THE AMOUNT OF DISCRETE SPECTRUM OF A PERTURBED PERIODIC SCHRÖDINGER OPERATOR INSIDE A FIXED INTERVAL (λ_1, λ_2)

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ABSTRACT. In this paper we extend the results of [4] for a more general class of perturbations. Let A be a periodic Schrödinger operator and let $V \geq 0$ be a decaying potential. We study the number $\tilde{N}(\alpha)$ of the eigenvalues of the operator $A(\alpha) = A - \alpha V$ inside a fixed interval (λ_1, λ_2). We obtain an asymptotic formula for $\tilde{N}(\alpha)$ as $\alpha \to \infty$.

0. INTRODUCTION

In a simple quantum mechanical model for impurities in a crystal, one studies the spectrum of a periodic Schrödinger operator $A = -\Delta + p(x)$, perturbed by a decaying potential $-\alpha V$, which models the impurity ($\alpha > 0$ is a coupling constant). The basic assumptions are that A has a spectral gap Λ in its essential spectrum; eigenvalues of $A(\alpha) = A - \alpha V$ inside Λ correspond to the eigenstates of the electrons observed in solid state physics. We study the number of eigenvalues of $A(\alpha)$ inside an interval $(\lambda_1, \lambda_2) \subset \Lambda$. The main purpose of our study is to obtain an asymptotics of this number, as $\alpha \to \infty$. We point out that the problem has been solved only for one-dimensional Schrödinger operators in [6] and for the case of sphericaly symmetric potential V in [4]. The results of the present paper deal with multidimensional Schrödinger operators perturbed by non-symmetric potentials $-\alpha V$. Note that the eigenvalues of $A(\alpha)$ in Λ are continuous and monotone functions of α. Therefore it makes sense to ask how many eigenvalues of $A(t)$ cross a fixed point $\lambda \in \Lambda$ as t grows from 0 to α (excluding α). Let us denote this quantity by $N_0(\lambda, A(\alpha))$. Then the difference $N_0(\lambda_2, A(\alpha)) - N_0(\lambda_1, A(\alpha))$ coincides with the number of eigenvalues

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inside the interval $[\lambda_1, \lambda_2)$. For slowly decaying potentials

$$V(x) \sim \Phi(\theta)|x|^{-s}, \ \theta = \frac{x}{|x|} \qquad |x| \to \infty, \tag{1}$$

with $s < 2$, it was shown in [1] that

$$N_0(\lambda, A(\alpha)) \sim C(\lambda)\alpha^{d/s}, \qquad \alpha \to \infty, \tag{2}$$

where the coefficient $C(\lambda)$ can be expressed in terms of the integrated density of states. The same situation arises when $V \le 0$. Since the coefficient $C(\lambda)$ in (2) depends on λ, one can easily obtain that the leading term in the asymptotic formula for the number of the eigenvalues inside (λ_1, λ_2) equals $(C(\lambda_2) - C(\lambda_1))\alpha^{d/s}$. However, if V satisfies (1) with $s > 2$ then we have the standard Weyl's asymptotics:

$$N_0(\lambda, A(\alpha)) \sim \omega_d \alpha^{d/2} \int V^{d/2}\, dx, \qquad \alpha \to \infty, \tag{3}$$

where the asymptotic coefficient *does not* depend on λ. So if one knew the second term in the asymptotics (3), one would be able to calculate the asymptotics of the difference $N_0(\lambda_2, A(\alpha)) - N_0(\lambda_1, A(\alpha))$. Unfortunately the second term in (3) has not been obtained. Therefore in order to solve the problem we need a completely different approach.

1. Formulation of the main result. Preliminary information

1. Basic notation. a. Below M is a densely defined linear operator in a Hilbert space $\mathfrak{H}$. We denote by $D(M)$, M^*, $\sigma(M)$ the domain, the adjoint operator and the spectrum of the operator respectively. In the case when $M = M^*$, the symbol $E_M(\cdot)$ denotes the spectral measure for M. Let the spectrum of the operator $M = M^*$ on $\mathfrak{H}$ be bounded from below and be discrete to the left from the point $\lambda_0 \in \mathbb{R}$. Then the operator M is semibounded from below and is generated by a certain quadratic form $m[\cdot,\cdot]$ in $\mathfrak{H}$. We define the spectral distribution function of M:

$$N(\lambda, M) = \mathrm{rank} E_M(-\infty, \lambda), \quad \lambda < \lambda_0.$$

Furthermore, let $d[m]$ be the domain of the form m. Then for the quantity $N(\lambda, M)$ the following relation holds:

$$N(\lambda, M) = \max(\mathrm{dim} F), \quad F \subset d[m], m[u,u] < \lambda(u,u), \forall u \in F, u \neq 0.$$

b. Below $d \ge 2, \mathbb{Q} = [0,1)^d$ and $B_r = \{\ x \in \mathbb{R}^d : |x| < r\}, r > 0$. Notation $H^s(\Omega)$, $\Omega \subset \mathbb{R}^d$, is taken for the Sobolev classes of order $s \in \mathbb{N}$; $H^s_0(\Omega)$ is the closure of the class $C_0^\infty(\Omega)$ in the metric of $H^s(\Omega)$. For Laplace operators on domains $\Omega \subset \mathbb{R}^d$ we use the notation Δ. The gradient operator is denoted by ∇. Different estimate constants are denoted by C, c (sometimes with indices).

2. Let a function $p \in L^\infty(\mathbb{R}^d)$ be real valued and let

$$p(x+n) = p(x), \quad x \in \mathbb{R}^d, n \in \mathbb{Z}^d.$$

Consider the selfadjoint operator

$$Au = -\Delta u + pu, \quad u \in D(A) = H^2(\mathbb{R}^d),$$

on $L^2(\mathbb{R}^d)$. Obviously A is an elliptic $\mathbb{Z}^d$-periodic operator. The spectrum $\sigma(A)$ can have gaps besides the semi-infinite one. Let $\Lambda = (\lambda_-, \lambda_+)$ be a fixed gap in the spectrum $\sigma(A)$ (if $\lambda_+ = \inf\sigma(A)$, then $\lambda_- = -\infty$). Below we perturb the operator A so, that the spectrum of the perturbed operator is discrete in Λ.

Let $V : \mathbb{R}^d \to \mathbb{R}_+$ be a smooth function and for every $\theta \in \mathbb{S}$

$$\lim_{r\to\infty} r^s V(\theta r) = \Phi(\theta), \quad 2 < s < 2 + 2/(d-1), \Phi(\theta) > 0. \tag{4}$$

The limit in (4) is assumed to be uniform with respect to θ. Below we systematically use the following notations for the operators on a domain. Let $\Omega \subset \mathbb{R}^d$ be an open domain. We consider the following operator on $L^2(\Omega)$

$$A^{(\varepsilon)}(\alpha, \Omega) = -\Delta + p - \alpha V, \quad \alpha > 0;$$

the index ε takes the values $\mathcal{D}$ or $\mathcal{N}$ and characterizes the boundary (Dirichlet or Neumann) conditions on $\partial\Omega$. In order to formulate the main result we need to introduce the density of states for the operator A. Let $x_1, x_2 \in \mathbb{R}^d$ and $l > 0$. Let also $\Omega(l) := \Omega(l; x_1, x_2) = l(\mathbb{Q} + x_1) + x_2$. Then for an arbitrary $\lambda \in \mathbb{R}$ the following limit exists

$$\rho(\lambda) = \lim_{l\to\infty} l^{-d} N(\lambda, A^{(\varepsilon)}(0, \Omega(l))).$$

It does not depend on x_1, x_2 and on the values of the index ε. This limit is called the integrated density of states for the operator A at the point λ.

We impose additional conditions on the function V:

$$V \in C^\infty(\mathbb{R}^d), \qquad |(\nabla^l V)(x)| \le C_l(1+|x|)^{-s-|l|}, \ |l| \ge 0. \tag{5}$$

The main result of the present work gives an asymptotics of the entire spectrum multiplicity of the operator $A(\alpha)$ in the interval (λ_1, λ_2) as $\alpha \to \infty$.

Theorem 1.1. *Let $d \ge 3$ and $\lambda_- < \lambda_1 < \lambda_2 < \lambda_+$. Assume that (4)* *and* (5) *are fulfilled. Then*

$$\lim_{\alpha\to\infty} \alpha^{-d/s}\mathrm{rank}E_{A(\alpha)}(\lambda_1,\lambda_2) = \int_{\mathbb{R}^d} (\rho(\lambda_2+\Phi(\theta)|x|^{-s}) - \rho(\lambda_1+\Phi(\theta)|x|^{-s}))dx. \tag{6}$$

Remark 1.1. For a fixed real number μ

$$\rho(\lambda+\mu)-\rho(\lambda) = O(\lambda^{(d-1)/2}), \qquad \lambda\to\infty.$$

This relation shows that the integral in the right hand side of (6) is finite.

Remark 1.2. Now we discuss the condition $s < 2+2/(d-1)$. The "Dirichlet-Neumann bracketing" method (used in the present paper) has a limit of accuracy. Namely it can be applied only to the asymptotics of the order α^q, $q > (d-1)/2$. In Theorem 1.1 $q = d/s$. This leads to the requirements on s discussed above.

2. Auxiliary result

1. Let $\mathcal{Y} = \{x\in\mathbb{R}^d:\ |x| < \eta\alpha^{1/s}\}$, $d\geq 3$. We write $A^{(\varepsilon)}(\alpha) = A^{(\varepsilon)}(\alpha,\mathcal{Y})$ for brevity. From the inequality $A^{(\mathcal{N})}(\alpha) < A^{(\mathcal{D})}(\alpha)$ we obtain that $N(\lambda, A^{(\mathcal{D})}(\alpha)) \leq N(\lambda, A^{(\mathcal{N})}(\alpha))$ for an arbitrary $\lambda\in\mathbb{R}$. The following statement gives an estimate of the difference of the distribution functions of the spectrum of the operators $A^{(\mathcal{D})}(\alpha)$ and $A^{(\mathcal{N})}(\alpha)$.

Proposition 2.1. *Let* $p\in L^\infty(\mathbb{R}^d)$, $2 < s < 2+s/(d-1)$ *and* $\lambda_1 < \lambda_2$. *Assume that* V *satisfies the conditions of Theorem 1.1. Then there is* $\eta_0 > 0$ *such that for all* $0<\eta<\eta_0$

$$\mu(\eta) := \limsup_{\alpha\to\infty} \alpha^{-d/s}(N(\lambda_2, A^{(\mathcal{N})}(\alpha)) - N(\lambda_1, A^{(\mathcal{D})}(\alpha))) < \infty,$$

where

$$\lim_{\eta\to 0}\mu(\eta) = 0.$$

In particular one has the following estimate

$$\liminf_{\alpha\to\infty} \alpha^{-d/s}(N(\lambda_2, A^{(\mathcal{D})}(\alpha)) - N(\lambda_1, A^{(\mathcal{N})}(\alpha))) \geq -\mu(\eta),$$

Proof. We set

$$\hat{N}(\lambda,\alpha) = (2\pi)^{-d}\omega_d\int_{\mathcal{Y}}(\alpha V(x)+\lambda)_+^{d/2}$$

(ω_d is the volume of the unit ball),

$$F_1(\lambda,\alpha) = |\lambda|^{(d-1)/2}\int_{\mathcal{Y}}(1+|x|)^{-1}dx,$$

$$F_2(\lambda,\alpha) = \int_{\mathcal{Y}} (\alpha V(x)+\lambda)_+^{(d-1)/2}(1+|x|)^{-1}dx$$

and

$$F_3(\lambda,\alpha) = \int_{\mathcal{Y}} (\alpha(1+|x|)^{-s})^{(d-1)/2}(1+|x|)^{-1}dx.$$

Let $N(\lambda,\varepsilon,\alpha)$ denote the number of negative eigenvalues of the operator

$$-\Delta-\lambda-\alpha V, \qquad \lambda\in\mathbb{R}$$

on the domain $\mathcal{Y}$ with the ε-boundary conditions. The following estimate for the distribution function has been established by V. Ivrii. In the present form this estimate follows from Proposition 3 of [5].

Theorem 2.1. *Let $d\geq 3$ and V satisfies (5) for $s>2$. Then for all $\alpha>\alpha_0>0$, $\lambda\in\mathbb{R}$ we have the estimate*

$$|N(\lambda,\mathcal{D},\alpha)-\hat{N}(\lambda,\alpha)| \leq F_1(\lambda,\alpha)+F_2(\lambda,\alpha)+F_3(\lambda,\alpha).$$

We also need the following result:

Lemma 2.1. *Let $d\geq 3$ and V satisfies (5) for $s>2$. Then*

$$\limsup_{\alpha\to\infty}\alpha^{d/s}(N(\lambda,\mathcal{N},\alpha)-N(\lambda,\mathcal{D},\alpha)) \leq C_1(\eta,\lambda).$$

where $C_1(\eta,\lambda)\to 0$ as $\eta\to 0$.

Proof of Lemma 2.1. Let β be a C^∞-function of one variable and

$$\beta(t)=\begin{cases}1 & \text{if} \quad t\leq -1/2,\\ 0 & \text{if} \quad t\geq 0.\end{cases}$$

Define

$$\zeta_1(x)=\beta(|x|-\eta\alpha^{1/s}+1), \qquad x\in\mathbb{R},$$

and find ζ_2 so that $\zeta_1^2+\zeta_2^2=1$. Obviously β can be chosen so that ζ_2 is differentiable. Then

$$\begin{aligned}\int_{\mathcal{Y}}|\nabla u|^2\,dx = \int_{\mathcal{Y}}|\nabla\zeta_1 u|^2\,dx+\int_{\mathcal{Y}}|\nabla\zeta_2 u|^2\,dx - \\ \int_{\mathcal{Y}}(|\nabla\zeta_1|^2+|\nabla\zeta_2|^2)|u|^2\,dx.\end{aligned} \tag{7}$$

Since the mapping

$$L^2(\mathcal{Y})\ni u\mapsto(\zeta_1 u,\,\zeta_2 u)\in L^2(\mathcal{Y})\times L^2(\mathcal{Y})$$

is a one to one correspondence onto the range and for any function W

$$\int_{\mathcal{Y}}W(x)|u|^2\,dx=\int_{\mathcal{Y}}W(x)|\zeta_1 u|^2\,dx+\int_{\mathcal{Y}}W(x)|\zeta_2 u|^2\,dx, \tag{8}$$

we obtain from the variational principle that

$$N(\lambda, \mathcal{N}, \alpha) \leq N(\lambda + c, \mathcal{D}, \alpha) + N(\lambda + c, \mathcal{D}, \mathcal{N}, \alpha) \tag{9}$$

where $c = \sup |\nabla\zeta_1|^2 + |\nabla\zeta_2|^2$ and $N(\lambda + c, \mathcal{D}, \mathcal{N}, \alpha)$ is the number of the negative eigenvalues of the operator $-\Delta - \lambda - c - \alpha V$ with mixed boundary conditions on the boundary of the domain $\Omega_0(\alpha) = \{x \in \mathbb{R}^d : \eta\alpha^{1/s} - 2 < |x| < \eta\alpha^{1/s}\}$. It is interesting to give a more lucid proof of the inequality (9). For that purpose we denote by $\mathfrak{H}_1$ and $\mathfrak{H}_2$ the spaces of eigenfunctions corresponding to negative eigenvalues of the operators $-\Delta - \lambda - c - \alpha V$ on the domains $\mathcal{Y}$ and $\Omega_0(\alpha)$ respectively. In the first case we impose the Dirichlet boundary condition and in the second case the conditions on the boundary are mixed. If (9) is false, then there is a function $u \in H^1(\mathcal{Y})$ which is orthogonal to $\zeta_1\mathfrak{H}_1$ and $\zeta_2\mathfrak{H}_2$ and satisfyies the inequality

$$\int_{\mathcal{Y}} |\nabla u|^2 \, dx \leq \int_{\mathcal{Y}} (\alpha V(x) + \lambda)|u|^2 \, dx,$$

which is impossible because of (7) and (8). Indeed, in this case the function $\zeta_j u$ will be orthogonal to $\mathfrak{H}_j$, which means that

$$\int_{\mathcal{Y}} |\nabla\zeta_j u|^2 \, dx > \int_{\mathcal{Y}} (\alpha V(x) + \lambda + c)|\zeta_j u|^2 \, dx, \qquad j = 1, 2.$$

One can show that

$$N(\lambda + c, \mathcal{D}, \mathcal{N}, \alpha) = O(\mathrm{vol}\, \Omega_0(\alpha)) = O(\alpha^{(d-1)/s}),$$

as $\alpha \to \infty$. Therefore

$$\limsup_{\alpha\to\infty} \alpha^{-d/s}(N(\lambda, \mathcal{N}, \alpha) - N(\lambda, \mathcal{D}, \alpha)) \leq$$

$$\limsup_{\alpha\to\infty} \alpha^{-d/s}(N(\lambda + c, \mathcal{D}, \alpha) - N(\lambda, \mathcal{D}, \alpha)).$$

Since for any λ and index j it holds $F_j(\lambda, \alpha) = o(\alpha^{d/s}), \quad \alpha \to \infty$, and starting from a certain $0 < \eta < \eta_0$

$$\limsup_{\alpha\to\infty} \alpha^{-d/s}\{\hat{N}(\lambda + c, \alpha) - \hat{N}(\lambda, \alpha)\} \leq C_1(\eta, \lambda),$$

$$C_1(\eta, \lambda) \to 0, \quad \text{as} \quad \eta \to 0,$$

the statement of Lemma 2.1 follows. □

The rest of the proof of Proposition 2.1 Since $F_j(\lambda \pm ||p||_\infty, \alpha) = o(\alpha^{d/s}), \quad \alpha \to \infty$, and starting from a certain $0 < \eta < \eta_0$

$$\limsup_{\alpha\to\infty} \alpha^{-d/s}\{\hat{N}(\lambda_2 + ||p||_\infty, \alpha) - \hat{N}(\lambda_1 - ||p||_\infty, \alpha)\} \leq C(\eta),$$

$$C(\eta) \to 0, \quad \text{as} \quad \eta \to 0,$$

we come to the statement of Proposition 2.1 with $\mu(\eta) = C(\eta) + C_1(\eta, \lambda_2 + ||p||_\infty)$. $\square$

3. Proof of Theorem 1.1

1. In this section we prove the relation (6). Necessary auxiliary statements for this are given in Subsections 1 and 2. In Subsection 3 the proof of Theorem 1.1 is completed. Here we follow the way of the article [2] but additionally we use Proposition 2.1.

For $\lambda \in \Lambda$ and $\alpha > 0$ we define a quantity $N_0(\lambda, A(\alpha))$ as a number of eigenvalues of the operator $A(t)$ which cross the point λ with t growing from zero to α (excluding α). It is clear that for $\lambda_1, \lambda_2 \in \Lambda, \lambda_1 < \lambda_2$

$$\mathrm{rank} E_{A(\alpha)}[\lambda_1, \lambda_2) = N_0(\lambda_2, A(\alpha)) - N_0(\lambda_1, A(\alpha)).$$

Therefore it is enough to obtain the asymptotics of the right-hand side, as $\alpha \to \infty$. The following statement plays an important role in the proof of Theorem 1.1.

Theorem 3.1. *Let $\lambda \in \Lambda$ and $\varepsilon > 0$. Assume that $\lambda-\varepsilon, \lambda+\varepsilon \in \Lambda$. Then there exist positive numbers r_0, α_0 and c such that for $r > r_0, \alpha > \alpha_0$ the following inequalities are fulfilled:*

$$\begin{aligned} N_0(\lambda, A(\alpha)) \le N((\lambda+\varepsilon)+0, A^{(\mathcal{N})}(\alpha, \alpha^{1/s}B_r)) - \\ N(\lambda+\varepsilon, A^{(\mathcal{N})}(0, \alpha^{1/s}B_r)) + cr^{d-1}\alpha^{(d-1)/s}, \end{aligned} \tag{10}$$

$$\begin{aligned} N_0(\lambda, A(\alpha)) \ge N(\lambda-\varepsilon, A^{(\mathcal{D})}(\alpha, \alpha^{1/s}B_r)) - \\ N(\lambda-\varepsilon, A^{(\mathcal{D})}(0, \alpha^{1/s}B_r)) - cr^{d-1}\alpha^{(d-1)/s}, \end{aligned} \tag{11}$$

where c does not depend on r and α.

We postpone the proof of Theorem 3.1 until Subsection 4. Actually all the ideas of the proof are given in [2]. In §4 we reproduce them in the exact form.

The following statement gives an asymptotics of the spectral distribution function of the operator on the increasing with the growth of α spherical layer.

Proposition 3.1. *Let $0 < \delta_1 < \delta_2 < \infty$, $\Omega = \{x \in \mathbb{R}^d : \delta_1 < |x| < \delta_2\}$ and $\lambda \in \mathbb{R}$. Then*

$$\lim_{\alpha\to\infty} \alpha^{-d/s} N(\lambda, A^{(\varepsilon)}(\alpha, \alpha^{1/s}\Omega)) = \int_{\delta_1<|x|<\delta_2} \rho(\lambda_2 + \Phi(\theta)|x|^{-s})dx.$$

The *proof* of this statement is similar to the proof of Proposition 3.3 in [4], where the case of spherically symmetric V is considered. In fact all the ideas of this statement are given in Theorem 2.1 of [1]. See also the first remark after Theorem 2.1 [1].

Let $\Omega \subset \mathbb{R}^d$ be a domain with a piecewise smooth boundary and $\lambda_j \in \mathbb{R}, j = 1, 2$. We put

$$n(\lambda_1, \lambda_2, \Omega, \alpha) = N(\lambda_2, A^{(\mathcal{N})}(\alpha, \alpha^{1/s}\Omega)) - N(\lambda_1, A^{(\mathcal{D})}(\alpha, \alpha^{1/s}\Omega)), \ \alpha > 0.$$

Obviously if $\overline{\Omega} = \overline{\Omega_1 \cup \Omega_2}$, then the following inequalities are fulfilled

$$n(\lambda_1, \lambda_2, \Omega, \alpha) \geq -n(\lambda_2, \lambda_1, \Omega_1, \alpha) - n(\lambda_2, \lambda_1, \Omega_2, \alpha) \tag{12}$$

$$n(\lambda_1, \lambda_2, \Omega, \alpha) \leq n(\lambda_1, \lambda_2, \Omega_1, \alpha) + n(\lambda_1, \lambda_2, \Omega_2, \alpha). \tag{13}$$

Proposition 3.2. *Let $r > 0$ and $\lambda_1 < \lambda_2$. Then*

$$\begin{gathered} \lim_{\alpha \to \infty} \alpha^{-d/s} n(\lambda_1, \lambda_2, B_r, \alpha) = \\ \int_{|x|<r} (\rho(\lambda_2 + \Phi(\theta)|x|^{-s}) - \rho(\lambda_1 + \Phi(\theta)|x|^{-s}))dx. \end{gathered} \tag{14}$$

Proof. Let us take $\Omega = B_r, \Omega_1 = B_\eta, \ 0 < \eta < r$ and $\Omega_2 = \Omega \setminus \overline{\Omega_1}$ in (13). Now we multiply the obtained inequality by $\alpha^{-d/s}$ and pass to the upper limit as $\alpha \to \infty$. According to Proposition 2.1 and 3.3 we get

$$\begin{gathered} \limsup_{\alpha \to \infty} \alpha^{-d/s} n(\lambda_1, \lambda_2, \Omega, \alpha) \leq \mu(\eta) + \\ \int_{\eta<|x|<r} (\rho(\lambda_2 + \Phi(\theta)|x|^{-s}) - \rho(\lambda_1 + \Phi(\theta)|x|^{-s}))dx. \end{gathered} \tag{15}$$

Taking $\eta \to 0$ we come to the following estimate

$$\begin{gathered} \limsup_{\alpha \to \infty} \alpha^{-d/s} n(\lambda_1, \lambda_2, \Omega, \alpha) \leq \\ \int_{|x|<r} (\rho(\lambda_2 + \Phi(\theta)|x|^{-s}) - \rho(\lambda_1 + \Phi(\theta)|x|^{-s}))dx. \end{gathered} \tag{16}$$

In the same way we obtain the lower estimate from the relation (12)

$$\begin{gathered} \liminf_{\alpha \to \infty} \alpha^{-d/s} n(\lambda_1, \lambda_2, \Omega, \alpha) \geq \\ \int_{|x|<r} (\rho(\lambda_2 + \Phi(\theta)|x|^{-s}) - \rho(\lambda_1 + \Phi(\theta)|x|^{-s}))dx. \end{gathered} \tag{17}$$

□

Let us combine Proposition 3.1 and 3.4. We get the following result.

Proposition 3.3. *Let* $\lambda_j \in \Lambda, j = 1, 2$, *and* $\varepsilon > 0$. *Assume that* $\lambda_j + \varepsilon \in \Lambda, \lambda_j - \varepsilon \in \Lambda, j = 1, 2$. *Then there exists* $r > 0$ *such that*

$$\limsup_{\alpha\to\infty} \alpha^{-d/s}(N_0(\lambda_2, A(\alpha)) - N_0(\lambda_1, A(\alpha))) \leq \int_{|x|<r} (\rho(\lambda_2 + \varepsilon + \Phi(\theta)|x|^{-s}) - \rho(\lambda_1 - \varepsilon + \Phi(\theta)|x|^{-s}))dx, \tag{18}$$

$$\liminf_{\alpha\to\infty} \alpha^{-d/s}(N_0(\lambda_2, A(\alpha)) - N_0(\lambda_1, A(\alpha))) \geq \int_{|x|<r} (\rho(\lambda_2 - \varepsilon + \Phi(\theta)|x|^{-s}) - \rho(\lambda_1 + \varepsilon + \Phi(\theta)|x|^{-s}))dx \tag{19}$$

Proof. The function $\rho(\lambda)$ is constant for $\lambda \in \Lambda$. Therefore if we put $\Phi(\theta) = 0$ in (14), then we get

$$\lim_{\alpha\to\infty} \alpha^{-d/s}(N(\lambda_2 \pm \varepsilon, A(0, \alpha^{1/s} B_r)) - N(\lambda_1 \mp \varepsilon, A(0, \alpha^{1/s} B_r))) = \mathrm{mes}(B_r)(\rho(\lambda_2 \pm \varepsilon) - \rho(\lambda_1 \mp \varepsilon)) = 0. \tag{20}$$

We subtract the inequality (11) with $\lambda = \lambda_1$ from (10) with $\lambda = \lambda_2$. We multiply the result by $\alpha^{-d/s}$ and pass to the upper limit at $\alpha \to \infty$. According to (14) and (20) we obtain

$$\limsup_{\alpha\to\infty} \alpha^{-d/s}(N_0(\lambda_2, A(\alpha)) - N_0(\lambda_1, A(\alpha))) \leq \limsup_{\alpha\to\infty} \alpha^{-d/s} n(\lambda_1 - \varepsilon, (\lambda_2 + \varepsilon) + 0, B_r, \alpha) = \int_{|x|<r} (\rho(\lambda_2 + \varepsilon + \Phi(\theta)|x|^{-s}) - \rho(\lambda_1 - \varepsilon + \Phi(\theta)|x|^{-s}))dx. \tag{21}$$

This coincides with (18). The inequality (19) can be obtained in the same way. We just have to subtract (10) with $\lambda = \lambda_1$ from (11) with $\lambda = \lambda_2$. □

Let us complete the proof of Theorem 1.1. In (18) and (19) we pass to the limit as $r \to \infty$. Then we take $\varepsilon \to 0$. The passage to the limit as $\varepsilon \to 0$ under the integral is valid since the integrated functions in (18), (19) have an integrable dominanting function for $\varepsilon < \varepsilon_0$. As a result we get

$$\limsup_{\alpha\to\infty} \alpha^{-d/s}(N_0(\lambda_2, A(\alpha)) - N_0(\lambda_1, A(\alpha))) \leq \int (\rho(\lambda_2 + \Phi(\theta)|x|^{-s}) - \rho(\lambda_1 + \Phi(\theta)|x|^{-s}))dx, \tag{22}$$

$$\liminf_{\alpha\to\infty} \alpha^{-d/s}(N_0(\lambda_2, A(\alpha)) - N_0(\lambda_1, A(\alpha))) \geq \int (\rho(\lambda_2 + \Phi(\theta)|x|^{-s}) - \rho(\lambda_1 + \Phi(\theta)|x|^{-s}))dx. \tag{23}$$

It remains to note that $\operatorname{rank} E_{A(\alpha)}(\lambda_1,\lambda_2) = N_0(\lambda_2, A(\alpha)) - N_0(\lambda_1, A(\alpha))$ for $\lambda_1 \in \rho(A(\alpha))$. $\square$

4. Proof of Theorem 3.1

1. We fix an interval $[a,b]$ satisfying the following conditions

$$\lambda-\varepsilon, \lambda+\varepsilon \in (a,b)$$

$$[a,b] \subset \Lambda. \tag{24}$$

Obviously there exist $a', b' \in \mathbb{R}$ such that

$$a' < a,\ \ b' > b,\ \ [a',b'] \subset \Lambda. \tag{25}$$

Let the operators $A^{(\varepsilon)}(\alpha,\Omega)$ be the same as in the section 1. We put $\Pi_r = E_{A^{(\mathcal{D})}(0,B_r)}(a',b'), r>0$. Also let the function $\psi \in C^\infty(\mathbb{R}^d)$ satisfy the following conditions

$$0 \le \psi(x) \le 1, \quad \psi(x) = \begin{cases} 0 & \text{for } \ |x| < 1/2, \\ 1 & \text{for } \ |x| > 3/4. \end{cases}$$

We denote by Ψ_r the multiplication operator by the function $\psi_r(x) = \psi(x/r), r > 0$, on $L^2(\mathbb{R}^d)$ and we put

$$\tilde{A}_r = A^{(\mathcal{D})}(0,B_r) + (b'-a')\Psi_r \Pi_r \Psi_r,\ \ r>0. \tag{26}$$

In [1] the following statement concerning $\tilde{A}_r$ is proved.

Proposition 4.1. *Let the conditions* (24), (25) *be fulfilled and let $\tilde{A}_r$ be defined as in* (26). *Then there exists $\tilde{r_0} > 0$ such that*

$$[a,b] \subset \rho(\tilde{A}_r), \quad \forall r > \tilde{r_0}.$$

We put

$$\tilde{A}_r(\alpha) = \tilde{A}_r - \alpha V. \tag{27}$$

The next statement allows to approximate the quantity $N_0(\lambda, A(\alpha))$ by similar quantities for operators on a ball $B_r, r > 0$. We refer to [2] for its proof.

Proposition 4.2. *Let $\lambda \in (a,b)$ and the conditions* (24), (25) *are fulfilled. Assume that the operator $\tilde{A}_r$ in* (27) *is defined in* (26). *Then*

$$N_0(\lambda, A(\alpha)) \ge \limsup_{r\to\infty}(N(\lambda, \tilde{A}_r(\alpha')) - N(\lambda, \tilde{A}_r)), \qquad 0 < \alpha' < \alpha; \tag{28}$$

$$N_0(\lambda, A(\alpha)) \le \liminf_{r\to\infty}(N(\lambda, \tilde{A}_r(\alpha')) - N(\lambda, \tilde{A}_r)), \quad 0 < \alpha < \alpha'. \tag{29}$$

We introduce the selfadjoint operators $A_{R,r}^{\varepsilon_1,\varepsilon_2} = -\Delta + p$ on $L^2(B_r \setminus B_R)$, $r > R > 0$, where the indexes ε_1 and ε_2, which take the values $\mathcal{D}$ or $\mathcal{N}$, characterize boundary conditions (Dirichlet or Neumann) on ∂B_R and ∂B_r correspondingly. We put also $\tilde{A}_{R,r}^{\varepsilon_1,\varepsilon_2} = A_{R,r}^{\varepsilon_1,\varepsilon_2} + (b' - a')\Psi_r\Pi_r\Psi_r$, $r > R > 0$. The next statement is contained in Lemma 3.2 and Lemma 4.2 of [2].

Proposition 4.3. *Let $\tilde{r_0}$ be as in Proposition 4.1 and $\lambda \in (a, b)$. Then for $r > \tilde{r_0}, 1 \le R \le r/2$ the following inequalities hold*

$$N(\lambda, A^{(\mathcal{D})}(0, B_R)) + N(\lambda, \tilde{A}_{R,r}^{\mathcal{D},\mathcal{D}}) \ge N(\lambda, \tilde{A}_r) - C_1 R^{d-1}, \quad C_1 = C_1(\lambda) \tag{30}$$

$$N(\lambda, A^{(\mathcal{N})}(0, B_R)) + N(\lambda, \tilde{A}_{R,r}^{\mathcal{N},\mathcal{D}}) \le N(\lambda, \tilde{A}_r) + C_2 R^{d-1}, \quad C_2 = C_2(\lambda), \tag{31}$$

where C_1 and C_2 do not depend on r and R.

2. Now let us complete the proof of Theorem 3.1. First of all we establish the relation (10). Denote by v_+ the quantity $\sup_{|x|>1} |x|^{-s}V(x)$. Let $\alpha' > \alpha$ and $r_0 = (v_+/\varepsilon)^{1/s}$. We put $R = m\alpha^{1/s}, m > r_0$. We note that $\alpha V(x) < \varepsilon$ for $|x| > R$. Therefore from the variational principle it follows that

$$N(\lambda, \tilde{A}_r(\alpha')) \le N(\lambda + \varepsilon, A(\alpha', B_R)) + N(\lambda + \varepsilon, \tilde{A}_{R,r}^{\mathcal{N},\mathcal{D}}), \quad R < r \tag{32}$$

Since $\lambda + \varepsilon \in (a, b)$, we obtain from (31) that

$$N(\lambda, \tilde{A}_r) = N(\lambda + \varepsilon, \tilde{A}_r) \ge N(\lambda + \varepsilon, A^{(\mathcal{N})}(0, B_R)) + N(\lambda + \varepsilon, \tilde{A}_{R,r}^{\mathcal{N},\mathcal{D}}) - C_2 R^{d-1}, \quad C_2 = C_2(\lambda + \varepsilon), r > \tilde{r_0}, 1 \le R \le r/2. \tag{33}$$

Estimating the right-hand side of (29) with the help of (32), (33) we get

$$N_0(\lambda, A(\alpha)) \le N(\lambda + \varepsilon, A^{(\mathcal{N})}(\alpha', B_R)) - N(\lambda + \varepsilon, A^{(\mathcal{N})}(0, B_R)) + C_2 R^{d-1}, \; R = m\alpha^{1/s} \ge 1. \tag{34}$$

Taking $\alpha' \to \alpha$ we come to the upper estimate

$$N_0(\lambda, A(E\alpha)) \leq N((\lambda+\varepsilon)+0, A^{(\mathcal{N})}(\alpha, B_R)) - N(\lambda+\varepsilon, A^{(\mathcal{N})}(0, B_R)) + C_2 R^{d-1},\ R = m\alpha^{1/s}, \tag{35}$$

which coincides with (10) for $r = m$ and $c = C_2$. Moreover (35) holds for all $m > r_0, \alpha > \alpha_0 = r_0^{-s}$. The relation (11) can be obtained in the same way from (28) and (30). $\square$

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