Adiabatic charge pumping in open quantum systems

J.E. Avron[(a)], A. Elgart[(b)], G.M. Graf[(c)], L. Sadun[(d)], K. Schnee[(e)]

[(a)] *Department of Physics, Technion, 32000 Haifa, Israel*
[(b)] *Jadwin Hall, Princeton University, Princeton, NJ 08544, USA*
[(c)] *Theoretische Physik, ETH-Hönggerberg, 8093 Zürich, Switzerland*
[(d)] *Department of Mathematics, University of Texas, Austin, TX 78712, USA*
[(e)] *Department of Mathematics, Caltech, Pasadena, CA 91125, USA*

September 16, 2002

Abstract

We introduce a mathematical setup for charge transport in quantum pumps connected to a number of external leads. It is proved that under rather general assumption on the Hamiltonian describing the system, in the adiabatic limit the current through the pump is given by a formula of Büttiker, Prêtre, and Thomas, relating it to the frozen S-matrix and its time derivative.

1 Introduction

Transport in quantum pumps has been investigated in relation to various properties and from many perspectives [8, 7, 1, 21, 14, 16]. The goal of this article is to provide a rigorous setting for a single but important aspect of these devices, namely the charge transport or, more precisely, its expectation value. The idealized setting is as follows: a pump, whose internal configuration varies slowly in time in a prescribed manner, is connected to n leads, or channels, along each of which independent electrons can enter or leave the pump. We assume that the electron in the lead has no transverse or spin degrees of freedom and may be thought of as a (non-relativistic) particle moving on a half line[1]. The incoming electron distribution, at zero temperature, is a Fermi sea with Fermi energy μ common to all leads. As a rule, this does not apply to the distribution of the outgoing electrons, as their energies may have been shifted while scattering at the pump. Because of this imbalance a net current is flowing in the leads. The expected charge transport is

[1]Such extra degrees of freedom can be represented by adding channels. In general, the different channels may then have different propagation speeds.

expressed by the formula [8, 7]

$$dQ_j = \frac{e}{2\pi}(\mathrm{i}(dS)S^*)_{jj} \ . \tag{1.1}$$

Here $S = (S_{ij})$ is the $n \times n$ scattering matrix at energy μ computed as if the pump were *frozen* into its instantaneous configuration. A change of the configuration is accompanied by a change $S \to S + dS$ of the scattering matrix and by a net charge dQ_j leaving the pump through lead j. Finally e is the electron charge, which is henceforth set equal to 1.

We shall next present a mathematical framework in which (1.1) can be phrased as a theorem.

- The *single-particle Hilbert space* is given as

  $$\mathcal{H} = \mathcal{H}_0 \oplus L^2(\mathbb{R}_+, \mathbb{C}^n) \ , \tag{1.2}$$

  where states in $L^2(\mathbb{R}_+, \mathbb{C}^n) = \oplus_{j=1}^n L^2(\mathbb{R}_+)$, resp. in $\mathcal{H}_0$, describe an electron in one of the leads $j = 1, \ldots . n$, resp. in the pump proper. The latter Hilbert space is not further specified, but the hypothesis A2 below confers on the pump the role of an abstract finite box [23]. Let $\Pi_j : \mathcal{H} \to \mathcal{H}$ denote the projection onto $\mathcal{H}_0$ for $j = 0$ and on j-th copy of $L^2(\mathbb{R}_+)$ for $j = 1 \ldots n$.

- Since the pump configuration is supposed to change slowly in time t, we will eventually consider the evolution of the electrons in an adiabatic limit, where $s = \varepsilon t$ is kept fixed as $\varepsilon > 0$ tends to 0. In terms of the rescaled time coordinate s, called *epoch*, the propagator $U_\varepsilon(s, s')$ on $\mathcal{H}$ satisfies the non-autonomous Schrödinger equation

  $$\mathrm{i}\partial_s U_\varepsilon(s, s') = \varepsilon^{-1} H(s) U_\varepsilon(s, s') \ , \tag{1.3}$$

  where $H(s)$ is a family of self-adjoint *Hamiltonians* on $\mathcal{H}$ enjoying the following properties:

  (A1) $H(s) - H(s')$ is bounded and smooth in s ,
  (A2) $\|(H(s) + \mathrm{i})^{-m}\Pi_0\|_1 < C$ for all s and some $m \in \mathbb{N}$
  (A3) $H(s)\psi = -d^2\psi/dx^2$ for $\psi \in C_0^\infty(\mathbb{R}_+, \mathbb{C}^n)$,
  (A4) $\sigma_{\mathrm{pp}}(H(s)) \cap (0, \infty) = \emptyset$,
  (A5) $H(s) = H_-$ for $s \le 0$.

  Here $\|\cdot\|_1$ denotes the trace class norm over $\mathcal{H}$, while the operator norm will be written as $\|\cdot\|$. Assumption A3 states that a particle in the leads is free; in particular, together with A1, it implies that changes in the Hamiltonian are confined to the pump proper:

  $$H(s) - H(s') = (H(s) - H(s'))\Pi_0 \ . \tag{1.4}$$

  Assumption A4 requires that there are no positive embedded eigenvalues; A5 states that the pump is at rest for $s \le 0$.

- The initial state of the electrons should be an equilibrium state. This is achieved thanks to Assumption A5 by positing that the *1-particle density matrix* at some (and hence any) epoch $s_- < 0$ is of the form $\rho(H_-)$, where $\rho(\lambda)$ is a function of bounded variation with $\operatorname{supp} d\rho \subset (0, \infty)$. A good example is the Fermi sea, where $\rho(\lambda) = \theta(E - \lambda)$. The time evolution then acts as

$$\rho(H_-) \mapsto U_\varepsilon(s, s_-)\rho(H_-)U_\varepsilon(s_-, s) \ . \tag{1.5}$$

- We define a generator of exterior scaling w.r.t. (1.2) by

$$A = 0 \oplus \frac{1}{2\mathrm{i}}\left(\frac{d}{dx}v(x) + v(x)\frac{d}{dx}\right) \ , \tag{1.6}$$

where $v(x) : [0, \infty) \to \mathbb{R}$ is smooth with $v(x) = 0$, resp. $= x$ for small, resp. large x, and $v'(x) \geq 0$ everywhere. We note that $A = A^*$ commutes with Π_j, and set $A_j = A\Pi_j$. The operator A distinguishes between incoming and outgoing states, respectively associated with spectral subspaces $A < -a$ and $A > a$ with some large $a > 0$. Detection of a particle, and hence of its charge, deep inside lead j may be realized as the operator $Q_j(a) = f(A_j - a) + f(-A_j - a)$, where $f \in C^\infty(\mathbb{R})$ is a switch function: $f(\alpha) = 0$ for $\alpha < -1$, resp. $= 1$ for $\alpha > 1$. The *current operator* then consistently is

$$I_j(a) = \mathrm{i}[H(s), f(A_j - a) + f(-A_j - a)] =: I_{j+}(a) + I_{j-}(a) \ . \tag{1.7}$$

(In what follows we shall sometimes suppress the index j for the sake of notational simplicity). One feature of this choice of current operator is that the "ammeter" is located not at a fixed distance from the pump, but rather at a fixed number of wavelengths, a, from it: The longer the wavelength the more distant the "ammeter". In the case that one focuses on a narrow energy interval, say near the Fermi energy, the "ammeter" is also at essentially fixed distance from the pump.

We remark that by the support property of v and by A3, the above commutator does not depend on s. The expectation value of the current at epoch s, i.e., in the state (1.5), is then given as

$$\langle I\rangle_j(s, a, \varepsilon) = \operatorname{tr}\left(U_\varepsilon(s, s_-)\rho(H_-)U_\varepsilon(s_-, s)I_j(a)\right) \ . \tag{1.8}$$

In contrast to $Q_j(a)$, which clearly has an infinite expectation value in that state, $I_j(a)$ is inclined to have a finite one. Moreover, it should behave as ε if, in accordance with (1.1), the charge dQ_j transferred during $ds = \varepsilon^{-1}dt$ is to have a non trivial limit as $\varepsilon \to 0$.

Other realizations of the current operator are possible[2], such as $\mathrm{i}[H(s), f(x_j - a)]$, or the example based on the precession of a spin proposed in [16], and the result (1.1) should be independent of the choice. Our definition (1.7) has the property that $I_j(a)$ naturally splits into two parts distinguished by their Heisenberg dynamics: The outgoing current $I_{j+}(a)$ which is essentially free in the future and the incoming current $I_{j-}(a)$ which is free

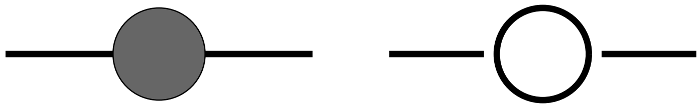

Figure 1: A scatterer with two channels (left) and its disconnected analog (right). Neumann boundary conditions are imposed at the edges of the channels.

in the past. As the initial condition is set in the past of the measurement, only $I_{j+}(a)$ will be affected by scattering.

- Finally we ought to state the reference dynamics H_0 to which $H(s)$ is compared in the definition of the (frozen) *scattering operator* [20, 24]

$$S(s) = S(H(s), H_0) \;, \tag{1.9}$$

whose fibers $S(s, E)$, called *scattering matrices*, appear in (1.1). While it turns out that the choice of H_0 is largely irrelevant, for the sake of simplicity let H_0 be the Laplacian, acting on $L^2(\mathbb{R}_+, \mathbb{C}^n)$, with a Neumann boundary condition at $x = 0$. $S(s, k^2)$, $(k > 0)$, then agrees with the familiar definition based on generalized eigenfunctions incident through lead i: $(\delta_{ji}\mathrm{e}^{-\mathrm{i}kx} + S_{ji}\mathrm{e}^{\mathrm{i}kx})_{j=1}^n$. In particular, this reproduces the standard form of the scattering matrix of two channels:

$$S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}$$

with r, r' the right and left reflection amplitudes and t, t' the corresponding transmission amplitudes.

The fundamental equation (1.1) may thus be given the following reformulation:

$$\lim_{a\to\infty}\lim_{\varepsilon\to 0} \varepsilon^{-1}\langle I\rangle_j(s, a, \varepsilon) = -\frac{\mathrm{i}}{2\pi}\int_0^\infty d\rho(E)\Big(\frac{dS}{ds}S^*\Big)_{jj} \;. \tag{1.10}$$

In particular, for the Fermi sea $\rho(H_-) = \theta(\mu - H_-)$ as initial state, we recover (1.1) from $d\rho(E) = -\delta(E-\mu)dE$. The limit $a \to \infty$ is taken so as to have the current measurement made well outside of the scattering region, but after the adiabatic limit $\varepsilon \to 0$. By doing so, a current is still measured within the same epoch as the scattering process which generated it, though at a different time.

Unfortunately, we are not able to prove the result in quite this form, for reasons that have to do with problems that arise both in the infrared and the ultraviolet. The adiabatic limit is realized [5] in the regime where the dimensionless quantity given by ε times the dwelling time of an electron in the pump is small. Low energy particles have a large dwell time in the pumps and, in addition, may get trapped indefinitely as new bound states

[2] The canonical choice of the current operators one normally finds in textbooks corresponds to $f(x) = \theta(x)$.

are born at the threshold $E = 0$. This means that no scattering description in terms of a single epoch is adequate at low energies. Similarly, at high energies resonances may become increasingly sharp with correspondingly long dwelling times [3]. On the other hand, low, resp. high energy states do not contribute to the net current, since they are filled, resp. empty in both the incoming and the outgoing flow. We shall therefore concentrate on the contribution to the current coming from states in any intermediate energy range, as selected by the function χ below.

Theorem 1.1. *Let $\chi \in C_0^\infty(0, \infty)$ with $\chi = 1$ on $\operatorname{supp} d\rho$. Redefine the current operator (with a UV and IR cutoff)*

$$I_{j,\pm}(s, a) = \chi\big(H(s)\big)\mathrm{i}\,[H(s), f(\pm A_j - a)]\chi\big(H(s)\big)\ , \qquad (a > 1)\ , \tag{1.11}$$

in (1.7). Then

$$\lim_{a\to\infty}\lim_{\varepsilon\to 0}\varepsilon^{-1}\langle I\rangle_j(s, a, \varepsilon) = -\frac{\mathrm{i}}{2\pi}\int_0^\infty d\rho(E)\Big(\frac{dS}{ds}S^*\Big)_{jj}\ , \tag{1.12}$$

where $S = S(s, E)$. The double limit is uniform in $s \in I$, I being a compact interval, whence it carries over to the transfered charge $\int_0^{s/\varepsilon} dt'\langle I\rangle_j(\varepsilon t', a, \varepsilon) = \varepsilon^{-1}\int_0^s ds'\langle I\rangle_j(s', a, \varepsilon)$.

Remark 1.2. *a. The result about the current in a pump should be contrasted with related results regarding the noise generated by it. By a formula of Lee, Lesovik and Levitov, [16] the noise generated by quantum pumps (as well as other moments) is again related to the scattering data. However, the relations regarding noise are not local in time. Rather these are integral relations that hold for a cycle of the pump.*

b. The auxiliary objects v and f affect the current operator. Nevertheless, they disappear from its expectation value in the adiabatic and the large a limits. This may be phrased as the statement that different ammeters measure the same current.

Theorem 1.1 is an example of an adiabatic theorem for open, gapless systems. In contrast to other results of that kind, such as [6, 3] which examine the evolution associated with an embedded eigenstate, ours is about the evolution associated with an infinite dimensional spectral subspace, e.g. the Fermi sea. The adiabatic scattering of wave packets, rather than of spectral subspaces, has been discussed in [18, 17], and the present paper makes use of related techniques.

The plan of the rest of the paper is as follows. In Sects. 2 we shall verify that $\langle I\rangle_j(s, a, \varepsilon)$ and the Stieltjes integral in (1.12) are well-defined. Further preliminaries, like propagation estimates, will be addressed in Sect. 3. In Sect. 4 we shall compute the limit $\varepsilon \to 0$ in terms of "frozen" data, which will further simplify in Sect. 5 where the limit $a \to \infty$ is taken. In the Appendix we establish some trace class estimates for operators related to (1.11). The main ideas are contained at the beginning of Sects. 4, 5.

[3] If the pump is a chaotic billiard, the classical dynamics will, in general, have arbitrarily long periodic orbits. It is natural to expect that such orbits give rise to resonances.

We conclude with a remark on notation. Multiple commutators are denoted by $\mathrm{ad}_A^{(k)}(B) = [\mathrm{ad}_A^{(k-1)}(B), A]$ with $\mathrm{ad}_A^{(0)}(B) = B$. By $F(A \geq a)$ we mean the spectral projection of A onto $[a, \infty)$. The trace class ideal is denoted as $\mathcal{J}_1$. Generic constants are indicated by C.

2 The current operator and the scattering matrix

The state of independent quantum particles is described by a density matrix $0 \leq \rho$ normalized so that $\operatorname{tr} \rho$ is the number of particles. In the case of Fermions, $\rho \leq 1$. Thermodynamic systems have $\operatorname{tr} \rho = \infty$ and observables that are otherwise innocent, and bounded operators in particular, may fail to have finite expectation values. For example, the charge associated with the "box" $f(A-a)$ in phase space is infinite. Nevertheless, the current flowing into the box should have a finite expectation. We begin by showing that the incoming and outgoing current operators of Eq. (1.7) are trace class and consequently, the expectation value $\langle I \rangle_j(s, a, \varepsilon)$ in any Fermionic state is well defined.

A classical interpretation of the trace class condition is for the observable to be associated with a localized (bounded) function in phase space. The heuristic reason why the current is trace class is then as follows: The commutator $[H(s), f(A_j - a)]$ is localized near the boundary of the box: a curve (hyperbola) in phase space. This is where the ammeter is. The ultraviolet and infrared cutoff $\chi(H(s))$ then further delineate a compact region of phase space near the hyperbola.

Before establishing the trace class properties of the current operator we want to discuss a feature introduced by the UV and IR cutoff in the definition of the regularized current, Eq. (1.11) which is both minor and annoying. Namely, that the regularized current operator is a-priori dependent on the epoch s and the state of the pump as expressed by $H(s)$.

Observe that the cutoff may be absorbed into the Hamiltonian and the regularized current may be written as

$$I_{j\pm}(s, a) = \chi(H(s))\mathrm{i}[H_b(s), f(\pm A_j - a)]\chi(H(s)) \ , \tag{2.1}$$

where $H_b(s) = H(s)b(H(s))$ with $b \in C_0^\infty(\mathbb{R})$ and $b\chi = \chi$.

The unregularized current, Eq. (1.7) is independent of s by A3 and the commutation

$$[H(s), \Pi_j] = 0 \ . \tag{2.2}$$

Since the commutation fails when $H(s)$ is replaced by $H_b(s)$ the regularized current is a-priori epoch dependent. However, the commutation is essentially recovered on the "box" $f(\pm A - a)$ when a is large. More precisely:

Lemma 2.1. *For $j = 0, \dots n$ and $a \geq -1$ we have*

$$\|[H_b(s), \Pi_j] f(\pm A - a)(\pm A - a + \mathrm{i})^2\| \leq C \tag{2.3}$$

Proof. Since $(A + \mathrm{i})^{-2} f(\pm A - a)(\pm A - a + \mathrm{i})^2$ is uniformly bounded in $a \geq 1$ we may prove instead that

$$\|[H_b(s), \Pi_j](A + \mathrm{i})^2\| \leq C \ . \tag{2.4}$$

We begin by showing that $\Pi_i H_b \Pi_j (A+\mathrm{i})^2 = \Pi_i H_b \Pi_j (A_j+\mathrm{i})^2$ is bounded for $i \neq j$. Indeed,

$$H_b(A_j + \mathrm{i})^2 = (A_j + \mathrm{i})^2 H_b + 2(A_j + \mathrm{i})[H_b, A_j] + [[H_b, A_j], A_j] \ ,$$

from which we infer that $(A_j + \mathrm{i})^{-2} H_b (A_j + \mathrm{i})^2$ is bounded. Hence so is (use $\Pi_i A_j = 0$)

$$\Pi_i H_b \Pi_j (A_j + \mathrm{i})^2 = -\Pi_i (A_j + \mathrm{i})^{-2} H_b (A_j + \mathrm{i})^2 \Pi_j \ .$$

Now, setting $\bar{\Pi}_j = 1 - \Pi_j = \sum_{k=0,k\neq j}^{n} \Pi_k$, we have

$$[H_b, \Pi_j](A + \mathrm{i})^2 = (\bar{\Pi}_j H_b \Pi_j - \Pi_j H_b \bar{\Pi}_j)(A + \mathrm{i})^2 \ ,$$

and the result follows. □

This implies that although the regularized current a-priori depends on the epoch and the pump, the dependence disappears in the limit of large a. The main use of this lemma will be in the proof of lemma 3.2 below.

Now that we have justified our choice of current operator we come to a basic result:

Proposition 2.2. *The operator of incoming and outgoing current in the j-th channel, $I_{j\pm}(s, a)$, is a trace class operator which is localized near $\pm a$ in the sense that*

$$\|F(|A \mp a| \geq \alpha) I_{j\pm}(s, a)\|_1 \leq C_N (1 + \alpha)^{-N} \tag{2.5}$$

for all $N \in \mathbb{N}$, $\alpha \geq 0$.

Proof. This proposition is a direct consequence of Lemma 3.2 below. □

When two reservoirs are out of equilibrium a current flows even if the scatterer is time independent. The pumping formula of Thm. 1.1 says that there are no such currents. The "adiabatic" state $\rho\big(H(s)\big)$, although not necessarily a thermal state, like a thermal state, has no such persistent currents.

Lemma 2.3. *The currents in the "adiabatic" state $\rho\big(H(s)\big)$ vanish, namely*

$$\mathrm{tr}\,\big(\rho(H(s))(I_{j+}(s, a) + I_{j-}(s, a))\big) = 0 \ . \tag{2.6}$$

Proof. We omit s from the notation. Since $I_{\pm}(a) \in \mathcal{J}_1$ it suffices to prove (2.6) for smooth ρ, since one may approximate the general case by a sequence $\mathrm{s}\!-\!\lim_{n\to\infty} \rho_n(H) = \rho(H)$.

First we show that $\mathrm{tr}(\rho(H) I_{\pm}(a))$ is independent of a. Note that $I_{\pm}(a_1) - I_{\pm}(a_2) = \chi(H)\mathrm{i}[H, g(A_j)]\chi(H)$ with $g(A_j) = f(\pm A_j - a_1) - f(\pm A_j - a_2)$. Since $\chi(H) g(A_j) = \chi(H) g(A) \Pi_j \in \mathcal{J}_1$, see (A.17), we have

$$\mathrm{tr}(\rho(H)(I_{\pm}(a_1) - I_{\pm}(a_2))) = \mathrm{i}\,\mathrm{tr}(\rho\chi(H) H g(A_j)\chi(H) - \rho\chi(H) g(A_j) H \chi(H)) = 0 \tag{2.7}$$

by cyclicity. It thus suffices to prove (2.6) in the limit of $a \to \infty$. To this end, let H_0 be the Neumann Hamiltonian on the leads introduced below Eq. (1.9), and let $J : L^2(\mathbb{R}_+, \mathbb{C}^n) \to \mathcal{H}$ be the embedding given by (1.2). We maintain that

$$\lim_{a\to\infty} \|(\rho(H) I_{\pm}(a) - J\rho(H_0) I^0_{\pm}(a) J^*\|_1 = 0 \ , \tag{2.8}$$

where $I^0_\pm(a)$ is defined as in (1.11) with H replaced by H_0. Indeed, by expanding that commutator we reduce matters to two estimates, both of the form

$$\lim_{a\to\infty} \|\chi_1(H)f(\pm A_j - a)\chi_2(H) - J\chi_1(H_0)f(\pm A_j - a)\chi_2(H_0)J^*\|_1 = 0 ,$$

with $\chi_i \in C_0^\infty(\mathbb{R})$, $(i = 1, 2)$. Then we write the difference as

$$(\chi_1(H) - J\chi_1(H_0)J^*)f(\pm A_j - a)\chi_2(H) + J\chi_1(H_0)J^*f(\pm A_j - a)(\chi_2(H) - J\chi_2(H_0)J^*) .$$

Since $\chi_i(H) - J\chi_i(H_0)J^* \in J_1$ by (A.6) and $\mathrm{s-lim}_{a\to\infty} f(\pm A_j - a) = 0$, Eq. (2.8) follows. At this point we only need

$$\mathrm{tr}(\rho(H_0)(I^0_+(a) + I^0_-(a)) = 0 ,$$

which holds by time reversal invariance: For $K\psi(x) = \bar{\psi}(x)$ we have $KH_0K = H_0$, $KAK = -A$ and hence $KI^0_\pm(a)K = -I^0_\mp(a)$. □

Remark 2.4. *One can establish the result without explicitly using time reversal and using instead the pull through formula Eq. (A.10).*

Mourre theory, see e.g. [2], plays a double role in our analysis. First, it is at the heart of time dependent scattering theory and the propagation estimates that we shall discuss in the next section. At the same time, the theory also plays a role in time independent methods and we shall use it to establish the differentiability of the scattering matrix which appears on the right hand side of Eq. (1.12).

Proposition 2.5. *Under the Assumptions A1-A5, the fibers of the frozen S matrix, $S(s, E)$, are continuously differentiable in $E > 0$. In particular, the integral on the right hand side of Eq. (1.12) is well defined for ρ with bounded variation.*

Proof. The Hamiltonians $H(s)$ and $H_b(s)$ are dilation analytic of type (A) with respect to the conjugate operator A. In particular,

$$\mathrm{ad}_A^{(k)}(H) \text{ is } H(s)\text{-bounded.} \tag{2.9}$$

More importantly, for any energy $E > 0$ the Mourre estimate holds

$$E_\Delta(H(s))\mathrm{i}[H(s), A]E_\Delta(H(s)) \geq \theta_0 E_\Delta(H(s)) \tag{2.10}$$

for some open interval $\Delta \ni E$ and $\theta_0 > 0$. Note that a compact term on the r.h.s can be dismissed by Assumption A4. The Mourre estimate (2.10) is stable under small bounded perturbations of s, which is seen from (A.5) and from the fact that $\mathrm{i}[H(s), A]$ is independent of s, just as the commutator in (1.7). Therefore, (2.10) holds uniformly in $s \in I$ and $E \in J$, with $J \subset (0, \infty)$ compact, and so do the usual consequences of these assumptions. They comprise:

i) Resolvent smoothness. Let $\langle A\rangle = (1+A^2)^{\frac{1}{2}}$ and $r > 1/2$. Then

$$B(z,s) = \langle A\rangle^{-r}(H(s)-z)^{-1}\langle A\rangle^{-r}$$

has smooth boundary values at $z = E + \mathrm{i}0, (E \in J)$, satisfying

$$\|\partial_E^k B(E+\mathrm{i}0,s)\| \le C_k \tag{2.11}$$

for $k = 0, 1, \ldots$, see [15]. Since

$$\partial_s B(z,s) = -\langle A\rangle^{-r}(H(s)-z)^{-1}\dot{H}(s)(H(s)-z)^{-1}\langle A\rangle^{-r}$$

and $\dot{H}(s) = \langle A\rangle^{-r}\dot{H}(s)\langle A\rangle^{-r}$, the function $B(E+\mathrm{i}0,s)$ is jointly continuously differentiable in (E,s).

ii) H-smoothness. $\langle A\rangle^{-r}$ is $H(s)$-smooth [20, 24], as a consequence of (2.11) for $k=0$.

iii) Stationary representation of the scattering matrix. Let

$$E_J(H(s))\mathcal{H} \to \int_J^{\oplus} \mathbb{C}^n \, dE\,, \qquad \psi \mapsto \{\psi(E)\} \tag{2.12}$$

be the spectral representation for $H(s)$ on J, and set $\Gamma_0(E)\psi = \psi(E)$. Then $\Gamma_0(E)\langle A\rangle^{-r} : \mathcal{H} \to \mathbb{C}^n$ is bounded by (ii). Let $S(s',s)$ be the scattering operator for the pair $(H(s'),H(s))$. Its fibers $S(E) : \mathbb{C}^n \to \mathbb{C}^n$ with respect to (2.12) admit the representation [24]

$$S(E) = 1 - 2\pi\mathrm{i}\Gamma_0(\lambda)(V - V(H(s') - (E+\mathrm{i}0))^{-1}V)\Gamma_0(\lambda)^*\,, \tag{2.13}$$

with $V = H(s') - H(s)$. The r.h.s. is defined pointwise because of $V = \langle A\rangle^{-r}V\langle A\rangle^{-r}$ and of (i). From this, Eq. (2.13), and the statements about $B(E+\mathrm{i}0,s)$ we see that $S(s',s)E_J(H(s))$ is continuously differentiable in $s' \in I$, and that the derivative can be computed fiber-wise. The same applies to the scattering operator $S(s) = S(H(s),H_0)$, though (2.13) appears slightly modified.

□

3 Propagation estimates

Propagation estimates play a key and multiple role in our analysis. One role is that they guarantee that particles do not get stuck in the pump. Consequently, the scattered particle indeed sees a frozen scatterer to lowest order in the adiabatic limit, and a linearly changing scatterer to first order. At the same time propagation estimates also play a role in establishing that the current measured by the ammeter at epoch s is determined by the state of the pump at the same epoch, in the adiabatic limit. This may be interpreted as a statement that the particles neither linger nor disperse too badly in the channels [4].

[4]Dispersion arises because we use a non-relativistic (quadratic) kinetic energy in the leads. In much of the literature on pumps dispersion is circumvented by making the kinetic energy linear in the momentum.

Propagation estimates play yet another role in establishing the "rigidity" of the current: The expectation value for the current a-priori depends on a choice of a switch function f, the function $v(x)$ in the generator of dilation and a choice of the initial configuration of the pump, H_-. This dependence is suppressed from our notation because Thm. 1.1 implies that the dependence disappears in the limit. Ultimately, this independence is a consequence of propagation estimates.

Eq. (2.10) implies a minimal escape velocity estimate [11, 13] for the autonomous dynamics generated by $H(s)$. The constants involved are understood as being uniform in $s \in I$ and in the stated range for a. Within proofs we shall abbreviate $H \equiv H(s)$.

Lemma 3.1. *Let χ in $C_0^\infty(0,\infty)$ (in particular with support away from $E = 0$). Then, for some $\theta > 0$, for all $a \in \mathbb{R}$, $b, t \geq 0$ and $N \in \mathbb{N}$:*

$$\|F(A \leq a - b + \theta t)\mathrm{e}^{-\mathrm{i}H(s)t}\chi(H(s))F(A \geq a)\| \leq C_N(\theta)(b + \theta t)^{-N} . \tag{3.1}$$

Similarly, if $b, t < 0$, then

$$\|F(A \geq a - b + \theta t)\mathrm{e}^{-\mathrm{i}H(s)t}\chi(H(s))F(A \leq a)\| \leq C_N(\theta)|b + \theta t|^{-N} . \tag{3.2}$$

Proof. The case of $b = 0$ is covered by Thm. 1.1 in [13], since its hypothesis (besides of (2.10)) that $\mathrm{ad}_A^{(k)}(f(H))$ is bounded for $f \in C_0^\infty(\mathbb{R})$ and $k \geq 1$ holds true by (2.9). To be precise, the result is formulated there for $\operatorname{supp}\chi \subset \Delta$, where Δ is as in (2.10) and $0 < \theta < \theta_0$, but it extends to our case by a covering argument. Actually, the proof given there essentially covers the general case $b \neq 0$. More precisely, let

$$A_{t\tau} = \tau^{-1}(A - a + \frac{b}{2} - \theta_0 t)$$

and $f \in C^\infty(\mathbb{R})$ be a function with $|f^{(k)}(x)| \leq C\langle x\rangle^{-k}, f' \leq 0$, and $f(x) = 0$ for $x \geq 0$. Then

$$\|f(A_{t\tau})\mathrm{e}^{-\mathrm{i}Ht}\chi(H)F(A \geq a)\| \leq C_N\tau^{-N}$$

uniformly in $0 \leq t \leq \tau$ and $a \in \mathbb{R}$. For $b = 0$, this is equation (2.11) in [13], whose proof applies to $b \geq 0$ as well. Let $\tau = b + t$. Since

$$\frac{b - \theta t}{b + t} \geq \frac{b/2 - \theta_0 t}{b + t} + \delta$$

for some $\delta > 0$ and all b, $t \geq 0$ we have

$$F(A \leq a - b + \theta t) \leq F\Big(\frac{A - a + (b/2) - \theta_0 t}{b + t} \leq -\delta\Big) \leq f(A_{t\tau})$$

for some f of the required type. $\square$

Lemma 3.2.

$$\begin{aligned}
\|\mathrm{i}[H_b(s), f(\pm A_j - a)]\chi(H(s))\|_1 &\leq C , &(3.3)\\
\|F(A < \pm a - \alpha)\mathrm{i}[H_b(s), f(\pm A_j - a)]\chi(H(s))\|_1 &\leq C_N(1 + \alpha)^{-N} , &(3.4)\\
\|F(A > \pm a + \alpha)\mathrm{i}[H_b(s), f(\pm A_j - a)]\chi(H(s))\|_1 &\leq C_N(1 + \alpha)^{-N} &(3.5)
\end{aligned}$$

for a, $\alpha \geq 1$. The same bounds hold in operator norm if $\chi(H(s))$ is replaced with $(\pm A - a + \mathrm{i})^2$.

The estimate (3.5) prevents the current operators from being located very far in the outgoing region of phase space. This will play a role in the next lemma and in Sect. 4. Actually, instead of $(3.5)_-$ we shall use there the weaker statement with characteristic function $F(A > a+\alpha)$. The pair $(3.4)_+$ and $(3.5)_-$ keeps the current operators away from the pump, a property used in Sect. 5.

Proof. The estimates to be proven are of the form

$$\begin{aligned}\|T[H_b, f(\pm A_j - a)]\chi(H)\|_1 &\le \|T[H_b, f(\pm A_j - a)](\pm A - a + \mathrm{i})^2\| \|(\pm A - a + \mathrm{i})^{-2}\chi(H)\|_1 \\ &\le C\|T[H_b, f(\pm A_j - a)](\pm A - a + \mathrm{i})^2\| \,, \end{aligned} \tag{3.6}$$

where we used (A.16).

We then have to establish the corresponding bounds for the remaining operator norm. Since $g(A_j) = g(0)+(g(A)-g(0))\Pi_j$ we have $f(\pm A_j - a) = f(\pm A - a)\Pi_j$ due to $f(-a) = 0$ for $a > 1$. Writing $f = f\tilde{f}$, where $\tilde{f}(\cdot) = f(\cdot + 2)$, the commutator in (3.6) is

$$[H_b, f\Pi_j\tilde{f}] = [H_b, f]\Pi_j\tilde{f} + f\Pi_j[H_b, \tilde{f}] + f[H_b, \Pi_j]\tilde{f} \,,$$

with $\tilde{f} = \tilde{f}(\pm A - a)$. In the contribution to (3.6) of the first two terms the projections Π_j may be moved out to the right or to the left, using $[T, \Pi_j] = 0$. As for the last term, we use (2.3). At this point, (3.6) reduces to corresponding estimates for

$$\|T[H_b, f(A)](A + \mathrm{i})^2\| + \|Tf(A)\| \,, \tag{3.7}$$

where we replaced $\pm A - a$ by A, as the argument given below applies to the more general case.

In the case of (3.3), where $T = 1$, the second term (3.7) is clearly bounded. On the first term we use the commutator expansion, see [12], eq. (B.16), or [10], based on the Helffer-Sjöstrand representation of $f(A)$.

$$\begin{aligned}[H_b, f(A)] &= \sum_{k=1}^{m-1} \frac{1}{k!}(-1)^{k-1}\mathrm{ad}_A^{(k)}(H_b)f^{(k)}(A) + R_m \,, \\ R_m &= -\frac{1}{\pi}\int dx\, dy \partial_{\bar{z}}\tilde{f}(z)(A - z)^{-1}\mathrm{ad}_A^{(m)}(H_b)(A - z)^{-m} \,, \end{aligned} \tag{3.8}$$

where $\partial_{\bar{z}} = (\partial_x + \mathrm{i}\partial_y)/2$ and $\tilde{f}$ is an almost analytic extension of f, which may be chosen in such a way that

$$\int dx\, dy |\partial_{\bar{z}}\tilde{f}(z)||y|^{-p-1} \le C\sum_{k=0}^{m+2} \|f^{(k)}\|_{k-p-1}$$

for $p = 0, \ldots n$, the norms being defined as $\|f\|_k = \int dx\langle x\rangle^k|f(x)|$. The choice of [12] is such that if $\operatorname{supp} f'$ is compact, as it is in our case, then $|y| \ge C_1|x| - C_2$, (C_1, $C_2 > 0$), for $z = x + \mathrm{i}y \in \operatorname{supp} \tilde{f}$, which thus implies

$$\|(A - z)^{-1}(A + \mathrm{i})\| \le C(|y|^{-1} + 1) \tag{3.9}$$

The expanded terms remain bounded if multiplied by $(A+\mathrm{i})^2$ from the right. For the remainder we obtain

$$\|R_m(A+\mathrm{i})^2\| \le C\|\mathrm{ad}_A^{(m)}(H_b)\| \sum_{k=0}^{m+2}(\|f^{(k)}\|_{k-m-1} + \|f^{(k)}\|_{k-m+1}) , \tag{3.10}$$

which is finite for $m \ge 3$.

We shall next prove (3.4) with $F(A < -\alpha)$ replaced by $F(A < -3\alpha)$ for easing notion below. This may be further replaced by $f(-\alpha^{-1}A-2)$ because of $F(A < -3\alpha) = F(A < -3\alpha)f(-\alpha^{-1}A-2)$. Since $f(-\alpha^{-1}A-2)f(A) = 0$, the claim Eq. (3.7) reads

$$\|[H_b, f(-\alpha^{-1}A-2)]f(A)(A+\mathrm{i})^2\| \le C_N\alpha^{-N} . \tag{3.11}$$

We now apply the expansion (3.8) to $-\alpha^{-1}A-2$ instead of A. Since $f^{(k)}(-\alpha^{-1}A-2)f(A) = 0$ only the remainder contributes. To bound $\|R_m(A+\mathrm{i})^2\|$ we use $\|(-\alpha^{-1}A-2-z)^{-1}(A+\mathrm{i})\| \le C\alpha(|y|^{-1}+1)$ instead of (3.9) and collect the powers of α^{-1} generated by each commutator. We so obtain the bound of (3.10) times α^{-m+2}.

Finally, (3.5) can be brought into a form similar to (3.4) by replacing f with $f-1$. Both functions have the same commutator with H_b but essentially complementary supports. □

In Sect. 4 we shall describe the dynamics (1.3) in terms of the autonomous dynamics generated by the Hamiltonians $H(s)$. Once this is achieved, the choice of an argument in initial 1-particle density matrix ρ, be it H_- or $H(s)$, does not matter much for the current measurement, as defined by (1.11). This is the content of the following lemma.

Proposition 3.3. *Let ρ, χ be as in Theorem 1.1. Then, for $a \ge 1$,*

$$\lim_{t\to-\infty} \|(\rho(H(s)) - \rho(H_-))\mathrm{e}^{-\mathrm{i}H(s)t}I_\pm(s,a)\|_1 = 0 . \tag{3.12}$$

We remark that no claim of uniformity w.r.t. s, a is made here. The statement for $t \to +\infty$ is also true, but not needed.

Proof. We first consider the case where ρ is smooth, and since H is bounded below we may assume $\rho \in C_0^\infty(\mathbb{R})$. Then

$$\rho(H) - \rho(H_-) \in \mathcal{J}_1 \tag{3.13}$$

by Eq. (A.5). To estimate the trace norm in (3.12) we use (2.1), insert $1 = F(A \le a+\alpha) + F(A > a+\alpha)$ to the left of the commutator in (1.11), and use (3.3, 3.5). We so obtain the bound

$$\|(\rho(H) - \rho(H_-))\mathrm{e}^{-\mathrm{i}Ht}\chi(H)F(A \le a+\alpha)\| \cdot C + C_N\alpha^{-N} . \tag{3.14}$$

The remaining operator norm is bounded for $t \le 0$ by

$$\|(\rho(H) - \rho(H_-))F(A \le a+\alpha+\theta t)\| + C_N t^{-N} , \tag{3.15}$$

where we used (3.2) with $b = 0$. We pick $\alpha = -\theta t/2$, so that the remainder in (3.14) tends to zero, and $\mathrm{s-lim}_{t\to-\infty} F(\dots) = 0$ in (3.15). Since $\rho(H) - \rho(H_-)$ is compact, the norm vanishes in the limit.

In the general case, where ρ is of bounded variation, let $\rho_n \in C_0^\infty(\mathbb{R})$, $\sup_n \|\rho_n\|_\infty < \infty$, be a sequence such that $\rho_n(\lambda) \to \rho(\lambda)$ pointwise, whence $\mathrm{s-lim}_{n\to\infty} \rho_n(H) = \rho(H)$, and the same for H_- instead of H. Also, $\mathrm{s-lim}_{t\to-\infty}(\mathrm{e}^{-\mathrm{i}Ht}P_{\mathrm{ac}}(H) - \mathrm{e}^{-\mathrm{i}H_-t}\Omega_-) = 0$ by definition of the wave operator $\Omega_- = \Omega_-(H_-, H)$ [20, 24]. Since $I_\pm(s,a)$ is trace class by (3.3) and since

$$\mathrm{s-}\lim_{n\to\infty} X_n = 0\,, \quad Y \in \mathcal{J}_1 \;\Rightarrow\; \lim_{n\to\infty} \|X_n Y\|_1 = 0\,, \tag{3.16}$$

we have

$$\begin{aligned} \lim_{n\to\infty} \|(\rho_n(H) - \rho(H))I_\pm(s,a)\|_1 &= 0\,, \\ \lim_{n\to\infty} \|(\rho_n(H_-) - \rho(H_-))\Omega_- I_\pm(s,a)\|_1 &= 0\,, \\ \lim_{t\to-\infty} \|(\mathrm{e}^{-\mathrm{i}Ht} - \mathrm{e}^{-\mathrm{i}H_-t}\Omega_-)I_\pm(s,a)\|_1 &= 0\,, \end{aligned} \tag{3.17}$$

where we dropped $P_{\mathrm{ac}}(H)$ due to $P_{\mathrm{ac}}(H)\chi(H) = \chi(H)$. We then estimate ($I_\pm \equiv I_\pm(s,a)$)

$$\begin{aligned} &\|(\rho(H) - \rho(H_-))\mathrm{e}^{-\mathrm{i}Ht} I_\pm\|_1 \\ &\le \|(\rho(H) - \rho_n(H))\mathrm{e}^{-\mathrm{i}Ht} I_\pm\|_1 + \|(\rho_n(H) - \rho_n(H_-))\mathrm{e}^{-\mathrm{i}Ht} I_\pm\|_1 \\ &\quad + \|(\rho_n(H_-) - \rho(H_-))(\mathrm{e}^{-\mathrm{i}Ht} - \mathrm{e}^{-\mathrm{i}H_-t}\Omega_-)I_\pm\|_1 + \|(\rho_n(H_-) - \rho(H_-))\mathrm{e}^{-\mathrm{i}H_-t}\Omega_- I_\pm\|_1 \\ &\le \|(\rho(H) - \rho_n(H))I_\pm\|_1 + \|(\rho_n(H) - \rho_n(H_-))\mathrm{e}^{-\mathrm{i}Ht} I_\pm\|_1 \\ &\quad + C\|(\mathrm{e}^{-\mathrm{i}Ht} - \mathrm{e}^{-\mathrm{i}H_-t}\Omega_-)I_\pm\|_1 + \|(\rho_n(H_-) - \rho(H_-))\Omega_- I_\pm\|_1 \end{aligned}$$

Given $\epsilon > 0$ we first pick n large enough such that the first and the last term together are less than $\epsilon/2$. For large negative t the same is true for the second and the third term by (3.17) and the first part of the proof. □

4 The adiabatic limit

The current generated by adiabatic pumps can be understood within the general framework of the theory of linear response: The adiabatic limit corresponds to weak driving, and the formal derivation of Eq. (1.1) in [8] is a perturbation expansion. Formally, the change in the state ρ of system, obtained by linearizing the Hamiltonian around epoch s is

$$U_\varepsilon(s, s_-)\rho(H(s))U_\varepsilon(s_-, s) - \rho(H(s)) \approx -\mathrm{i}\varepsilon \int_{-\infty}^0 dt\, t\, \mathrm{e}^{\mathrm{i}H(s)t}[\dot H(s), \rho(H(s))]\mathrm{e}^{-\mathrm{i}H(s)t}. \tag{4.1}$$

One immediate difficulty with this expression is that the integrand on the right hand side grows linearly with time. As an operator identity the above equation does not make sense, and the right hand side is not recognizably of order ε.

One of the reasons why a perturbation expansion can nevertheless be made is that the current of Eq. (1.11) has good localization in phase-space and so distinguishes a region of phase space where ρ is to be evaluated. Therefore, only a restricted range of times contribute to the integral: The time associated with propagation from the pump to the ammeter. This makes the expectation value of the current well defined. Estimates of this kind are known as propagation estimates and are controlled by Mourre theory.

In this section the limit $\varepsilon \to 0$ in (1.12) is taken as a is kept fixed. The main result is the following.

Proposition 4.1. *For fixed $a \geq 1$, we have*

$$\lim_{\varepsilon\to 0} \varepsilon^{-1} \operatorname{tr}\big([U_\varepsilon(s,s_-)\rho(H_-)U_\varepsilon(s_-,s) - \rho(H(s))]I_\pm(s,a)\big) = \operatorname{tr}([\Omega_-^{(1)}(s), \rho(H(s))]I_\pm(s,a)) \tag{4.2}$$

uniformly in $s \in I$, where

$$\Omega_-^{(1)}(s) = -\mathrm{i}\int_{-\infty}^{0} dt\, t\, \mathrm{e}^{\mathrm{i}H(s)t}\dot{H}(s)\mathrm{e}^{-\mathrm{i}H(s)t}\ . \tag{4.3}$$

Moreover,

$$\lim_{\varepsilon\to 0} \varepsilon^{-1} \operatorname{tr}\big(U_\varepsilon(s,s_-)\rho(H_-)U_\varepsilon(s_-,s)(I_+(s,a)+I_-(s,a))\big) = \operatorname{tr}([\Omega_-^{(1)}(s), \rho(H(s))](I_+(s,a)+I_-(s,a)))\ . \tag{4.4}$$

Remarks.

- The integral (4.3) is a trace class norm convergent integral once multiplied from the left or from the right by $I_\pm(s,a)$, as in (4.2).
- Eq. (4.2) separately describes, to leading order, the incoming and outgoing currents in an adiabatically evolved state as compared to the corresponding instantaneous state $\rho\big(H(s)\big)$. Eq. (4.4) is then an immediate consequence of (2.6).
- $\Omega_-^{(1)}(s)$ is formally the first order in ε term in the expansion of the wave operator which for fixed s relates $U_\varepsilon(s',s)$ to the autonomous dynamics $u_s(s'-s)$ generated by the Hamiltonian $H(s)$:

  $$u_s(\sigma) := \mathrm{e}^{-iH(s)\varepsilon^{-1}\sigma}\ ,$$

  where ε has been suppressed from the notation. In the next section we shall also meet the first order term in the expansion of the scattering operator relating these two dynamics,

  $$S^{(1)}(s) = \Omega_-^{(1)}(s) - \Omega_+^{(1)}(s) = -\mathrm{i}\int_{-\infty}^{\infty} dt\, t\, \mathrm{e}^{\mathrm{i}H(s)t}\dot{H}(s)\mathrm{e}^{-\mathrm{i}H(s)t}\ . \tag{4.5}$$

  For more details, see [5]. The reason $\Omega_-^{(1)}(s)$, rather than $\Omega_+^{(1)}(s)$, appears in (4.2) is that the initial condition was set in the past of the current measurement.

- All estimates in this section hold true uniformly in $s \in I$, $s_- \le 0$. Constants may however depend on a.

The two dynamics, U_ε and u_s, are compared by means of the Duhamel formula

$$U_\varepsilon(s', s) = u_s(s' - s) + \mathrm{i}\varepsilon^{-1} \int_{s'}^{s} ds_1 U_\varepsilon(s', s_1)(H(s_1) - H(s))u_s(s_1 - s) \ . \tag{4.6}$$

Starting at epoch s, the Heisenberg dynamics of the currents carries them through the scatterer within a finite time, i.e., essentially still at same epoch s. When applying (4.6) to $I_\pm(s, a)$ it is thus appropriate to linearize $H(s_1) - H(s)$ around s. The next lemma essentially says that calculating the current to first order in ε the error one makes is second order in ε:

Lemma 4.2. *We have*

$$\varepsilon^{-1} \|[U_\varepsilon(s_-, s) - (u_s(s_- - s) + X_\varepsilon(s_-, s))]I_\pm(s, a)\|_1 \le C\varepsilon \ , \tag{4.7}$$

where

$$X_\varepsilon(s_-, s) = \mathrm{i}\varepsilon^{-1} \int_{s_-}^{s} ds_1 (s_1 - s)U_\varepsilon(s_-, s_1)\dot{H}(s)u_s(s_1 - s) \ .$$

In turn this yields the following expression for the (rescaled) currents at epoch s:

$$\begin{aligned}
\varepsilon^{-1} \operatorname{tr}(\rho(H_-)U_\varepsilon(s_-, s)I_\pm(s, a)U_\varepsilon(s, s_-)) = \\
&\varepsilon^{-1} \operatorname{tr}(u_s(s - s_-)\rho(H_-)u_s(s_- - s)I_\pm(s, a)) + \\
&\varepsilon^{-1} \operatorname{tr}(X_\varepsilon(s_-, s)^*\rho(H_-)u_s(s_- - s)I_\pm(s, a)) + \\
&\varepsilon^{-1} \operatorname{tr}(u_s(s - s_-)\rho(H_-)X_\varepsilon(s_-, s)I_\pm(s, a)) + \\
&\varepsilon^{-1} \operatorname{tr}(\rho(H_-)X_\varepsilon(s_-, s)I_\pm(s, a)X_\varepsilon(s_-, s)^*) + \mathrm{O}(\varepsilon) \ .
\end{aligned} \tag{4.8}$$

Proof. Using Duhamel's formula (4.6) and

$$\begin{aligned}
H(s_1) - H(s) - (s_1 - s)\dot{H}(s) &= (H(s_1) - H(s) - (s_1 - s)\dot{H}(s))F(A = 0) \ , \\
\|H(s_1) - H(s) - (s_1 - s)\dot{H}(s)\| &\le C|s_1 - s|^2 \ ,
\end{aligned}$$

see (1.4), Assumption A1 and $\Pi_0 = F(A = 0)\Pi_0$, we are left with showing that

$$\varepsilon^{-2} \int_{s_-}^{s} ds_1 \, |s_1 - s|^2 \cdot \|F(A = 0)u_s(s_1 - s)I_\pm(s, a)\|_1 \le C\varepsilon \ . \tag{4.9}$$

We insert $1 = F(A \le a + \alpha) + F(A > a + \alpha)$ to the left of the commutator in (1.11). By (2.1, 3.5) the trace norm appearing under the integral is bounded as

$$\begin{aligned}
&\|F(A = 0)u_s(s_1 - s)I_\pm(s, a)\|_1 \\
&\qquad \le C\|F(A = 0)u_s(s_1 - s)\chi(H(s))F(A \le a + \alpha)\| + C_N(1 + \alpha)^{-N} \ ,
\end{aligned}$$

where we also used (3.3). By (3.2) with $b=0$ the latter norm is estimated as

$$\begin{aligned}&\|F(A=0)u_s(s_1-s)\chi(H(s))F(A\le a+\alpha)\|\\&\qquad\le C\|F(A=0)F(A\le a+\alpha+\theta\varepsilon^{-1}(s_1-s))\|+C_N(1+\varepsilon^{-1}|s_1-s|)^{-N}\ .\end{aligned}\tag{4.10}$$

Picking $\alpha=\frac{1}{2}\theta\varepsilon^{-1}(s-s_1)>0$, the first term vanishes for $s-s_1>2a\theta^{-1}\varepsilon$. Now Eq. (4.9) holds because the l.h.s. is estimated by a constant times

$$\varepsilon^{-2}\int_0^\infty d\sigma\,\sigma^2(F(\sigma\le 2a\theta^{-1}\varepsilon)+(1+\varepsilon^{-1}\sigma)^{-N})\le C(\varepsilon a^3+\varepsilon)\ ,\tag{4.11}$$

which proves the lemma. □

For later purposes we retain the following estimate from the above proof:

$$\|F(A=0)\mathrm{e}^{-\mathrm{i}H(s)t}I_\pm(s,a)\|\le F(|t|\le 2a\theta^{-1})+C_N(1+|t|)^{-N}\ .\tag{4.12}$$

In particular, it proves the first remark after Prop. 4.1.

The last term in Eq. (4.8) is also $\mathrm{O}(\varepsilon)$, as shown by the next estimate.

Lemma 4.3. *We have*

$$\|X_\varepsilon(s_-,s)I_\pm(s,a)X_\varepsilon(s_-,s)^*\|_1\le C\varepsilon^2\ .\tag{4.13}$$

Proof. As in the proof of the previous lemma, the norm in (4.13) is bounded by

$$\varepsilon^{-2}\int_{s_-}^s ds_1\,ds_2|s_1-s||s_2-s|\|F(A=0)u_s(s_1-s)I_\pm(s,a)u_s(s_2-s)^*F(A=0)\|_1\ .\tag{4.14}$$

Using again (2.1, 3.5), this last norm by itself is bounded by

$$\Big(\|F(A=0)u_s(s_1-s)\chi(H(s))F(A\le a+\alpha)\|+C_N(1+\alpha)^{-N}\Big)\cdot\Big(s_1\to s_2\Big)$$

where $(s_1\to s_2)$ is shorthand for the previous expression with s_1 replaced by s_2. We pick $\alpha=\frac{1}{2}\theta\varepsilon^{-1}\min(s-s_1,s-s_2)\ge 0$. Then the previous expression is estimated as

$$\Big(F(\sigma_1\le 2a\theta^{-1}\varepsilon)+(1+\frac{1}{2}\theta\varepsilon^{-1}\min(\sigma_1,\sigma_2))^{-N}\Big)\cdot\Big(s_1\to s_2\Big)\ ,$$

where $\sigma_i=s-s_i$. Using

$$\int_0^\infty d\sigma_2\,\sigma_2(1+\frac{1}{2}\theta\varepsilon^{-1}\min(\sigma_1,\sigma_2))^{-N}\le C(\sigma_1^2+\varepsilon^2)\ ,$$
$$\int_0^\infty d\sigma_1 d\sigma_2\,\sigma_1\sigma_2(1+\frac{1}{2}\theta\varepsilon^{-1}\min(\sigma_1,\sigma_2))^{-N}\le C\varepsilon^4\ ,$$

the claimed bound is established for (4.14). □

We now turn to the second and third term in Eq. (4.8), which will account for the current in the adiabatic limit.

Lemma 4.4.

$$\lim_{\varepsilon\to 0}\|\big(U_\varepsilon(s_-,s_1)-u_s(s_--s_1)\big)u_s(s_1-s)\rho(H(s))\chi(H(s))(\pm A-a+\mathrm{i})^{-2}\|_1=0\ , \quad (4.15)$$

uniformly also in $s_1\in[s_-,s]$.

Proof. We first consider the case where ρ is smooth. Since $T=\chi(H(s))(\pm A-a+\mathrm{i})^{-2}\in\mathcal{J}_1$ by (A.16) and $\mathrm{s-lim}_{\alpha\to\infty}F(A>\alpha)=0$ we have $\lim_{\alpha\to\infty}\|F(A>\alpha)T\|_1=0$. By approximation we may thus prove

$$\lim_{\varepsilon\to 0}\|(U_\varepsilon(s_-,s_1)-u_s(s_--s_1))u_s(s_1-s)\rho(H(s))\tilde\chi(H(s))F(A\le\alpha)\|=0\ ,$$

where $\tilde\chi\in C_0^\infty(0,\infty)$ with $\tilde\chi\chi=\chi$. To estimate this operator norm we use (4.6) together with

$$\|H(s_2)-H(s)\|\le C|s_2-s| \quad (4.16)$$

to obtain the bound

$$\varepsilon^{-1}\int_{s_-}^{s_1}ds_2|s_2-s|\|F(A=0)u_s(s_2-s)\rho(H(s))\tilde\chi(H(s))F(A\le\alpha)\|\le C\varepsilon(1+\alpha^2)\ ,$$

where we used, see the argument in connection with (4.10), that the norm under the integral is bounded by $F(s-s_2\le\varepsilon\theta^{-1}\alpha)+C_N(1+\varepsilon^{-1}|s_2-s|)^{-N}$. The general case, where ρ is of bounded variation, is also dealt with by approximating $\rho(H)=\mathrm{s-lim}_{n\to\infty}\rho_n(H)$ with ρ_n smooth. In fact we can pick n so that $\|(\rho_n(H)-\rho(H))T\|_1$ is arbitrarily small (uniformly in s). □

Proof of Prop. 4.1. As mentioned in the introduction, $U_\varepsilon(s,s_-)\rho(H_-)U_\varepsilon(s_-,s)$ is independent of $s_-\le 0$. We may thus evaluate the r.h.s. of (4.8) in the limit $s_-\to\infty$. By estimate (3.12) and its adjoint, this amounts to replacing $\rho(H_-)$ by $\rho(H(s))$, i.e.,

$$\begin{aligned}\varepsilon^{-1}\operatorname{tr}(\rho(H_-)U_\varepsilon(s_-,s)I_\pm(s,a)U_\varepsilon(s,s_-))=\varepsilon^{-1}\operatorname{tr}(\rho(H(s))I_\pm(s,a))+&\\ \varepsilon^{-1}\operatorname*{Lim}_{s_-\to-\infty}\operatorname{tr}(X_\varepsilon(s_-,s)^*u_s(s_--s)\rho(H(s))I_\pm(s,a))+&\\ \varepsilon^{-1}\operatorname*{Lim}_{s_-\to-\infty}\operatorname{tr}(u_s(s-s_-)X_\varepsilon(s_-,s)I_\pm(s,a)\rho(H(s)))+\mathrm{O}(\varepsilon)\ ,&\quad (4.17)\end{aligned}$$

where we also used $[u_s(s-s_-),\rho(H(s))]=0$ and (4.13). The first term one the r.h.s. is just the equilibrium value of the current subtracted on the l.h.s. of Eq. (4.2). As we are not going to show that the limits in (4.17) exist, they are just to be understood as sets of limit points. We next claim that

$$\varepsilon^{-1}\operatorname*{Lim}_{s_-\to\infty}\operatorname{tr}((X_\varepsilon(s_-,s)^*u_s(s_--s)-\varepsilon\Omega_-^{(1)}(s))\rho(H(s))I_\pm(s,a))\to 0\ ,\qquad(\varepsilon\to 0)\ . \quad (4.18)$$

By taking the complex conjugate this implies

$$\varepsilon^{-1} \operatorname*{Lim}_{s_-\to\infty} \operatorname{tr}(I_\pm(s,a)\rho(H(s))(u_s(s-s_-)X_\varepsilon(s_-,s)+\varepsilon\Omega_-^{(1)}(s))) \to 0\ , \qquad (\varepsilon\to 0)\ ,$$

where we used $\Omega_-^{(1)}(s)^* = -\Omega_-^{(1)}(s)$. Used together in (4.17) they prove (4.2).

We next note that, by a change of variables,

$$\varepsilon\Omega_-^{(1)}(s) = -\mathrm{i}\varepsilon^{-1}\int_{-\infty}^{s} ds_1(s_1-s)u_s(s-s_1)\dot{H}(s)u_s(s_1-s)\ .$$

As noted before, the integral is convergent in trace class norm upon multiplication from either side with $I_\pm(s,a)$. For the purpose of proving (4.18) we may thus assume it to have s_- as its lower limit of integration. Then the trace there, including the factor ε^{-1} in front, equals

$$-\mathrm{i}\varepsilon^{-2}\int_{s_-}^{s} ds_1(s_1-s)\times$$
$$tr(u_s(s-s_1)\dot{H}(s)\big(U_\varepsilon(s_1,s_-)u_s(s_--s)-u_s(s_1-s)\big)\rho(H(s))I_\pm(s,a))\ .$$

We use

$$U_\varepsilon(s_1,s_-)u_s(s_--s)-u_s(s_1-s) = U_\varepsilon(s_1,s_-)\big(u_s(s_--s_1)-U_\varepsilon(s_-,s_1)\big)u_s(s_1-s)$$

and turn $u_s(s-s_1)\dot{H}(s)$ around the trace, so that the previous expression estimated as

$$\varepsilon^{-2}\int_{s_-}^{s} ds_1|s_1-s|\|\big(U_\varepsilon(s_-,s_1)-u_s(s_--s_1)\big)u_s(s_1-s)\rho(H(s))\chi(H(s))(\pm A-a+\mathrm{i})^{-2}\|_1\times$$
$$\|(\pm A-a+\mathrm{i})^2[H_b(s),f(\pm A-a)]\chi(H(s))u_s(s-s_1)F(A=0)\|\ .$$

The first factor tends to zero uniformly as $\varepsilon\to 0$ by (4.15), so we are left to show

$$\varepsilon^{-2}\int_{s_-}^{s} ds_1|s_1-s|\|F(A=0)u_s(s_1-s)\chi(H(s))[H_b(s),f(\pm A-a)](\pm A-a+\mathrm{i})^2\|\le C\ . \tag{4.19}$$

We insert $1 = F(A\le a+\alpha)+F(A>a+\alpha)$ to the left of the commutator. By (3.3, 3.5) and the remark following them we obtain the bound

$$\varepsilon^{-2}\int_{s_-}^{s} ds_1|s_1-s|\big(\|F(A=0)u_s(s_1-s)\chi(H(s))F(A\le a+\alpha)\|+C_N(1+\alpha)^{-N}\big)\ ,$$

where we take $\alpha=\frac{1}{2}\theta\varepsilon^{-1}(s-s_1)>0$ as in (4.10). The resulting bound differs from (4.11) by having σ instead of σ^2. Hence the bound (4.19). □

5 The scattering limit

In the previous section we saw that the currents can be computed from frozen data in the adiabatic limit. These data were not directly related to the frozen scattering data and involved objects like $H(s)$ and $\Omega_-^{(1)}(s)$. In this section we show that in the limit that the ammeter is far, $a \to \infty$, then all we need to know is the frozen, scattering operator S and the initial state ρ.

First we show that in the large a limit the incoming, resp. outgoing currents have no scattering in the past, resp. the future. As usual, all statements are uniform in $s \in I$.

Lemma 5.1. *We have*

$$\lim_{a\to\infty} \|\Omega_-^{(1)}(s) I_-(s,a)\|_1 = 0 ,$$
$$\lim_{a\to\infty} \|(\Omega_-^{(1)}(s) - S^{(1)}(s)) I_+(s,a)\|_1 = 0 .$$

We recall that $S^{(1)}$ was defined in (4.5). These facts will yield a first expression for the scattering limit of the current, Eq. (4.4).

Lemma 5.2. *We have*

$$\lim_{a\to\infty} \mathrm{tr}([\Omega_-^{(1)}(s), \rho(H(s))](I_+(s,a) + I_-(s,a))) = \mathrm{tr}\big([S^{(1)}(s), \rho(H(s))] I_+(s,a)\big) , \qquad (5.1)$$

where a on the r.h.s. is arbitrary.

We then express the latter result in terms of "frozen" scattering data, such as the scattering operator $S(s)$, defined in (1.9). Notice that it acts on the asymptotic Hilbert space $L^2(\mathbb{R}_+, \mathbb{C}^n)$ of the channels, rather than to $\mathcal{H}$. Further distinguished operators acting there are the Neumann Hamiltonian H_0, introduced in the introduction, and the generator of dilations,

$$A_0 = \frac{1}{2\mathrm{i}}\Big(\frac{d}{dx}x + x\frac{d}{dx}\Big) ,$$

which may be regarded as a model for (1.6). The trace (5.1) can then finally be computed exactly.

Proposition 5.3. *Suppose, in addition to the hypotheses of Thm. 1.1, that ρ is smooth. Then,*

$$\begin{aligned} \mathrm{tr}\big([S^{(1)}(s), \rho(H(s))] I_+(s,a)\big) &= -\mathrm{i}\,\mathrm{tr}\big(\dot S(s) S(s)^* \rho'(H_0)\Pi_j \mathrm{i}[H_0, f(A_0 - a)]\big) , & (5.2)\\ &= -\frac{\mathrm{i}}{2\pi}\int_0^\infty d\rho(E)\Big(\frac{dS}{ds}S^*\Big)_{jj} , & (5.3) \end{aligned}$$

where $A_{0j} = A_0\Pi_j$, and the scattering operator $S(s)$ is defined in Eq. (1.9).

For ρ which is not smooth an approximation argument will complete the task.

Proof of Lemma 5.1. Both claims of this proposition are the consequence of the stronger bound:

$$\|\Omega_{\pm}^{(1)}(s)I_{\pm}(s,a)\|_1 \le C_N a^{-(N-2)} ,$$

for large enough N. Let us proof this bound for a case of Ω_-, the proof for Ω_+ follows the same lines. It is clear from Eq. (4.3) that we may establish that bound for

$$\int_{-\infty}^{0} dt\,|t|\cdot\|F(A=0)\mathrm{e}^{-\mathrm{i}H(s)t}I_-(s,a)\|_1 .$$

The norm under this integral also appeared under the integral (4.9). We estimate it similarly by means of (2.1, 3.5), except that we insert $F(A \le -a+\alpha)$ (and the complementary projection) instead of $F(A \le a+\alpha)$. By (3.2) we get

$$\begin{aligned}\|F(A=0)\mathrm{e}^{-\mathrm{i}H(s)t}I_-(s,a)\|_1 \;\le\;& C\|F(A=0)F(A\le -a+\alpha-b+\theta t)\| \\ &+C_N(|b|+|t|)^{-N}+C_N\alpha^{-N} .\end{aligned}$$

Choosing $b=-a/2$, $\alpha=(a/2-\theta t)/2>0$, we see that the first term vanishes and we obtain the desired bound since

$$\int_{-\infty}^{0} dt\,|t|(\frac{a}{2}+\theta|t|)^{-N} \le C_N a^{-(N-2)} .$$

□

An ingredient to the proof of Lemma 5.3 and, to a minor extent, of Lemma 5.2 is the formal relation

$$[S^{(1)}(s),\rho(H(s))] = -\mathrm{i}\partial_{s'}S(s',s)_{s'=s}\rho'(H(s)) , \tag{5.4}$$

with

$$\partial_{s'}S(s',s)_{s'=s} = -\mathrm{i}\int_{-\infty}^{\infty} dt\,\mathrm{e}^{\mathrm{i}H(s)t}\dot{H}(s)\mathrm{e}^{-\mathrm{i}H(s)t} . \tag{5.5}$$

The importance of Eq. (5.4) is twofold. First it reduces matters to "frozen" scattering data and its derivatives, namely the scattering operator $S(s',s)$ for the pair $(H(s'),H(s))$, already introduced in the context of (2.13). Second, it makes it clear why for the Fermi sea $\rho(\lambda)=\theta(E-\lambda)$ only states at the Fermi energy E contribute to the current. Formally, (5.5) follows from the Born approximation for $S(s',s)\approx -\mathrm{i}\int_{-\infty}^{\infty} dt\,\mathrm{e}^{\mathrm{i}H(s)t}(H(s')-H(s))\mathrm{e}^{-\mathrm{i}H(s)t}$, which is exact to leading order as s' tends to s. Eq. (5.4) follows for $\rho(\lambda)=\mathrm{e}^{\mathrm{i}\lambda t}$, $(t\in\mathbb{R})$, by a change of variables in (4.5) and hence for general functions ρ. For our purposes we shall need a somewhat more precise statement, given by the first part of the following lemma (cf. the remark after Prop. 4.1).

Lemma 5.4. *(a) Eq. (5.5) and, for ρ smooth, Eq. (5.4) hold true if multiplied from either side by $I_+(s,a)$.*

(b) The wave operators $\Omega_\pm(s',s) = \mathrm{s}{-}\lim_{T\to\pm\infty} \mathrm{e}^{\mathrm{i}H(s')T}\mathrm{e}^{-\mathrm{i}H(s)T}P_{\mathrm{ac}}(H(s))$ *for the pair* $(H(s'),H(s))$ *satisfy the intertwining property*

$$f(H(s'))\Omega_\pm(s',s) = \Omega_\pm(s',s)f(H(s)) \tag{5.6}$$

for any (Borel) function f.
(c) The scattering operators for the pairs $(H(s),H(s_i))$, $(i=1,2)$ *are related as*

$$S(s,s_2) = \Omega_+(s_2,s_1)S(s,s_1)\Omega_-(s_2,s_1)^* \,. \tag{5.7}$$

Proof. Parts (b, c) are standard [20, 5]. The wave operators exist and are complete under our assumptions. The chain rule for wave operators,

$$\Omega_\pm(s,s_1) = \Omega_\pm(s,s_2)\Omega_\pm(s_2,s_1)\,,$$

and $S(s,s_i) = \Omega_+(s,s_i)^*\Omega_-(s,s_i)$ imply $S(s,s_1) = \Omega_+(s_2,s_1)^*S(s,s_2)\Omega_-(s_2,s_1)$. From this (5.7) follows by the completeness of scattering, $\Omega_\pm(s_2,s_1)^*\Omega_\pm(s_2,s_1) = P_{\mathrm{ac}}(H(s_2))$.

We begin the proof of part (a) by claiming that

$$\|(\Omega_+(s',s)-1)\chi(H(s))\| \le C|s'-s|\,, \tag{5.8}$$

where $\chi \in C_0^\infty(0,\infty)$. Indeed, let $\tilde\chi \in C_0^\infty(0,\infty)$ with $\tilde\chi\chi = \chi$. Then, by (5.6), $\|(\Omega_+ - 1)\chi(H(s))\| \le \|\tilde\chi(H(s'))(\Omega_+ - 1)\chi(H(s))\| + \|\tilde\chi(H(s'))\chi(H(s)) - \chi(H(s))\|$. The second term is bounded by $\|\tilde\chi(H(s')) - \tilde\chi(H(s))\|$, and fits the bound (5.8) by (A.5). For the first term we use the fundamental theorem of calculus:

$$\tilde\chi(H(s'))(\Omega_+(s',s)-1)\chi(H(s)) = \mathrm{s}{-}\lim_{T\to\infty} \mathrm{i}\int_0^T dt\,\tilde\chi(H(s'))\mathrm{e}^{\mathrm{i}H(s')t}V\mathrm{e}^{-\mathrm{i}H(s)t}\chi(H(s))\,, \tag{5.9}$$

where $V = H(s') - H(s)$. We write $V = \langle A\rangle^{-r}V\langle A\rangle^{-r}$ and use (4.16) for V, as well the $H(s)$-smoothness of $\langle A\rangle^{-r}\chi(H(s))$ for $r > 1/2$ (and similarly for s'), see item (ii) in Sect. 2. An application of the Cauchy-Schwartz inequality on matrix elements of (5.9) yields (5.8).

We can now prove the statement about Eq. (5.5), whose r.h.s. we denote by T. Then $TI_+(s,a)$ is a convergent integral by (4.12). Moreover, $T\mathrm{e}^{-\mathrm{i}H(s)t}I_+(s,a) = \mathrm{e}^{-\mathrm{i}H(s)t}TI_+(s,a)$, whence $T\tilde\chi(H(s))I_+(s,a) = \tilde\chi(H(s))TI_+(s,a)$ for $\tilde\chi \in C_0^\infty(0,\infty)$. Since $\tilde\chi(H(s))I_+(s,a) = I_+(s,a)$ for $\tilde\chi\chi = \chi$ and χ as in (1.11), we may thus prove

$$\tilde\chi(H(s))\partial_{s'}S(s',s)_{s'=s}I_+(s,a) = -\mathrm{i}\tilde\chi(H(s))\int_{-\infty}^{\infty} dt\,\mathrm{e}^{\mathrm{i}H(s)t}\dot H(s)\mathrm{e}^{-\mathrm{i}H(s)t}I_+(s,a)\,. \tag{5.10}$$

We write $S - 1 = \Omega_+^*(\Omega_- - \Omega_+)$, so that

$$\begin{aligned}
\tilde\chi(H(s))(S(s',s)-1)I_+(s,a) &= -\mathrm{i}\tilde\chi(H(s))\Omega_+^*\int_{-\infty}^{\infty} dt\,\mathrm{e}^{\mathrm{i}H(s')t}V\mathrm{e}^{-\mathrm{i}H(s)t}I_+(s,a)\\
&= -\mathrm{i}\int_{-\infty}^{\infty} dt\,\mathrm{e}^{\mathrm{i}H(s)t}\tilde\chi(H(s))(\Omega_+^*-1)V\mathrm{e}^{-\mathrm{i}H(s)t}I_+(s,a)\\
&\quad -\mathrm{i}\tilde\chi(H(s))\int_{-\infty}^{\infty} dt\,\mathrm{e}^{\mathrm{i}H(s)t}V\mathrm{e}^{-\mathrm{i}H(s)t}I_+(s,a)\,.
\end{aligned}$$

The first integral is estimated as $C(a)|s'-s|^2$ due to Eqs. (5.8, 4.16, 4.12). In the second one, the contribution coming from the remainder in $V = \dot{H}(s)(s'-s) + O((s'-s)^2)$ is estimated the same way. This implies (5.10).

As for Eq. (5.4) in the case $\rho(\lambda) = \mathrm{e}^{\mathrm{i}\lambda t}$, the change of variables mentioned before is legitimate, as the integrals are convergent once multiplied with $I_+(s,a)$ due to (4.12). The result extends to $\rho(H(s))$ with $\rho \in C_0^\infty(\mathbb{R})$ or $\rho' \in C_0^\infty(\mathbb{R})$, which amounts to the same since $H(s)$ is bounded below. □

Proof of Lemma 5.2. By Lemma 5.1 the l.h.s. of (5.1) equals $\lim_{a\to\infty} \operatorname{tr}\bigl([S^{(1)}(s), \rho(H(s))]\, I_+(s,a)\bigr)$, provided this limit exists. It does, since the expression is independent of a. For ρ smooth this follows from Lemma 5.4. Indeed, the r.h.s. of (5.4) commutes with $H(s)$, so that the independence is seen as in (2.7). For general ρ the conclusion follows by approximation by a sequence $\{\rho_n\}$ of smooth functions, such that $\mathrm{s-}\lim \rho_n(H(s)) = \rho(H(s))$. The traces converge by (3.16). □

Proof of Lemma 5.3. For the sake of precision we recall that the scattering operator $S(s) = \Omega_+(s)^*\Omega_-(s)$ depends on wave operators defined in a two-Hilbert space setting: $\Omega_\pm(s) = \mathrm{s-}\lim_{T\to\pm\infty} \mathrm{e}^{\mathrm{i}H(s)T} J \mathrm{e}^{-\mathrm{i}H_0T}$, where $J : L^2(\mathbb{R}_+, \mathbb{C}^n) \to \mathcal{H}$ is the embedding given by (1.2). We may also replace J by a smooth function $\tilde{J}$ on $\mathbb{R}$, which equals 0 near $x = 0$, and 1 near $x = \infty$. This is without effect on $\Omega_\pm(s)$, since $\mathrm{s-}\lim_{T\to\pm\infty}(J - \tilde{J})\mathrm{e}^{-\mathrm{i}H_0T} = 0$. In particular, we may pick $\tilde{J}$ so that $\tilde{J} = 1$ on $\operatorname{supp} v$, see (1.6). In the present context (5.7) reads $S(s', s) = \Omega_+(s)S(s')\Omega_-(s)^*$ and implies

$$\partial_{s'} S(s', s)_{s'=s} = \Omega_+(s)\dot{S}(s)\Omega_-(s)^* \,,$$

where the differentiability is granted e.g. after multiplication with $I_+(s,a)$.

We now prove Eq. (5.2) with $\Pi_j \mathrm{i}[H_0, f(A_0 - a)]$ replaced by $\Pi_j \mathrm{i}[H_0, f(A - a)] = \mathrm{i}[H_0, f(A_j - a)]$. Its l.h.s. equals, by Lemma 5.4,

$$\begin{aligned}
&\operatorname{tr}\bigl(\Omega_+(s)\dot{S}(s)\Omega_-(s)^*\rho'(H(s))\chi(H(s))I_+(a)\chi(H(s))\bigr)\\
&\qquad = \operatorname{tr}\bigl(\dot{S}(s)S(s)^*\rho'(H_0)\chi(H_0)I_+(a)\chi(H_0)\bigr)\\
&\qquad\quad + \operatorname{tr}\bigl(\dot{S}(s)\Omega_-(s)^*\rho'(H(s))\chi(H(s))[I_+(a), \Omega_+(s)]\chi(H_0)\bigr) \,.
\end{aligned}$$

Here $I_+(a)$ is defined in Eq. (1.7). We turned $\Omega_+(s)$ around the trace, which thereby moved from $\mathcal{H}$ to $L^2(\mathbb{R}_+, \mathbb{C}^n)$. The first term corresponds to the claim, since $\chi\rho'\chi = \rho'$. It is again independent of a. The second term will thus vanish as soon as it does in limit $a \to \infty$. The commutator there is $[I_+(a), \Omega_+(s)] = [I_+(a), \Omega_+(s) - \tilde{J}]$ by the above choice $\tilde{J}$. We are so left to show that

$$\lim_{a\to\infty} \|\chi(H(s))(\Omega_+(s) - \tilde{J})\chi(H_0)I_+(a)\|_1 = 0 \,,$$

and also with $\Omega_+(s)$ replaced by $\Omega_+(s)^*$ and $\chi(H_0)$, $\chi(H(s))$ interchanged. Using the integral representation for $\Omega_+(s) - \tilde{J}$ the estimate reduces to

$$\int_0^\infty dt \, \|\chi(H(s)[H_0, \tilde{J}]\chi(H_0)\mathrm{e}^{-\mathrm{i}H_0 t} I_+(a)\|_1$$
$$\leq C \int_0^\infty dt \, \|F(A = 0)\chi(H_0)\mathrm{e}^{-\mathrm{i}H_0 t} I_+(a)\|_1 \to 0 \, , \qquad (a \to \infty) \, ,$$

(and with H_0, $H(s)$ interchanged) which holds true by the estimates in the proof of Lemma 5.1.

We next replace A by A_0 on the r.h.s. of (5.2). Since both traces are independent of a we may, in each of them, replace f by a smeared switch function $\tilde{f}'$, such that $\tilde{f} \in H^2(d)$ with $d > 2$ (see the Appendix for notation). Since $(\tilde{f}(A_j) - \tilde{f}(A_{0j}))\rho'(H_0) \in \mathcal{J}_1$ by (A.12), the difference is seen to vanish by cyclicity.

Finally, to prove (5.3), we may keep f replaced by $\tilde{f}$ as before. It follows from Eqs. (A.10, A.2) that

$$\begin{aligned}
&\mathrm{tr}\big(\dot{S}(s)S(s)^*\rho'(H_0)\Pi_j[H_0, \tilde{f}(A_0 - a)]\big) \\
&\qquad = \mathrm{tr}\big(\dot{S}(s)S(s)^*\rho'(H_0)\Pi_j(\tilde{f}(A_0 - a - 2\mathrm{i}) - \tilde{f}(A_0 - a))H_0\big) \\
&\qquad = \frac{1}{2\pi}\int_0^\infty \frac{dE}{2E}\rho'(E)E\Big(\frac{dS}{ds}S^*\Big)_{jj} \cdot \int_{-\infty}^\infty d\lambda \, (\tilde{f}(\lambda - a - 2\mathrm{i}) - \tilde{f}(\lambda - a)) \, .
\end{aligned}$$

To compute the last integral, note that $\int_{-\infty}^\infty d\lambda \, (\tilde{f}(\lambda - c) - \tilde{f}(\lambda)) = -c$ for any $c \in \mathbb{R}$ which extends by analyticity to $|\operatorname{Im} c| < d$. Hence the result. □

Proof of Theorem 1.1. We have shown that

$$\mathrm{tr}\big([S^{(1)}(s), \rho(H(s))]I_+(s,a)\big) = -\frac{\mathrm{i}}{2\pi}\int_0^\infty d\rho(E)\Big(\frac{dS}{ds}S^*\Big)_{jj} \tag{5.11}$$

holds true if ρ is smooth. If ρ is of bounded variation it can be approximated by a sequence $\rho_n \in C_0^\infty(\mathbb{R})$ with $\rho_n(\lambda) \to \rho(\lambda)$ pointwise and uniformly bounded total variation $\sup_n V(\rho_n) < \infty$. Then $\mathrm{s-}\lim_n \rho_n(H(s)) = \rho(H(s))$ and $d\rho_n(\lambda) \to d\rho(\lambda)$ in the sense of weak* convergence. Hence (5.11) is inherited by the limit. The proof is completed by Eqs. (4.4) and (5.1). □

A Appendix. Trace class properties of $g(H)f(A)$

In this section we will discuss different properties of the operator product $g(H(s))f(A)$. We first prove that operators of the form $g(H_0)f(A_0)$ are Hilbert-Schmidt under suitable conditions on the functions g and f. For operator of this type one can essentially write down the integral kernel, and use it in order to compute the Hilbert-Schmidt norm. Heuristically the trace of an operator is the integral over phase space of the symbol (divided by 2π). Since

$$\frac{1}{2\pi}\int dx dp = \frac{1}{2\pi}\int \frac{dE}{2E} da \, ,$$

under the map $(x,p) \mapsto (E = p^2, a = px)$, we introduce the norms

$$\|f\|^2 = \int_{-\infty}^{\infty} da\, |f(a)|^2 \;, \qquad |||g|||^2 = \int_0^{\infty} \frac{dE}{2E} |g(E)|^2 \;,$$

on functions f on the real line, resp. g on the half–line. The Fourier transform is $\hat{f}(t) = (2\pi)^{-1} \int e^{-i\lambda t} f(\lambda) d\lambda$.

Proposition A.1. *We have*

$$\|g(H_0)f(A_0)\|_2 = (2\pi)^{-1/2} |||g||| \cdot \|f\| \;. \tag{A.1}$$

If g and $\hat{f}$ are continuous and the operator $g(H_0)f(A_0)$ is trace class, then

$$\mathrm{tr}\,(g(H_0)f(A_0)) = \frac{1}{2\pi} \int_0^{\infty} \frac{dE}{2E} g(E) \cdot \int_{-\infty}^{\infty} da\, f(a) \;. \tag{A.2}$$

Proof. The space $L^2(\mathbb{R}_+)$ can be identified with the even functions in $L^2(\mathbb{R})$ by means of the map $(J\varphi)(x) = \varphi(|x|)$, with $\|J\varphi\|^2 = 2\|\varphi\|^2$. Consider the operator $g(p^2)f(A_0)$ on $L^2(\mathbb{R})$. Since it preserves parity and p^2 reduces to H_0 on the (so identified) subspace of even functions (with projection P_+) we have

$$\begin{aligned} \|g(H_0)f(A_0)\|_2^2 &= \mathrm{tr}_{L^2(\mathbb{R}_+)}(\bar{g}(H_0)|f|^2(A_0)g(H_0)) \\ &= \frac{1}{2}\,\mathrm{tr}_{L^2(\mathbb{R})}(P_+\bar{g}(p^2)|f|^2(A_0)g(p^2)P_+) \\ &= \frac{1}{2}\,\mathrm{tr}_{L^2(\mathbb{R})}(U^* P_+\bar{g}(p^2)|f|^2(A_0)g(p^2)P_+U) \\ &= \frac{1}{2}\,\mathrm{tr}_{L^2(\mathbb{R})}(P_+\bar{g}(x^2)|f|^2(-A_0)g(x^2)P_+) \\ &= \mathrm{tr}_{L^2(\mathbb{R}_+)}(\bar{g}(x^2)|f|^2(-A_0)g(x^2)) \;, \end{aligned} \tag{A.3}$$

where $(U\psi)(k) = (2\pi)^{1/2}\hat{\psi}(k)$ is the Fourier transform: $[U, P_+] = 0$, $P_+ = (1+U^2)/2$, $pU = -Ux$, and $A_0U = -A_0U$. Using the Mellin transform $M : L^2(\mathbb{R}_+) \to L^2(\mathbb{R})$,

$$(M\varphi)(a) = (2\pi)^{-1/2} \int_0^{\infty} \frac{dx}{x^{1/2}} x^{-ia} \varphi(x) \;,$$

which diagonalizes $A_0 = M^* a M$, one obtains that $h(a)$ has integral kernel

$$h(A)(x,y) = (2\pi)^{-1}(xy)^{-\frac{1}{2}} \hat{h}(\log \frac{y}{x}) \tag{A.4}$$

for $h \in L^1(\mathbb{R})$. Since the kernel is continuous in $x, y > 0$, we can write for (A.3) (cf. Thm. 3.9. [22])

$$(2\pi)^{-1} \int_0^{\infty} \frac{dx}{x} |g(x^2)|^2 \int_{-\infty}^{\infty} da\, |f|^2(-a) \;.$$

By means of the change of variable $x^2 = E$, this is the square of the r.h.s. of (A.1).

Using the Fourier transform as in (A.3) we find $\mathrm{tr}\big(g(H_0)f(A_0)\big) = \mathrm{tr}\big(g(x^2)f(-A_0)\big)$. The integral kernel of the latter operator is obtained from (A.4) and the trace is the integral over its diagonal, due to Thm. 3.9 [22]. □

In the rest of this appendix we shall give an example of trace class operator involving A_0, H_0 and use it to establish $g(H(s))f(A) \in \mathcal{J}_1$ for a large enough class of operators.

Lemma A.2. *Let $g \in C_0^\infty(\mathbb{R})$, then*

$$\begin{aligned} \|g(H(s)) - g(H(s'))\|_1 &\leq C|s - s'| \,, && \text{(A.5)} \\ g(H(s)) - Jg(H_0)J^* &\in \mathcal{J}_1 \,, && \text{(A.6)} \end{aligned}$$

where $J : L^2(\mathbb{R}_+, \mathbb{C}^n) \to \mathcal{H}$ is the embedding given by (1.2).

Proof. We first prove (A.5) for $g(\lambda) = (\lambda + \mathrm{i})^{-2m}$, where m is as in Assumption A2. Setting $R(s) = (H(s) + \mathrm{i})^{-1}$ we have by (1.4) and the resolvent identity

$$\begin{aligned} R(s)^{2m} - R(s')^{2m} &= \sum_{k=1}^{2m} R(s)^{2m-k}(R(s) - R(s'))R(s')^{k-1} \\ &= \sum_{k=1}^{2m} R(s)^{2m-k+1}\Pi_0(H(s') - H(s))\Pi_0 R(s')^k \,. \end{aligned}$$

The desired bound now follows from A1 by using A2 either for s or s'. For general $g \in C_0^\infty(\mathbb{R})$, as well as for the rest of this proof, we use the Helffer-Sjöstrand representation,

$$g(H) = \frac{1}{\pi}\int_{\mathbb{R}^2} (H - z)^{-1}\partial_{\bar{z}}\tilde{g}(z)dxdy \,, \tag{A.7}$$

where $\partial_{\bar{z}} = (\partial_x + \mathrm{i}\partial_y)/2$, and $\tilde{g}$ is an almost analytic extension of g with $\tilde{g}(z)$ vanishing to a high power near the real axis. Before doing that, we set $G(\lambda) = g(\lambda)(\lambda + \mathrm{i})^{2m}$, so that the first term in

$$g(H(s)) - g(H(s')) = (R(s)^{2m} - R(s')^{2m})G(H(s')) + R(s)^{2m}(G(H(s)) - G(H(s'))$$

is taken care of. For the second, we apply (A.7) to G and are led to estimate

$$\begin{aligned} &\|R(s)^{2m}[(H(s) - z)^{-1} - (H(s') - z)^{-1}]\|_1 \leq \\ &\quad \|(H(s) - z)^{-1}\|\|R(s)^{2m}\Pi_0\|_1\|H(s') - H(s)\|\|(H(s') - z)^{-1}\| \leq C|\operatorname{Im} z|^{-2}|s - s'| \,, \end{aligned}$$

which completes the proof of (A.5).

Before proving (A.6) we claim that

$$g(H)h \in \mathcal{J}_1 \,, \tag{A.8}$$

where h acts as multiplication by $h \in C_0^\infty[0, \infty)$ on $L^2(\mathbb{R}_+, \mathbb{C}^n)$ and by $h(0)$ on $\mathcal{H}_0$. Since $g(H)\Pi_0 \in \mathcal{J}_1$ it will be enough to show $(1 - \Pi_0)g(H)(1 - \Pi_0)h \in \mathcal{J}_1$ or actually just $\|J(H - z)^{-1}J^*h(x)\|_1 \leq C|\operatorname{Im} z|^{-1}$. Since the kernel of $(H - z)^{-1}$ is decaying exponentially [9], matters are further reduced to

$$\|J_L(H - z)^{-1}J_L^*\|_1 \leq C|\operatorname{Im} z|^{-1} \,,$$

as an operator on $L^2([0,L],\mathbb{C}^n)$, where J_L is the corresponding embedding operator into $\mathcal{H}$. The initial piece $[0,L]$ has to be taken large enough, so that $\operatorname{supp} h \subset [0,L)$. Let now B be the quadratic form of the Bilaplacian,

$$B(\varphi,\psi) = \int_0^L \bar{\varphi}''(x)\psi''(x)dx \ , \tag{A.9}$$

with domain given by the Sobolev space $W^2(0,L)$. We maintain that

$$\begin{gathered}\|(1+B)^{1/2}J_L(H-z)^{-1}J_L^*\| \le C|\operatorname{Im} z|^{-1} \ , \\ (1+B)^{-1/2} \in \mathcal{J}_1 \ ,\end{gathered}$$

which imply the claim. The first statement follows by A3 and (A.9) from

$$J_L(H-\bar{z})^{-1}J_L^* B J_L(H-z)^{-1}J_L^* = T^*T \ ,$$

with $T = 1 + zJ_L(H-z)^{-1}J_L^*$. The second statement can be seen by computing the eigenvalues k^4 of the operator B associated to (A.9). The latter is given as $B = d^4/dx^4$ with boundary conditions $\varphi'' = \varphi''' = 0$ at $x = 0,\, L$. From this one computes the eigenvalues as the zeros of $1 - \cosh kL \cos kL$.

To prove (A.6) we make use of a smooth embedding $\tilde{J}$ as in the proof of Lemma 5.3. Since $(J-\tilde{J})g(H_0) \in \mathcal{J}_1$ (see [19], Thm. XI.20) we will prove the trace class property for

$$g(H) - \tilde{J}g(H_0)\tilde{J}^* = g(H)(1 - \tilde{J}\tilde{J}^*) + (g(H)\tilde{J} - \tilde{J}g(H_0))\tilde{J}^* \ .$$

The first term is trace class by (A.8). For the second term, Eq. (A.7) and the resolvent identity reduce matters to

$$\|(H-z)^{-1}[\tilde{J},H](H_0-z)^{-1}\tilde{J}^*\|_1 \le \|(H-z)^{-1}[\tilde{J},H]\|\|h(x)(H_0-z)^{-1}\|_1 \le C|\operatorname{Im} z|^{-2} \ ,$$

where $h \in C_0^\infty(\mathbb{R})$ with $h\tilde{J}' = \tilde{J}'$. □

The Hardy class $H^2(d)$ consists of all functions f, analytic in the strip $\{z \mid |\operatorname{Im} z| < d\}$, which satisfy

$$\sup_{-d<y<d} \int_{-\infty}^{\infty} |f(x+\mathrm{i}y)|^2 dx < \infty \ .$$

We recall that $f \in H^2(d)$ is also characterized by $\sup_{0<y<d} \|\mathrm{e}^{y|k|}\hat{f}(k)\|_2 < \infty$.

Proposition A.3 (The pull-through formula). *For any $f \in H^2(d)$ with $d > 2$*

$$H_0 f(A_0) = f(A_0 - 2\mathrm{i})H_0 \ . \tag{A.10}$$

Proof. We first prove the statement for $f(x) = \mathrm{e}^{-\mathrm{i}tx}$. Under conjugation with the dilation operator, $(\mathrm{e}^{-\mathrm{i}tA_0}\psi)(x) = \mathrm{e}^{-t/2}\psi(e^{-t}x)$, the Neumann Laplacian becomes

$$\mathrm{e}^{\mathrm{i}tA_0} H_0 \mathrm{e}^{-\mathrm{i}tA_0} = e^{-2t} H_0 \ , \tag{A.11}$$

hence $H_0\mathrm{e}^{-\mathrm{i}tA_0} = \mathrm{e}^{-\mathrm{i}t(A_0-2\mathrm{i})}H_0$. Now, for $f \in H^2(d)$,

$$H_0 f(A_0) = \int_{-\infty}^{\infty} H_0 \mathrm{e}^{\mathrm{i}tA_0}\hat{f}(t)dt = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}t(A_0-2\mathrm{i})}H_0\hat{f}(t)dt = f(A_0 - 2\mathrm{i})H_0 \ ,$$

where the last step is justified by the above mentioned property of the Hardy class functions. □

Lemma A.4. *Let $f' \in H^2(d)$ with $d > 2$ and $g \in C_0^\infty(\mathbb{R})$. Then*

$$(f(A) - f(A_0))g(H_0) \in J_1 \ . \tag{A.12}$$

Proof. As mentioned, the Fourier transform of $f'(\lambda) = \int_{-\infty}^{\infty} dt\, \mathrm{e}^{i\lambda t}\widehat{f'}(t)$ is bounded as

$$|\widehat{f'}(t)| \le C\mathrm{e}^{-y|t|} \ , \tag{A.13}$$

for any $y < d$. We represent f as

$$f(\lambda) = f(0) + \int_{-\infty}^{\infty} dt \left(\frac{\mathrm{e}^{\mathrm{i}\lambda t} - 1}{\mathrm{i}t}\right)\widehat{f'}(t) \ ,$$

and write

$$\frac{1}{\mathrm{i}t}(\mathrm{e}^{\mathrm{i}At} - \mathrm{e}^{\mathrm{i}A_0t}) = \frac{1}{t}\int_0^t ds\, \mathrm{e}^{\mathrm{i}A(t-s)}(A - A_0)\mathrm{e}^{\mathrm{i}A_0s} \ , \mathrm{e}^{\mathrm{i}A_0s}g(H_0) = g(H_0\mathrm{e}^{-2s})\mathrm{e}^{\mathrm{i}A_0s} \ ,$$

which follows from (A.11). Applying Thm. XI.20 [19] to

$$(A - A_0)g(H_0\mathrm{e}^{-2s}) = (w(x)p - \frac{i}{2}w'(x))g(H_0\mathrm{e}^{-2s}) \ ,$$

where $w(x) = v(x) - x$, one finds

$$\|(A - A_0)g(H_0\mathrm{e}^{-2s})\|_1 \le C_\delta(\mathrm{e}^{s(\frac{3}{2}+\delta)} + \mathrm{e}^{s(\frac{1}{2}+\delta)}) \ , \tag{A.14}$$

for any $\delta > \frac{1}{2}$. We obtain $\|t^{-1}(\mathrm{e}^{\mathrm{i}At} - \mathrm{e}^{\mathrm{i}A_0t})g(H_0)\|_1 \le C(1 + \mathrm{e}^{\tilde{d}t})$ with $\tilde{d} = \frac{3}{2} + \delta$. Picking δ so that $\tilde{d} < d$, we obtain the claim from Eq. (A.13). □

The following example will be useful.

Lemma A.5.

$$\|H_0(H_0 + 1)^{-2}(A_0 - z)^{-2}\|_1 \le C|\operatorname{Im} z|^{-2} \ . \tag{A.15}$$

Proof. It suffices to prove the claim for $|\operatorname{Im} z|$ large enough since $\|(A_0 - z + \mathrm{i}y)(A_0 - z)^{-1}\| \le C|\operatorname{Im} z|^{-2}$ for small $|\operatorname{Im} z|$. We shall do that for $|\operatorname{Im} z| > 4$. By the pull-through formula, Eq. (A.10), we compute

$$\begin{aligned} H_0(A_0 - z)^{-1} &= (A_0 - z - 2\mathrm{i})^{-1} H_0 \\ &= (A_0 - z - 2\mathrm{i})^{-1}(H_0^2 + 2H_0 + 1)H_0(H_0+1)^{-2} \\ &= \left[H_0^2(A_0 - z + 2\mathrm{i})^{-1} + 2H_0(A_0 - z)^{-1} + (A_0 - z - 2\mathrm{i})^{-1}\right] H_0(H_0+1)^{-2} \ . \end{aligned}$$

We multiply this expression from the left by $(H_0+1)^{-2}$ and from the right by $(A_0 - z)^{-2}$, so as to obtain the bound

$$\|H_0(H_0+1)^{-2}(A_0 - z)^{-2}\|_1 \le \sum_{k=-1}^{1} \|(A_0 - z + 2k\mathrm{i})^{-1} H_0(H_0+1)^{-2}(A_0 - z_2)^{-1}\|_1 \ .$$

Eq. (A.1) now yields $\|H_0^{1/2}(H_0+1)^{-1}(A_0 - z)^{-1}\|_2 = (2|\operatorname{Im} z|)^{-1/2}$. Hence (A.15) follows from the Hölder inequality. □

Lemma A.6. *For $g \in C_0^\infty(0,\infty)$ we have*

$$\sup_{a\in\mathbb{R}} \|(A - a \pm \mathrm{i})^{-2} g(H)\|_1 < \infty \ . \tag{A.16}$$

In particular, for any $f \in C^\infty(\mathbb{R})$ with $f(x)(x+\mathrm{i})^2$ bounded,

$$f(A)g(H) \in \mathcal{J}_1 \ . \tag{A.17}$$

Proof. Since $g(H_0)H_0^{-1}(H_0+1)^2$ is bounded, by (A.15) the bound holds for A_0, H_0 in place of A, H. We first undo the replacement in A and write

$$(A - a \pm \mathrm{i})^{-2} g(H_0) = (A - a \pm \mathrm{i})^{-2}\phi(x)g(H_0) + (A - a \pm \mathrm{i})^{-2}(1-\phi(x))g(H_0) \ ,$$

where $\phi \in C_0^\infty[0,\infty)$ has $v(x) = x$ on $\operatorname{supp}(1-\phi)$. Then $\phi(x)g(H_0) \in J_1$ (see [19], Thm. XI.20). For the second term on the r.h.s. we write, dropping a, $(A+\mathrm{i})^{-2}(1-\phi(x))(A_0 + \mathrm{i})^2 \cdot (A_0+\mathrm{i})^{-2} g(H_0)$. Since the last factor is trace class, we are left with showing that the first factor is bounded. This follows from

$$(1-\phi)(A_0+\mathrm{i})^2 = (A+\mathrm{i})^2(1-\phi) - (A+\mathrm{i})[\phi, A_0] - [\phi, A_0](A+\mathrm{i}) \ ,$$

where $[\phi, A] = \mathrm{i}x\phi'(x)$ bounded.

The bound (A.16) clearly also holds for $(A+\mathrm{i})^{-2} J g(H_0) J^* = J(A+\mathrm{i})^{-2} g(H_0) J^*$ and hence, by (A.6), as claimed. □

Acknowledgments. This work is supported in part by the ISF; the Fund for promotion of research at the Technion; the Texas Advanced Research Program and NSF Grant PHY-9971149, and the Swiss National Science Foundation. G.M.G. thanks T. Spencer for hospitality at the Institute for Advanced Study, where part of this work was done.

References

[1] L. Aleiner and A.V. Andreev, Adiabatic charge pumping in almost open dots, Phys. Rev. Lett. **81**, 1286 (1998).

[2] W.O. Amrein, A. Boutet de Monvel-Berthier, V. Georgescu, *C_0-groups, commutator methods and spectral theory of N-body Hamiltonians*, Birkhäuser (1996).

[3] J.E. Avron, A. Elgart, Adiabatic theorem without a gap condition. Comm. Math. Phys. **203**, 445 (1999).

[4] J.E. Avron, A. Elgart, G.M. Graf, L. Sadun, Optimal quantum pumps, Phys. Rev. Lett. **87**, 236601 (2001).

[5] J.E. Avron, A. Elgart, G.M. Graf, L. Sadun, Time-energy coherent states and adiabatic scattering, J. Math. Phys. **43**, 3415 (2002).

[6] F. Bornemann, *Homogenization in time of singularly perturbed mechanical systems*, Lecture Notes in Mathematics, Vol. 1687, Springer (1998).

[7] P.W. Brouwer, Scattering approach to parametric pumping, Phys. Rev. B**58**, 10135 (1998).

[8] M. Büttiker, H. Thomas, A. Prêtre, Current partition in multi-probe conductors in the presence of slowly oscillating external potentials, Z. Phys. B**94**, 133 (1994).

[9] J.M. Combes and L. Thomas, Asymptotic behaviour of eigenfunctions for multiparticle Schrödinger operators, Comm. Math. Phys. **34**, 251 (1973).

[10] E.B. Davies, *Spectral theory of differential operators*, Cambridge Univ. Press (1995).

[11] J. Dereziński, C. Gérard, *Scattering theory of classical and quantum N-particle systems*, Springer (1997).

[12] W. Hunziker, I.M. Sigal, Time-dependent scattering theory of N-body quantum systems. Rev. Math. Phys. **12**, 1033-1084 (2000).

[13] W. Hunziker, I.M. Sigal, A. Soffer, Minimal escape velocities. Comm. Partial Differential Equations **24**, 2279 (1999).

[14] D.A. Ivanov, H.W. Lee, and L.S. Levitov, Coherent states of alternating current, Phys. Rev. B**56**, 6839 (1997).

[15] A. Jensen, E. Mourre, P.A. Perry, Multiple commutator estimates and resolvent smoothness in quantum scattering theory, Ann. Inst. H. Poincaré, Physique Théorique **41**, 207 (1984).

[16] L.S. Levitov, H. Lee, B. Lesovik, Electron counting statistics and coherent states of electric current, J. Math. Phys. **37**, 4845 (1996).

[17] P.A. Martin, M. Sassoli de Bianchi, On the low- and high-frequency limit of quantum scattering by time-dependent potentials. J. Phys. A**28**, 2403 (1995).

[18] A. Martinez, S. Nakamura, Adiabatic limit and scattering, C. R. Acad. Sci. Paris, **318**, Série II, 1153 (1994).

[19] M. Reed, B. Simon, *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*, Academic Press (1975).

[20] M. Reed, B. Simon, *Methods of modern mathematical physics. III. Scattering theory*, Academic Press (1979).

[21] T.A. Shutenko, I.L. Aleiner, B.L. Altshuler, Mesoscopic fluctuations of adiabatic charge pumping in quantum dots, Phys. Rev. B**61**, 10366 (2000).

[22] B. Simon, *Trace ideals and their applications*, Cambridge University Press (1979).

[23] J. Sjöstrand, M. Zworski, Complex scaling and the distribution of scattering poles, J. of AMS **4**, 729 (1991).

[24] D.R. Yafaev, *Mathematical Scattering Theory*, AMS (1992).