theta}(H_0)\mathrm{e}^{\mathrm{i}tH_0}\mathbf{1}_{\overline{0}}\mathrm{e}^{-\mathrm{i}tH}\mathbf{1}_I(H) = 0.$$

Since Θ is a countable union of closed intervals, we conclude that

$$\mathrm{s} - \lim_{t\to\pm\infty} \mathbf{1}_{\mathbb{R}\setminus\Theta}(H_0)\mathrm{e}^{\mathrm{i}tH_0}\mathbf{1}_{\overline{0}}\mathrm{e}^{-\mathrm{i}tH}\mathbf{1}_\Theta(H) = 0. \tag{3.11}$$

We are now ready to prove that the limits (3.8) exist. Let

$$\zeta(t) := \mathbf{1}_\Theta(H_0)\mathrm{e}^{\mathrm{i}tH_0}\mathbf{1}_{\overline{0}}\mathrm{e}^{-\mathrm{i}tH}\phi.$$

By (3.9) and (3.11), it suffices to show that $\lim_{t\to\pm\infty}\zeta(t)$ exist. Let $\psi \in \mathcal{H}$ be arbitrary. It follows from (3.10) that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\psi|\zeta(t)) = \mathrm{i}(\mathrm{e}^{-\mathrm{i}tH_0}\mathbf{1}_\Theta(H_0)\psi|T\mathrm{e}^{-\mathrm{i}tH}\phi),$$

and so, for $t > s$,

$$|(\psi|(\zeta(t) - \zeta(s))| \leq \|T\| \left(\int_s^t \|\mathbf{1}_1 \mathrm{e}^{-\mathrm{i}\tau H_0} \mathbf{1}_\Theta(H_0)\psi\|^2 \mathrm{d}\tau \right)^{1/2} \left(\int_s^t \|\mathbf{1}_1 \mathrm{e}^{-\mathrm{i}\tau H} \phi\|^2 \mathrm{d}\tau \right)^{1/2}.$$

Since $\mathbf{1}_1$ is H_0-smooth on Θ, there is a constant C such that for all $\psi \in \mathcal{H}$,

$$\int_{\mathbb{R}} \|\mathbf{1}_1 \mathrm{e}^{-\mathrm{i}\tau H_0} \mathbf{1}_\Theta(H_0)\psi\|^2 \mathrm{d}t \leq C\|\psi\|^2.$$

Hence, for some constant C,

$$\|\zeta(t) - \zeta(s)\| \leq C \left(\int_s^t \|\mathbf{1}_1 \mathrm{e}^{-\mathrm{i}\tau H} \phi\|^2 \mathrm{d}\tau \right)^{1/2}.$$

By (3.7), the sequence $\zeta(t)$ is Cauchy as $t \to \pm\infty$, and the limits $\lim_{t\to\pm\infty} \zeta(t)$ exist. This finishes the proof that if (b) holds, then (1) $\Rightarrow$ (2).

Assume now that in addition, (c) and (d) also hold. If (2) holds, then $W^\pm : \mathrm{Ran}\, \mathbf{1}_\Theta(H_0) \mapsto \mathrm{Ran}\, \mathbf{1}_\Theta(H)$ are norm-preserving bijections. Hence, it suffices to show that for all $\psi \in \mathcal{D}$,

$$\int_{\mathbb{R}} \|\mathbf{1}_1 \mathrm{e}^{-\mathrm{i}tH} W^+\psi\|^2 \mathrm{d}t < \infty. \tag{3.12}$$

By (3.8),

$$W^+ = \lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH} \mathbf{1}_{\overline{0}} \mathrm{e}^{-\mathrm{i}tH_0},$$

and so

$$\begin{aligned} W^+\psi - \psi &= \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}\tau} \mathrm{e}^{\mathrm{i}\tau H} \mathbf{1}_{\overline{0}} \mathrm{e}^{-\mathrm{i}\tau H_0}\psi \\ &= \mathrm{i} \int_0^\infty \mathrm{e}^{\mathrm{i}\tau H} T \mathrm{e}^{-\mathrm{i}\tau H_0}\psi \mathrm{d}\tau. \end{aligned}$$

Hence,

$$\begin{aligned} \|\mathbf{1}_1 \mathrm{e}^{-\mathrm{i}tH} W^+\psi\|^2 &= \|\mathbf{1}_1 W^+ \mathrm{e}^{-\mathrm{i}tH_0}\psi\|^2 \\ &\leq L(t) + 2\|\mathbf{1}_1 \mathrm{e}^{-\mathrm{i}tH_0}\psi\|^2, \end{aligned}$$

where

$$L(t) \leq C \left(\int_0^\infty \|\mathbf{1}_1 \mathrm{e}^{-\mathrm{i}(t+\tau)H_0}\psi\| \mathrm{d}\tau \right)^2.$$

By the definition of $\mathcal{D}$, $L(t) = O(|t|^{-2})$, and (3.12) follows. □

We have used the assumption that Θ is an open set only in the proof of the estimate (3.11). If $\mathbf{1}_1$ is H_0-smooth this estimate is not needed and the proposition holds for any Borel set Θ.

The next proposition is of an independent interest and we prove it in a more general setting.

Proposition 3.5 *Let A be a self-adjoint opeartor on a Hilbert space $\mathfrak{H}$ and $\{\delta_n\}_{n\in\mathcal{F}}$ a countable orthonormal set in $\mathfrak{H}$. Assume that $\{\delta_n\}_{n\in\mathcal{F}}$ is a cyclic family for A. Let $\Theta \subset \mathbb{R}$ be a Borel set of positive Lebesgue measure. Consider the following assumptions:*
(a) *The operator $A\mathbf{1}_\Theta(A)$ has purely absolutely continuous spectrum.*
(b) *The spectral measure for A and δ_n, δ_m is real-valued for all $n, m \in \mathcal{F}$.*
(c) *For Lebesgue a.e. $E \in \Theta$ and all $n \in \mathcal{F}$,*

$$\mathrm{Im}\,(\delta_n|(A - E - \mathrm{i}0)^{-1}\delta_n) > 0.$$

Consider the following statements:
(1) *For Lebesgue a.e. $E \in \Theta$ and for all $n \in \mathcal{F}$,*

$$\sum_{m\in\mathcal{F}} |\mathrm{Im}\,(\delta_n|(A - E - \mathrm{i}0)^{-1}\delta_m)|^2 < \infty.$$

(2) *For a dense set of $\phi \in \mathbf{1}_\Theta(A)\mathfrak{H}$,*

$$\sum_{m\in\mathcal{F}} \int_{\mathbb{R}} |(\delta_m|\mathrm{e}^{-\mathrm{i}tA}\phi)|^2 \mathrm{d}t < \infty. \tag{3.13}$$

If (a) *and* (b) *hold, then* (1) $\Rightarrow$ (2)*. If* (b) *and* (c) *hold, then* (2) $\Rightarrow$ (1)*.*

Proof. We first assume that (a) and (b) hold and show that (1) $\Rightarrow$ (2). For $n \in \mathcal{F}$ let $\mathfrak{H}_n$ be the cyclic space spanned by A and δ_n. It suffices to show that for all n there is a dense set of $\phi \in \mathbf{1}_\Theta(A)\mathfrak{H}_n$ for which (3.13) holds. In what follows we fix n.

Let

$$\Theta_j = \left\{ E \in \Theta : \sum_{m\in\mathcal{F}} |\mathrm{Im}\,(\delta_n|(A - E - \mathrm{i}0)^{-1}\delta_m)|^2 < j \right\}.$$

The set $\{\chi(A)\mathbf{1}_{\Theta_j}(A)\delta_n : \chi \in L^\infty(\mathbb{R}), j > 0\}$ is dense in $\mathbf{1}_\Theta(A)\mathfrak{H}_n$ and so it suffices to show that (3.13) holds for ϕ's in this set. In what follows we fix $\phi = \chi(H)\mathbf{1}_{\Theta_j}(H)\delta_n$.

The spectral theorem and the assumption (b) yield,

$$(\delta_m|\mathrm{e}^{-\mathrm{i}tA}\phi) = \int_{\Theta_j} \mathrm{e}^{-\mathrm{i}tE}\chi(E)\pi^{-1}\mathrm{Im}\,(\delta_m|(A - E - \mathrm{i}0)^{-1}\delta_n)\mathrm{d}E.$$

Hence

$$\int_{\mathbb{R}} |(\delta_m|\mathrm{e}^{-\mathrm{i}tH}\phi)|^2 \mathrm{d}t = 2\pi^{-1} \int_{\Theta_j} |\chi(E)|^2 |\mathrm{Im}\,(\delta_n(A - E - \mathrm{i}0)^{-1}\delta_m)|^2 \mathrm{d}E,$$

and

$$\sum_{m\in\mathcal{F}} \int_{\mathbb{R}} |(\delta_m|e^{-\mathrm{i}tA}\phi)|^2 \mathrm{d}t = \sum_{m\in\mathcal{F}} 2\pi^{-1} \int_{\Theta_j} |\chi(E)|^2 |\mathrm{Im}\,(\delta_n|(H - E - \mathrm{i}0)^{-1}\delta_m)|^2 \mathrm{d}E \\ \leq 4(d+1)j\pi^{-1}\|\chi\|_\infty.$$

Assume now that (b) and (c) hold. Assume that (2) holds but (1) does not (note that (2) implies (a)). Then there is $n \in \mathcal{F}$ and a Borel set $\tilde{\Theta} \subset \Theta$ of positive Lebesgue measure such that for $E \in \tilde{\Theta}$,

$$\sum_{m\in\mathcal{F}} |\mathrm{Im}\,(\delta_n|(A - E - \mathrm{i}0)^{-1}\delta_m)|^2 = \infty.$$

By assumption (c), $\mathbf{1}_{\tilde{\Theta}}(H)\mathfrak{H}_n$ is a non-trivial subspace of $\mathfrak{H}$. Let ν_n be the spectral measure for A and δ_n. By the spectral theorem, for every $\phi \in \mathbf{1}_{\tilde{\Theta}}(A)\mathfrak{H}_n$ there is a Borel function $\chi_\phi \in L^2(\mathbb{R}, \mathrm{d}\nu_{\delta_n})$ such that

$$(\phi|\phi) = \int_{\mathbb{R}} |\chi_\phi(E)|^2 \mathrm{d}\nu_{\delta_n} = \pi^{-1} \int_{\tilde{\Theta}} |\chi_\phi(E)|^2 \mathrm{Im}\,(\delta_n|(A - E - \mathrm{i}0)^{-1}\delta_n)\mathrm{d}E. \tag{3.14}$$

Obviously, if $\phi \neq 0$, then $\chi_\phi(E) \neq 0$ for a set of E's in $\tilde{\Theta}$ of positive Lebesgue measure. Moreover,

$$(\delta_m|\mathrm{e}^{-\mathrm{i}tA}\phi) = \pi^{-1} \int_{\tilde{\Theta}} \mathrm{e}^{-\mathrm{i}tE}\chi(E)\mathrm{Im}\,(\delta_m|(A - E - \mathrm{i}0)^{-1}\delta_n)\mathrm{d}E,$$

and so for all non-zero $\phi \in \mathbf{1}_{\tilde{\Theta}}(A)\mathfrak{H}_n$,

$$\sum_{m\in\mathcal{F}} \int_{\mathbb{R}} |(\delta_m|\mathrm{e}^{-\mathrm{i}tA}\phi)|^2 \mathrm{d}t = 2\pi^{-1} \sum_{m\in\mathcal{F}} \int_{\tilde{\Theta}} |\chi_\phi(E)|^2 |\mathrm{Im}\,(\delta_n|(A - E - \mathrm{i}0)^{-1}\delta_m)|^2 \mathrm{d}E \\ = \infty.$$

This contradicts (2). □

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