Charge as the Stereographic Projection of Geometric Precession on Pseudospheres

Bernd Binder
88682 Salem, Germany, binder@quanics.com ©*2002*
(Dated: 30.9.2002)

In this paper geometric phases (Berry and Aharonov-Bohm) are generalized to nonlinear topological phase fields on pseudospheres, where the coordinate vector field is parallel transported along the signal/soliton vector field with Levi–Civita connection. Projective $PSL(2,\mathbb{R})$ symmetry describes the relativistic self-interacting bosonic sine-Gordon field. A Coulomb potential can be induced as the stereographic projection of a harmonic oscillator potential mapping angles or phases to distances and vice versa resulting in mutual coupling with a generalized coupling constant given by a nonlinear iteration. With single-valuedness requirement in 137-gonal symmetry it fits within a few ppb uncertainty to the Sommerfeld fine structure constant.

PACS numbers: 02.10.De, 03.65.Bz, 03.65.Vf, 03.65-w, 03.67.Lx, 05.45.Yv, 06.20.Jr, 12.20.-m, 42.65.T

Introduction. The geometric origin of the sine-Gordon equation SG can be assigned to the study of Riemannian geometry on surfaces of constant negative scalar curvature, also known as pseudospherical surfaces that can be considered as varieties embedded (with the induced natural Riemannian metric) into the three dimensional Euclidean space $\mathbb{R}^3$ [1, 2] (very similar to the Liouville equation). The curvature and coupling parameter are given by the Levi–Civita connection, where the SG appears by fixing special local coordinate frames on pseudospherical surfaces with a Riemann tensor that has only one independent component R_{1212}. The SG has soliton solutions which describe elastic collision of localizable waves, that can be interpreted as particles of non–perturbative nature. This is relevant for spatial and spectral localization of energy, intrinsic nonlinear modes, self-induced transparency, and supra effects (energy propagation in the forbidden band gap by means of nonlinear modes) [3].

Topological and/or geometric phases are subject of concepts in differential-geometry and topology associated with abelian and non–abelian groups [4, 5, 6]. Generally, phase factors or phases representing the 'holonomy' provide for important boundary conditions while reducing the degree of redundancy in variables. This is one of the reasons why phases and gauge theories are not unimportant in quantum mechanics, despite of the central role of amplitude densities. Berry showed that the geometric phase has the same mathematical (gauge) structure as the Aharonov-Bohm (AB) phase [5] and is the integral of an effective vector potential along a closed path. Both phases even combine [7] especially if a charged particle is in an spatial extended quantum state, i.e. if the orbital loop includes spatially extended sub-loops. Both, the local non–abelian Berry phase evolution on $S^2 = SU(2)/U(1)$ and the nonlocal abelian AB scattering effect on $\mathbb{R}^2$ with conic metric provide for phase evolutions and deficit angles that can be combined. In a previous work it was shown how the deficit angle of the AB conic metric and the geometric precession cone vertex angle of the Berry phase can be mutually adjusted to restore single-valuedness. The resulting interplay between both phases provides a non–linear iterative system providing for generalized fine structure constants [8]. Here we start to generalize the Berry and Aharonov-Bohm phases to nonlinear topological phase fields on pseudospheres, where projective $PSL(2,\mathbb{R})$ symmetry describes the relativistic self-interacting bosonic sine-Gordon field, and where the iterative interplay can be assigned to a harmonic oscillator potential.

The central news of this paper is, that oscillators on the sphere and the pseudosphere are related by the Bohlin–Levi–Civita transformation with the Coulomb system on the pseudosphere [9, 10]. Consequently, we can introduce the Coulomb potential and the corresponding coupling constant as the stereographic projection of the harmonic oscillator potential mapping angles or phases to distances. With this relations the previous work [8] can be fundamentally confirmed and generalized, SG relativistic soliton dynamics can be related to electrodynamics.

Strategy. This paper outlines the 4 essential steps. Step 1: starting with a Riemannian geometry of surfaces of constant scalar curvature (pseudospherical and spherical) embedded into the three dimensional Euclidean space with symmetry group $SU(2)$ or $SU(1,1)$. Step 2: fixing local coordinates that have only one independent component given by the curvature scalar (sine-Gordon and Liouville). Step 3: introducing feedback coupling by the stereographic projection onto (conformal-flat) coordinates of the two-dimensional oscillator with symmetry group $PSL(2,\mathbb{R})$ and identifying the dual potentials with $su(d)$ and $so(d+1)$ symmetry (for $d = 2$). Step 4: fixing the coupling constant by a nonlinear iteration.

Bäcklund transformations. The nonlinear SG phase field evolves with a pseudospherical curvature constraint. This property is found with generalized Chebyshev coordinates on a plane $\mathcal{S}$ embedded in $\mathbb{R}^3$

$$ds^2 = M^{-2}(dx)^2 + M^2(dy)^2 + 2\cos\theta dxdy \qquad (1)$$

with scalar curvature $R = 2R_{1212}/\det(g_{ij})$ [2] of the generalized Chebyshev metric, where

$$\partial_x\partial_y\theta = (\pi M_g)^2 = -R\sin\theta/2 \tag{2}$$

is the SG. As a generator of the SG eq.(2) and manifestation of integrability, the Bäcklund transformations (BT) maps the SG into itself and enables to build new surfaces of constant negative curvature from old [1]. From a given solution of the SG [1, 2] we can construct new solutions by solving the ordinary differential equations for a family of elementary BT $\theta \mapsto \widetilde{\theta}$

$$\begin{aligned}(\partial_x\widetilde{\theta} + \partial_x\theta)/M &= 2\pi M_g \sin[(\widetilde{\theta} - \theta)/2],\\ (\partial_y\widetilde{\theta} - \partial_y\theta)M &= 2\pi M_g \sin[(\widetilde{\theta} + \theta)/2].\end{aligned} \tag{3}$$

The second order equation eq.(2) arises as the integrability conditions of a pair of first order equations eq.(3), i.e. $\partial_y(\partial_x\widetilde{\theta}) = \partial_x(\partial_y\widetilde{\theta})$. Provided θ is a solution of the SG, then $\widetilde{\theta}$ is also a solution. For simplicity, $\widetilde{\theta}$ will serve as the special reference field of constant phase given by the rather trivial case $\widetilde{\theta} = 4\pi(\frac{1}{2} + n)$, with quantum gauge (or spin) dependent winding number $n = 0, 1, 2, \ldots$. This provides for a simplification and the dimensional reduction $\partial_x = M^2\partial_y$ in eq.(3) with

$$\partial_x\theta/M = M\partial_y\theta = 2\pi M_g \sin(\theta/2). \tag{4}$$

This form corresponds to travelling waves-like solutions with $\xi = \pi M_g(ax + by)$, where the SG can be reduced to the ordinary differential equation $ab\partial_\xi^2\psi(\xi) = \sin\psi(\xi)$, in our case $b = 1/a = M$. The stationary SG soliton solutions are expressed by elliptic functions, the generalized pseudosphere solution follows immediately from integrating eq.(4) $\psi(\xi) = 4\arctan\exp(\xi/\sqrt{ab})$. With $\cos\theta = 1 - 2\sin^2(\theta/2)$ and introducing $r^2 = x^2 + y^2 = (1 + 1/M^4)x^2 = (M^4 + 1)y^2$ with $\partial_r^2 = \partial_x^2 + \partial_y^2$ the potential is usually given by

$$\begin{aligned}2V(\theta) &= (\partial_x\theta/M)^2 = (M\partial_y\theta)^2\\ &= (M^2 + 1/M^2)(\partial_r\theta)^2 = 2\pi^2M_g^2(1 - \cos\theta),\end{aligned} \tag{5}$$

From eq.(5) the self-energy term can be identified as a constant θ-independent Riemann curvature scalar $R = -2/\rho^2$, with eq.(2) $\pi M_g\rho = 1$. Therefore, it is plausible to decompose energy in eq.(5) into at least two terms: a self-energy term $\pi^2M_g^2$ and a dynamic coupling term $\pi^2M_g^2\cos\theta$ that accounts for the field evolution based on the BT.

Coupling space and phase coordinate by harmonic oscillation. Searching for external coupling and synchronization, eq.(5) allows to force global harmonic oscillations via potential

$$V_o(r) = \frac{1}{2}\left(\frac{r}{\rho}\right)^2 = \frac{1}{2}(\pi M_g r)^2 = -\frac{1}{4}Rr^2. \tag{6}$$

Regarding eq.(4) and the square roots of eq.(5) and eq.(6), space and phase coordinate become directly coupled

$$r = \mp\rho\sqrt{M^2 + 1/M^2}\partial_r\theta = \mp 2\sin(\theta/2), \tag{7}$$

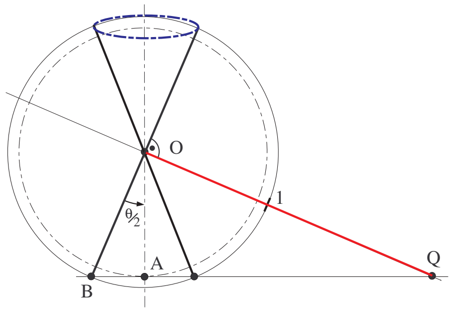

FIG. 1: Oscillation by precession on the sphere S^2 with amplitude $r = 2|A - B|$ (blue) stereographically projected to the Coulomb distance $r_c = |O - Q|$ (red) with $r = 2\sin(\theta/2)$ and $r_c = \cot(\theta/2)$. The horizontal distance $d = |A - Q| = |A - O|^2/|A - B| = \cos(\theta/2)r_c = (1 + \cos\theta)/r$.

which provides for an additional dimensional reduction. The coupling space and phase coordinate provides for a coupling constant since integration of eq.(7) provides for adynamic coupling term $\mp\pi M_g\sqrt{M^2 + 1/M^2}\theta$ that can be combined with a self-energy term and integration constant to

$$V(\theta) = V_o(y) = \pi^2M_g^2 \mp \pi M_g\sqrt{M^2 + 1/M^2}\theta. \tag{8}$$

Comparing the correspondent parts of self-energy and dynamic coupling in eq.(8) and eq.(5), we immediately obtain an iterative equation of cooperative macroscopic phase shift driven by stereographic feedback

$$\sqrt{M^2 + 1/M^2}\theta = \pm\pi M_g\cos\theta, \tag{9}$$

where the coupling allows for two possible signs.

Coulomb potential from fractional linear transformations. The relation between r and θ in eq.(7) is a stereographic projection with (pseudo)spherical angle $\theta/2$ onto the conformally-flat (x, y)-plane, see fig.1. Defining Lobachevskian planes and constructing a Lie–Bäcklund transformation which relates the Liouville equation to the SG [2], these systems possess nonlinear hidden symmetries providing for properties similar to those of conventional oscillator and Coulomb systems. In the previous work is was proposed, that the iterative solution $\alpha(M) = \theta(M)/\pi$ could be interpreted as a generalized spin-orbit or fine structure constant, since α enters in [8] as a Newton-type coupling constant of the conic metric. Let where r_c and r denote the radial coordinates of Coulomb and oscillator systems, respectively. Under stereographic projection the conventional Bohlin transformation $r_c = r^2$ plus inversion relates the harmonic oscillator potential eq.(6) on the (pseudo)sphere to the Coulomb system on pseudosphere, as well as those interacting with specific external magnetic fields [10]. Since the group of the isometries of the Lobachevsky and Kaehler metric coincides with $PSL(2, \mathbb{R})$, it acts by projective fractional linear (or Möbius) transformations [2]

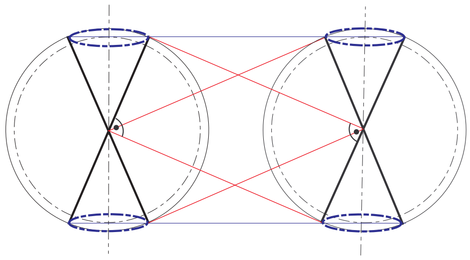

FIG. 2: Precessional coupling by stereographic projection: mutual geometric interaction of oscillators.

and allows to obtain the typical electromagnetic field patterns. In our case, see fig.1 and eq.(7), the parameterization is nothing but the stereographic projection of the two-dimensional (pseudo)sphere

$$z = r_c e^{i\varphi} = \begin{cases} \cot\frac{\theta}{2}e^{i\varphi} \text{ sphere;} \\ \coth\frac{\theta}{2}e^{i\varphi} \text{ pseudosphere,} \end{cases} \quad (10)$$

where θ, φ are the (pseudo)spherical coordinates [10]. The radial dependence of the Coulomb potential is $V_c \propto 1/r_c$, the oscillator potential $V_o \propto r^2$. In both, classical and quantum cases, the fractional linear transformation and successive transformation $r_c = r^2$ converts the $su(d)$ symmetry algebra of the oscillator to the $so(d+1)$ symmetry of the Coulomb system [10]. Fig.2 extends the principle to mutual interaction of geometric precession.

The coupling strength. The coupling strength obtained with the radial distance on the projective plan r differs from the coupling strength obtained from a pure one-dimensional definition [8, 11] by a factor $\sqrt{1+1/M^4}$, for $M = 137$ a relative reduction in coupling strength of about $1.42 \cdot 10^{-9}$. The iteration eq.(9) is now invariant with respect to the inversion and duality $M \leftrightarrow 1/M$. Inversion seems to be in our case the central linear fractional transformation between local and non–local holonomy relating the Coulomb and oscillator potential. M-type inversion could also characterize the relations between the electric and magnetic monopole charge $ge = 1$ with $g/e = M^2$, and also between group and phase velocity of a wave packet in the ground state $v_g v_p = 1$ with $v_p/v_g = M^2$. The coupling strength is balanced by the orbital degree of degeneracy M or $1/M$ of the precessional field, the Bäcklund parameter introduced in eq.(3). M as an integral quantum number describes the phase-locked and single-valued field [8] and provides for integrability. The coupling constant and special θ-value or oscillation range is iteratively obtained in eq.(9), where $M = 137$ or $M = 1/137$ provides with $M_g = 1$ for $1/\alpha = 137.03600960$ that fits within some ppb's to the Sommerfeld fine structure constant obtained in neutron interferometry. The meaning of the number 137 remains unclear.

Conclusion. There is a clear geometrical interpretation: the coordinate vector field is parallel transported along the signal/soliton vector field with respect to the Levi–Civita connection. A "privileged" surface $\mathbb{H}$ of scalar curvature $R = -2$ is given by the Lobachevskian plane and Poincaré disks. The potential eq.(6) provides for a global harmonic precession balanced by the topological phase shift $\theta(r)$, where the usual SG coupled pendulum interpretation is extended to a macroscopically coupled spin interpretation. The situation becomes stable and self-consistent if the Coulomb feedback synchronizes to local soliton oscillations (breather) that generate the Coulomb potential by stereographic projection. Regarding the recent work of [3], the nonlinear mechanism behind $V[\theta(r)]$ could have a strong relevance for self-induced transparency and nonlinear supratransmission. Eq.(9) is an chaotic algorithm, bifurcation starts above a special values of M_g. In [3] the bifurcation of energy transmission is demonstrated numerically and experimentally on the chain of coupled pendula (sine-Gordon and nonlinear Klein-Gordon equations). Energy propagation in the forbidden band gap by means of nonlinear modes requires a degree of macroscopic coherence initiated i.e. by eq.(6). It appears, that both $\alpha(M)$ and SG-solitons could be simultaneously observed in Josephson ladders [9] in the context of supratransmission and supraconductivity.

[1] M. J. Ablowitz and H. Segur,*Solitons and the Inverse Scattering Transform*, SIAM, Philadelphia, (1981).

[2] H. Belich, G. Cuba, R. Paunov,*Surfaces of Constant negative Scalar Curvature*, solv-int/9909018.

[3] F. Geniet, J. Leon, Phys. Rev. Lett.**89**, 134102 (2002) nlin.PS/0209020.

[4] M. V. Berry, Proc. Roy. Soc. Lond. A**392**, 45 (1984).

[5] Y. Aharonov, D. Bohm, Phys. Rev.**11**, 485-91 (1959).

[6] F. Wilczek and A. Zee., Phys. Rev. Lett.**52** , 2111 (1984).

[7] Y. Aharonov, S. Coleman, Goldhaber, S. Nussinov, S. Popescu, B. Reznik, D. Rohrlich, and L. Vaidman, Phys. Rev. Lett.**73**, 918 (1994).

[8] B. Binder,*Iterative Interplay between Aharonov-Bohm Deficit Angle and Berry Phase* (2002); PITT-PHIL-SCI00000810.

[9] E. Trias, J.J. Mazo, T.P. Orlando, Phys. Rev. Lett.**84**, 741 (2000); P. Binder, D. Abraimov, A.V. Ustinov, S. Flach, Y. Zolotaryuk, Phys. Rev. Lett.**84**, 745 (2000).

[10] A. Nersessian, G. Pogosyan, Phys.Rev. **D63**, 020103, (2001); quant-ph/0006118; math-ph/0010049.

[11] B. Binder,*Geometric Phase Locked in Fine Structure* (2002); PITT - PHIL-SCI00000782.