Directed Polymers in Random Environment: Path Localization and Strong Disorder

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Abstract

We consider directed polymers in random environment. Under mild assumptions on the environment, we prove here: (i) equivalence of decay rate of the partition function with some natural localization properties of the path, (ii) quantitative estimates of the decay of the partition function in dimensions one or two, or at sufficiently low temperature, (iii) existence of quenched free energy. In particular, we generalize to general environments, some of the results recently obtained by P. Carmona and Y. Hu for a Gaussian environment. We do not discuss here superdiffusivity or critical exponents.

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1 Introduction and Main Results

1.1 Directed Polymers in Random Environment

The models we consider in this paper are defined in terms of a random walk and of a random environment, that we introduce now:

- *The random walk:* $(\{S_n\}_{n\geq 0}, \{P^x\}_{x\in\mathbb{Z}^d})$ is a simple random walk on d-dimensional integer lattice $\mathbb{Z}^d$. More precisely, let Ω be the path space $\Omega = \{\omega = (\omega_n)_{n\geq 0}; \omega_n \in \mathbb{Z}^d, n \geq 0\}$, let $\mathcal{F}$ be the cylindric σ-field on Ω, and, for all $n \geq 0$, $S_n : \omega \mapsto \omega_n$ the projection map. For all $x \in \mathbb{Z}^d$ we consider the unique probability measure P^x on $(\Omega, \mathcal{F})$ such that $S_1 - S_0, \dots, S_n - S_{n-1}$ are independent and

$$P^x\{S_0 = x\} = 1, \quad P^x\{S_n - S_{n-1} = \pm\delta_j\} = (2d)^{-1}, \quad j = 1, 2, \dots, d,$$

  where $\delta_j = (\delta_{kj})_{k=1}^d$ is the j-th vector of the canonical basis of $\mathbb{Z}^d$. For $x = 0$ we will write shortly $P = P^0$.

- *The random environment:* $\xi = \{\xi(x,n) : x \in \mathbb{Z}^d,\ n \geq 1\}$ is a real, non-constant, i.i.d. sequence of random variables defined on a probability space $(\Xi, \mathcal{E}, Q)$ such that

$$Q[\exp(\beta\xi(x,n))] < \infty \quad \text{for all } \beta \in \mathbb{R}. \tag{1.1}$$

  (All through, $Q[Y]$ denotes the Q-expectation of a r.v. Y.) Let $\lambda(\beta)$ be the logarithmic moment generating function of $\xi(x,n)$,

$$\lambda(\beta) = \ln Q[\exp(\beta\xi(x,n))], \quad \beta \in \mathbb{R}. \tag{1.2}$$

For all $n > 0$ define the probability measure μ_n on the path space $(\Omega, \mathcal{F})$

$$\mu_n(d\omega) = P[e_n : d\omega]/P[e_n], \tag{1.3}$$

where

$$e_n = e_n(\xi, S) = \exp\left(\sum_{1\leq j\leq n} (\beta\xi(S_j, j) - \lambda(\beta))\right) \tag{1.4}$$

with a parameter $\beta \in \mathbb{R}$. Here, the graph $\{(S_j, j)\}_{j\geq 0}$ may be interpreted as a polymer chain living in the $(d+1)$-dimensional space, constrained to stretch in the $(d+1)$-th direction, and governed by the Hamiltonian

$$-\sum_{j\geq 1} (\beta\xi(S_j, j) - \lambda(\beta)) \, ,$$

i.e. the so-called directed polymer in the environmnent ξ. If $\beta > 0$, then the parameter $\beta > 0$ plays the role of temperature inverse in this interpretation. Since this Hamiltonian is parametrized by ξ, the polymer measure μ_n is random. Here are two standard choices for ξ.

Example 1.1 *Gaussian environment* (Carmona and Hu, 2001) This is the case in which the distribution of $\xi(x,n)$ is given by standard normal distribution, so that

$$\lambda(\beta) = \tfrac{1}{2}\beta^2.$$

Example 1.2 *Bernoulli environment* (Bolthausen 1989, Imbrie and Spencer 1988, Song and Zhou 1996): This is the case in which $\xi(x,n)$ takes two different values a and b with probability $p > 0$ and $1-p > 0$, respectively, so that

$$\lambda(\beta) = \ln(pe^{\beta a} + (1-p)e^{\beta b}).$$

As discussed by Johansson (2000, Remark 1.8), directed percolation can be thought as the case of $0 = a > b$ and zero-temperature ($\beta \to \infty$), which however is outside the scope of this paper.

We are interested in the large time behavior of the path $\{S_k\}_{k=1}^n$ under the (sequence of) polymer measure μ_n. As is the case in many other models in statistical mechanics, one of the fundamental question to be asked is the asymptotic behavior of the partition function

$$Z_n = Z_n(\xi) = P[e_n] \ . \tag{1.5}$$

Since Z_n is a positive martingale on $(\Xi, \mathcal{E}, Q)$, the following limit exists Q-a.s.:

$$Z_\infty \stackrel{\text{def.}}{=} \lim_{n \nearrow \infty} Z_n \ . \tag{1.6}$$

The event $\{Z_\infty = 0\}$ is measurable with respect to the tail σ-field

$$\bigcap_{n \geq 1} \sigma[\xi(x,j) \ ; \ j \geq n, \ x \in \mathbb{Z}^d]$$

and therefore by Kolmogorov's 0-1 law

$$Q\{Z_\infty = 0\} = 0 \text{ or } 1. \tag{1.7}$$

We refer to the former case as **weak disorder** and the latter as **strong disorder**. It is known (e.g., Song and Zhou 1996) that for $d \geq 3$,

$$Q\{Z_\infty = 0\} = 0 \ \text{ if } \lambda(2\beta) - 2\lambda(\beta) < -\ln(1-q) \tag{1.8}$$

where $q = P\{S_n \neq 0 \text{ for all } n \geq 1\}$; Similar results for weak disorder were obtained by Bolthausen (1989) and Sinai (1995). Note that the condition in (1.8) does hold for small β. In dimension $d \geq 3$, this condition amounts to L^2-convergence in (1.6), and it allows using the so-called second moment method: For small β and $d \geq 3$, Imbrie and Spencer (1988) first, then Bolthausen (1989) with martimgales techniques, proved that the polymer is diffusive; more recently Albeverio and Zhou (1996) showed that the invariance principle holds for almost every environment. On the other hand, for strong disorder, it can be seen that

$$Q\{Z_\infty = 0\} = 1 \ \text{ if } \beta\lambda'(\beta) - \lambda(\beta) \geq \ln(2d). \tag{1.9}$$

This was shown by Kahane and Peyrière (1976) for a different model called Mandelbrot martingale (or, equivalently, multiplicative chaos), where graphs $\{(S_j, j)\}_{j\geq 0}$ are replaced by infinite paths, without loops and starting from the root, on the d-ary tree. Although the directed polymer we are considering here is more intricate due to correlations, the same argument applies as far as to deduce (1.9). Recently, P. Carmona and Y. Hu (2001) proved for Gaussian environment that for all β,

$$Q\{Z_\infty = 0\} = 1, \quad d = 1, 2, \tag{1.10}$$

which, together with (1.9), displays a non-trivial dependence on the dimension.

In the present paper, we consider general environment and present some results for strong disorder case: $Q\{Z_\infty = 0\} = 1$, including the extension of (1.10) to non-Gaussian case. Using martingale analysis, we also obtain natural localization properties which characterize the strong disorder regime. More precisely, decay of the partition function is equivalent to concentration of the path on favourite sites. Though, we will not discuss here superdiffusivity or critical exponents, we refer to Johansson (2000), Licea *et al* (1996), Petermann (2000) and Piza (1997) for rigourous results in this direction.

1.2 Results

On the product space $(\Omega^2, \mathcal{F}^{\otimes 2})$, we consider the probability measure $\mu_n^{\otimes 2} = \mu_n^{\otimes 2}(d\omega, d\tilde{\omega})$, that we will view as the distribution of the couple $(S, \widetilde{S})$ with $\widetilde{S} = \{\widetilde{S}_k\}_{k\geq 0}$ an independent copy of $S = \{S_k\}_{k\geq 0}$ with law μ_n. An important role in the analysis is played by the random sequence

$$I_n = \mu_{n-1}^{\otimes 2}(S_n = \widetilde{S}_n)\ , \tag{1.11}$$

which conveys some information on the localization of paths under μ_n, see (1.18) below. Our basic result relates the partition function Z_n and the expected intersection time $\sum_{j\leq n} I_j$ of two independent polymers in the same environment.

Theorem 1.1 *Let $\beta \neq 0$. Then,*

$$\{Z_\infty = 0\} = \left\{\sum_{n\geq 1} I_n = \infty\right\}, \quad Q\text{-a.s.}. \tag{1.12}$$

Moreover, if $Q\{Z_\infty = 0\} = 1$, then there exist $c_1, c_2 \in (0, \infty)$ such that

$$-c_1 \ln Z_n \leq \sum_{1\leq j\leq n} I_j \leq -c_2 \ln Z_n \qquad \text{for large enough } n\text{'s}, \quad Q\text{-a.s.}. \tag{1.13}$$

As a consequence we see that fast decay property of Z_n is reflected in some specific localization property of the path $\{S_n\}_{n\geq 1}$ under the random measure (1.3). Our main interest in the following corollary is the case of $a_n = cn$, $n \geq 1$ for a positive constant $c > 0$.

Corollary 1.2 *For $\beta \neq 0$ and a sequence $a_n \nearrow \infty$ of positive numbers, the following properties are equivalent:*

(Z1) *There exists $c > 0$ such that*

$$Q\left\{\varliminf_{n\nearrow\infty} -\frac{1}{a_n}\ln Z_n \geq c\right\} = 1. \tag{1.14}$$

(I1) *There exists $c > 0$ such that*

$$Q\left\{\varliminf_{n\nearrow\infty} \frac{1}{a_n}\sum_{1\leq j\leq n} I_j \geq c\right\} = 1. \tag{1.15}$$

Remark 1.1 The equivalence presented in Theorem 1.1 was shown first by Carmona and Hu (2001, Theorem 1.1 and Proposition 5.1) in the Gaussian case.

Proposition 1.3 *(i) Assume that Property (Z1) in Corollary 1.2 holds with $a_n = n$. Then, there is a constant $c \in (0, \infty)$ such that*

$$\varlimsup_{n\nearrow\infty} I_n \geq c \quad Q\text{-a.s.}. \tag{1.16}$$

(ii) If $Q\{Z_\infty > 0\} = 1$, then

$$\lim_{n\nearrow\infty} I_n = 0, \quad Q\text{-a.s.}$$

Assume now that the (stronger) condition in (1.8) holds, i.e. $\lambda(2\beta) - 2\lambda(\beta) < -\ln(1-q)$. Then, there is a constant $c > 0$ such that

$$I_n = O(n^{-c}) \tag{1.17}$$

in Q-probability.

Remark 1.2 A natural quantity of interest here, relating to localization phenomenon, is the favorite site for the path at time n. First observe that

$$\max_{x\in\mathbb{Z}^d} \mu_{n-1}(S_n = x)^2 \leq I_n \leq \max_{x\in\mathbb{Z}^d} \mu_{n-1}(S_n = x)\ . \tag{1.18}$$

Therefore, all the statement we obtained for I_n can be translated into those for $\max_{x\in\mathbb{Z}^d} \mu_{n-1}(S_n = x)$. In particular, we showed in Proposition 1.3 that the probability of the favorite site vanishes for weak disorder, but does not vanish for strong disorder. In the latter case the polymer localizes (in a set of lattice points depending on the evironment), though in the former it spreads out somewhat alike the usual simple random walk.

Some sufficient conditions for (Z1) and (I1) in Corollary 1.2 are provided by the following result

Theorem 1.4 *Consider the following property:*

(Z2) *There exist constants* $c \in (0, \infty)$ *and* $\theta \in (0, 1)$ *such that*

$$Q[Z_n^\theta] \le c \exp(-a_n) \quad n \ge 1. \tag{1.19}$$

Then,

(a) *(Z2) implies (Z1) in Corollary 1.2 if*

$$\sum_{n \ge 1} \exp(-\delta a_n) < \infty \quad \textit{for some } \delta \in (0,1).$$

(b) *(Z2) holds for* $a_n = cn$ *for some* $c > 0$ *if*

$$\beta \lambda'(\beta) - \lambda(\beta) > \ln(2d). \tag{1.20}$$

(c) *(Z2) holds for* $d = 1, 2$ *with*

$$a_n = \begin{cases} c_1 n^{1/3} & \textit{if } d = 1 \\ c_2 \sqrt{\ln n} & \textit{if } d = 2 \end{cases} \tag{1.21}$$

where $c_1, c_2 \in (0, \infty)$ *are some constants. In particular, for Q-a.s.,*

$$Z_n \begin{cases} = \mathcal{O}\left(\exp(-\frac{c_1}{2} n^{1/3})\right) & \textit{as } n \nearrow \infty \textit{ if } d = 1 \\ \longrightarrow 0 & \textit{as } n \nearrow \infty \textit{ if } d = 2. \end{cases}$$

Remark 1.3 Theorem 1.4(c) generalizes Theorem 1.1 in Carmona and Hu (2001) to non-gaussian environments. Moreover, we give here a quantitative bound for the rate of decay.

Finally, we remark that the "quenched free energy"

$$\lim_{n \nearrow \infty} \frac{1}{n} \ln Z_n$$

exists Q-a.s. under our assumption (1.1).

Proposition 1.5 *The limit*

$$\psi(\beta) = \lim_{n \nearrow \infty} \frac{1}{n} Q[\ln Z_n] \in (-\infty, 0]$$

exists. Moreover, for any $\varepsilon > 0$*, there is an* $n_0 = n_0(\beta, \varepsilon) < \infty$ *such that*

$$Q\left\{ \left| \frac{1}{n} \ln Z_n - \psi(\beta) \right| > \varepsilon \right\} \le \exp\left(-\frac{\varepsilon^{2/3} n^{1/3}}{4} \right), \quad n \ge n_0. \tag{1.22}$$

As a consequence,

$$\lim_{n \nearrow \infty} \frac{1}{n} \ln Z_n = \psi(\beta), \quad Q\textit{-a.s.}.$$

Remark 1.4 The inequality (1.22) is a concentration inequality with the streched exponential decay rate. An inspection of our proof reveals that an exponential concentration can be obtained by a slightly stronger assumption. In fact, if we assume that there is $\delta > 0$ such that

$$Q\left[\exp(\delta|\xi(x,n)|^2)\right] < \infty, \tag{1.23}$$

then, we obtain the following; for any $\varepsilon > 0$, there is an $n_0 = n_0(\beta, \varepsilon) < \infty$ such that

$$Q\left\{\left|\frac{1}{n}\ln Z_n - \psi(\beta)\right| > \varepsilon\right\} \le \exp\left(-\frac{\varepsilon^2 n}{c}\right), \quad n \ge n_0. \tag{1.24}$$

where $c = c(\beta) > 0$. See Remark 5.1 below.

Remark 1.5 We can define a similar model by considering a Markov chain $(\{S_n\}_{n\ge 0}, \{P^x\}_{x\in\Gamma})$ on a certain state space Γ instead of the random walk on $\mathbb{Z}^d$. The proofs presented in this paper apply without change to this generalization.

2 Proof of Theorem 1.1

We begin by proving the following general estimate.

Lemma 2.1 *Let X_i, $1 \le i \le m$ be non-constant, square integrable i.i.d. random variables on a probability space $(\Xi, \mathcal{E}, Q)$ such that*

$$Q[\exp(X_1)] = 1, \quad Q[\exp(4X_1)] < \infty.$$

For a probability distribution $\{\alpha_i\}_{1\le i\le m}$ on $\{1, \dots, m\}$, define a centered random variable $U > -1$ by

$$U = \sum_{1\le i\le m} \alpha_i \exp(X_i) - 1.$$

Then, there exists a constant $c \in (0, \infty)$, independent of $\{\alpha_i\}_{1\le i\le m}$ such that

$$\frac{1}{c}\sum_{1\le i\le m} \alpha_i^2 \le Q\left[\frac{U^2}{2+U}\right], \tag{2.1}$$

$$\frac{1}{c}\sum_{1\le i\le m} \alpha_i^2 \le -Q\left[\ln(1+U)\right] \le c\sum_{1\le i\le m} \alpha_i^2, \tag{2.2}$$

$$Q\left[\ln^2(1+U)\right] \le c\sum_{1\le i\le m} \alpha_i^2. \tag{2.3}$$

Remark 2.1 These estimates are proved in (Carmona and Hu 2001) for Gaussian case by runing Brownian motion and making use of Itô's formula. Here, we present a simple argument which works for general case.

Proof of Lemma 2.1: In this proof, we let $c_1, c_2, \ldots$ stand for constants which are independent of $\{\alpha_i\}_{1\le i\le m}$. We have by a standard argument

$$\begin{aligned} Q[|U|^2] &= c_1 \sum_{1\le i\le m} \alpha_i^2, \\ Q[|U|^p] &\le c_2 \sum_{1\le i\le m} \alpha_i^2, \quad 2\le p\le 4. \end{aligned}$$

Indeed, explicit computations yield such estimates when $p = 2, 4$, and the general case follows by interpolating by Hölder's inequality $Q[|U|^p] \le Q[|U|^2]^{(4-p)/2} Q[|U|^4]^{(p-2)/2}$. Therefore,

$$\begin{aligned} c_1 \sum_{1\le i\le m} \alpha_i^2 &= Q[U^2] \\ &= Q\left[\frac{|U|}{\sqrt{2+U}} |U| \sqrt{2+U}\right] \\ &\le Q\left[\frac{U^2}{2+U}\right]^{1/2} Q\left[2U^2 + |U|^3\right]^{1/2} \\ &\le c_3 Q\left[\frac{U^2}{2+U}\right]^{1/2} \left(\sum_{1\le i\le m} \alpha_i^2\right)^{1/2}, \end{aligned}$$

which proves (2.1).

To prove the other inequalities, it is convenient to introduce a function $\varphi : (-1,\infty) \to [0,\infty)$ as follows;

$$\varphi(u) = u - \ln(1+u),$$

so that

$$-Q\left[\ln(1+U)\right] = Q\left[\varphi(U)\right].$$

Since

$$\frac{1}{4}\frac{u^2}{2+u} \le \varphi(u), \quad u > -1,$$

the left-hand-side inequality of (2.2) follows from (2.1). The right-hand-side inequality can be seen as follows. We have for any $\varepsilon \in (0,1)$,

$$\begin{aligned} Q\left[\varphi(U)\right] &= Q[\varphi(U) : 1+U \ge \varepsilon] + Q[\varphi(U) : 1+U \le \varepsilon] \\ &\le Q[\varphi(U) : 1+U \ge \varepsilon] - Q[\ln(1+U) : 1+U \le \varepsilon]. \end{aligned}$$

Since $\varphi \le \frac{1}{2}(u/\varepsilon)^2$ if $1+u \ge \varepsilon$,

$$\begin{aligned} Q[\varphi(U) : 1+U \ge \varepsilon] &\le \tfrac{1}{2}\varepsilon^{-2} Q[U^2] \\ &= \tfrac{1}{2}\varepsilon^{-2} c_1 \sum_{1\le i\le m} \alpha_i^2. \end{aligned} \tag{2.4}$$

We now set $\gamma = -Q[X_1] \geq 0$ and choose $\varepsilon > 0$ so small that $\ln(1/\varepsilon) - \gamma \geq 1$. We introduce another centered random variable

$$V = \sum_{1 \leq i \leq m} \alpha_i (X_i + \gamma).$$

We then see from Jensen's inequality that

$$\begin{aligned}\{1 + U \leq \varepsilon\} &\subset \{V - \gamma \leq \ln(1+U) \leq \ln \varepsilon\} \\ &\subset \{-\ln(1+U) \leq -V + \gamma\} \cap \{1 \leq -V\}.\end{aligned}$$

We consequently have that

$$\begin{aligned}-Q[\ln(1+U) : 1 + U \leq \varepsilon] &\leq Q[-V : 1 \leq -V] + \gamma Q\{1 \leq -V\} \\ &\leq (1+\gamma) Q[V^2] \\ &= c_4 \sum_{1 \leq i \leq m} \alpha_i^2.\end{aligned}$$

This, together with (2.4) proves the right-hand-side inequality of (2.2). The proof of (2.3) is similar as above. In fact, since $|\ln(1+u)| \leq \varepsilon^{-1} \ln(\varepsilon^{-1})|u|$ if $\varepsilon \leq 1 + u$, we have that

$$Q[\ln^2(1+U) : \varepsilon \leq 1 + U] \leq \varepsilon^{-2} \ln^2(\varepsilon^{-1}) Q[U^2].$$

We see on the other hand that

$$\begin{aligned}\{1 + U \leq \varepsilon\} &\subset \{V - \gamma \leq \ln(1+U) \leq \ln \varepsilon\} \\ &\subset \{\ln^2(1+U) \leq 2V^2 + 2\gamma^2\} \cap \{1 \leq -V\}.\end{aligned}$$

We obtain as a consequence that

$$\begin{aligned}Q[\ln^2(1+U) : 1 + U \leq \varepsilon] &\leq 2Q[V^2] + 2\gamma^2 Q\{1 \leq -V\} \\ &\leq c_5 \sum_{1 \leq i \leq m} \alpha_i^2.\end{aligned}$$

□

To prove Theorem 1.1, it is enough to prove the following (2.5) and (2.6):

$$\{Z_\infty = 0\} \subset \left\{ \sum_{n \geq 1} I_n = \infty \right\}, \quad Q\text{-a.s.}, \tag{2.5}$$

There are $c_1, c_2 \in (0, \infty)$ such that

$$\left\{ \sum_{n \geq 1} I_n = \infty \right\} \subset \left\{ -c_1 \ln Z_n \leq \sum_{1 \leq j \leq n} I_j \leq -c_2 \ln Z_n \ \text{ for large enough } n\text{'s.} \right\}, \quad Q\text{-a.s..} \tag{2.6}$$

The proof of (2.5) and (2.6) is based on Doob's decomposition for the process $-\ln Z_n$. It is convenient to introduce some more notations. For a sequence $(a_n)_{n\geq 0}$ (random or non-random), we set $\Delta a_n = a_n - a_{n-1}$ for $n \geq 1$. We denote by $\mathcal{E}_n$ the σ-field generated by $\{\xi(x,j) \; ; \; 1 \leq j \leq n, \; x \in \mathbb{Z}^d\}$, and we denote by Q_n^ξ the conditional expectation with respect to Q given $\mathcal{E}_n$. Let us now recall Doob's decomposition in this context; any $(\mathcal{E}_n)$-adapted process $X = \{X_n\}_{n\geq 0} \subset L^1(Q)$ can be decomposed in a unique way as

$$X_n = M_n(X) + A_n(X), \quad n \geq 1,$$

where $M(X)$ is an $(\mathcal{E}_n)$-martingale and

$$A_0 = 0, \;\; \Delta A_n = Q_{n-1}^\xi[\Delta X_n], \quad n \geq 1.$$

$M_n(X)$ and $A_n(X)$ are called respectively, the martingale part and the compensator of the process X. If X is a square integrable martingale, then the compensator $A_n(X^2)$ of the process $X^2 = \{(X_n)^2\}_{n\geq 0} \subset L^1(Q)$ is denoted by $\langle\, X \,\rangle_n$ and is given by the following formula;

$$\Delta\langle\, X \,\rangle_n = Q_{n-1}^\xi[(\Delta X_n)^2]$$

Here, we are interested in the Doob's decomposition of $X_n = -\ln Z_n$, whose martingale part and the compensator will be henceforth denoted M_n and A_n respectively;

$$-\ln Z_n = M_n + A_n. \tag{2.7}$$

To compute M_n and A_n, we introduce

$$U_n = \mu_{n-1}[\exp(\beta\xi(S_n,n) - \lambda(\beta))] - 1 \; .$$

It is then clear that

$$Z_n/Z_{n-1} = 1 + U_n \tag{2.8}$$

and hence that

$$\Delta A_n = -Q_{n-1}^\xi \ln(1+U_n), \quad \Delta M_n = -\ln(1+U_n) + Q_{n-1}^\xi \ln(1+U_n). \tag{2.9}$$

In particular,

$$\Delta\langle\, M \,\rangle_n \leq Q_{n-1}^\xi \ln^2(1+U_n). \tag{2.10}$$

On the other hand, we have that

$$I_n = \sum_{|z|_1 \leq n} \mu_{n-1}(S_n = z)^2.$$

We now claim that there is a constant $c \in (0,\infty)$ such that

$$\frac{1}{c} I_n \;\; \leq \;\; \Delta A_n \leq c I_n, \tag{2.11}$$

$$\Delta\langle\, M \,\rangle_n \;\; \leq \;\; c I_n. \tag{2.12}$$

In fact, we obtain these from (2.9), (2.10) and Lemma 2.1; $\{X_i\}$, $\{\alpha_i\}$ and Q in the lemma play roles of $\{\beta\xi(z,n)-\lambda(\beta)\}_{|z|_1\le n}$, $\{\mu_{n-1}(S_n=z)\}_{|z|_1\le n}$ and Q^{ξ}_{n-1}.

We now conclude (2.5) from (2.11), (2.12) as follows (the equality and the inclusions here being understood as Q-a.s.);

$$\begin{aligned}
\left\{\sum_{n\ge 1} I_n<\infty\right\} &= \{A_\infty<\infty\}\\
&\subset \{A_\infty<\infty,\ \langle\, M\,\rangle_\infty<\infty\}\\
&\subset \{A_\infty<\infty,\ \lim_{n\nearrow\infty} M_n \text{ exists and is finite}\}\\
&\subset \{Z_\infty>0\}.
\end{aligned}$$

Here, in the third line, we have used a well-known property of a martingale, e.g. (4.9) page 255 in Durrett (1995).

Finally we prove (2.6). By (2.11), it is enough to show that

$$\{A_\infty=\infty\}\subset\left\{\lim_{n\nearrow\infty}-\frac{\ln Z_n}{A_n}=1\right\},\quad Q\text{-a.s.}. \tag{2.13}$$

Let us thus suppose that $A_\infty=\infty$. If $\langle\, M\,\rangle_\infty<\infty$, then again by (4.9) page 255 in Durrett (1995), $\lim_{n\nearrow\infty} M_n$ exists and is finite and therefore (2.13) holds. If $\langle\, M\,\rangle_\infty=\infty$, then

$$-\frac{\ln Z_n}{A_n}=\frac{M_n}{\langle\, M\,\rangle_n}\frac{\langle\, M\,\rangle_n}{A_n}+1\to 1\quad Q\text{-a.s.}$$

by (2.11), (2.12) and the law of large numbers for martingales (see (4.10) page 255 in Durrett 1995). This completes the proof. □

3 Proof of Proposition 1.3

The statement in (i) directly follows from (1.15). The first statement in (ii) is derived using the the convergence of I_n to 0. We now prove (1.17). Since $Q\{Z_\infty>0\}=1$ in the present case -see (1.8)-,it is enough to show that

$$Z^2_{n-1}I_n=O(n^{-c}) \tag{3.1}$$

in Q-probability. With $\gamma=\lambda(2\beta)-2\lambda(\beta)<-\ln(1-q)$, we compute

$$\begin{aligned}
Q\left[Z^2_{n-1}I_n\right] &= Q\left[P^{\otimes 2}(e_{n-1}(\xi,S)e_{n-1}(\xi,\widetilde{S}):S_n=\widetilde{S}_n)\right]\\
&= P^{\otimes 2}(Q[e_{n-1}(\xi,S)e_{n-1}(\xi,\widetilde{S})]:S_n=\widetilde{S}_n)\\
&= P^{\otimes 2}\left(\exp\left\{\gamma\sum_{j=1}^{n-1}\mathbf{1}_{S_j=\widetilde{S}_j}\right\}:S_n=\widetilde{S}_n\right)\\
&\le P^{\otimes 2}\left(\exp\left\{\alpha\gamma\sum_{j=1}^{n-1}\mathbf{1}_{S_j=\widetilde{S}_j}\right\}\right)^{1/\alpha}P^{\otimes 2}(S_n=\widetilde{S}_n)^{1/\alpha'},
\end{aligned}$$

using Hölder's inequality with conjugate exponent α, α'. Since $\sum_{j\geq 1} \mathbf{1}_{S_j=\widetilde{S}_j}$ is geometrically distributed with failure probability $1-q \in (0,1)$ with q as in (1.8), the first factor in the right-hand side is bounded for $\alpha\gamma < -\ln(1-q)$. The second factor is $O(n^{-d/(2\alpha')})$. From this we obtain (1.17) for arbitrary $c < d[1+\gamma/\ln(1-q)]/2$. □

4 Proof of Theorem 1.4

4.1 Proof of part (a)

This follows easily by the Borel-Cantelli lemma. □

4.2 Proof of part (b)

For $\theta \in (0,1)$, by the subadditive estimate $(u+v)^\theta \leq u^\theta + v^\theta$, $u, v > 0$, we get

$$Z_n^\theta \leq (2d)^{-\theta} \sum_{x,|x|_1=1} \exp(\theta\beta\xi(x,1) - \theta\lambda(\beta))(Z_{1,n}^x)^\theta$$

where

$$Z_{1,n}^x = P^x \exp\left(\sum_{1\leq j\leq n-1} (\beta\xi(S_j, j+1) - \lambda(\beta))\right).$$

Since $Z_{1,n}^x$ has the same law as Z_{n-1},

$$Q[Z_n^\theta] \leq r(\theta) Q[Z_{n-1}^\theta],$$

where

$$r(\theta) = (2d)^{1-\theta} Q[\exp(\theta\beta\xi(x,1) - \theta\lambda(\beta))]$$

Note that $\theta \mapsto \ln r(\theta)$ is convex and that $\ln(2d) = \ln r(0) > \ln r(1) = 0$. Therefore $r(\theta) < 1$ for some $\theta \in (0,1)$ if and only if $0 < \left.\frac{d\ln r(\theta)}{d\theta}\right|_{\theta=0}$, which is equivalent to (1.20). □

4.3 Proof of part (c)

Lemma 4.1 *For $\theta \in [0,1]$ and $\Lambda \subset \mathbb{Z}^d$,*

$$Q\left[Z_{n-1}^\theta I_n\right] \geq \frac{1}{|\Lambda|} Q\left[Z_{n-1}^\theta\right] - \frac{2}{|\Lambda|} P(S_n \notin \Lambda)^\theta. \tag{4.1}$$

Proof: Following the argument in Liggett (1985, page 453), we see that

$$\begin{aligned} I_n &\geq \sum_{z\in\Lambda} \mu_{n-1}(S_n = z)^2 \\ &\geq \frac{1}{|\Lambda|} \mu_{n-1}(S_n \in \Lambda)^2 \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{|\Lambda|}\left(1-\mu_{n-1}(S_n \notin \Lambda)\right)^2 \\
&\geq \frac{1}{|\Lambda|}\left(1-2\mu_{n-1}(S_n \notin \Lambda)\right) \\
&\geq \frac{1}{|\Lambda|}\left(1-2\mu_{n-1}(S_n \notin \Lambda)^\theta\right).
\end{aligned}
$$

Note also that

$$
\begin{aligned}
Q\left[Z_{n-1}^\theta \mu_{n-1}(S_n \notin \Lambda)^\theta\right] &\leq Q\left[Z_{n-1}\mu_{n-1}(S_n \notin \Lambda)\right]^\theta \\
&= P(S_n \notin \Lambda)^\theta.
\end{aligned}
$$

We therefore see that

$$
\begin{aligned}
Q\left[Z_{n-1}^\theta I_n\right] &\geq \frac{1}{|\Lambda|} Q\left[Z_{n-1}^\theta\right] - \frac{2}{|\Lambda|} Q\left[Z_{n-1}^\theta \mu_{n-1}(S_n \notin \Lambda)^\theta\right] \\
&\geq \frac{1}{|\Lambda|} Q\left[Z_{n-1}^\theta\right] - \frac{2}{|\Lambda|} P(S_n \notin \Lambda)^\theta.
\end{aligned}
$$

□

Define a function $f : (-1, \infty) \to [0, \infty)$ by

$$
f(u) = 1 + \theta u - (1+u)^\theta.
$$

It is then clear that there are constants $c_1, c_2 \in (0, \infty)$ such that

$$
\frac{c_1 u^2}{2+u} \leq f(u) \leq c_2 u^2 \quad \text{for all } u \in (-1, \infty). \tag{4.2}
$$

We see from (2.8), (4.2) and (2.1) that

$$
\begin{aligned}
Q_{n-1}^\xi \Delta Z_n^\theta &= Z_{n-1}^\theta Q_{n-1}^\xi \left((1+U_n)^\theta - 1\right) \\
&= -Z_{n-1}^\theta Q_{n-1}^\xi f(U_n) \\
&\leq -c_3 Z_{n-1}^\theta I_n.
\end{aligned}
$$

We therefore have by (4.1) that

$$
Q Z_n^\theta \leq \left(1 - \frac{c_3}{|\Lambda|}\right) Q\left[Z_{n-1}^\theta\right] - \frac{2c_3}{|\Lambda|} P(S_n \notin \Lambda)^\theta. \tag{4.3}
$$

For $d = 1$, set $\Lambda = (-n^{2/3}, n^{2/3}]$. Then,

$$
P(S_n \notin \Lambda) = P\left(\left|\frac{S_n}{n^{1/2}}\right| \geq n^{1/6}\right) \leq 2\exp\left(-\frac{n^{1/3}}{2}\right),
$$

so that (4.3) reads,

$$
Q Z_n^\theta \leq \left(1 - \frac{c_3}{2n^{2/3}}\right) Q\left[Z_{n-1}^\theta\right] + 4c_3 \exp\left(-\frac{n^{1/3}}{2}\right).
$$

It is not difficult to conclude (1.21) for $d = 1$ from the above.

For $d = 2$, we set
$$\Lambda = (-n^{1/2} \ln^{1/4} n, n^{1/2} \ln^{1/4} n]^2$$
to get (1.21) for $d = 2$ in a similar way as above. □

5 Proof of Proposition 1.5

Though the first statement is well known, we give a proof here for definiteness. Note that for $m, n \geq 1$,
$$Z_{n+m} = Z_n \sum_x \mu_n\{S_n = x\} Z^x_{n,m} ,$$
where $Z^x_{n,m} = P^x[\exp\{\sum_{1\leq j\leq m} \beta\xi(S_j, n+j) - \lambda(\beta)\}]$ has the same law as Z_m. By Jensen's inequality,
$$\ln Z_{n+m} \geq \ln Z_n + \sum_x \mu_n\{S_n = x\} \ln Z^x_{n,m} .$$
Recall the notations $\mathcal{E}_n$ and Q^ξ_n introduced in the proof of Theorem 1.1. Taking expectation and using independence, we obtain
$$\begin{aligned} Q[\ln Z_{n+m}] &\geq Q[\ln Z_n] + Q[\sum_x \mu_n\{S_n = x\} Q^\xi_n[\ln Z^x_{n,m}]] \\ &= Q[\ln Z_n] + Q[\ln Z_m] , \end{aligned}$$
i.e., $Q[\ln Z_n]$ is super-additive. From the super-additive Lemma we see that
$$\frac{1}{n} Q[\ln Z_n] \to \psi(\beta) = \sup_n \frac{1}{n} Q[\ln Z_n]$$
as $n \to \infty$.

To prove (1.22) we write $\ln Z_n - n\psi(\beta)$ as a sum of $(\mathcal{E}_j)_{1\leq j\leq n}$-martingale differences,
$$\ln Z_n - Q[\ln Z_n] = \sum_{j=1}^n V_{n,j}$$
with $V_{n,j} = Q^\xi_j[\ln Z_n] - Q^\xi_{j-1}[\ln Z_n]$. Set
$$e_{n,j} = \exp\left(\sum_{1\leq k\leq n, k\neq j} (\beta\xi(S_k, k) - \lambda(\beta)) \right) , \quad Z_{n,j} = P[e_{n,j}].$$
Clearly $Q^\xi_j[\ln Z_{n,j}] = Q^\xi_{j-1}[\ln Z_{n,j}]$, and hence,
$$V_{n,j} = Q^\xi_j\left[\ln \frac{Z_n}{Z_{n,j}}\right] - Q^\xi_{j-1}\left[\ln \frac{Z_n}{Z_{n,j}}\right] .$$

By (2.2) in Lemma 2.1 with $X_\cdot = \beta\xi(\cdot, j) - \lambda(\beta)$ and $\alpha_\cdot = \frac{P[e_{n,j}:S_j=\cdot]}{Z_{n,j}}$, we see that

$$\begin{aligned} -Q^{\xi}_{j-1}\left[\ln \frac{Z_n}{Z_{n,j}}\right] &= -Q^{\xi}_{j-1}\left[Q\left[\ln\left(\sum_x \alpha_x \exp(X_x)\right)\Big| \mathcal{E}_{n,j}\right]\right] \\ &\in [0, c] \,, \end{aligned}$$

where $\mathcal{E}_{n,j} = \sigma[\xi(\cdot, k) \ ; \ 1 \le k \le n, k \ne j]$; note that $\alpha_\cdot$ is $\mathcal{E}_{n,j}$-measurable. Therefore, for $\theta \in \mathbb{R}$, we have by Jensen's inequality that

$$\begin{aligned} Q[\exp \theta V_{n,j} | \, \mathcal{E}_{n,j}] &\le e^{c\theta^-} Q\left[\exp \theta Q^{\xi}_j \left[\ln \frac{Z_n}{Z_{n,j}}\right] \Big| \, \mathcal{E}_{n,j}\right] \\ &\le e^{c\theta^-} Q\left[\left(\frac{Z_n}{Z_{n,j}}\right)^{\theta} \Big| \, \mathcal{E}_{n,j}\right] \\ &= e^{c\theta^-} Q\left[\left(\sum_x \alpha_x \exp(X_x)\right)^{\theta} \Big| \, \mathcal{E}_{n,j}\right] \\ &\le e^{c\theta^-} \sum_x \alpha_x Q\left[\exp(\theta X_x) | \, \mathcal{E}_{n,j}\right] \\ &= e^{c\theta^-} \sum_x \alpha_x Q[\exp(\theta X_x)] \\ &= \exp\{c\theta^- + \lambda(\theta\beta) - \theta\lambda(\beta)\}, \end{aligned}$$

where $\theta^- = \max\{0, -\theta\}$. Finally we conclude that

$$Q[\exp |V_{n,j}|] \le Q[\exp(V_{n,j}) + \exp(-V_{n,j})] \le c_1 < \infty \,.$$

Therefore, the large deviation estimate for sum of martingale-differences of Lesigne and Volný (2001, Theorem 3.2) applies to our case, yielding (1.22). The final statement in Proposition 1.5 is now a simple consequence of Borel-Cantelli lemma. □

Remark 5.1 Let us remark that the stronger assumption (1.23) implies the exponential concentration (1.24). In what follows, $c_i = c_i(\beta, \delta)$ $(i = 1, 2, \dots)$ are positive constants. Note first that (1.23) implies that $\lambda(\beta) \le c_1 + c_2\beta^2$ for all $\beta \in \mathbb{R}$ and hence that

$$Q\left[\exp(\theta V_{n,j}) | \mathcal{E}_{n,j}\right] \le \exp\left(c_3 + c_4\theta^2\right), \quad \text{for all } \theta \in \mathbb{R}, \tag{5.1}$$

by the computations in our proof of Proposition 1.5. By expanding the exponential and using the fact $Q[V_{n,j}|\mathcal{E}_{n,j}] = 0$, we can improve (5.1) into the following stronger form;

$$Q\left[\exp(\theta V_{n,j}) | \mathcal{E}_{n,j}\right] \le \exp\left(c_5\theta^2\right), \quad \text{for all } \theta \in \mathbb{R}. \tag{5.2}$$

It is then, not difficult to conclude (1.24) by (5.2) and a standard Gaussian estimate for a martingale.

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