Curvature for the Sake of Single-Valuedness

Iterative Interplay between Aharonov-Bohm Deficit Angle and Berry Phase

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Geometric phases can be observed by interference as preferred scattering directions in the Aharonov-Bohm (AB) effect or as Berry phase shifts leading to precession on cyclic paths. Without curvature single-valuedness is lost in both case. It is shown how the deficit angle of the AB conic metric and the geometric precession cone vertex angle of the Berry phase can be adjusted to restore single-valuedness. The resulting interplay between both phases confirms the non–linear iterative system providing for generalized fine structure constants obtained in the preliminary work. Topological solitons of the scalar coupling field emerge as localized, non-dispersive and non-singular solutions of the (complex) sine-Gordon equation with a relation to the Thirring coupling constant and non-linear optics.

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Introduction

Local confinement requires proper phase relationships defined by topology and geometry, i.e. to guarantee single-valuedness of the wave-function. Geometric phases are subject of concepts in differential-geometry and topology associated with abelian and non–abelian groups [1, 2, 3]. Generally, phase factors or phases representing the 'holonomy' provide for important boundary conditions while reducing the degree of redundancy in variables. This is one of the reasons why phases and gauge theories are not unimportant in quantum mechanics, despite of the central role of amplitude densities. Berry showed that the geometric phase has the same mathematical (gauge) structure as the Aharonov-Bohm (AB) phase [2] and is the integral of an effective vector potential along a closed path. Both phases even combine [4] especially if a charged particle is in an spatial extended quantum state, i.e. if the orbit loop includes spatially extended sub-loops. Both, the local non–abelian Berry phase evolution on $S^2 = SU(2)/U(1)$ and the nonlocal abelian AB scattering effect on R^2 with conic metric provide for phase evolutions and deficit angles that can be combined. This paper provides for an exact solution to the most simple and central case, where the AB deficit angle exactly compensates the Berry phase shift. The purpose of this paper is not to outrage conventional wisdom concerning geometric phases, rather to discover hidden connections on a purely topological ground.

Berry's Topological Phase

Two time- or lengthscales are natural requirements of geometric phases, especially in the adiabatic Berry case: the long scale is given by the path of a vector signal on a curved manifold, i.e. the orbital coupling period, the short scale defines the vector signal, i.e. it's spinning period. The non-adiabatic generalization of [5] defines a geometric phase factor for any cyclic evolution of a quantum system, where the geometric phase is an eigenstate of the time evolution operator describing the holonomy of a natural connection in the Hilbert space of states. Consider a T-periodic cyclic vector $|\,\psi(\tau)\,\rangle$ that evolves on a closed path $\mathcal{C}$ according to

$$|\,\psi(T)\,\rangle = e^{i\varphi(T)}\,|\,\psi(0)\,\rangle, \tag{1}$$

where the total phase $\varphi(T)$ acquired by the cyclic vector can naturally be decomposed into a geometric $\varphi_g(T)$ and dynamical phase $\varphi_d(T)$

$$\varphi(T) = \varphi_g(T) + \varphi_d(T). \tag{2}$$

The dynamical phase for one loop $t \in [0;T]$ is with the Schrödinger equation and hamilton operator H given by

$$\varphi_d(T) = -\frac{1}{\hbar}\int_0^T \langle\,\psi(\tau)\,|H(\tau)|\,\psi(\tau)\,\rangle d\tau. \tag{3}$$

The Berry phase or geometric phase depends not on the explicit time dependence of the trajectory and is for one loop given by

$$\varphi_g(T) = i\oint_{\mathcal{C}} \langle\,\psi(\tau)\,|d|\,\psi(\tau)\,\rangle. \tag{4}$$

The 'parallel transported' spin vector will come back after every loop with a directional change $\varphi_g(T)$ equal to the curvature enclosed by the path $\mathcal{C}$. On the unit sphere $S^2 = SU(2)/U(1)$ the curvature increment is proportional to the area increment that can be a spherical triangle with area given by

$$d\Omega := [1 - \cos\theta_B(\tau)]d\varphi(\tau), \tag{5}$$

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the area on the Bloch sphere enclosed by the closed evolution loop of the eigenstate equal to

$$\Omega = \oint_{\mathcal{C}} d\Omega := \int_0^T d\tau [1 - \cos\theta_B(\tau)] \dot{\varphi}(\tau). \tag{6}$$

The Berry phase $\varphi_g(T) = J\Omega$ and the total phase are proportional to spin J. In the standard case of conic precession on the sphere

$$\varphi_g(T) = 2\pi J(1 - \cos\theta_B), \quad \varphi(T) = 2\pi J, \tag{7}$$

where θ_B is the vertex cone semiangle, $\varphi_d(T) = 2\pi J \cos\theta_B$, see fig.1. Introducing a total phase evolution frequency $\dot{\varphi}(\tau) = \omega$, the phase evolution of the geometric phase is given by the angular frequency

$$\omega_p = J\frac{d\Omega}{dt} = \frac{d\varphi_g}{dt} = (1 - \cos\theta_B)\gamma\omega, \tag{8}$$

where $\gamma = d\tau/dt$ is a relativistic factor. Usually, ω_p can be assigned to precession, the difference $\gamma\omega - \omega_p$ is given by the evolution of the dynamical phase that will be defined by

$$\omega_d = M\omega_M = \gamma\omega\cos\theta_B, \tag{9}$$

where M is the quantum number of regular M-gonal symmetry that divides all loops phases in M equal sub-loop parts, i.e. $\varphi_d(T) = M\triangle\varphi_d(T)$. With n parameters $\lambda_\mu(t)$, $\mu = 1, 2, ..., n$ that span a closed curve $\mathcal{C}$ in the T-periodic parameter space $\lambda_\mu(0) = \lambda_\mu(T)$, the Berry phase may be represented in terms of the 'gauge potential' A with connection matrix

$$(A_\mu)^{\alpha\beta} := \langle\psi^\alpha(\lambda)|\, \partial/\partial\lambda^\mu \,|\psi^\beta(\lambda)\rangle \tag{10}$$

where $A = \sum_\mu A_\mu \, d\lambda_\mu$, and

$$\varphi_g(T) = \oint_{\mathcal{C}} A = \int_{\mathcal{S}_\mathcal{C}} F, \quad F = dA. \tag{11}$$

A is the non–abelian gauge potential that can be regarded as a winding number density and allows for parallel transport of vectors over $\mathcal{S}_\mathcal{C}$, an arbitrary surface in the parameter space bounded by the contour $\mathcal{C}$. For more details regarding monopoles and Wilson loops on the lattice in non–abelian gauge theories, see e.g. [6].
Generalized spin-orbit coupling strength can be defined as

$$\alpha(M) = \frac{J\triangle\varphi_d(T)}{\varphi(T)} = \frac{J\omega_M}{\gamma\omega}, \tag{12}$$

where the coupling is proportional to the evolution of the dynamical part $\triangle\varphi_d(T)/\varphi(T)$ and spin J. With eq.(9) in eq.(12) the dynamical part of eq.(8) provides for

$$M\alpha = J\cos\theta_B. \tag{13}$$

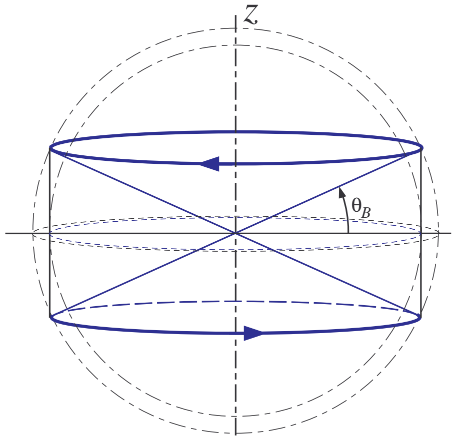

FIG. 1: The radial projection from the conic z-offset on S^2 onto the polar plane for $M = 7$.

The AB topological phase

Nonlinear AB effects will arise if the pattern of the scattered particle/wave couples back to the magnetic flux that generates the AB phase, see fig.2. In this case the magnetic field is not given externally, it is created by the orbital loop of the charge in cylindrical symmetry of orbital spin and magnetic moment where the geometry of the magnetic flux can be thought to be infinitely long and straight thin with constant magnetic flux and mass density. The magnet field is usually interpreted as the curvature associated with the notion of parallel transport defined by the vector potential. The Aharonov-Bohm phase

$$\theta_{AB} = \frac{q}{\hbar}\oint_{\mathcal{C}_{\mathcal{AB}}} \mathbf{A}_{AB}(\mathbf{r}) \cdot d\mathbf{r} = 2\pi n \frac{q}{e}\frac{\Phi}{\Phi_0} \tag{14}$$

collected by a charge q, moving in a closed path $\mathcal{C}_{\mathcal{AB}}$ about a line of magnetic flux Φ, is purely topological. n is the winding number of $\mathcal{C}_{\mathcal{AB}}$ around the magnetic flux, e the elementary charge, and Φ_0 the corresponding flux unit $\Phi_0 = 2\pi\hbar/e$. In contrast to the Berry phase, the AB phase is independent of the shape of the path $\mathcal{C}_{\mathcal{AB}}$ and of the history of motion along it. In the case of strong currents a flux tube model will have a non-vanishing mass density and will introduce considerable topological coupling effects in the motion of the scattered particles [7]. The AB-scattering process is essentially two-dimensional. The field of the string (here rather cosmic, later solitonic) can be described in $2 + 1$ dimensions [8, 9]. As the third spatial dimension can be taken to be parallel to the flux and will decouple from the problem, the time-component of the metric does not play any role. A convenient characterization of this conical space is based on embedded coordinates [9]

$$(dl)^2 = (\alpha_{AB})^2(dr)^2 + r^2(d\theta)^2,$$
$$-\pi \le \theta \le \pi, \quad \alpha_{AB} = 1 - 4mG/(R_0c^2), \tag{15}$$

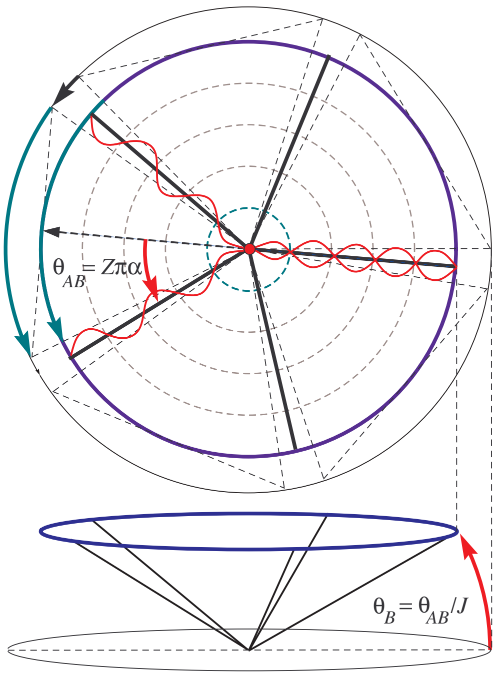

FIG. 2: Aharonov-Bohm scattering of a wave on a cone with radial resonance for $M = 5$ and $Z = J = M_g = 1$. The Berry phase is shown by the short black arrow with length $\triangle\varphi_g(T)$, the dynamical phase by the two green and longer arrows with length $\triangle\varphi_d(T)$, the deficit angle on the conic metric is $Z\pi\alpha$. In this picture the conic AB-deficit angle and Berry's phase have been adjusted according to eq.(17) via iteration eq.(19).

where G is a "Newton-type" coupling constant, m the mass, and R_0 a characteristic orbital radius or horizon (in the original work $1/\alpha_{AB} \to \alpha$). All of the curvature of the cone is located at the tip of the cones but is experienced indirectly by interference effects revealing the deficit angle as a classical AB-scattering angle θ_{AB} [7, 8, 9], see fig.2:

$$\theta_{AB\pm} = \pm\left(\alpha_{AB}\pi - \pi\right). \tag{16}$$

Coupling AB with Berry

Aharonov et al.[4] have investigated the phase accumulated by a charged particle in an extended quantum state and revealed by topological analysis a possible interplay between the AB phase and Berry phase. In contrast to the non–abelian Berry phase on a local path $\mathcal{C}$ the abelian AB phase is independent of the shape of the path $\mathcal{C}_{AB}$, it has a nonlocal character. This allows in the AB case, to adjust a path $\mathcal{C}_{AB}$ that supports a special Berry phase constellation forced by topology. The interplay between the Berry phase and the AB effect can essentially be described as a reaction of the curvature to the additional geometric phase and vice versa. One could argue, that the AB effect is due to the non triviality of the holonomy (Dirac phase) even though the curvature of the connection $F_{\mu\nu}$ vanishes. The problem with this reasoning is that the wave function is not single-valued [10]. To restore single-valuedness and to generate a closed string state, curvature could generate a deficit angle that exactly compensates the Berry phase in a non–linear iterative process. In other words: in contrast to the flat metric the conic metric provides for single-valuedness, where the conic opening angle θ_{AB} is directly related to the Berry phase leading to the precession angle θ_B. The coupling is driven by the spatial extension of the subloop pattern. Regarding the scattering geometry shown in figs.2, those arguments can directly be transformed into a geometrical resonance condition as follows:
The cone vertex angle of Berry precession multiplied by spin equals the absolute value of the cone deficit angle of AB-scattering on the conic 2+1 dimensional metric in polygonal axial symmetry, where

$$\theta_{AB\pm} = \mp J\theta_B, \quad \alpha_{AB} = 1 - Z\alpha. \tag{17}$$

The difference between the two couplings for the same regular M-gonal symmetry is obviously given by the angle π and a geometric coupling factor $M_g = Z/J$. Now, eq.(17) provides with eq.(15) for the Newton-type coupling

$$Z\alpha = \frac{4mG}{R_0 c^2} = \mp\frac{J\theta_B}{\pi}, \quad Z = JM_g. \tag{18}$$

eq.(18) reveals the role of Z as a charge unit that is integral with integral M_g and J. The factor $\cos\theta_B$ provides for the radial projection from the z-offset on S^2 onto the polar plane, see fig.1.

Proceeding with eq.(17) and eq.(16) in eq.(13) provides for the conic iteration

$$\mp\,\theta_{B,i+1} = \frac{M_g}{M}\pi\cos\theta_{B,i}, \tag{19}$$

where the sign of the conic deficit angle $\theta_{AB\pm}$ of eq.(16) related to eq.(7) can realize a bi-stable configuration that can be characterized by two states:

$$\theta_{AB+} : \frac{M_g}{M} > 0, \quad \theta_{AB-} : \frac{M_g}{M} < 0. \tag{20}$$

This non–linear iteration converges for integral $|M/M_g| > 2.4$ outside the chaotic regime after a few steps to a special fixed point $\theta_{B\pm}$ [11, 12, 13, 14]. In any case, it provides for a nice and fast converging non–linear iterative system where the scattered wave packet couples back to the scattering conic singularity in resonance with the orbital Berry phase evolution. The iterative solution provides for a AB deficit angle that compensates the Berry phase, where the conic metric provides for single-valuedness despite of the phase shift.

Solitons and monopoles

The relations for the phase shift and scattering angle are derived under fairly general conditions based on fundamental topological principles. Topological solitons are spatially confined (localized), non-dispersive and non-singular solution of a non–linear field theory. In 2+1-dimensional gauge vortex scattering it follows from purely geometric considerations that the head-on scattering of M topological solitons (like monopoles, vortices, skyrmions, ...) distributed symmetrically around the point of scattering (relative angular separations $2\pi/M$) is by an angle π/M, independent of various details of the scattering [24]. In this case the initial configuration has the symmetry group of a regular M-gon, the "moduli space" of M vortices, $\mathcal{M}_M$[25].

The Berry and AB phases are intimately related to Dirac magnetic monopoles and arise naturally in the field of a monopole [1], [16, 17] with M-gonal scattering symmetry where $M = 137$ ('t Hooft-Polyakov and Dirac monopoles will be discussed elsewhere). In this case the Newton-type coupling constant of eq.(18) is given by $\alpha \approx 1/137.03600941164$ and fits within error range to a neutral and theory independent determination of the Sommerfeld fine structure constant, see the Java simulation [14]. This provides for evidence that α is indeed a topological spin-orbit coupling constant. The origin of special M-values is up to this stage unclear but could be found in non–linear topological quantum field theory. A non–linear non–pertubative quantum field theory seems to be generally relevant for information processing, storage, and transport [13].

A scalar field for the deficit angle

The conic metric eq.(15) corresponds to a 2+1-dimensional closed-loop a priori open in the z-direction. Recently, [19] discussed coupling gravitomagnetism-spin and Berry's phase and pointed out, that the geometric phase changes should depend exclusively upon the solid angle of a field, and not on the strength of the field. In this context it should be possible to decouple angular properties from the coupling field strength and localize it on the z-axis with a confinement potential. For this purpose, there are candidates for soliton-type solutions based on a non–pertubative non–linear topological quantum field theory of a scalar field. In this case the field must describe the coupling or charge density $\phi(z)$, i.e. via $\theta_B(z) = \mp\alpha\phi(z)$ as a travelling geometric precession (or local conic deficit angle) moving with radial symmetry in z-direction according to eq.(18) with $M_g = \phi(0)$. In 1+1–dimensional space–time there are two non–trivial minimal quantum field theories which describe non–perturbative phenomena: the sine–Gordon (SG) model [20] and the massive Thirring model [21] (a self-coupled Dirac field, see the Lagrangians [20]), both are intimately related [22]. These types emerge from the most simple relation between field strength and potential, where the potential is given by the square of the z gradient of $\phi(z)$. Based on the relative precession frequency and energy eq.(8) at $z = z_0$ the potential $V(z)$ of a relativistic SG kink is given by

$$\hbar\omega_p \to V(z) = \frac{1}{2}\left(\frac{\partial\phi}{\partial z}\right)^2 = \frac{k_z^2}{\alpha^2}\left[1 - \cos(\alpha\phi)\right], \quad (21)$$

where k_z is a characteristic wave number and $E = 8k_z/\alpha^2$ the self-energy. The equation of motion is given by the SG equation

$$\Box\phi + \frac{k_z^2}{\alpha}\sin(\alpha\phi) = 0. \quad (22)$$

Boosting the static solution gives the indestructible moving solution

$$\phi_\pm(z) = \pm 4\arctan\left(\exp\left[\frac{\gamma(z - z_0 - vt)}{\sqrt{\alpha}}\right]\right),$$
$$\gamma = \frac{1}{\sqrt{1 - v^2}}. \quad (23)$$

For the SG system colliding solitons preserve their (z)-shape, a desirable property regarding the stability of α and the iteration. For $\alpha \to 0$ or $M \to \infty$ the relativistic 1-d wave equation is recovered. Solitons can annihilate with anti–solitons, many–soliton solutions obey Pauli's exclusion principle. As pointed out by Skyrme [23] this can be interpreted as a fermion–like behavior. Regarding this work, there is a relation between the real Thirring coupling parameter $g > 0$ to the SG parameter α [22] given by

$$\frac{4\pi}{\alpha^2} = 1 + \frac{g}{\pi}. \quad (24)$$

Reduced to the basic mechanism it is oversimplified regarding electrodynamics. But it is possible to generalize to non–abelian group structures and their physical properties hidden in nonlinear interactions to a group theoretic effective field theory which generalizes the sine-Gordon theory to the non–abelian cases on a coset $SU(N)/U(N-1)$ in terms of a deformed gauged Wess-Zumino-Witten action [26]. The $SU(2)/U(1)$ case can be identified with the so-called complex SG theory that is applied to the phenomenon of anomalously low energy loss in coherent optical pulse propagation. In this case of nonlinear optics there are coupled nonlinear partial differential equations in 1+1-dimensional spacetime given by the Maxwell and Bloch equations with two complex fields and one real field. The optical Bloch equation can be used to describe dipole spin precession [27]. The chain of reasoning can be closed by observing, that the Maxwell equation determines the strength of the torque vector along the z-axis. This agrees not only with the conventional mechanical interpretation of the SG equation as a continuum limit of the infinite chain of coupled pendulum equations, it allows also to interpret the SG field or pendulum amplitude as an precession angle, in our case $\theta_B = \mp\alpha\phi = \mp\theta_{AB}/J$.

Conclusion

Orbital (x, y)-stability is guaranteed by the single-valuedness requirement with α-iteration that could be assigned to a regular M-gonal coupled soliton system. The strength of the field is given by deficit angle induced by the closed-loop around the z-axis, a rotating target surely "sees" a deficit modified by its relative angular velocity with two possible signs. The deficit strength is clearly quantized with ϕ mod 2π, so it seems that electromagnetic properties could be traced down to topological phases. The AB effect can now be understood as a path-dependent deficit angle that can be found by passing on two different sides of a "deficit flux", the Berry phase is the deficit induced by parallel transport. Both deficits can be combined, the interplay is nonlinear and can restore single-valuedness. Topological solitons of the scalar coupling field emerge as localized, non-dispersive and non-singular solutions of the sine-Gordon equation with a relation to the Thirring coupling constant. SG solitons could serve as carriers of gauge dependent conic "deficit energy" originated by topological phases.

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