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Classical solutions of quasilinear parabolic systems on two dimensional domains

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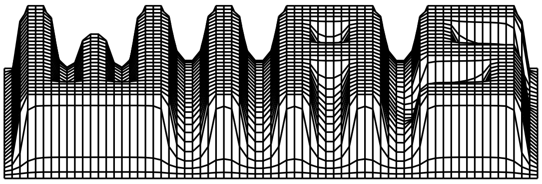

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Key words and phrases. Partial differential equations, quasilinear parabolic systems, nonsmooth domains, mixed boundary conditions, discontinuous coefficients, local classical solutions, reaction–diffusion systems..

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Abstract

Using a classical theorem of Sobolevskii on equations of parabolic type in a Banach space and recently obtained results on elliptic operators with discontinuous coefficients including mixed boundary conditions we prove that quasilinear parabolic systems in diagonal form admit a local, classical solution in the space of p–integrable functions, for some $p > 1$, over a bounded two dimensional space domain. As applications we have in mind systems of reaction diffusion equations, e.g. van Roosbroeck's system. The treatment of such equations in a space of integrable functions enables us to define the normal component of the flow across any part of the Dirichlet boundary by Gauss' theorem.

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Contents

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1 Introduction

In this paper we will study quasilinear parabolic systems of the form

$$
\begin{aligned}
\frac{\partial u_k}{\partial t} - E_k\,(t,\mathbf{u})\,\nabla\cdot\left(F_k\,(t,\mathbf{u})\,\mu_k\nabla u_k\right) &= G_k\,(t,\mathbf{u},\nabla\mathbf{u}) && \text{on } [T_0,T_1]\times\Omega,\\
u_k &= \phi_k && \text{on } [T_0,T_1]\times\partial\Omega\setminus\Gamma_k,\\
\nu\cdot(\mu_k\nabla u_k) &= 0 && \text{on } [T_0,T_1]\times\Gamma_k,\\
u_k(T_0) &= u_k^0 && \text{on } \Omega,\\
&& k = 1,\ldots,l, \quad (1.1)
\end{aligned}
$$

where $\mathbf{u} = (u_1,\ldots,u_l)$ is the vector of unknowns depending on $t \in [T_0,T_1]$ and $x \in \Omega \subset \mathbb{R}^2$, and ν is the outer unit normal to the boundary $\partial\Omega$ of the bounded spatial domain Ω. The functions u_k are subject to initial conditions $u_k^0 = u_k^0(x)$ and in general mixed Neumann and Dirichlet boundary conditions on the parts Γ_k and $\partial\Omega\setminus\Gamma_k$ of the boundary of Ω, respectively; the functions ϕ_k may depend on t. E_k, F_k, and G_k are mappings from a space of $\mathbb{R}^l$ valued functions into a space of real valued functions, depending on t parametrically. In particular they may be Nemytzkii operators

$$E_k\,(t,\mathbf{w})\,(x) := e_k\big(t,x,\mathbf{w}(x)\big), \quad t\in[T_0,T_1], \quad x\in\Omega$$

with suitable functions e_k, $k \in \{1,\ldots,l\}$; mutatis mutandis for F_k and G_k. The μ_k are essentially bounded real 2×2 matrix functions on Ω with an ellipticity condition. We will specify the data of the problem in Assumption 2.1, Assumption 2.3, and Assumption 4.1. Then §4 presents the precise formulation of the problem.

Equations of the form (1.1) are occurring frequently in physics, chemistry and biology, cf. [1] for an overview on the subject. In these fields of applications one often is confronted with nonsmooth domains, discontinuous coefficients and mixed boundary conditions cf. e.g. AMANN [1] and GAJEWSKI/GRÖGER [10]. In this situation equations of the type (1.1) have usually been regarded in negatively indexed Sobolev spaces, cf. [10] and the references cited there. Though this approach often provides a weak solution on $[T_0,T_1]$ one does not know in the end that for any t the divergence of the flow is an integrable function; one only obtains that it is a distribution. However, it would be highly satisfactory to define the normal flow over any part of the Dirichlet boundary by Gauss' theorem. The continuity of the normal component of the flow plays an essential role in connecting potential flow systems

(1.1) to adjacent systems. This problem occurs e.g. in electronic device simulation, cf. GAJEWSKI [4]. In modelling heterogeneous semiconductors it is inevitable to deal with nonsmooth domains, discontinuous coefficients and mixed boundary conditions. Even more, the corresponding problem for pure Dirichlet or pure Neumann boundary conditions is physically irrelevant, cf. e.g. [14] and [4]. Thus, it seems adequate to treat equations (1.1) not only in negatively indexed Sobolev spaces, but also in spaces of integrable functions.

We will demonstrate, cf. Theorem 5.10, that (1.1) admits a local, classical solution which is unique. This solution is once continuously differentiable in t and its derivative with respect to t is integrable over Ω to some exponent $p > 1$. For proving our result we use the classical theorem of SOBOLEVSKII [15] on equations of parabolic type in a Banach space. The problem is that with the discontinuous coefficients μ_k and the mixed boundary conditions we have in mind one cannot use standard regularity results. However, this difficulty can be overcome by means of recently obtained resolvent estimates for second order elliptic differential operators with essentially bounded coefficients, cf. [7], and interpolation results for function spaces related to mixed boundary value problems, cf. [6]. In this paper we will not deal with global existence of solutions to (1.1). Indeed, it cannot be generally expected in such a wide spread class of problems.

Of course, the question arises why our investigation is restricted to the case of two spatial dimensions and how relevant this case is for applications. Systems of the form (1.1) on two dimensional spatial domains are very useful models for the simulation of many processes in natural and technical sciences. In engineering the essentially two dimensional structure of the problem in space often follows the applied technology. Inspecting the proofs in this paper readers will realise that similar proofs in three spatial dimensions would require (at least) that the gradients of solutions for the elliptic boundary value problem associated to (1.1) are integrable to an exponent $p > 3$. However, this cannot be expected in general, not even for arbitrary step functions μ_k whose—finitely many—constancy domains are arbitrary polyhedral subdomains of Ω, cf. e.g. [3].

2 Notations and general assumptions

We start this section by specifying the spatial domain under consideration.

2.1 Assumption. Let $\Omega \subset \mathbb{R}^2$ be a fixed bounded Lipschitz domain. Further, for any $k \in \{1, \ldots, l\}$ there is a boundary part $\Gamma_k \subset \partial\Omega$ which may be empty and which is the union of a finite set of open arc pieces such that no component of $\partial\Omega \setminus \Gamma_k$ consists only of a single point.

This class of domains Ω and parts Γ_k of its boundary $\partial\Omega$ to be contacted from the outside covers almost all spatially two dimensional problems encountered in science and engineering.

2.2 Remark. Assumption 2.1 implies the following condition on Ω and the sets Γ_k: For every point $x \in \partial\Omega$ there exist an open neighbourhood $\mathcal{U} \subset \mathbb{R}^2$ of x, another open set $\mathcal{O} \subset \mathbb{R}^2$, and a bi–Lipschitz transformation $\mathcal{L}$ from $\mathcal{U}$ onto $\mathcal{O}$ with almost everywhere constant absolute value of the functional determinant such that $\mathcal{L}\big(\mathcal{U} \cap (\Omega \cup \Gamma_k)\big)$ coincides either with the open unit half ball in R^2, the open unit half ball united with its ground plate, or the open unit half ball united with half of its ground plate, cf. [18, Ch. I Thm. 2.5]. These three model sets may be replaced by just two model sets, namely the open unit ball and the open unit half ball united with its ground plate, cf. [6]. Thus Assumption 2.1 implies for each $k \in \{1, \ldots, l\}$ the regularity of the set $\Omega \cup \Gamma_k$ in the sense of GRÖGER, cf. [8], [9].

If p is from $[1, \infty[$, then we denote the space of real, Lebesgue measurable, p–integrable functions on Ω by L^p. Moreover, we denote the space of real Lebesgue measurable, essentially bounded functions on Ω by L^∞. By $\mathbf{L}^p = L^p \times \ldots \times L^p$ we denote the l times direct product of the spaces L^p, $p \in [1, \infty]$. If $s \in [0, 1]$ and $q \in]1, \infty[$, then we denote—as usual, cf. [17]—by $H^{s,q}$ the space of real Bessel potentials with the differentiability index s and the integrability index q on the set Ω. N.B. for $s = 1$ these spaces coincide with the Sobolev spaces $W^{1,q}(\Omega)$. We define $H^{s,q}_k$ as the closure in $H^{s,q}$ of the set

$$\left\{u|_\Omega : u \in C_0^\infty(\mathbb{R}^2),\, \operatorname{supp} u \cap \big(\overline{\Omega} \setminus (\Omega \cup \Gamma_k)\big) = \emptyset\right\}.$$

If $s \in [-1, 0[$, then $H^{s,q}_k$ denotes the dual to $H^{-s,q'}_k$, where $\frac{1}{q} + \frac{1}{q'} = 1$. We abbreviate by $\mathbf{H}^{s,q} = H^{s,q} \times \ldots \times H^{s,q}$ the l times direct product of the spaces $H^{s,q}$ and by $\mathbf{H}^{s,q}_0 = H^{s,q}_1 \times \ldots \times H^{s,q}_l$ the direct product of the spaces $H^{s,q}_k$.

Apart of the real function spaces we introduce their complex analoga: If Z is a real Banach space, we denote by $\widetilde{Z}$ its complexification.

For two Banach spaces X and Y we denote the space of linear, bounded operators from X into Y by $\mathcal{B}(X; Y)$. If $X = Y$, then we abbreviate $\mathcal{B}(X)$. If $X \subset Y$

continuously embeds into Y, then we write $X \hookrightarrow Y$.

$$C([t_1,t_2];X), \quad C^\eta([t_1,t_2];X), \quad C^1([t_1,t_2];X), \qquad \eta \in]0,1[$$

denote the spaces of continuous, Hölder continuous, and once continuously differentiable functions, respectively, on the closed interval $[t_1,t_2]$ with values in the Banach space X. The symbols $\langle\cdot,\cdot\rangle_{\mathbb{R}^2}$ and $\langle\cdot,\cdot\rangle_{\mathbb{C}^2}$ denote the canonic bilinear forms on $\mathbb{R}^2$ and $\mathbb{C}^2$, respectively.

2.3 Assumption. $\mu_k : \Omega \longrightarrow \mathcal{B}(\mathbb{R}^2)$, $k \in \{1,\dots,l\}$ are measurable mappings into the set of real, symmetric 2×2 matrices, satisfying the relations

$$\begin{aligned} &\max_{k\in\{1,\dots,l\}} \operatorname*{vraimax}_{x\in\Omega} \|\mu_k(x)\|_{\mathcal{B}(\mathbb{R}^2)} \le \mu^\bullet \\ &\min_{k\in\{1,\dots,l\}} \operatorname*{vraimin}_{x\in\Omega} \langle\mu_k(x)\xi,\xi\rangle_{\mathbb{R}^2} \ge \mu_\bullet \|\xi\|^2_{\mathbb{R}^2} \qquad \forall\xi\in\mathbb{R}^2 \end{aligned} \tag{2.1}$$

for two strictly positive constants $\mu_\bullet$ and $\mu^\bullet$.

2.4 Definition. For each $k \in \{1,\dots,l\}$ we define the operator

$$a_k : \widetilde{H}^{1,2}_k \longrightarrow \widetilde{H}^{-1,2}_k \tag{2.2}$$

by

$$\langle a_k\psi_1,\psi_2\rangle = \int_\Omega \langle\mu_k\nabla\psi_1,\nabla\psi_2\rangle_{\mathbb{C}^2}\,dx, \tag{2.3}$$

where $\langle\cdot,\cdot\rangle$ on the left hand side denotes the dual pairing between $\widetilde{H}^{1,2}_k$ and $\widetilde{H}^{-1,2}_k$. Further, we denote by $\mathbf{a}$ the operator

$$\widetilde{\mathbf{H}}^{1,2}_0 \ni (\psi_1,\dots,\psi_l) \longmapsto (a_1\psi_1,\dots,a_l\psi_l) \in \widetilde{\mathbf{H}}^{-1,2}_0. \tag{2.4}$$

The restrictions of the operators a_k, $k\in\{1,\dots,l\}$ and $\mathbf{a}$ to the real domains $H^{1,2}_k$ and $\mathbf{H}^{1,2}_0$, respectively, map into the (real) spaces $H^{-1,2}_k$ and $\mathbf{H}^{-1,2}_0$, respectively, and will be denoted by the same symbols.

2.5 Definition. Suppose p to be a fixed number from the interval $]1,\infty[$. Then for each $k\in\{1,\dots,l\}$ we define the following restriction of the operator a_k from Definition 2.4:

$$A_k : \psi \mapsto a_k\psi \qquad \psi \in \operatorname{dom} A_k = \left\{\psi\in\widetilde{H}^{1,2}_k \,:\, a_k\psi\in\widetilde{L}^p\right\}. \tag{2.5}$$

Further, we define

$$\begin{aligned} &\mathbf{A} : \mathcal{D} := \operatorname{dom}\mathbf{A} = \operatorname{dom}A_1\times\ldots\times\operatorname{dom}A_l \longrightarrow \widetilde{\mathbf{L}}^p \\ &\mathcal{D} \ni (\psi_1,\dots,\psi_l) \longmapsto \mathbf{A}(\psi_1,\dots,\psi_l) := (A_1\psi_1,\dots,A_l\psi_l) \in \widetilde{\mathbf{L}}^p \end{aligned} \tag{2.6}$$

The domain of the operator A_k will be equipped with the norm $\|(A_k + 1)\cdot\|_{\widetilde{L}^p}$ which is equivalent to the graph norm of A_k. The restrictions of the operators A_k, $k \in \{1,\dots,l\}$ and $\mathbf{A}$ to the real domains map into the (real) spaces L^p and $\mathbf{L}^p$, respectively, and will be denoted by the same symbols.

2.6 Remark. Definition 2.4 incorporates in the usual way, cf. [5] or [2], homogeneous Neumann boundary conditions on Γ_k and homogeneous Dirichlet boundary conditions on $\partial\Omega \setminus \Gamma_k$ for functions from $\operatorname{dom} A_k$, $k \in \{1,\dots,l\}$.

3 Functional analytic tools

We outline in this section the functional analytic background of our investigation. In particular, we specify the adequate Banach spaces for the mathematical treatment of the system formally introduced above. We start by quoting a recently obtained interpolation result for spaces representing mixed boundary conditions and gather some regularity results on elliptic and parabolic equations. Throughout this section we suppose $k \in \{1,\dots,l\}$ and Assumption 2.1, and Assumption 2.3 apply.

3.1 Proposition. [6]. *If*

$$\theta \in]0,1[, \qquad q \in]1,\infty[, \qquad s_0,\, s_1 \in [-1,1] \setminus \left\{-\tfrac{1}{q'},\, \tfrac{1}{q}\right\},$$

then there is the following complex interpolation identity:

$$\left[\widetilde{H}_k^{s_0,q}, \widetilde{H}_k^{s_1,q}\right]_\theta = \widetilde{H}_k^{s,q}, \qquad s = (1-\theta)s_0 + \theta s_1. \tag{3.1}$$

3.2 Proposition. [11]. *For each $k \in \{1,\dots,l\}$ there is a number*

$$q_k = q_k(\Omega, \Gamma_k, \mu_k) > 2$$

such that for any $q \in [2, q_k]$

i) the maximal restriction of the operator a_k from Definition 2.4 which still maps into $\widetilde{H}_k^{-1,q}$ provides a topological isomorphism—we denote it also by a_k—between $\widetilde{H}_k^{1,q}$ and $\widetilde{H}_k^{-1,q}$; and

ii) the negative of this isomorphism generates an analytic semigroup on $\widetilde{H}_k^{-1,q}$, i.e.

$$\sup_{\Re z \ge 0} (1+|z|)\left\|(a_k + 1 + z)^{-1}\right\|_{\mathcal{B}(\widetilde{H}_k^{-1,q})} < \infty. \tag{3.2}$$

3.3 Remark. The numbers q_k may be arbitrarily close to 2, even in the case of pure, homogeneous Dirichlet boundary conditions, cf. [13, §5].

3.4 Proposition. [7, §5]. *Let $U \in L^\infty$ be a function with strictly positive essential upper and lower bounds $U^\bullet$ and $U_\bullet$, respectively. If $p \in]1,\infty[$, and A_k, $k \in \{1,\dots,l\}$ is one of the operators from Definition 2.5, then the operator UA_k is densely defined and satisfies the following resolvent estimate for $z \in \mathbb{C}$ with $\Re z \geq 0$:*

$$\left\|(UA_k + 1 + z)^{-1}\right\|_{\mathcal{B}(\widetilde{L}^p)} \leq \frac{U^\bullet}{U_\bullet}\,\Lambda\left(\frac{\mu^\bullet}{\mu_\bullet}, p\right)\,\frac{1}{1+|z|}, \tag{3.3}$$

with some explicitly given continuous function

$$\Lambda \;:\; [1,\infty[\times]1,\infty[\longrightarrow]0,\infty[.$$

3.5 Proposition. *Let X be a Banach space and let B be a densely defined operator on X satisfying the resolvent estimate*

$$\sup_{t\in[0,\infty[} (1+t)\left\|(B+t)^{-1}\right\|_{\mathcal{B}(X)} < \infty.$$

If $0 \leq \alpha < \beta \leq 1$, then

$$[X, \operatorname{dom} B]_\beta \hookrightarrow \operatorname{dom} B^\alpha, \tag{3.4}$$

and

$$\operatorname{dom} B^\beta \hookrightarrow [X, \operatorname{dom} B]_\alpha, \tag{3.5}$$

the domains of B, B^α, and B^β being topologized by a norm equivalent to the graph norm of the corresponding operator.

Proof. The assertions are obtained from [17, 1.15.2, 1.10.3, and 1.3.3]. □

3.6 Lemma. *Let X and Y be Banach spaces, let us assume that Y embeds densely and continuously into X, and let $B : Y \to X$ be a linear, topological isomorphism which satisfies*

$$\sup_{t\in[0,\infty[} (1+t)\left\|(B+t)^{-1}\right\|_{\mathcal{B}(X)} < \infty. \tag{3.6}$$

Furthermore, let $X_0 \supset Y$ be another Banach space which embeds continuously into the complex interpolation space $[X,Y]_\theta$ for some $\theta \in]0,1[$. We denote the maximal restriction of B which maps into X_0 by B_0. As $Y \subset X_0$, the domain of B_0 is a subspace of X_0. We assume that this subspace is dense in X_0 and we suppose

$$\sup_{t\in[0,\infty[} (1+t)\left\|(B_0+t)^{-1}\right\|_{\mathcal{B}(X_0)} < \infty. \tag{3.7}$$

Then both, B^α and B_0^α are well defined for any $\alpha \in [0,1]$. Moreover, if $\alpha > 1-\theta$, then the complex interpolation space $[X_0, \operatorname{dom} B_0]_\alpha$ continuously embeds into Y.

Proof. The first assertion is well known, cf. [17, 1.15.1]. In order to prove the second assertion we regard an arbitrary $\alpha \in]1-\theta, 1[$ and some β with $\alpha > 1-\beta > 1-\theta$. Then there are constants $\delta_1, \ldots, \delta_7$ such that for any w from the complex interpolation space $[X_0, \operatorname{dom} B_0]_\alpha$ the following is true (read it from right to left):

$$
\begin{aligned}
\|w\|_Y \le \left\|B^{-1}\right\|_{\mathcal{B}(X;Y)} \|Bw\|_X = \delta_1 \left\|B^\beta B^{1-\beta} w\right\|_X \le \delta_2 \left\|B^{1-\beta} w\right\|_{\operatorname{dom} B^\beta} \\
\le \delta_3 \left\|B^{1-\beta} w\right\|_{[X, \operatorname{dom} B]_\theta} \le \delta_4 \left\|B^{1-\beta} w\right\|_{[X,Y]_\theta} \le \delta_5 \left\|B^{1-\beta} w\right\|_{X_0} \\
= \delta_5 \left\|B_0^{1-\beta} w\right\|_{X_0} \le \delta_6 \left\|w\right\|_{\operatorname{dom} B_0^{1-\beta}} \le \delta_7 \left\|w\right\|_{[X_0, \operatorname{dom} B_0]_\alpha},
\end{aligned}
$$

cf. (3.4) from Proposition 3.5, the presupposed continuity of the embedding $X_0 \hookrightarrow [X, Y]_\theta$, and the equivalence of $\|\cdot\|_Y$ with the graph norm of B due to B being a topological isomorphism from Y into X. N.B. under the preconditions of Lemma 3.6 B_0^δ is the maximal restriction of B^δ which maps into X_0, for any $\delta \in]0, 1[$. Indeed, there is

$$
(B + t)^{-1}\big|_{X_0} = (B_0 + t)^{-1} \qquad \text{for all } t \in [0, \infty[
$$

according to the definition of the operator B_0; hence,

$$
B^{-\delta}\big|_{X_0} = B_0^{-\delta} \qquad \text{for all } \delta \in]0, 1[, \tag{3.8}
$$

cf. [17, 1.15.1]. Now one passes over to the inverse in (3.8). □

3.7 Proposition. SOBOLEVSKII [15]. *Let $\mathcal{A}_0$ be a linear operator on a (complex) Banach space X with dense domain* $\operatorname{dom} \mathcal{A}_0$*, and let us assume that $\mathcal{A}_0$ admits the resolvent estimate*

$$
\sup_{\Re z \ge 0} (1 + |z|) \|(\mathcal{A}_0 + z)^{-1}\|_{\mathcal{B}(X)} < \infty. \tag{3.9}
$$

Suppose $1 \ge \beta > \alpha \ge 0$ and $v_0 \in \operatorname{dom} \mathcal{A}_0^\beta$. Additionally, let

$$
[T_0, T_1] \times \operatorname{dom} \mathcal{A}_0^\alpha \ni (t, v) \longmapsto \mathcal{A}(t, v) \in \mathcal{B}(\operatorname{dom} \mathcal{A}_0, X)
$$

be a mapping satisfying $\mathcal{A}(T_0, v_0) = \mathcal{A}_0$ and for some number $R > 0$:

$$
\begin{aligned}
&\left\|\left(\mathcal{A}(t_1, \mathcal{A}_0^{-\alpha} v_1) - \mathcal{A}(t_2, \mathcal{A}_0^{-\alpha} v_2)\right) \mathcal{A}_0^{-1}\right\|_{\mathcal{B}(X)} \le C_R \left(|t_1 - t_2|^\eta + \|v_1 - v_2\|_X\right) \\
&\quad \text{for all } t_1,\, t_2 \in [T_0, T_1] \text{ and all } v_1,\, v_2 \in X \text{ with } \|v_1\|_X \le R,\ \|v_2\|_X \le R,
\end{aligned} \tag{3.10}
$$

where C_R is a constant. Finally, let

$$
[T_0, T_1] \times \operatorname{dom} \mathcal{A}_0^\alpha \ni (t, v) \longmapsto f(t, v) \in X
$$

be a mapping obeying

$$\left\|f(t_1,\mathcal{A}_0^{-\alpha}v_1)-f(t_2,\mathcal{A}_0^{-\alpha}v_2)\right\|_X \le C_R\left(|t_1-t_2|^\eta+\|v_1-v_2\|_X\right)$$
$$\text{for all } t_1,\, t_2\in[T_0,T_1] \text{ and all } v_1,\, v_2\in X \text{ with } \|v_1\|_X\le R,\ \|v_2\|_X\le R. \tag{3.11}$$

If $\|\mathcal{A}_0^\alpha v_0\|_X < R$, *then there is an interval* $[T_0,T]$, $T>T_0$, *such that the equation*

$$\frac{\partial v}{\partial t}-\mathcal{A}(t,v(t))v=f(t,v), \qquad v(T_0)=v_0 \tag{3.12}$$

admits exactly one solution on $[T_0,T]$ *which belongs to the space*

$$C([T_0,T];\operatorname{dom}\mathcal{A}_0^\alpha)\cap C(]T_0,T];\operatorname{dom}\mathcal{A}_0)\cap C^1(]T_0,T];X). \tag{3.13}$$

4 Exact formulation of the problem

At first we will make precise assumptions about the operators E_k, F_k, and G_k, $k\in\{1,\dots,l\}$, and the boundary conditions. We always make Assumption 2.1 and Assumption 2.3.

4.1 Assumption. With respect to the numbers q_k from Proposition 3.2 there are numbers q, η, and γ with

$$2<q\le\min_{k\in\{1,\dots,l\}}q_k, \quad 0<\eta\le1, \quad \frac{1}{2}+\frac{1}{q}<\gamma<1, \qquad p:=\frac{q}{2} \tag{4.1}$$

such that:

i) E_k, $k\in\{1,\dots,l\}$, maps $[T_0,T_1]\times\mathbf{H}^{1,q}$ into the set $L^\infty\cap\{\psi\ :\ \psi>0\}$. For any bounded set $M\subset\mathbf{H}^{1,q}$ there is

$$\min_{k\in\{1,\dots,l\}}\ \inf_{t\,\in\,[T_0,T_1],\ \mathbf{w}\,\in\,M}\ \operatorname*{vraimin}_{x\in\Omega}E_k\left(t,\mathbf{w}\right)(x)>0, \tag{4.2}$$

and a constant $\mathcal{E}_M$ such that

$$\|E_k\left(t_1,\mathbf{w}_1\right)-E_k\left(t_2,\mathbf{w}_2\right)\|_{L^\infty}\le\mathcal{E}_M\left(|t_1-t_2|^\eta+\|\mathbf{w}_1-\mathbf{w}_2\|_{\mathbf{H}^{1,q}}\right) \tag{4.3}$$

for all $k\in\{1,\dots,l\}$, t_1, $t_2\in[T_0,T_1]$ and $\mathbf{w}_1$, $\mathbf{w}_2\in M$.

ii) F_k, $k\in\{1,\dots,l\}$, maps $[T_0,T_1]\times\mathbf{H}^{1,q}$ into the set $H^{1,q}\cap\{\psi\ :\ \psi>0\}$. For any bounded set $M\subset\mathbf{H}^{1,q}$ there is

$$\min_{k\in\{1,\dots,l\}}\ \inf_{t\,\in\,[T_0,T_1],\ \mathbf{w}\,\in\,M}\ \inf_{x\in\Omega}F_k\left(t,\mathbf{w}\right)(x)>0. \tag{4.4}$$

and a constant $\mathcal{F}_M$ such that

$$\|F_k(t_1, \mathbf{w}_1) - F_k(t_2, \mathbf{w}_2)\|_{H^{1,q}} \le \mathcal{F}_M \left(|t_1 - t_2|^\eta + \|\mathbf{w}_1 - \mathbf{w}_2\|_{\mathbf{H}^{1,q}}\right) \tag{4.5}$$

for all $k \in \{1, \dots, l\}$, t_1, $t_2 \in [T_0, T_1]$ and $\mathbf{w}_1$, $\mathbf{w}_2 \in M$.

iii) G_k, $k \in \{1, \dots, l\}$, maps $[T_0, T_1] \times \mathbf{H}^{1,q}$ into L^p. Moreover, for any bounded set $M \subset \mathbf{H}^{1,q}$ there is a constant $\mathcal{G}_M$ such that

$$\|G_k(t_1, \mathbf{w}_1) - G_k(t_2, \mathbf{w}_2)\|_{L^p} \le \mathcal{G}_M \left(|t_1 - t_2|^\eta + \|\mathbf{w}_1 - \mathbf{w}_2\|_{\mathbf{H}^{1,q}}\right) \tag{4.6}$$

for all $k \in \{1, \dots, l\}$, t_1, $t_2 \in [T_0, T_1]$, and $\mathbf{w}_1$, $\mathbf{w}_2 \in M$.

iv) There are functions $\Phi_k \in C^\eta([T_0, T_1]; H^{1,q})$, $k \in \{1, \dots, l\}$, representing the inhomogeneous Dirichlet boundary conditions for the u_k. For all $t \in [T_0, T_1]$ the distributional derivatives $\nabla \cdot (\mu_k \nabla \Phi_k(t))$ belong to L^p and there is in the sense of traces, cf. [16, Ch. 1],

$$\Phi_k(t)|_{\partial\Omega \setminus \Gamma_k} = \phi_k(t) \quad \text{and} \quad \nu \cdot (\mu_k \nabla \Phi_k(t)) = 0 \quad \text{on } \Gamma_k.$$

Moreover, the mapping

$$[T_0, T_1] \ni t \mapsto \nabla \cdot (\mu_k \nabla \Phi_k(t)) \in L^p \qquad k \in \{1, \dots, l\}$$

is from $C^\eta([T_0, T_1]; L^p)$. Further,

$$\Phi_k \in C^1([T_0, T_1]; L^p), \qquad \frac{\partial \Phi_k}{\partial t} \in C^\eta([T_0, T_1]; L^p), \qquad k \in \{1, \dots, l\}.$$

In the sequel we will denote by $\mathbf{\Phi}(t)$ the vector $\big(\Phi_1(t), \dots, \Phi_l(t)\big)$.

v) If $\mathbf{u}_0 = (u_1^0, \dots, u_l^0)$, then $\mathbf{u}_0 - \mathbf{\Phi}(T_0)$ belongs to the complex interpolation space $\big[\widetilde{\mathbf{L}}^p, \mathcal{D}\big]_\gamma$ for some number γ, cf. (4.1). N.B. $\mathcal{D}$ is the domain of the operator $\mathbf{A}$ from Definition 2.5.

4.2 Remark. In the formal equation (1.1) G_k was allowed to depend on the gradient of the solution while this dependence here seems to be forbidden. In fact, this is not the case as operators such as e.g. $G_k(t, \mathbf{u}) := \|\nabla \mathbf{u}\|^2_{\mathbb{R}^{2l}}$ are covered by Assumption 4.1.

4.3 Example. Let $\xi : [T_0, T_1] \times \mathbb{R} \to]0, \infty[$ be a mapping which satisfies the following conditions:

i) for any $R > 0$ there is

$$\inf_{(t,s)\in[T_0,T_1]\times[-R,R]} \xi(t,s) > 0;$$

ii) for any fixed $t \in [T_0, T_1]$ the function $\xi(t,\cdot)$ is twice differentiable;

iii) there is a number $\eta \in]0,1]$ and for any $R > 0$ there is a constant c_R such that

$$|\xi(t_1,s) - \xi(t_2,s)| + |\partial_2\xi(t_1,s) - \partial_2\xi(t_2,s)| \le c_R|t_1 - t_2|^\eta$$

for all $s \in [-R, R]$ and all t_1, $t_2 \in [T_0, T_1]$;

iv) for any $R > 0$ there is

$$\sup_{(t,s)\in[T_0,T_1]\times[-R,R]} \left|\partial_2^2\xi(t,s)\right| < \infty.$$

If $w \in H^{1,q}$, $2 < q \le q_1$, and $F_1(t,w)$ denotes the Nemytzkii operator $F_1(t,w)(x) := \xi\big(t, w(x)\big)$, induced by the function $\xi(t,\cdot)$, then the mapping $(t,w) \mapsto E_1(t,w)$ satisfies Assumption 4.1.ii.

One can generalize Example 4.3 in a straightforward way to a Nemytzkii operator $F_1(t,w)(x) := \xi\big(t,x,w(x)\big)$ induced by a function $\xi : [T_0,T_1]\times\Omega\times\mathbb{R} \to]0,\infty[$. The next example demonstrates that Assumption 4.1 does not restrict the mappings E_k, F_k, and G_k to Nemytzkii operators.

4.4 Example. Let Γ_* and μ_* be a part of $\partial\Omega$ and a real 2×2 matrix function to which apply mutatis mutandis Assumption 2.1 and Assumption 2.3, respectively, and let $a_* : H^{1,2}_* \to H^{-1,2}_*$ be the corresponding operator as in Definition 2.4 and let q_* be the associated number from Proposition 3.2. Suppose $\mathfrak{F}_1$ and $\mathfrak{F}_2$ to be mappings from $\mathbb{R}$ into $]0,\infty[$ which are either multiples of exponentials or strictly monotonous, twice continuously differentiable, polynomially bounded functions (e.g. Fermi's integral to the index one half). Further, let

$$w_1,\, w_2 \in H^{1,q}, \qquad d \in H^{-1,q}_*, \qquad 2 < q \le \min\{q_*,\, q_1,\, q_2\}$$

be given, and let $\mathcal{P}$ be the operator which assigns to the tuple (w_1, w_2) the solution φ of the equation

$$a_*\varphi + \mathfrak{F}_1(w_1 + \varphi) - \mathfrak{F}_2(w_2 - \varphi) = d.$$

This solution exists and is unique because the operator

$$H^{1,2}_* \ni \varphi \mapsto a_*\varphi + \mathfrak{F}_1(w_1 + \varphi) - \mathfrak{F}_2(w_2 - \varphi) \in H^{-1,2}_*$$

is well defined and strongly monotone. Moreover, as a_* provides an isomorphism between $H^{1,q}_*$ and $H^{-1,q}_*$, cf. Proposition 3.2, φ even belongs to $H^{1,q}_* \subset H^{1,q}$. If one defines

$$F_1\big(t,(w_1,w_2)\big) := \mathfrak{F}_1\big(w_1 + \mathcal{P}(w_1,w_2)\big), \quad F_2\big(t,(w_1,w_2)\big) := \mathfrak{F}_2\big(w_2 - \mathcal{P}(w_1,w_2)\big),$$

then F_1, F_2 satisfy Assumption 4.1.ii.

4.5 Remark. The importance of the first example is evident. The second example plays a role in the theory of van Roosbroeck's system which describes the flow of electrons and holes in semiconductors, cf. e.g. [14] and [4]. Based upon Theorem 5.10 we shall derive a local, unique solution of van Roosbroeck's system on two dimensional domains in a forthcoming paper [12].

4.6 Definition. We fix from now on the numbers q, η, γ, and $p := \frac{q}{2}$ according to Assumption 4.1 and the operators $A_k : \operatorname{dom} A_k \to \widetilde{L}^p$ and $\mathbf{A} : \mathcal{D} \to \widetilde{\mathbf{L}}^p$ from Definition 2.5 always will refer to this number p. Furthermore, let $\mathbf{P} : \widetilde{\mathbf{H}}^{1,q} \to \mathbf{H}^{1,q}$ be the mapping onto the real part of $\widetilde{\mathbf{H}}^{1,q}$ which takes componentwise the real part of the function, and let $Q : L^p \to \widetilde{L}^p$ be the canonic embedding of the real space into the complex one.

Now we present a precise formulation of the formal equations (1.1).

4.7 Definition. We say that $\mathbf{u} = \mathbf{v} + \mathbf{\Phi}$ is a local solution of (1.1), if there is a solution $\mathbf{v}$ to the following

4.8 Problem. Determine a number $T \in]T_0, T_1]$ and a function

$$\mathbf{v} \in C([T_0,T],\mathbf{L}^p) \cap C(]T_0,T],\mathbf{P}\mathcal{D}) \cap C^1(]T_0,T],\mathbf{L}^p) \tag{4.7}$$

such that

$$\mathbf{v}(T_0) = \mathbf{u}_0 - \mathbf{\Phi}(T_0), \tag{4.8}$$

and

$$\begin{aligned}
\frac{\partial v_k}{\partial t}(t) &+ \left(E_k\left(t,\mathbf{v}(t)+\mathbf{\Phi}(t)\right) F_k\left(t,\mathbf{v}(t)+\mathbf{\Phi}(t)\right) A_k + 1\right) v_k(t) \\
&= v_k(t) - \frac{\partial \Phi_k}{\partial t}(t) \\
&+ E_k\left(t,\mathbf{v}(t)+\mathbf{\Phi}(t)\right) \langle \nabla F_k\left(t,\mathbf{v}(t)+\mathbf{\Phi}(t)\right), \mu_k \nabla v_k(t)\rangle_{\mathbb{R}^2} \\
&+ G_k\left(t,\mathbf{v}(t)+\mathbf{\Phi}(t)\right) \\
&- E_k\left(t,\mathbf{v}(t)+\mathbf{\Phi}(t)\right) F_k\left(t,\mathbf{v}(t)+\mathbf{\Phi}(t)\right) \nabla\cdot\left(\mu_k \nabla \Phi_k(t)\right) \\
&+ E_k\left(t,\mathbf{v}(t)+\mathbf{\Phi}(t)\right) \langle \nabla F_k\left(t,\mathbf{v}(t)+\mathbf{\Phi}(t)\right), \mu_k \nabla \Phi_k(t)\rangle_{\mathbb{R}^2}
\end{aligned} \tag{4.9}$$

for all $k \in \{1,\dots,l\}$ on $]T_0,T]$.

4.9 Remark. We have split up

$$\begin{aligned}-E_k(t,\mathbf{v}+\mathbf{\Phi})\,\nabla\cdot\left(F_k(t,\mathbf{v}+\mathbf{\Phi})\,\mu_k\nabla v_k\right)&\\ = E_k(t,\mathbf{v}+\mathbf{\Phi})\,F_k(t,\mathbf{v}+\mathbf{\Phi})\,A_k v_k&\\ - E_k(t,\mathbf{v}+\mathbf{\Phi})\,\langle\nabla F_k(t,\mathbf{v}+\mathbf{\Phi}),\mu_k\nabla v_k\rangle_{\mathbb{R}^2}\,.&\end{aligned}$$

This corresponds to the definition of the mappings a_k and A_k, cf. Definition 2.4 and Definition 2.5, respectively, and the differentiation rules for distributions.

In Problem 4.8 an initial value problem for a system of operator differential equations in the real space $\mathbf{L}^p$ has been formulated. However, the methods for its solution operate in complex Banach spaces, cf. Proposition 3.7. That's why we now pass over to a complex version of the problem.

4.10 Definition. For any $(t,\mathbf{w})\in[T_0,T_1]\times\widetilde{\mathbf{H}}^{1,q}$ we define

$$\widetilde{E}_k(t,\mathbf{w}):=QE_k(t,\mathbf{P}\mathbf{w})\,,\quad \widetilde{F}_k(t,\mathbf{w}):=QF_k(t,\mathbf{P}\mathbf{w})\,,\quad \widetilde{G}_k(t,\mathbf{w}):=QG_k(t,\mathbf{P}\mathbf{w})\,,$$

where $\mathbf{P}$ and Q are the operators from Definition 4.6.

4.11 Lemma. *If Assumption 4.1 applies to E_k, F_k, G_k, and Φ_k, $k\in\{1,\dots,l\}$, then Assumption 4.1 applies to $\widetilde{E}_k$, $\widetilde{F}_k$, $\widetilde{G}_k$, and $Q\Phi_k$ with the complexified function spaces.*

For the sake of simplicity, we denote the complexified functions $Q\Phi_k$ and the vector $\big(Q\Phi_1,\dots,Q\Phi_l\big)$ again by Φ_k and $\mathbf{\Phi}$, respectively. Furthermore, in referring to Assumption 4.1 we implicitly also refer to Lemma 4.11.

4.12 Problem. Determine a number $T\in]T_0,T_1]$ and a function

$$\mathbf{v}\in C([T_0,T],\widetilde{\mathbf{L}}^p)\cap C(]T_0,T],\mathcal{D})\cap C^1(]T_0,T],\widetilde{\mathbf{L}}^p), \tag{4.10}$$

such that

$$\mathbf{v}(T_0)=\mathbf{u}_0-\mathbf{\Phi}(T_0), \tag{4.11}$$

and

$$\frac{\partial v_k}{\partial t}(t)+\Big(\widetilde{E}_k(t,\mathbf{v}(t)+\mathbf{\Phi}(t))\,\widetilde{F}_k(t,\mathbf{v}(t)+\mathbf{\Phi}(t))\,A_k+1\Big)\,v_k(t)=f_k(t,\mathbf{v}(t))$$
$$\text{for all } k\in\{1,\dots,l\} \text{ and } t\in]T_0,T], \tag{4.12}$$

where

$$\begin{aligned} f_k(t,\mathbf{v}) = {} & v_k - \frac{\partial \Phi_k}{\partial t}(t) + \widetilde{E}_k\left(t,\mathbf{v}+\mathbf{\Phi}(t)\right)\left\langle \nabla \widetilde{F}_k\left(t,\mathbf{v}+\mathbf{\Phi}(t)\right), \mu_k \nabla v_k \right\rangle_{\mathbb{C}^2} \\ & + \widetilde{G}_k\left(t,\mathbf{v}+\mathbf{\Phi}(t)\right) - \widetilde{E}_k\left(t,\mathbf{v}+\mathbf{\Phi}(t)\right)\widetilde{F}_k\left(t,\mathbf{v}+\mathbf{\Phi}(t)\right)\nabla\cdot\left(\mu_k\nabla\Phi_k(t)\right) \\ & + \widetilde{E}_k\left(t,\mathbf{v}+\mathbf{\Phi}(t)\right)\left\langle \nabla \widetilde{F}_k\left(t,\mathbf{v}+\mathbf{\Phi}(t)\right), \mu_k \nabla \Phi_k(t) \right\rangle_{\mathbb{C}^2}. \end{aligned} \tag{4.13}$$

5 Existence and uniqueness of a solution

In this section we prove the local existence and uniqueness of a solution to Problem 4.8 under Assumption 2.1, Assumption 2.3, and Assumption 4.1. To that end we prove, by means of Proposition 3.7, the local existence and uniqueness of a solution to Problem 4.12. Proposition 3.7 applies with $X := \widetilde{\mathbf{L}}^p$; at first we specify the operator $\mathcal{A}_0$.

5.1 Definition. We define the mapping

$$(\mathfrak{A}_1,\ldots,\mathfrak{A}_l) \ : \ [T_0,T_1]\times\widetilde{\mathbf{H}}^{1,q} \longrightarrow \mathcal{B}(\mathcal{D};\widetilde{\mathbf{L}}^p),$$

$$\mathcal{D} \ni (\psi_1,\ldots,\psi_l) \longmapsto (\mathfrak{A}_1(t,\mathbf{w})\psi_1,\ldots,\mathfrak{A}_l(t,\mathbf{w})\psi_l) \in \widetilde{\mathbf{L}}^p,$$

where

$$\mathfrak{A}_k(t,\mathbf{w}) := 1 + \widetilde{E}_k\left(t,\mathbf{w}+\mathbf{\Phi}(t)\right)\widetilde{F}_k\left(t,\mathbf{w}+\mathbf{\Phi}(t)\right)A_k, \qquad k\in\{1,\ldots,l\}.$$

In particular, we denote by $\mathcal{A}_0$ the operator

$$\mathcal{A}_0 = \left(\mathfrak{A}_1(T_0,\mathbf{u}_0-\mathbf{\Phi}(T_0)),\ldots,\mathfrak{A}_l(T_0,\mathbf{u}_0-\mathbf{\Phi}(T_0))\right). \tag{5.1}$$

Furthermore, we abbreviate

$$U_k := \widetilde{E}_k\left(T_0,\mathbf{u}_0\right)\widetilde{F}_k\left(T_0,\mathbf{u}_0\right) \qquad k\in\{1,\ldots,l\};$$

thus,

$$\mathcal{A}_0 = \left(1+U_1A_1,\ldots,1+U_lA_l\right). \tag{5.2}$$

5.2 Remark. Definition 5.1 is justified because according to Assumption 4.1 and Lemma 4.11 the functions $\widetilde{E}_k\left(t,\mathbf{w}+\mathbf{\Phi}(t)\right)$ and $\widetilde{F}_k\left(t,\mathbf{w}+\mathbf{\Phi}(t)\right)$ have strictly positive essential lower and upper bounds for all $(t,\mathbf{w})\in[T_0,T_1]\times\widetilde{\mathbf{H}}^{1,q}$; N.B. $\widetilde{H}^{1,q}\hookrightarrow\widetilde{L}^\infty$. Hence, the multiplication operator induced by

$$\widetilde{E}_k\left(t,\mathbf{w}+\mathbf{\Phi}(t)\right)\widetilde{F}_k\left(t,\mathbf{w}+\mathbf{\Phi}(t)\right)$$

is a linear homeomorphism on $\widetilde{L}^p$ which implies

$$\operatorname{dom}\mathfrak{A}_k = \operatorname{dom}\bigl(1 + \widetilde{E}_k\,(t,\mathbf{w})\,\widetilde{F}_k\,(t,\mathbf{w})\,A_k\bigr) = \operatorname{dom}\bigl(1 + A_k\bigr) = \operatorname{dom}A_k \tag{5.3}$$

for all $t \in [T_0, T_1]$ and $\mathbf{w} \in \widetilde{\mathbf{H}}^{1,q}$ including the equivalence of the norms

$$\|\mathfrak{A}_k \cdot\|_{\widetilde{L}^p} \quad \text{and} \quad \bigl\|\bigl(1 + A_k\bigr)\ \cdot\ \bigr\|_{\widetilde{L}^p}.$$

In particular, there is $\mathcal{D} = \operatorname{dom}\mathbf{A} = \operatorname{dom}\mathcal{A}_0$.

Having an application of Proposition 3.7 in mind, our first goal is to prove the precondition (3.9) for the operator $\mathcal{A}_0$ from Definition 5.1. Thus, we also will be able to make use of Proposition 3.5 for the operator $B := \mathcal{A}_0$.

5.3 Theorem. *The domain $\mathcal{D}$ of the operator $\mathcal{A}_0$ from Definition 5.1 is dense in $\widetilde{\mathbf{L}}^p$. With respect to the Banach space $X := \widetilde{\mathbf{L}}^p$ the operator $\mathcal{A}_0$ satisfies the resolvent estimate* (3.9)*.*

Proof. Proposition 3.4 applies to the k^{th} component of the operator $\mathcal{A}_0$ with $U := U_k = \widetilde{E}_k\,(T_0,\mathbf{u}_0)\,\widetilde{F}_k\,(T_0,\mathbf{u}_0)$. According to Assumption 4.1.i and Assumption 4.1.ii the function U_k has strictly positive essential lower and upper bounds; N.B. $\widetilde{H}^{1,q} \hookrightarrow \widetilde{L}^\infty$. By Proposition 3.4 now follows that U_kA_k is densely defined and obeys the resolvent estimate (3.3). Hence, $\mathcal{A}_0$ is densely defined and obeys the resolvent estimate (3.9). □

5.4 Theorem. *For every $\alpha \in]\frac{1}{2} + \frac{1}{q}, 1[$ the space $\operatorname{dom}\mathcal{A}_0^\alpha$, equipped with the norm $\|\mathcal{A}_0^\alpha \cdot\|_{\widetilde{\mathbf{L}}^p}$ continuously embeds into $\widetilde{\mathbf{H}}_0^{1,q}$.*

Proof. It suffices to prove, cf. (5.2),

$$\operatorname{dom}\left(1 + U_kA_k\right)^\alpha \hookrightarrow \widetilde{H}_k^{1,q} \quad \text{for all } k \in \{1,\dots,l\}. \tag{5.4}$$

Lemma 3.6 applies to this situation with

$$Y := \widetilde{H}_k^{1,q}, \quad X := \widetilde{H}_k^{-1,q}, \quad X_0 := \widetilde{L}^p, \quad B := a_k + 1;$$

N.B. $p = \frac{q}{2}$ and Assumption 4.1 applies. We verify the preconditions of Lemma 3.6. At first, from Proposition 3.1 one obtains

$$\widetilde{H}_k^{-\frac{1}{p},q} = \Bigl[\widetilde{H}_k^{-1,q}, \widetilde{H}_k^{1,q}\Bigr]_{\frac{q-2}{2q}}. \tag{5.5}$$

According to [17, 4.6.1] the embeddings

$$\widetilde{H}_k^{\frac{1}{p},q'} \hookrightarrow \widetilde{H}^{\frac{1}{p},q'} \hookrightarrow \widetilde{L}^{p'}, \qquad \tfrac{1}{p} + \tfrac{1}{p'} = \tfrac{1}{q} + \tfrac{1}{q'} = 1$$

are continuous, hence,

$$\widetilde{L}^p \hookrightarrow \widetilde{H}_k^{-\frac{1}{p},q}. \tag{5.6}$$

The operators $B = a_k + 1$ and $B_0 = A_k + 1$, cf. Definition 2.4 and Definition 2.5, respectively, obey the resolvent estimates (3.6) and (3.7), respectively. Lemma 3.6 yields

$$\left[\widetilde{L}^p, \operatorname{dom} A_k\right]_\alpha \hookrightarrow \widetilde{H}_k^{1,q} \qquad \text{for all } \alpha \in \left]\tfrac{1}{2} + \tfrac{1}{q}, 1\right[. \tag{5.7}$$

(5.3) and (5.7) imply

$$\left[\widetilde{L}^p, \operatorname{dom}(1 + U_k A_k)\right]_\alpha \hookrightarrow \widetilde{H}_k^{1,q} \qquad \text{for all } \alpha \in]\tfrac{1}{2} + \tfrac{1}{q}, 1[. \tag{5.8}$$

Now the assertion (5.4) follows from (5.8) by means of (3.5), cf. Proposition 3.5, thereby observing Theorem 5.3. □

5.5 Definition. Let α be an arbitrary but from now on fixed number from the interval $\left]\frac{1}{2} + \frac{1}{q}, \gamma\right[$, where q and γ are the numbers from Assumption 4.1. Then we denote by $\mathcal{A}$ the restriction of the operator $(\mathfrak{A}_1, \ldots, \mathfrak{A}_l)$ from Definition 5.1 to the domain

$$\operatorname{dom} \mathcal{A} := [T_0, T_1] \times \operatorname{dom} \mathcal{A}_0^\alpha.$$

Moreover, we denote by κ the embedding constant from $\operatorname{dom} \mathcal{A}_0^\alpha$ into $\widetilde{\mathbf{H}}^{1,q}$

$$\|\mathbf{y}\|_{\widetilde{\mathbf{H}}_0^{1,q}} = \|\mathbf{y}\|_{\widetilde{\mathbf{H}}^{1,q}} \leq \kappa \, \|\mathcal{A}_0^\alpha \mathbf{y}\|_{\widetilde{\mathbf{L}}^p} \qquad \text{for all } \mathbf{y} \in \operatorname{dom} \mathcal{A}_0^\alpha \tag{5.9}$$

and by λ the embedding constant from $\widetilde{H}^{1,q}$ into $\widetilde{L}^\infty$

$$\|\psi\|_{\widetilde{L}^\infty} \leq \lambda \, \|\psi\|_{\widetilde{H}^{1,q}} \qquad \text{for all } \psi \in \widetilde{H}^{1,q}. \tag{5.10}$$

5.6 Remark. The definition of $\mathcal{A}$ and κ is justified because of Theorem 5.4.

5.7 Lemma. *Let M be a bounded set in* $\operatorname{dom} \mathcal{A}_0^\alpha$*. Then there are constants $\widehat{\mathcal{E}}_M$ and $\widehat{\mathcal{F}}_M$ such that*

$$\begin{aligned} \max_{k \in \{1,\ldots,l\}} \sup_{(t,\mathbf{y}) \in [T_0,T_1] \times M} \left\| \widetilde{E}_k\left(t, \mathbf{y} + \mathbf{\Phi}(t)\right) \right\|_{\widetilde{L}^\infty} \leq \widehat{\mathcal{E}}_M < \infty, \\ \max_{k \in \{1,\ldots,l\}} \sup_{(t,\mathbf{y}) \in [T_0,T_1] \times M} \left\| \widetilde{F}_k\left(t, \mathbf{y} + \mathbf{\Phi}(t)\right) \right\|_{\widetilde{H}^{1,q}} \leq \widehat{\mathcal{F}}_M < \infty, \end{aligned} \tag{5.11}$$

and

$$\left\| \widetilde{E}_k\left(t_1, \mathbf{y}_1 + \mathbf{\Phi}(t_1)\right) - \widetilde{E}_k\left(t_2, \mathbf{y}_2 + \mathbf{\Phi}(t_2)\right) \right\|_{\widetilde{L}^\infty} \\ \le \widehat{\mathcal{E}}_M\left(|t_1 - t_2|^\eta + \|\mathcal{A}_0^\alpha \mathbf{y}_1 - \mathcal{A}_0^\alpha \mathbf{y}_2\|_{\widetilde{\mathbf{L}}^p}\right), \quad (5.12)$$

$$\left\| \widetilde{F}_k\left(t_1, \mathbf{y}_1 + \mathbf{\Phi}(t_1)\right) - \widetilde{F}_k\left(t_2, \mathbf{y}_2 + \mathbf{\Phi}(t_2)\right) \right\|_{\widetilde{H}^{1,q}} \\ \le \widehat{\mathcal{F}}_M\left(|t_1 - t_2|^\eta + \|\mathcal{A}_0^\alpha \mathbf{y}_1 - \mathcal{A}_0^\alpha \mathbf{y}_2\|_{\widetilde{\mathbf{L}}^p}\right) \quad (5.13)$$

for all $k \in \{1, \ldots, l\}$*, all* $t_1,\, t_2 \in [T_0, T_1]$*, and all* $\mathbf{y}_1,\, \mathbf{y}_2 \in M$*.*

Proof. According to Theorem 5.4 the set M is not only bounded in $\operatorname{dom} \mathcal{A}_0^\alpha$ but also in $\widetilde{\mathbf{H}}^{1,q}$. Thus, by means of Assumption 4.1.i, Lemma 4.11, and Assumption 4.1.iv we can estimate

$$\begin{aligned} &\left\| \widetilde{E}_k\left(t_1, \mathbf{y}_1 + \mathbf{\Phi}(t_1)\right) - \widetilde{E}_k\left(t_2, \mathbf{y}_2 + \mathbf{\Phi}(t_2)\right) \right\|_{\widetilde{L}^\infty} \\ &\le \mathcal{E}_M\left(|t_1 - t_2|^\eta + \|\mathbf{y}_1 + \mathbf{\Phi}(t_1) - \mathbf{y}_2 - \mathbf{\Phi}(t_2)\|_{\widetilde{\mathbf{H}}^{1,q}}\right) \\ &\le \mathcal{E}_M\left(|t_1 - t_2|^\eta + \|\mathbf{\Phi}\|_{C^\eta([T_0,T_1];\widetilde{\mathbf{H}}^{1,q})}|t_1 - t_2|^\eta + \|\mathbf{y}_1 - \mathbf{y}_2\|_{\widetilde{\mathbf{H}}^{1,q}}\right) \\ &\le \mathcal{E}_M\left((1 + \|\mathbf{\Phi}\|_{C^\eta([T_0,T_1];\widetilde{\mathbf{H}}^{1,q})})|t_1 - t_2|^\eta + \kappa\|\mathcal{A}_0^\alpha \mathbf{y}_1 - \mathcal{A}_0^\alpha \mathbf{y}_2\|_{\widetilde{\mathbf{L}}^p}\right). \end{aligned}$$

This proves (5.12). In order to show (5.13) we estimate by means of Assumption 4.1.ii, Lemma 4.11, and Assumption 4.1.iv:

$$\begin{aligned} &\left\| \widetilde{F}_k\left(t_1, \mathbf{y}_1 + \mathbf{\Phi}(t_1)\right) - \widetilde{F}_k\left(t_2, \mathbf{y}_2 + \mathbf{\Phi}(t_2)\right) \right\|_{\widetilde{H}^{1,q}} \\ &\le \mathcal{F}_M\left(|t_1 - t_2|^\eta + \|\mathbf{y}_1 + \mathbf{\Phi}(t_1) - \mathbf{y}_2 - \mathbf{\Phi}(t_2)\|_{\widetilde{\mathbf{H}}^{1,q}}\right) \\ &\le \mathcal{F}_M\left(|t_1 - t_2|^\eta + \|\mathbf{\Phi}\|_{C^\eta([T_0,T_1];\widetilde{\mathbf{H}}^{1,q})}|t_1 - t_2|^\eta + \|\mathbf{y}_1 - \mathbf{y}_2\|_{\widetilde{\mathbf{H}}^{1,q}}\right) \\ &\le \mathcal{F}_M\left((1 + \|\mathbf{\Phi}\|_{C^\eta([T_0,T_1];\widetilde{\mathbf{H}}^{1,q})})|t_1 - t_2|^\eta + \kappa\|\mathcal{A}_0^\alpha \mathbf{y}_1 - \mathcal{A}_0^\alpha \mathbf{y}_2\|_{\widetilde{\mathbf{L}}^p}\right). \end{aligned}$$

(5.11) follows from (5.12) and (5.13). ◻

5.8 Theorem. *The operator* $\mathcal{A}$ *from Definition 5.5 satisfies* (3.10) *with respect to the Banach space* $X := \widetilde{\mathbf{L}}^p$*.*

Proof. For an arbitrary $R > 0$ we regard the ball B_R with radius R in the space $\widetilde{\mathbf{L}}^p$. Then the set $\mathcal{A}_0^{-\alpha} B_R$ is identical with the R–ball in $\operatorname{dom} \mathcal{A}_0^\alpha$; hence, it is a bounded set in $\widetilde{\mathbf{H}}^{1,q}$, cf. Theorem 5.4. Let

$$t_1,\, t_2 \in [T_0, T_1], \qquad \mathbf{w}_1,\, \mathbf{w}_2 \in B_R \quad (5.14)$$

be arbitrarily chosen. We abbreviate

$$\begin{aligned} &\mathbf{y}_j = \left(y_k^j\right)_{k\in\{1,\dots,l\}} := \mathcal{A}_0^{-\alpha}\mathbf{w}_j, && \widetilde{E}_k^j := \widetilde{E}_k\left(t_j, \mathbf{y}_j + \mathbf{\Phi}(t_j)\right), \\ &\widetilde{F}_k^j := \widetilde{F}_k\left(t_j, \mathbf{y}_j + \mathbf{\Phi}(t_j)\right), && \widetilde{G}_k^j := \widetilde{G}_k\left(t_j, \mathbf{y}_j + \mathbf{\Phi}(t_j)\right), \end{aligned} \qquad j = 1, 2. \tag{5.15}$$

Then the k–th component of $\big(\mathcal{A}(t_1,\mathbf{y}_1) - \mathcal{A}(t_2,\mathbf{y}_2)\big)\mathcal{A}_0^{-1}$ can be estimated in the following way

$$\begin{aligned} &\left\|\left((1 + \widetilde{E}_k^1\widetilde{F}_k^1 A_k) - (1 + \widetilde{E}_k^2\widetilde{F}_k^2 A_k)\right)(1 + U_k A_k)^{-1}\right\|_{\mathcal{B}(\widetilde{L}^p)} \\ \le\;& \left\|\frac{\widetilde{E}_k^1\widetilde{F}_k^1 - \widetilde{E}_k^2\widetilde{F}_k^2}{U_k}\right\|_{\mathcal{B}(\widetilde{L}^p)} \left\|U_k A_k(1 + U_k A_k)^{-1}\right\|_{\mathcal{B}(\widetilde{L}^p)} \\ \le\;& \left\|\frac{\widetilde{E}_k^1\widetilde{F}_k^1 - \widetilde{E}_k^2\widetilde{F}_k^2}{U_k}\right\|_{L^\infty} \Big(1 + \left\|(1 + U_k A_k)^{-1}\right\|_{\mathcal{B}(\widetilde{L}^p)}\Big) \\ =\;& \left\|\widetilde{E}_k^1\widetilde{F}_k^1 - \widetilde{E}_k^2\widetilde{F}_k^2\right\|_{L^\infty} \frac{1 + \left\|(1 + U_k A_k)^{-1}\right\|_{\mathcal{B}(\widetilde{L}^p)}}{\operatorname{vraimin}_{x\in\Omega} U_k(x)}. \end{aligned}$$

The last fraction is finite according to Assumption 4.1 and Proposition 3.4. Thus, it remains to estimate by means of Lemma 5.7 and (5.10)

$$\begin{aligned} &\left\|\widetilde{E}_k^1\widetilde{F}_k^1 - \widetilde{E}_k^2\widetilde{F}_k^2\right\|_{L^\infty} \\ \le\;& \left\|\left(\widetilde{E}_k\left(t_1, \mathbf{y}_1 + \mathbf{\Phi}(t_1)\right) - \widetilde{E}_k\left(t_2, \mathbf{y}_2 + \mathbf{\Phi}(t_2)\right)\right)\widetilde{F}_k\left(t_1, \mathbf{y}_1 + \mathbf{\Phi}(t_1)\right)\right\|_{L^\infty} \\ &+ \left\|\widetilde{E}_k\left(t_2, \mathbf{y}_2 + \mathbf{\Phi}(t_2)\right)\left(\widetilde{F}_k\left(t_1, \mathbf{y}_1 + \mathbf{\Phi}(t_1)\right) - \widetilde{F}_k\left(t_2, \mathbf{y}_2 + \mathbf{\Phi}(t_2)\right)\right)\right\|_{L^\infty} \\ \le\;& 2\,\lambda\,\widehat{\mathcal{E}}_{\mathcal{A}_0^{-\alpha}B_R}\widehat{\mathcal{F}}_{\mathcal{A}_0^{-\alpha}B_R}\left(|t_1 - t_2|^\eta + \|\mathbf{w}_1 - \mathbf{w}_2\|_{\widetilde{\mathbf{L}}^p}\right). \end{aligned} \tag{5.16}$$

□

In order to apply Sobolevskii's Proposition 3.7 to the Problem 4.12 we still have to prove that the right hand side of (4.12) satisfies (3.11).

5.9 Theorem. *Each f_k from* (4.13) *provides a mapping from $[T_0, T_1] \times \widetilde{\mathbf{H}}^{1,q}$ into $\widetilde{L}^p$. Moreover, if B_R is again the ball with radius R in $\widetilde{L}^p$, and $\mathbf{f}$ denotes the vector $(f_1, \dots, f_l)$, then for each $R > 0$ there is a constant C_R such that*

$$\left\|\mathbf{f}(t_1, \mathcal{A}_0^{-\alpha}\mathbf{w}_1) - \mathbf{f}(t_2, \mathcal{A}_0^{-\alpha}\mathbf{w}_2)\right\|_{\widetilde{\mathbf{L}}^p} \le C_R\left(|t_1 - t_2|^\eta + \|\mathbf{w}_1 - \mathbf{w}_2\|_{\widetilde{\mathbf{L}}^p}\right) \tag{5.17}$$

for all t_1, $t_2 \in [T_0, T_1]$ and all $\mathbf{w}_1$, $\mathbf{w}_2 \in B_R$.

Proof. The first assertion immediately follows from Assumption 4.1. We suppose t_1, $t_2 \in [T_0, T_1]$ and $\mathbf{w}_1$, $\mathbf{w}_2 \in B_R$ to be arbitrarily chosen and adopt the notation

(5.15). As in the proof of Theorem 5.8 one obtains that $\mathcal{A}_0^{-\alpha} B_R$ is a bounded set in $\widetilde{\mathbf{H}}^{1,q}$; $\mathcal{A}_0^{-\alpha} B_R$ is contained in the κR ball of $\widetilde{\mathbf{H}}^{1,q}$, cf. (5.9). For each $k \in \{1, \ldots, l\}$ there is

$$\left\| f_k(t_1, \mathbf{y}_1) - f_k(t_2, \mathbf{y}_2) \right\|_{\widetilde{L}^p} \le \left\| \frac{\partial \Phi_k}{\partial t}(t_1) - \frac{\partial \Phi_k}{\partial t}(t_2) \right\|_{\widetilde{L}^p} + \left\| y_k^1 - y_k^2 \right\|_{\widetilde{L}^p} \tag{5.18}$$

$$+ \left\| \widetilde{E}_k^1 \left\langle \nabla \widetilde{F}_k^1, \mu_k \nabla y_k^1 \right\rangle_{\mathbb{C}^2} - \widetilde{E}_k^2 \left\langle \nabla \widetilde{F}_k^2, \mu_k \nabla y_k^2 \right\rangle_{\mathbb{C}^2} \right\|_{\widetilde{L}^p} \tag{5.19}$$

$$+ \left\| \widetilde{G}_k^1 - \widetilde{G}_k^2 \right\|_{\widetilde{L}^p} \tag{5.20}$$

$$+ \left\| \widetilde{E}_k^1 \widetilde{F}_k^1 \nabla \cdot (\mu_k \nabla \Phi_k(t_1)) - \widetilde{E}_k^2 \widetilde{F}_k^2 \nabla \cdot (\mu_k \nabla \Phi_k(t_2)) \right\|_{\widetilde{L}^p} \tag{5.21}$$

$$+ \left\| \widetilde{E}_k^1 \left\langle \nabla \widetilde{F}_k^1, \mu_k \nabla \Phi_k(t_1) \right\rangle_{\mathbb{C}^2} - \widetilde{E}_k^2 \left\langle \nabla \widetilde{F}_k^2, \mu_k \nabla \Phi_k(t_2) \right\rangle_{\mathbb{C}^2} \right\|_{\widetilde{L}^p}. \tag{5.22}$$

We estimate the addends on the right hand side separately and will show that each of them has an upper bound as claimed in (5.17). For the first term on the right hand side of (5.18) this follows directly from Assumption 4.1.iv and Lemma 4.11. The second addend can be estimated

$$\left\| y_k^1 - y_k^2 \right\|_{\widetilde{L}^p} \le \|\mathbf{y}_1 - \mathbf{y}_2\|_{\widetilde{\mathbf{L}}^p} \le \left\| \mathcal{A}_0^{-\alpha} \right\|_{\mathcal{B}(\widetilde{\mathbf{L}}^p)} \|\mathbf{w}_1 - \mathbf{w}_2\|_{\widetilde{\mathbf{L}}^p}.$$

The norm (5.19) can be estimated

$$\begin{aligned}
&\left\| \widetilde{E}_k^1 \left\langle \nabla \widetilde{F}_k^1, \mu_k \nabla y_k^1 \right\rangle_{\mathbb{C}^2} - \widetilde{E}_k^2 \left\langle \nabla \widetilde{F}_k^2, \mu_k \nabla y_k^2 \right\rangle_{\mathbb{C}^2} \right\|_{\widetilde{L}^p} \\
&\le \left\| (\widetilde{E}_k^1 - \widetilde{E}_k^2) \langle \nabla \widetilde{F}_k^1, \mu_k \nabla y_k^1 \rangle_{\mathbb{C}^2} \right\|_{\widetilde{L}^p} + \left\| \widetilde{E}_k^2 \langle \nabla (\widetilde{F}_k^1 - \widetilde{F}_k^2), \mu_k \nabla y_k^1 \rangle_{\mathbb{C}^2} \right\|_{\widetilde{L}^p} \\
&\qquad + \left\| \widetilde{E}_k^2 \langle \nabla \widetilde{F}_k^2, \mu_k \nabla (y_k^1 - y_k^2) \rangle_{\mathbb{C}^2} \right\|_{\widetilde{L}^p} \\
&\le \left\| \widetilde{E}_k^1 - \widetilde{E}_k^2 \right\|_{\widetilde{L}^\infty} \left\| \widetilde{F}_k^1 \right\|_{\widetilde{H}^{1,q}} \mu^\bullet \|\mathbf{y}_1\|_{\widetilde{\mathbf{H}}^{1,q}} + \left\| \widetilde{E}_k^2 \right\|_{\widetilde{L}^\infty} \left\| \widetilde{F}_k^1 - \widetilde{F}_k^2 \right\|_{\widetilde{H}^{1,q}} \mu^\bullet \|\mathbf{y}_1\|_{\widetilde{\mathbf{H}}^{1,q}} \\
&\qquad + \left\| \widetilde{E}_k^2 \right\|_{\widetilde{L}^\infty} \left\| \widetilde{F}_k^2 \right\|_{\widetilde{H}^{1,q}} \mu^\bullet \|\mathbf{y}_1 - \mathbf{y}_2\|_{\widetilde{\mathbf{H}}^{1,q}}.
\end{aligned}$$

Taking into account Lemma 5.7 and $\operatorname{dom} \mathcal{A}_0^\alpha \hookrightarrow \mathbf{H}^{1,q}$, cf. Theorem 5.4, we continue

$$\le \mu^\bullet \kappa (2R + 1) \widehat{\mathcal{E}}_{\mathcal{A}_0^{-\alpha} B_R} \widehat{\mathcal{F}}_{\mathcal{A}_0^{-\alpha} B_R} \left(|t_1 - t_2|^\eta + \|\mathbf{w}_1 - \mathbf{w}_2\|_{\widetilde{\mathbf{L}}^p} \right).$$

As for the term (5.20) one obtains according to Assumption 4.1.iii and (5.9)

$$\begin{aligned}
\left\| \widetilde{G}_k^1 - \widetilde{G}_k^2 \right\|_{\widetilde{L}^p} &\le \mathcal{G}_{\mathcal{A}_0^{-\alpha} B_R} \left(|t_1 - t_2|^\eta + \|\mathbf{y}_1 - \mathbf{y}_2\|_{\mathbf{H}^{1,q}} \right) \\
&\le \mathcal{G}_{\mathcal{A}_0^{-\alpha} B_R} (1 + \kappa) \left(|t_1 - t_2|^\eta + \|\mathbf{w}_1 - \mathbf{w}_2\|_{\widetilde{\mathbf{L}}^p} \right).
\end{aligned}$$

The term (5.21) can be estimated

$$\begin{aligned}
&\left\| \widetilde{E}_k^1 \widetilde{F}_k^1 \nabla \cdot (\mu_k \nabla \Phi_k(t_1)) - \widetilde{E}_k^2 \widetilde{F}_k^2 \nabla \cdot (\mu_k \nabla \Phi_k(t_2)) \right\|_{\widetilde{L}^p} \\
&\qquad \le \left\| \widetilde{E}_k^1 \widetilde{F}_k^1 - \widetilde{E}_k^2 \widetilde{F}_k^2 \right\|_{\widetilde{L}^\infty} \left\| \nabla \cdot (\mu_k \nabla \Phi_k(t_1)) \right\|_{\widetilde{L}^p} \\
&\qquad\qquad + \left\| \widetilde{E}_k^2 \widetilde{F}_k^2 \right\|_{\widetilde{L}^\infty} \left\| \nabla \cdot (\mu_k \nabla \Phi_k(t_1)) - \nabla \cdot (\mu_k \nabla \Phi_k(t_2)) \right\|_{\widetilde{L}^p}.
\end{aligned}$$

According to (5.10), (5.16) Lemma 5.7, and Assumption 4.1.iv this term is bounded from above by

$$3\,\lambda\,\widehat{\mathcal{E}}_{\mathcal{A}_0^{-\alpha}B_R}\widehat{\mathcal{F}}_{\mathcal{A}_0^{-\alpha}B_R}\big\|\nabla\cdot(\mu_k\nabla\Phi_k)\,\big\|_{C^\eta([T_0,T_1];\widetilde{L}^p)}\left(|t_1-t_2|^\eta+\|\mathbf{w}_1-\mathbf{w}_2\|_{\widetilde{\mathbf{L}}^p}\right).$$

Finally, the term (5.22) can be treated as follows:

$$\begin{aligned}
&\left\|\widetilde{E}_k^1\left\langle\nabla\widetilde{F}_k^1,\mu_k\nabla\Phi_k(t_1)\right\rangle_{\mathbb{C}^2}-\widetilde{E}_k^2\left\langle\nabla\widetilde{F}_k^2,\mu_k\nabla\Phi_k(t_2)\right\rangle_{\mathbb{C}^2}\right\|_{\widetilde{L}^p}\\
&\le\left\|\left(\widetilde{E}_k^1-\widetilde{E}_k^2\right)\left\langle\nabla\widetilde{F}_k^1,\mu_k\nabla\Phi_k(t_1)\right\rangle_{\mathbb{C}^2}\right\|_{\widetilde{L}^p}\\
&\qquad\qquad+\left\|\widetilde{E}_k^2\left\langle\nabla(\widetilde{F}_k^1-\widetilde{F}_k^2),\mu_k\nabla\Phi_k(t_1)\right\rangle_{\mathbb{C}^2}\right\|_{\widetilde{L}^p}\\
&\qquad\qquad\qquad\qquad+\left\|\widetilde{E}_k^2\left\langle\nabla\widetilde{F}_k^1,\mu_k\nabla(\Phi_k(t_1)-\Phi_k(t_2)\right\rangle_{\mathbb{C}^2}\right\|_{\widetilde{L}^p}.
\end{aligned}$$

Making use of (2.1) we continue

$$\begin{aligned}
&\le\big\|\widetilde{E}_k^1-\widetilde{E}_k^2\big\|_{\widetilde{L}^\infty}\big\|\widetilde{F}_k^1\big\|_{\widetilde{H}^{1,q}}\mu^\bullet\big\|\Phi_k(t_1)\big\|_{\widetilde{H}^{1,q}}\\
&\qquad\qquad+\big\|\widetilde{E}_k^2\big\|_{\widetilde{L}^\infty}\big\|\widetilde{F}_k^1-\widetilde{F}_k^2\big\|_{\widetilde{H}^{1,q}}\mu^\bullet\big\|\Phi_k(t_1)\big\|_{\widetilde{H}^{1,q}}\\
&\qquad\qquad\qquad\qquad+\big\|\widetilde{E}_k^2\big\|_{\widetilde{L}^\infty}\big\|\widetilde{F}_k^1\big\|_{\widetilde{H}^{1,q}}\mu^\bullet\big\|\Phi_k(t_1)-\Phi_k(t_2)\big\|_{\widetilde{H}^{1,q}}
\end{aligned}$$

and by means of Lemma 5.7 and Assumption 4.1.iv we finish this estimate

$$\le\widehat{\mathcal{E}}_{\mathcal{A}_0^{-\alpha}B_R}\widehat{\mathcal{F}}_{\mathcal{A}_0^{-\alpha}B_R}\mu^\bullet\big\|\mathbf{\Phi}\big\|_{C^\eta([T_0,T_1];\widetilde{\mathbf{H}}^{1,q})}\left(|t_1-t_2|^\eta+\|\mathbf{w}_1-\mathbf{w}_2\|_{\widetilde{\mathbf{L}}^p}\right).$$

□

5.10 Theorem. *Problem 4.12 admits exactly one solution*

$$\mathbf{v}\in C([T_0,T],\operatorname{dom}\mathcal{A}_0^\alpha)\cap C(]T_0,T],\mathcal{D})\cap C^1(]T_0,T],\widetilde{\mathbf{L}}^p).$$

The function $\mathbf{u}:=\mathbf{v}+\mathbf{\Phi}$ *is a local solution of* (1.1) *in the sense of Definition 4.7.*

Proof. The first assertion is implied by Proposition 3.7 which applies with respect to the Banach space $X:=\widetilde{\mathbf{L}}^p$ to the operators $\mathcal{A}_0$ and $\mathcal{A}(t,\mathbf{w})$ from Definition 5.5. According to Theorem 5.3, Theorem 5.8, and Theorem 5.9 the preconditions of Proposition 3.7 on the operators and the right hand side of the equation are fulfilled. As for the initial value, (3.4) from Proposition 3.5 implies $\mathbf{u}_0-\mathbf{\Phi}(T_0)\in\operatorname{dom}\mathcal{A}_0^\beta$ for any $\beta\in]\alpha,\gamma[$.

The complex conjugate $\overline{\mathbf{v}}$ of $\mathbf{v}$ also is a solution of (4.12) and has the same initial value. Hence, $\mathbf{v}=\overline{\mathbf{v}}$, i.e. $\mathbf{v}$ takes its values in $\mathbb{R}^l$ and satisfies (4.9). □

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