More ergodic billiards with an infinite cusp

Marco Lenci
Department of Mathematical Sciences
Stevens Institute of Technology
Hoboken, NJ 07030
U.S.A.
E-mail: mlenci@stevens-tech.edu

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Abstract

In [Le2] the following class of billiards was studied: For $f : [0, +\infty) \longrightarrow (0, +\infty)$ convex, sufficiently smooth, and vanishing at infinity, let the billiard table be defined by Q, the planar domain delimited by the positive x-semiaxis, the positive y-semiaxis, and the graph of f.

For a large class of f we proved that the billiard map was hyperbolic. Furthermore we gave an example of a family of f that makes this map ergodic. Here we extend the latter result to a much wider class of functions.

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1 Introduction

This note is a follow-up to [Le2], where we studied a certain family of billiards in the plane. A billiard is a dynamical system defined by the free motion of a material point inside a domain, called the *table*, with the prescription that when the point hits the boundary of the table it gets reflected at an angle equal to the angle of incidence.

The family introduced in [Le2] was defined as follows: To a three-times differentiable function $f : [0, +\infty) \longrightarrow (0, +\infty)$, convex, vanishing at $+\infty$ (and thus bounded), we associate the table Q delimited by the positive x-semiaxis, the positive y-semiaxis, and the graph of f, as in Fig. 1.

These billiards are *semi-dispersing*, which means that the boundaries of the tables are made of flat and convex parts, when seen from the inside. The main feature of

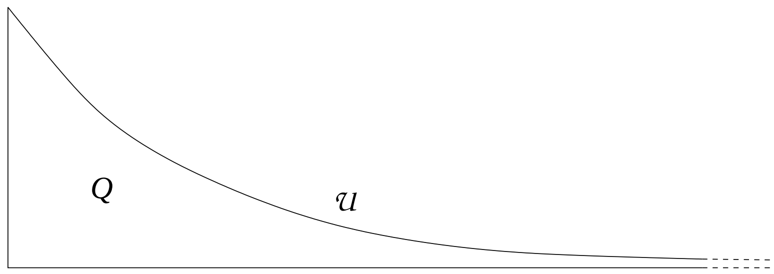

Figure 1: An example of a billiard table Q.

these domains, however, is that they are unbounded; in particular they possess a non-compact *cusp*.

There are several reasons why models of the like merit study. From a physical point of view they are on the borderline between *closed* and *open* systems. By definition, a closed system is one in which the evolution of any representative point in the relevant phase space remains confined within a compact set. An open system, on the contrary, possesses unbounded orbits. Our case is borderline in the sense that, albeit open, an arbitrarily small perturbation of the system will make it closed. Some suggest these systems be called *ajar*.

In general it is reasonable to expect ajar systems to have many unbounded orbits, in the sense of a positive-measure set of initial conditions, with respect to a suitable measure. Moreover, if the system enjoys some stochastic properties (e.g., ergodicity) one might expect that almost all orbits are unbounded. What might really discriminate between substantially different dynamics for two distinct ajar systems is the number of *escape orbits*. In a few words, an escape orbit is a trajectory that eventually leaves for good any given compact set of the phase space.

As an example of two opposite behaviors, consider a particle in a Newtonian potential with total energy zero. We know that all trajectories, apart from the singular ones that intersect the center of the potential, are parabolas, hence escape both in the past and in the future. On the other hand, the billiard in Fig. 1 has only one escape orbit—the obvious one that runs along the x-semiaxis [L, K, Le1].

From a more mathematical standpoint the relevance of our billiards lies in the fact that they give rise to infinite-measure hyperbolic dynamical systems, about which not much is known. We do not delve in this issue here, as a good part of the introduction of [Le2] is devoted to it. Let us just mention that, if one is interested in studying the stochastic properties of these systems (e.g., ergodicity, as in our case), one cannot apply the fundamental results of [KS], where a version of Pesin's theory is engineered to be used on *finite-measure* hyperbolic systems with singularities. Other standard results that are denied to us include the estimatates on the measure of the tubular neighborhoods of the singularities and, what's more, the local ergodicity theorem (also known as *fundamental theorem*).

The content of this paper is well summarized by its title: we are presenting a (fairly large) class of ergodic billiards like the above. In [Le2] we expounded the underpinning ideas and technical arguments, but produced only a one-parameter family of examples—those defined by $f(x) = Cx^{-p}$, with $C, p > 0$. Here we extend the result to general perturbations of the above family. To do so, we need to assume on the reader's side a rather strong grasp on [Le2] (which in turn is heavily based on the seminal article [LW]).

After some generalities, outlined in Section 2, we will verify that a large class of functions f satisfies all the general theorems of [Le2], the ones establishing hyperbolicity; this is done in Section 3. In Section 4 we will work on ergodicity. In particular, we will present a lemma that substitutes a corresponding lemma of [Le2] and extends its conclusions to our new class of functions.

Before moving on to the mathematics proper, it might be worthwhile to recall what we mean by ergodicity here, as this might lend itself to ambiguities in the case of phase spaces of infinite measure. We say that a dynamical system, endowed with an invariant measure μ, is ergodic when the time average of every integrable function is constant almost everywhere w.r.t. μ [Le2, Defn. 8.1]. Although this is a rather weak notion of ergodicity (it does not even prevent the existence of two complementary invariant subsets of infinite measure) we also have a much more satisfactory result: the ergodicity of the (finite-measure) Poincaré map corresponding to the returns onto the vertical side of the boundary [Le2, Prop. 8.11].

2 Main result and generalities

In the rest of this note we show that the class of functions defined hereafter yields ergodic billiards, in the sense mentioned above (see also [Le2, Sec. 8]).

Assume that $f : [0, +\infty) \longrightarrow (0, +\infty)$ is three times differentiable, convex, and vanishing at $+\infty$; and that there exists a function υ such that

$$f(x) = \upsilon(x)\, x^{-p}, \qquad \text{for some } p > 0; \tag{H1}$$

$$\upsilon(x)\, x^{-\varepsilon} \to 0, \qquad \text{for every } \varepsilon > 0; \tag{H2}$$

$$\frac{\upsilon^{(i)}(x) x^i}{\upsilon(x)} \to 0, \qquad i = 1, 2, 3. \tag{H3}$$

The limit in question is always $x \to +\infty$. In (H3), $\upsilon^{(i)}$ denotes the i-th derivative of υ.

In practical terms, this means that $f(x)$ is a well-behaved perturbation of x^{-p}, even at the level of a few derivatives (see Lemma 3.1). The family $f(x) = Cx^{-p}$, $C > 0$, was shown in [Le2] to give ergodic tables and is here superseded by this wider class. That this family is much more general is shown by the next few examples, that are easily checked against (H1)-(H3).

- $f(x) = \log^\gamma x\, x^{-p}$, with $\gamma \in \mathbb{R}$; thus $\upsilon(x) = \log^\gamma x$.

- $f(x) = ax^{-p} + bx^{-q}$, with $0 < p < q$, $a > 0$, $b \in \mathbb{R}$; take $\upsilon(x) = a + bx^{-q+p}$.
- $f(x) = \dfrac{1}{ax^p + bx^q}$, with $p > q > 0$, $a > 0$, $b \in \mathbb{R}$; use $\upsilon(x) = \dfrac{1}{a + bx^{-p+q}}$.

Remark 2.1 Not all functions listed above are necessarily positive and convex for $x \geq 0$, but certainly they become so asymptotically. The reader is therefore invited to think of them as shifted to the left as much as is needed.

The second of the above examples is actually an instance of a more general result about the family at hand:

Proposition 2.2 *The space of sufficiently smooth, positive, convex, vanishing functions that verify* (H1)-(H3) *is positive-linear; that is, a linear combination of elements in the space, with positive coefficients, belongs to the space.*

Proof. That the space in question is homogeneous is obvious. We only need prove that it is additive. Therefore, for $j = 1, 2$, take $f_j(x) = \upsilon_j(x)\, x^{-p_j}$. To fix the ideas, suppose that $p_1 < p_2$. (H3) amounts to saying that, for $i = 1, 2, 3$ and $j = 1, 2$, there exist six positive functions $g_j^{(i)}$ such that

$$\left| \upsilon_j^{(i)}(x) \right| x^i = g_j^{(i)}(x) \upsilon_j(x) \tag{2.1}$$

and $g_j^{(i)}(x) \to 0$, as $x \to +\infty$. In order to rewrite $f := f_1 + f_2$ in terms of (H1), we choose $p := p_1$ and

$$\upsilon(x) := \upsilon_1(x) + \upsilon_2(x)\, x^{-\alpha}, \tag{2.2}$$

with $\alpha := p_2 - p_1 > 0$. Evidently υ verifies (H2). We show that it verifies (H3) as well. Let us do it only for $i = 2$, the other cases just being analogous. Since

$$\upsilon''(x) = \upsilon_1''(x) + \upsilon_2''(x) x^{-\alpha} - 2\alpha \upsilon_2'(x) x^{-\alpha-1} + \alpha(\alpha+1)\upsilon_2(x) x^{-\alpha-2}, \tag{2.3}$$

then

$$\begin{aligned} |\upsilon''(x)| x^2 &\leq |\upsilon_1''(x)| x^2 + |\upsilon_2''(x)| x^2 x^{-\alpha} + 2\alpha |\upsilon_2'(x)| x x^{-\alpha} + \alpha(\alpha+1)\upsilon_2(x) x^{-\alpha} \\ &\leq g_1^{(2)}(x)\upsilon_1(x) + g_2^{(2)}(x)\upsilon_2(x) x^{-\alpha} + \\ &\qquad +2\alpha g_2^{(1)}(x)\upsilon_2(x) x^{-\alpha} + \alpha(\alpha+1)\upsilon_2(x) x^{-\alpha} \\ &\leq h(x)\left[\upsilon_1(x) + \upsilon_2(x)\right], \end{aligned} \tag{2.4}$$

with

$$h(x) := 3 \max\left\{ g_1^{(2)}(x), g_2^{(2)}(x) x^{-\alpha}, 2\alpha g_2^{(1)}(x) x^{-\alpha}, \alpha(\alpha+1) x^{-\alpha} \right\}. \tag{2.5}$$

But $h(x) \to 0$, as $x \to +\infty$; which gives (H3) when $i = 2$. Q.E.D.

The traditional way to study a billiard problem is to look at the system only when the point collides against the boundary of the table. Mathematically speaking, this means that we consider the Poincaré map defined by the billiard flow on the cross-section that comprises all vectors of the unit tangent bundle of ∂Q that point inside Q (i.e., immediately after the collision). The restriction to unit vectors is obvious as the original Hamiltonian system preserves the modulus of the velocity, which is conventionally fixed to 1. These vectors are traditionally called *line elements* [S, CFS].

In our case, the geometry of the table suggests that we further restrict the cross-section to vectors based in $\mathcal{U}$, the curved (dispersing) part of the boundary, as indicated in Fig. 1. In fact, setting $Q_4 := \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid |y| \le f(|x|)\}$ (see Fig. 3 later on), one can think of working with the billiard in Q_4 where the four smooth components of ∂Q_4 (together with all unit vectors thereupon based) have been identified. This is called a four-copy *unfolding* of Q.

Line elements can be parametrized by $z := (r, \varphi) \in \mathcal{M} := (0, +\infty) \times (0, \pi)$, where r, the arc-length along $\mathcal{U}$, specifies the base point of the vector, and φ is the angle that this vector makes with the tangent to $\mathcal{U}$ there. We choose the convention that φ close to zero denotes almost tangential vectors that point towards the infinite cusp. The billiard map T is defined at all points of $\mathcal{M}$ that would not end up in a vertex or hit ∂Q_4 tangentially; that is, we morally exclude the set "$T^{-1}\partial\mathcal{M}$". In fact, these points make up the discontinuity set of T—commonly called the singularity set, as T there is more often singular than not. It is easy to see that this set, denoted $\mathcal{S}^+$, is the union of two curves, $\mathcal{S}^{1+}$ and $\mathcal{S}^{2+}$, depicted in Fig. 2. To conclude this hasty description of the dynamical system at hand, we recall that T preserves the measure $d\mu(r, \varphi) = \sin\varphi \, dr d\varphi$. More details are available in [Le2].

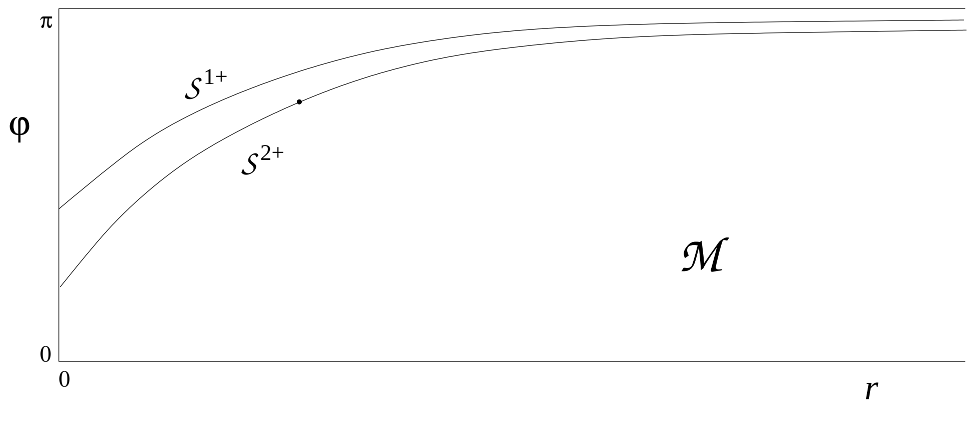

Figure 2: The singularity lines $\mathcal{S}^{1+}$ and $\mathcal{S}^{2+}$ in $\mathcal{M}$.

3 Hyperbolicity

In order to show that a given billiard is ergodic we must first prove that the associated dynamical system $(\mathcal{M}, T, \mu)$ has a hyperbolic structure. This means that for μ-almost every point $z \in \mathcal{M}$, there is a local stable and unstable manifold [Le2, Defn. 6.1], and the resulting foliations are absolutely continuous w.r.t. μ. Sections 6–7 of [Le2] do precisely that, provided f satisfies five assumptions that we list as soon as we have given some necessary notation.

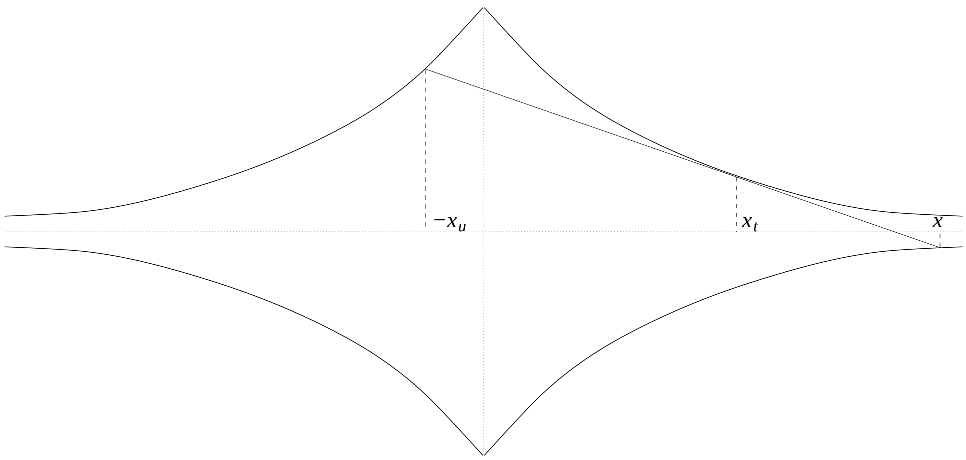

Figure 3: The definition of x_t and x_u.

In Q_4, for $x > 0$, consider the straight line passing through $(x, -f(x))$ and tangent to ∂Q (i.e., to the part of ∂Q_4 that lies in the first quadrant). Denote by $x_t = x_t(x)$ the abscissa of the tangency point. This is uniquely determined by the equation

$$\frac{f(x) + f(x_t)}{x - x_t} = -f'(x_t). \tag{3.1}$$

Then denote by $-x_u = -x_u(x) < 0$ the abscissa of the point at which this line intersects ∂Q_4 in the second quadrant. This number is given by the solution of

$$\frac{f(x) + f(x_u)}{x + x_u} = -f'(x_t). \tag{3.2}$$

In the sequel, for $f, g \geq 0$, we use the notation $f(x) \ll g(x)$ to indicate that there is constant C such that $f(x) \leq C\, g(x)$, as $x \to +\infty$; likewise for $\gg$. Also, $f(x) \sim g(x)$ means that, when $x \to +\infty$, $f(x)/g(x)$ is bounded away from 0 and $+\infty$; whereas $f(x) \simeq g(x)$ means that $f(x)/g(x) \to 1$.

The five assumptions read as follows:

$$f''(x) \to 0; \tag{A1}$$

$$|f'(x_t)| \ll |f'(x)|; \tag{A2}$$

$$\frac{f(x) f''(x)}{(f'(x))^2} \gg 1; \tag{A3}$$

$$\frac{|f'''(x)|}{f''(x)} \ll 1. \tag{A4}$$

$$|f'(x)| \gg (f(x))^\theta, \qquad \text{for some } \theta > 0. \tag{A5}$$

We now prove a result that will make it simpler to check that our functions verify (A1)-(A5).

Lemma 3.1 *For any positive f as in* (H1)-(H3)*,*

$$|f^{(i)}(x)| \simeq \left[\prod_{j=0}^{i-1} (p+j) \right] \upsilon(x)\, x^{-p-i}, \qquad i = 1, 2$$

and

$$|f'''(x)| \ll \upsilon(x)\, x^{-p-3}.$$

Proof. This is a simple verification. Let us work out explicitly the case $i = 2$. Via (H3),

$$f''(x) = \upsilon''(x)x^{-p} - 2p\upsilon'(x)x^{-p-1} + p(p+1)\upsilon(x)x^{-p-2} \simeq p(p+1)\upsilon(x)x^{-p-2}. \tag{3.3}$$

Recall that both f' and f'' have definite signs. For f''' we have only one asymptotic bound, as $f'''(x)$ might vanish somewhere. Q.E.D.

Now, checking (A3) and (A4) is immediate. For (A1) we apply (H2) to $f''(x) \sim \upsilon(x)x^{-p-2}$. For (A5) we may select any $\theta > (p+1)/p$: then the assertion is given by applying (H2) to $(\upsilon(x))^{\theta-1}x^{-\theta p+p+1}$.

Working on (A2) will be more complicated. We need a few lemmas.

Lemma 3.2 *Given any $c \in (0, 1)$, then,* uniformly *in $\xi \in [c, 1]$,*

$$\frac{\upsilon(x)}{\upsilon(\xi x)} \to 1,$$

as $x \to +\infty$.

Proof. Assume for simplicity that $\xi < 1$ (otherwise the above ratio is simply 1). Using Lagrange's Mean Value Theorem, there exists a $\widetilde{\xi} = \widetilde{\xi}(x, \xi) \in (\xi, 1)$ such that

$$\log \upsilon(x) - \log \upsilon(\xi x) = \frac{\upsilon'(\widetilde{\xi}x)}{\upsilon(\widetilde{\xi}x)} \frac{(1-\xi)x\widetilde{\xi}}{\widetilde{\xi}}. \tag{3.4}$$

This vanishes via (H3) and the fact that $(1-\xi)/\widetilde{\xi} < 1/c$. Q.E.D.

Lemma 3.3 *If $x_t(x)$ is defined by (3.1), then there exists a constant $\bar{\xi}$ such that*

$$\frac{x_t(x)}{x} \to \bar{\xi}.$$

Proof of Lemma 3.3. We plug (H1) in (3.1) and utilize Lemma 3.1: there is a function $\phi(x) \to 1$ such that

$$\frac{\upsilon(x)x^{-p} + \upsilon(x_t)x_t^{-p}}{x - x_t} = \phi(x)p\,\upsilon(x_t)x_t^{-p-1}. \tag{3.5}$$

Set $\xi := x_t/x \in (0,1)$. After some algebraic manipulations, this equation can be rewritten as

$$\frac{\upsilon(x)}{\upsilon(\xi x)}\,\xi^{p+1} + [1 + \phi(x)p]\,\xi - \phi(x)p = 0. \tag{3.6}$$

We know the genesis of (3.6) and so we know that, at least for x sufficiently large, it has one and only one solution ξ_x in $(0,1)$. We claim that $\lim_{x\to+\infty} \xi_x = \bar{\xi}$, the latter being the (unique) solution of

$$\xi^{p+1} + [1+p]\,\xi - p = 0 \tag{3.7}$$

in $(0,1)$. This would yield Lemma 3.3.

Lemma 3.4 *Assume that the real function $F_k(\xi)$ is smooth (in ξ) for $\xi \in \mathbb{R}$ and $k \in \mathbb{R}^+$. Suppose that one can find a $\bar{\xi}$ and one of its neighborhoods U such that*

$$\lim_{k\to+\infty} F_k(\bar{\xi}) = 0$$

and

$$\liminf_{k\to+\infty} \inf_{\xi\in U} |F_k'(\xi)| > 0.$$

Then, for large k, $F_k(\xi) = 0$ has a unique solution ξ_k in U, and

$$\lim_{k\to+\infty} \xi_k = \bar{\xi}.$$

We do not prove Lemma 3.4 for its proof is obvious, and we apply it immediately to our case, setting $F_x(\xi)$ to be l.h.s. of (3.6). We utilize x in lieu of k.

We check the first hypothesis of Lemma 3.4 by plugging $\xi = \bar{\xi}$ in (3.6) and using Lemma 3.2 and (3.7). As for the second hypothesis, taking the derivative of $F_x(\xi)$ w.r.t. ξ we get, after some rearrangement,

$$F_x'(\xi) = \frac{\upsilon(x)}{\upsilon(\xi x)}\,\xi^p \left[p + 1 - \frac{\upsilon'(\xi x)}{\upsilon(\xi x)}\,\xi x\right] + 1 + \phi(x)p. \tag{3.8}$$

Now choose an open interval U such that $\bar{\xi} \in U \subset (0,1)$. Via (H3), the only term in (3.8) that is preceded by a minus sign vanishes, as $x \to +\infty$. Taking into account the asymptotics of all other terms (which are positive, anyway) we obtain, for x sufficiently large,

$$\inf_{\xi \in U} F'_x(\xi) \geq 1 + p/2, \tag{3.9}$$

which implies the second condition of Lemma 3.4, and thus Lemma 3.3. Q.E.D.

By Lemma 3.1,

$$\frac{|f'(x_t)|}{|f'(x)|} \simeq \frac{\upsilon(x_t)}{\upsilon(x)} \left(\frac{x_t}{x} \right)^{-p-1}. \tag{3.10}$$

Therefore, through Lemmas 3.2 and 3.3, we find a constant $C > 1$ such that

$$|f'(x_t)| \simeq C|f'(x)|, \tag{3.11}$$

which implies (A2).

4 Ergodicity

If we look at [Le2, Sec. 9] we see that there are two results that are needed for the proof of the ergodicity that do not descend from (A1)-(A5). One is eqn. (9.2) and the other is Lemma 9.3. Proving the first result is easy, with the techniques developed in the previous section.

Proposition 4.1

$$|f'(x_u)| \ll |f'(x)|.$$

Proof. First of all, let us plug (3.11) into (3.2):

$$\frac{f(x) + f(x_u)}{x + x_u} \simeq C|f'(x)|. \tag{4.1}$$

Then we use (H1) as in the proof of Lemma 3.3. Setting $\xi := x_u/x$ and proceeding as in (3.5)-(3.6), we get

$$\phi(x)p\,\xi^{p+1} + [\phi(x)p - 1]\,\xi^p - \frac{\upsilon(\xi x)}{\upsilon(x)} = 0, \tag{4.2}$$

for some function $\phi(x) \to C$. Once again, we know that (4.2) has one and only one solution $\xi_x \in (0,1)$ and, once again, we claim that $\lim_{x\to+\infty} \xi_x = \bar{\xi}$. This time $\bar{\xi}$ is the solution of

$$Cp\,\xi^{p+1} + [Cp - 1]\,\xi^p - 1 = 0 \tag{4.3}$$

in $(0,1)$. Via a relation totally analogous to (3.10), this would prove that $|f'(x_u)| \simeq C_1|f'(x_u)|$, for some $C_1 > 1$, whence Proposition 4.1.

The idea is to use Lemma 3.4 one more time. Let $F_x(\xi)$ be given by the l.h.s. of (4.2). Substituting $\xi = \bar{\xi}$ in (4.2) gives the first condition of Lemma 3.4, through Lemma 3.2 and (4.3). Furthermore,

$$\begin{aligned} F_x'(\xi) &= \phi(x)p(p+1)\,\xi^p + [\phi(x)p-1]p\,\xi^{p-1} - \frac{\upsilon'(\xi x)\xi x}{\upsilon(\xi x)}\,\frac{\upsilon(\xi x)}{\xi \upsilon(x)} = \\ &= \phi(x)p\,\xi^p + \frac{p}{\xi}\left\{\phi(x)p\,\xi^{p+1} + [\phi(x)p-1]\,\xi^p\right\} - \sigma(x;\xi). \end{aligned} \tag{4.4}$$

Here $\sigma(x;\xi)$, whose definition is implicit in the above, vanishes as $x \to +\infty$ and ξ remains away from 0 (due to (H3) and Lemma 3.2).

Focussing our attention on the term inside the braces, we see that it goes to 1, if we take $\xi = \bar{\xi}$—which is the solution of (4.3). Therefore, $F_x'(\bar{\xi}) \to Cp\,\bar{\xi}^p + p/\bar{\xi}$. Furthermore, as all the terms containing x are well-behaved when $x \to +\infty$, there is a neighborhood U of $\bar{\xi}$ such that

$$\inf_{\xi\in U} F_x'(\xi) \geq \frac{1}{2}\left(Cp\,\bar{\xi}^p + \frac{p}{\bar{\xi}}\right), \tag{4.5}$$

for x large enough. This is the second hypothesis of Lemma 3.4. Proposition 4.1 is thus proved. Q.E.D.

Lemma 4.2 that we are going to give in the second part of Section 4 substitutes Lemma 9.3 of [Le2]. Since we present it out of context and the result is rather technical anyway, we had better precede it with a little preamble.

A point $(x,y) \in \mathcal{U}$ can be parametrized in two ways: r, the arc-length coordinate along $\mathcal{U}$, and x, the abscissa of the point in the plane. The transformation between the two coordinates is given by

$$r(x) := \int_0^x \sqrt{1+(f'(t))^2}\,dt. \tag{4.6}$$

Sometimes, like in the remainder, is convenient to use x. We do so without worrying too much about a rigorous notation. For example we will write (x,φ) to mean the point $(r(x),\varphi) \in \mathcal{M}$. With this in mind, define

$$h_2(x) := K_2|f'(x)|, \tag{4.7}$$

for a certain constant K_2 provided by the system itself (the subscript here has the sole purpose of keeping consistency of notation with the corresponding objects in [Le2]). For x large enough, $h_2(x) < \pi/2$ and it makes sense to consider the line element $w := w_0 := (x,\varphi_0) := (x,h_2(x))$. The associated unit vector points more and more towards the cusp, as $x \to +\infty$; thus it will take the material point more and more rebounds to "come back from the cusp". In other words, setting

$w_i := (x_i, \varphi_i) := T^i w$ and denoting by m the first time (i.e., the first T-iteration) that the billiard trajectory of w_i crosses the y-axis in Q_4, we have $m = m(x) \to +\infty$. The situation is illustrated in Fig. 4.

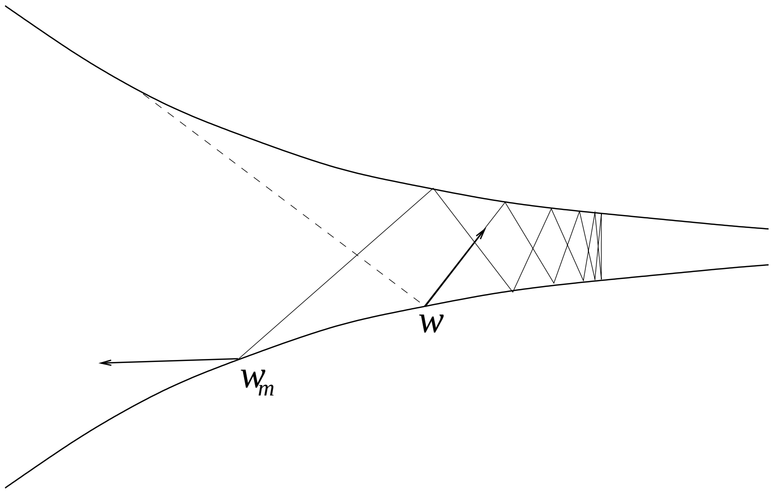

Figure 4: For φ sufficiently small, the trajectory of $w = (r, \varphi)$ travels toward the cusp and comes back. The number m of collisions the material point makes against ∂Q_4 before crossing the y-axis tends to ∞ as $\varphi \to 0$ and/or $r \to +\infty$.

The recursive formula to determine x_{n+1} is easily found out to be

$$\frac{f(x_{n+1}) + f(x_n)}{x_{n+1} - x_n} = \tan(\varphi_n + \alpha_n), \tag{4.8}$$

with $\alpha_n := \arctan |f'(x_n)|$. It is not difficult either to see that $\varphi_{n+1} = \varphi_n + \alpha_n + \alpha_{n+1}$ [Le2], whence

$$\varphi_n = \varphi_0 + \alpha_0 + 2 \sum_{i=1}^{n-1} \alpha_i + \alpha_n. \tag{4.9}$$

Notice that all quantities ultimately depend on x. We are now ready to state and prove our last result.

Lemma 4.2 *For f as in* (H1)-(H3)*, there exists an increasing sequence $\{\xi_n\}$ such that, for fixed n,*

$$\lim_{x \to +\infty} \frac{x_n(x)}{x} = \xi_n.$$

Furthermore

$$\lim_{n \to +\infty} \xi_n = +\infty.$$

PROOF. As concerns the first assertion, we prove it by induction. For $n=0$ there is nothing to prove, as $x_0 = x$ (whence $\xi_0 = 1$). So let us assume that the limit above exists for all $i = 1, ..., n$, and show that it exists for $n+1$, too.

As n is fixed, all $\alpha_i(x)$, $i = 1, ..., n$, tend to zero, when $x \to +\infty$. And so does $\varphi_n(x)$, from (4.9) and the fact that $\varphi_0(x) = h_2(x) = K_2|f'(x)|$—the latter coming from (4.7). Therefore,

$$\tan(\varphi_n + \alpha_n) \simeq \varphi_n + \alpha_n \simeq K_2|f'(x)| + |f'(x)| + 2\sum_{i=1}^{n} |f'(x_i)|. \tag{4.10}$$

Plugging (4.10) in (4.8) we get, with the aid of Lemma 3.1,

$$\frac{\upsilon(x_{n+1})x_{n+1}^{-p} + \upsilon(x_n)x_n^{-p}}{x_{n+1} - x_n} \simeq p\left[(K_2+1)\upsilon(x)x^{-p-1} + 2\sum_{i=1}^{n} \upsilon(x_i)x_i^{-p-1}\right]. \tag{4.11}$$

Divide both sides by $\upsilon(x)x^{-p-1}$:

$$\frac{\dfrac{\upsilon(x_{n+1})}{\upsilon(x)}\left(\dfrac{x_{n+1}}{x}\right)^{-p} + \dfrac{\upsilon(x_n)}{\upsilon(x)}\left(\dfrac{x_n}{x}\right)^{-p}}{\left(\dfrac{x_{n+1}}{x}\right) - \left(\dfrac{x_n}{x}\right)} = \phi(x)p\left[K_2 + 1 + 2\sum_{i=1}^{n} \frac{\upsilon(x_i)}{\upsilon(x)}\left(\frac{x_i}{x}\right)^{-p-1}\right], \tag{4.12}$$

with $\phi(x) \to 1$ to replace the $\simeq$ sign. Let us name $\phi_1(x)$ the above r.h.s. For $i = 1, ..., n$, by induction hypothesis, $x_i/x \to \xi_i$ and, by Lemma 3.2, $\upsilon(x_i)/\upsilon(x) \to 1$. Therefore $\phi_1(x) \to \widetilde{\phi}_1$, for some $\widetilde{\phi}_1 > 0$. Setting $\xi := x_{n+1}/x$ and rewriting (4.12) conveniently, we obtain

$$\frac{\upsilon(\xi x)}{\upsilon(x)}\,\xi^{-p} - \phi_1(x)\xi = -\phi_1(x)\left(\frac{x_n}{x}\right) - \frac{\upsilon(x_n)}{\upsilon(x)}\left(\frac{x_n}{x}\right)^{-p} =: -\phi_2(x). \tag{4.13}$$

For the same reasons as before, $\phi_2(x) \to \widetilde{\phi}_2 > 0$. Multiplying by ξ^p, we get the equation

$$F_x(\xi) := \phi_1(x)\,\xi^{p+1} - \phi_2(x)\,\xi^p - \frac{\upsilon(\xi x)}{\upsilon(x)} = 0. \tag{4.14}$$

It is now clear that we intend to apply again Lemma 3.4 and its associated arguments. Let ξ_{n+1} be the (unique) solution of

$$\widetilde{\phi}_1\,\xi^{p+1} - \widetilde{\phi}_2\,\xi^p - 1 = 0 \tag{4.15}$$

in $(0,1)$. (Here ξ_{n+1} takes on the role of $\bar{\xi}$ as in the statement of Lemma 3.4.) In view of the asymptotics of (4.14), one easily verifies that $F_x(\xi_{n+1}) \to 0$, as $x \to +\infty$. Furthermore

$$\begin{aligned} F_x'(\xi) &= \phi_1(x)(p+1)\,\xi^p - \phi_2(x)p\,\xi^{p-1} - \sigma(x;\xi) = \\ &= \phi_1(x)\,\xi^p + \frac{p}{\xi}\left\{\phi_1(x)\,\xi^{p+1} - \phi_2(x)\,\xi^p\right\} - \sigma(x;\xi) \end{aligned} \tag{4.16}$$

where $\sigma(x;\xi)$ is precisely the same as in (4.4), and thus vanishes uniformly for $\xi \in [c,1]$. Also as in (4.4), the term inside the braces goes to 1, when $\xi = \xi_{n+1}$. Proceeding exactly as in the proof of Proposition 4.1, we can take U to be a sufficiently small neighborhood of ξ_{n+1} such that

$$\inf_{\xi \in U} F'_x(\xi) \geq \frac{1}{2} \left(\widetilde{\phi}_1 p \, \xi_{n+1}^p + \frac{p}{\xi_{n+1}} \right), \tag{4.17}$$

for x large enough. This allows us to apply Lemma 3.4 and conclude that $x_{n+1}/x = \xi \to \xi_{n+1}$.

The sequence $\{\xi_n\}$ is increasing since $x_{n+1} > x_n$, at least for x sufficiently large (so that $\varphi_n < \pi/2$ and the material point moves to the right, after the n-th collision).

As concerns the second assertion of the lemma, we proceed by contradiction. Suppose that $\xi_n \nearrow \widetilde{\xi} < +\infty$, as $n \to +\infty$. Applying the first assertion to (4.12) one obtains

$$\frac{\xi_{n+1}^{-p} + \xi_n^{-p}}{\xi_{n+1} - \xi_n} = p \left[K_2 + 1 + 2 \sum_{i=1}^{n} \xi_i^{-p-1} \right]. \tag{4.18}$$

Then the r.h.s. of (4.18) grows asymptotically like n. But the numerator of the l.h.s. converges, implying that $\xi_{n+1} - \xi_n \sim n^{-1}$, which in turn contradicts the convergence of $\{\xi_n\}$. Q.E.D.

References

[CFS] I. P. Cornfeld, S. V. Fomin and Ya. G. Sinai, *Ergodic theory*, Springer-Verlag, New York, 1982.

[KS] A. Katok and J.-M. Strelcyn (in collaboration with F. Ledrappier and F. Przytycki), *Invariant manifolds, entropy and billiards; smooth maps with singularities*, LNM 1222, Springer-Verlag, Berlin-New York, 1986.

[K] J. L. King, *Billiards inside a cusp*, Math. Intelligencer **17**(1) (1995), 8-16.

[Le1] M. Lenci, *Escape orbits for non-compact flat billiards*, Chaos **6**(3) (1996), 428-431.

[Le2] M. Lenci, *Semi-dispersing billiards with an infinite cusp*, preprint, downloadable from `www.arxiv.org/abs/nlin.CD/0107041`.

[L] A. M. Leontovich, *The existence of unbounded oscillating trajectories in a problem of billiards*, Soviet Math. Dokl. **3**(4) (1962), 1049-1052.

[LW] C. Liverani and M. Wojtkowski, *Ergodicity in Hamiltonian systems*, in: Dynamics Reported: Expositions in Dynamical Systems (N.S.), **4**, Springer-Verlag, Berlin, 1995.

[S] Ya. G. Sinai, *Dynamical systems with elastic reflections*, Russ. Math. Surveys **25** (1970), 137-189.