THE SCATTERING MATRIX FOR THE SCHRÖDINGER OPERATOR WITH A LONG-RANGE ELECTROMAGNETIC POTENTIAL

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Abstract

We consider the Schrödinger operator $H = (i\nabla + A)^2 + V$ in the space $L_2(\mathbb{R}^d)$ with long-range electrostatic $V(x)$ and magnetic $A(x)$ potentials. Using the scheme of smooth perturbations, we give an elementary proof of the existence and completeness of modified wave operators for the pair $H_0 = -\Delta$, H. Our main goal is to study spectral properties of the corresponding scattering matrix $S(\lambda)$. We obtain its stationary representation and show that its singular part (up to compact terms) is a pseudodifferential operator with an oscillating amplitude which is an explicit function of V and A. Finally, we deduce from this result that, in general, for each $\lambda > 0$ the spectrum of $S(\lambda)$ covers the whole unit circle.

1. INTRODUCTION

Let the Schrödinger operator H be defined by differential expression

$$H = H(V, A) = (i\nabla + A(x))^2 + V(x), \tag{1.1}$$

where the scalar function $V(x)$ and the vector-valued function $A(x) = (A_1(x), \dots, A_d(x))$ are real valued and called the electrostatic and magnetic potentials, respectively. We assume that the potentials are C^∞-functions satisfying the assumption

$$|\partial_x^\alpha A(x)| + |\partial_x^\alpha V(x)| \leq C_\alpha (1 + |x|)^{-\rho - |\alpha|}, \quad \rho > 0, \tag{1.2}$$

for all multi-indices α. Then H is a self-adjoint operator in the space $\mathcal{H} = L_2(\mathbb{R}^d)$ on domain $\mathcal{D}(H) = \mathcal{D}(H_0)$ of the "free" operator $H_0 = -\Delta$. Our approach to the scattering theory for the pair H_0, H relies on wave operators with non-trivial identifications $J_\pm$ (depending on the sign of t)

$$W_\pm = W_\pm(H, H_0; J_\pm) = s - \lim_{t \to \pm\infty} e^{iHt} J_\pm e^{-iH_0 t}. \tag{1.3}$$

We construct $J_\pm$ as pseudodifferential operators (PDO). Such wave operators were first introduced by Isozaki and Kitada in [3], where the Enss method was used for the proof of their existence and completeness (the method of [3] was extended to long-range magnetic potentials in [13]). We show that the long-range scattering fits into the theory of smooth perturbations (in the sense of Kato). This yields, in particular, the stationary (in terms of the resolvent) representation for the corresponding scattering matrix $S(\lambda)$ which allows us to study its structure and spectral properties. Actually, we follow closely the paper [20] where the case $\rho > 1/2$ (and $A = 0$) was considered. However consideration of the general case $\rho > 0$ is technically much more involved.

The operators $J_\pm$ emerge naturally as PDO with symbols constructed in terms of approximate eigenfunctions $\Psi_\pm(x,\xi) = e^{i\varphi_\pm(x,\xi)}$ of the operator H. Substituting $\Psi = e^{i\varphi}$ in the Schrödinger equation $H\Psi = |\xi|^2\Psi$ and neglecting imaginary terms, we obtain the eikonal equation

$$|\nabla\varphi|^2 - 2 < A, \nabla\varphi > + V(x) + |A(x)|^2 - |\xi|^2 = 0, \quad \nabla = \nabla_x, \tag{1.4}$$

for the phase function $\varphi = \varphi_\pm$. We seek approximate solutions of this equation in the form

$$\varphi_\pm(x,\xi) = < \xi, x > + \Phi_\pm(x,\xi) \tag{1.5}$$

where $\Phi_\pm(x,\xi) = o(|x|)$ as $|x| \to \infty$. Such solutions can be constructed by an explicit procedure described in Section 4. Then the Schrödinger equation for $\Psi_\pm = e^{i\varphi_\pm}$ is satisfied up to short-range terms (that is, satisfying an estimate of the type (1.2) with $\rho > 1$) off any neighbourhood of the direction $\hat{x} = \mp\hat{\xi}$. To get rid of the "bad" direction $\hat{x} = \mp\hat{\xi}$, we introduce in the symbol $j_\pm(x,\xi)$ suitable cut-off functions $\zeta_\pm(x,\xi)$, such that $\zeta_\pm$ are asymptotically homogeneous in x of degree 0, $\zeta_\pm(x,\xi) = 0$ in a conical neighbourhood (in $\mathbb{R}^{2d}$) of the set $\hat{x} = \mp\hat{\xi}$ and $\zeta_\pm(x,\xi) = 1$ off somewhat larger neighbourhood of this set. At the same time we localize our considerations on a compact energy subinterval of $\mathbb{R}_+ = (0,\infty)$. The symbol of the PDO $J_\pm$ can be defined by the equality

$$j_\pm(x,\xi) = e^{i\Phi_\pm(x,\xi)}\zeta_\pm(x,\xi). \tag{1.6}$$

As in [20], from analytical point of view our approach relies on the limiting absorption principle and the radiation estimates. The first of these results means that the operator $\langle x\rangle^{-p}$, $p > 1/2$, is locally H-smooth on any bounded interval disjoint from zero (see Definition 3.1). The second, pertaining to the critical case $p = 1/2$, ensures local H-smoothness of the operators $G_j = \langle x\rangle^{-1/2}\nabla_j^\perp$, $j = 1,\dots d$, where

$$\nabla_j^\perp u = \partial_j u - |x|^{-2} < \nabla u, x > x_j, \quad j = 1,\dots,d. \tag{1.7}$$

Let $T_\pm = HJ_\pm - J_\pm H_0$ be the "effective perturbation". It follows from definition (1.6) that both operators $T = T_\pm$ factorize into a sum of products of locally H_0- and H-smooth operators

$$T = \langle x\rangle^{-p} B^{(r)} \langle x\rangle^{-p} + \sum_{j=1}^{d} G_j^* B^{(s)} G_j, \quad p > 1/2, \tag{1.8}$$

where the operators $B^{(r)} = B_\pm^{(r)}$, $B^{(s)} = B_\pm^{(s)}$ are bounded. This implies the existence of all wave operators $W_\pm(H,H_0,J_\gamma), W_\pm(H_0,H,J_\gamma^*)$, both for $\gamma = $ " + " and $\gamma = $ " − ".

The operators $W_\pm(H,H_0,J_\pm)$ are isometric since $J_\pm^* J_\pm e^{-iH_0t}f \sim e^{-iH_0t}f$ as $t\to\pm\infty$. For the proof of completeness of $W_\pm(H,H_0,J_\pm)$, we use that, for a suitable choice of the cut-off functions ζ_+ and ζ_-, the operator $J_+J_+^* + J_-J_-^* - I$ is essentially compact. Finally, we check that wave operators (1.3) coincide with wave operators obtained by a time-dependent modification of the free dynamics in the coordinate representation (see [22]). In the special case where $V=0$ and A satisfies assumption (1.2) with $\rho > 1/2$ and the additional condition of transversal gauge

$$< A(x), x >= 0 \tag{1.9}$$

(for $x\in\mathbb{R}^d$ large enough), the identifications $J_\pm$ can be replaced in (1.3) by the identity operator, that is the usual wave operators $W_\pm(H,H_0)$ exist. Thus, we recover the well known result [9] of Loss and Thaller about long-range magnetic scattering.

Since the scattering operator $\mathbf{S} = W_+^* W_-$ commutes with H_0, it reduces in the spectral representation of H_0 to the multiplication by the operator-function $S(\lambda)$, $\lambda>0$, acting in the space $\mathfrak{N} = L_2(\mathbb{S}^{d-1})$ and known as the scattering matrix (SM). Our study of $S(\lambda)$ relies on its stationary representation in terms of the resolvent $R(z)=(H-z)^{-1}$ of the operator H. To describe it, we need to introduce the auxiliary wave operator $W_+(H_0,H_0;J_+^*J_-)$. It commutes with H_0 and hence acts in the spectral representation of H_0 as multiplication by the operator-function $\mathcal{W}(\lambda):\mathfrak{N}\to\mathfrak{N}$. Then (see [4, 20, 22])

$$S(\lambda) = \mathcal{W}(\lambda) - 2\pi i \Gamma_0(\lambda)\left(J_+^* T_- - T_+^* R(\lambda+i0) T_-\right)\Gamma_0^*(\lambda), \tag{1.10}$$

where

$$(\Gamma_0(\lambda)f)(\omega) = 2^{-1/2} k^{(d-2)/2}\hat{f}(k\omega), \quad k=\lambda^{1/2}>0,\ \omega\in\mathbb{S}^{d-1}, \tag{1.11}$$

is, up to the numerical factor, the restriction of the Fourier transform $\hat{f} = \mathcal{F}f$ to the sphere of radius k. Formula (1.10) is actually of abstract nature and, roughly speaking, is valid always when its right-hand side is well defined.

In our particular case $\mathcal{W}(\lambda) = 0$ for all $\lambda > 0$. Then, using the resolvent estimates called usually propagation estimates (see [6, 7, 11]), we show that the part of $S(\lambda)$ containing in (1.10) the resolvent of H is a compact operator. Thus, the essential spectrum of $S(\lambda)$ is determined by the explicit operator $-2\pi i\Gamma_0(\lambda)J_+^*T_-\Gamma_0^*(\lambda)$ which we consider as a PDO on the unit sphere. In the general case $\rho>0$ this PDO is determined by its amplitude $a(y,\omega,\omega';\lambda)$ where $\omega,\omega'\in\mathbb{S}^{d-1}$ are close to some point ω_0 and y belongs to the hyperplane orthogonal to ω_0. It turns out that, essentially, $a(y,\omega,\omega';\lambda) = \exp(i\theta(y,\omega,\omega';\lambda))$ where $\theta(y,\omega,\omega;\lambda)\to\infty$ as $|y|\to\infty$. Actually, the asymptotics of $\theta(y,\omega,\omega;\lambda)$ as $|y|\to\infty$ is determined by the asymptotics of $V(x)$ and $A(x)$ as $|x|\to\infty$. For homogeneous potentials of degree $-\rho$, $\rho\in(0,1)$, the function θ is asymptotically homogeneous of degree $1-\rho$ and it has, generically, a logarithmic growth if $\rho=1$. In particular, this implies [21] that the spectrum of the SM covers the whole unit circle $\mathbb{T}$. This situation is completely different from the short-range case where the principal symbol of PDO $S(\lambda)$ equals 1 which corresponds to the Dirac-function in its kernel. In this case the spectrum of $S(\lambda)$ consists of eigenvalues accumulating at the point 1 only. Thus, in the long- and short-range cases the singularities and the spectral structure of $S(\lambda)$ are of qualitatively different nature.

A detailed analysis of singularities (as well as asymptotic expansion as $\lambda \to \infty$) of the SM $S(\lambda)$ can be found in [23] where a method, somewhat different from this paper, was used.

Technically, our paper relies on the PDO calculus. Thus, we treat the operators $J_\pm$ and $T_\pm$ as PDO. For any $\rho \in (0,1)$, they belong to the Hörmander classes $\mathcal{S}^0_{\rho,1-\rho}$ and $\mathcal{S}^{-1}_{\rho,1-\rho}$, respectively. In these classes a convenient PDO calculus can be developed for $\rho > 1/2$ only. This creates new difficulties for potentials satisfying (1.2) for $\rho \in (0, 1/2]$. Fortunately, the operators $J_\pm$ and $T_\pm$ fit into the framework of PDO with oscillating symbols developed in [21]. Auxiliary results concerning PDO calculus are discussed in Section 2.

2. PSEUDODIFFERENTIAL OPERATORS

2.1. We need some elementary facts about PDO defined by the equality

$$(Af)(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i<\xi,x>} a(x,\xi) \hat{f}(\xi) d\xi, \tag{2.1}$$

where

$$\hat{f}(\xi) = (\mathcal{F}f)(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i<\xi,x>} f(x) dx$$

is the Fourier transform of f from the Schwartz class $\mathcal{S} = \mathcal{S}(\mathbb{R}^d)$. We denote by $\mathcal{S}^m_{\rho,\delta}$ the class of symbols $a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ satisfying, for all multi-indices α and β, the estimates

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha,\beta} (1 + |x|)^{m - \rho|\alpha| + \delta|\beta|}. \tag{2.2}$$

Here and below C, c are different positive constants, whose precise values are of no importance. Moreover, we assume that $a(x,\xi) = 0$ for sufficiently large $|\xi|$. We always suppose that $\rho > 0$, $\delta < 1$ and set $\mathcal{S}^m = \mathcal{S}^m_{1,0}$. The operators A and A^* are well-defined as mappings of the Schwartz class $\mathcal{S}$ into itself. The calculus of PDO can be conveniently developed (see, e.g., [2] or [17]) in the classes $\mathcal{S}^m_{\rho,\delta}$ if $\rho > 1/2 > \delta$, but it fails in the general case.

To treat the case $\rho \leq 1/2$, we need a more special class of PDO with oscillating symbols introduced in [21]. Let us denote by $\mathcal{C}^m(\Phi)$ the class of symbols admitting the representation

$$a(x,\xi) = e^{i\Phi(x,\xi)} b(x,\xi), \quad \Phi \in \mathcal{S}^r, \quad b \in \mathcal{S}^m, \quad r \in [0,1). \tag{2.3}$$

Since $\mathcal{C}^m(\Phi) \subset \mathcal{S}^m_{1-r,r}$, only the case $r \geq 1/2$ requires a special study. Occasionally we use the notation $\mathcal{S}^m_{\rho,\delta}$ and $\mathcal{C}^m(\Phi)$ for PDO (2.1) with symbols from these classes. The operator of multiplication by $(1 + |x|^2)^{1/2}$ is denoted by $\langle x \rangle$.

Let us collect a necessary information on the calculus of PDO. Note that compared to the standard definition of PDO the roles of the variables x and ξ in (2.1) are interchanged. However, the operator $\mathcal{F}A^*\mathcal{F}^*$ satisfies the usual assumptions. This allows us to recover all results of PDO theory for operators (2.1). We start with an elementary assertion, which can be proven by the standard technique (see, e.g., [2] or [17]).

Proposition 2.1 *Suppose that* $a_1 \in \mathcal{C}^{m_1}(\Phi)$ *for some* $\Phi \in \mathcal{S}^r$, $r \in [0,1)$, $a_2 \in \mathcal{S}^{m_2}$ *and* $a_0(x,\xi) = a_1(x,\xi)a_2(x,\xi)$. *Let* A_j *be the PDO with symbol* a_j, $j = 0,1,2$. *Then* A_1A_2 *and* A_2A_1 *are PDO from the class* $\mathcal{C}^m(\Phi)$ *where* $m = m_1 + m_2$ *and the PDO* $A_1A_2 - A_0$ *and* $A_2A_1 - A_0$ *belong to the class* $\mathcal{C}^{m+r-1}(\Phi)$.

The following two assertions are borrowed directly from [21].

Proposition 2.2 *Let* $A \in \mathcal{C}^m(\Phi)$. *Then the operator* A *is bounded in the space* $L_2(\mathbb{R}^d)$ *if* $m = 0$, *and it is compact if* $m < 0$.

Proposition 2.3 *Let* A_j, $j = 1,2$, *be PDO with symbols* $a_j \in \mathcal{C}^0(\Phi)$ *and let* A *be the PDO with symbol* $a_1(x,\xi)\overline{a_2(x,\xi)}$. *Then both operators* $A - A_1A_2^*$ *and* $A - A_2^*A_1$ *are compact in the space* $L_2(\mathbb{R}^d)$.

To formulate further results on PDO, we need the following technical

Lemma 2.4 *Let* $\Omega_0 \in C_0^\infty(\mathbb{R}^d)$, $\Omega_0(\xi) = 1$ *for* $|\xi| \le R_0 - 1$ *and* $\Omega_0(\xi) = 0$ *for* $|\xi| \ge R_0$ *(for some* $R_0 > 1$*). Let* $\Omega_1 \in C^\infty(\mathbb{R}^d)$, $\Omega_1(\xi) = 0$ *for* $|\xi| \le R_1$ *and* $\Omega_1(\xi) = 1$ *for* $|\xi| \ge R_1 + 1$. *Assume that* $R_1 > R_0$. *We denote by* Ω_j, $j = 0,1$, *the multiplication operators (in the momentum representation) by the functions* $\Omega_j(\xi)$. *Let* g *be the multiplication operator by a function* $g \in C^\infty(\mathbb{R}^d)$ *such that* $\partial^\alpha g(x) = O(|x|^{n-|\alpha|})$ *as* $|x| \to \infty$ *for some* n *and all* α. *Then the operator* $\Omega_0 g \Omega_1 = \Omega_0 g_0 \Omega_1$ *where* $g_0 \in \mathcal{S}$.

Proof. – In the momentum representation g acts as the convolution with the function $\hat{g}(\xi)$ where $\hat{g} \in C^\infty$ except the point zero and decays at infinity, together with all its derivatives, faster that any power of $|\xi|^{-1}$. Therefore $\Omega_0 g \Omega_1$ is the integral operator with kernel

$$(2\pi)^{-d/2}\Omega_0(\xi)\hat{g}(\xi - \xi')\Omega_1(\xi'), \tag{2.4}$$

which equals 0 if $|\xi - \xi'| \le R_1 - R_0$. Let $\omega \in C^\infty(\mathbb{R}^d)$, $\omega(\xi) = 0$ for $|\xi| \le (R_1 - R_0)/2$ and $\omega(\xi) = 1$ for $|\xi| \ge R_1 - R_0$. Then kernel (2.4) is not changed if we replace $\hat{g}(\xi - \xi')$ by $\hat{g}_0(\xi - \xi')$ where $\hat{g}_0(\xi) = \hat{g}(\xi)\omega(\xi)$ belongs to $\mathcal{S}$. $\square$

Lemma 2.5 *Let* A *be a PDO with symbol* a *from the class* $\mathcal{S}^m_{\rho,\delta}$ *where* $\rho > 0$, $\delta < 1$ *and* m *are arbitrary and* $a(x,\xi) = 0$ *if* $|\xi| \ge R$. *Let* g *and* Ω_1 *be the same as in Lemma* 2.4 *and* $R_1 > R$. *Then for any* p *the operator* $Ag\Omega_1\langle x\rangle^p$ *is bounded in* $L_2(\mathbb{R}^d)$.

Proof. – Assuming that p is an even integer and commuting the operators Ω_1 and $\langle x\rangle^p$, we reduce the problem to the case $p = 0$. Let Ω_0 be the same as in Lemma 2.4 with $R_0 \in (R, R_1)$. Then $Ag\Omega_1 = A\Omega_0 g\Omega_1$, so that, by Lemma 2.4, we may suppose that $g \in \mathcal{S}$. In the coordinate representation, the operator Ag has kernel

$$(2\pi)^{-d}\int_{\mathbb{R}^d} e^{i<\xi,x-x'>}a(x,\xi)d\xi\, g(x'). \tag{2.5}$$

Using the formula $\langle x\rangle^{-k}\langle D_\xi\rangle^k e^{i<\xi,x>} = e^{i<\xi,x>}$ and integrating by parts, we estimate the kernel (2.5) by $\langle x\rangle^{m-(1-\delta)k}\langle x'\rangle^k|g(x')|$. For sufficiently large k, both functions of x and x' belong to $L_2(\mathbb{R}^d)$, and hence the operator Ag belongs to the Hilbert-Schmidt class. $\square$

Now it easy to check the following two assertions.

Proposition 2.6 *Let $A \in \mathcal{C}^m(\Phi)$ for some m. Then the operator $A \langle x \rangle^{-m}$ is bounded in the space $L_2(\mathbb{R}^d)$.*

Proof. – Assume that $a(x,\xi) = 0$ for $|\xi| \geq R$. Let Ω_1 be the same operator as in Lemma 2.4 with $R_1 > R$ and $\Omega_2 = I - \Omega_1$. Clearly, $\langle x \rangle^{-m} \Omega_2$ is a PDO from the class $\mathcal{S}^{-m}$. Therefore, by Proposition 2.1, the PDO $A(\langle x \rangle^{-m} \Omega_2)$ belongs to the class $\mathcal{C}^0(\Phi)$ and hence, by Proposition 2.2, it is bounded. The operator $A \langle x \rangle^{-m} \Omega_1$ is bounded according to Lemma 2.5. □

Proposition 2.7 *Suppose that A is a PDO with symbol $a \in \mathcal{C}^m(\Phi)$, for some $\Phi \in \mathcal{S}^r$ with $r \in [0,1)$, and G is a PDO (actually, a differential operator) with symbol $g(x,\xi) = \sum_{|s|\leq s_0} g_s(x)\xi^s$, where $g_s \in C^\infty(\mathbb{R}^d)$ and $\partial^\alpha g_s(x) = O(|x|^{n-|\alpha|})$ as $|x| \to \infty$ for some n and all α. Let T be the PDO with symbol $t(x,\xi) = |g(x,\xi)|^2 a(x,\xi)$. Then the operator $\langle x \rangle^p (G^* A G - T) \langle x \rangle^p$ is bounded if $2p = -m - 2n + 1 - r$.*

Proof. – Let $a_1(x,\xi)$ be the symbol of the PDO $A_1 = G^* A$. By direct calculation we show that

$$a_1(x,\xi) - \overline{g(x,\xi)} a(x,\xi) \in C^{m+n+r-1}(\Phi).$$

Assuming that $a(x,\xi) = 0$ for $|\xi| \geq R$, we introduce the same operators Ω_1, Ω_2 as in Proposition 2.6. Proposition 2.1 implies that, up to terms from the class $C^{m+2n+r-1}(\Phi)$, the operator $A_1 G \Omega_2$ has symbol $t(x,\xi)$. Therefore the operator $\langle x \rangle^p (G^* A G \Omega_2 - T) \langle x \rangle^p$ is bounded by Proposition 2.6. It remains to take into account that the operator $\langle x \rangle^p A_1 G \Omega_1 \langle x \rangle^p$ is bounded for any p according to Lemma 2.5. □

For the study of the SM we need to consider PDO defined by their amplitudes. We define such operators in terms of the corresponding sesquilinear forms on $\mathcal{S} \times \mathcal{S}$

$$(\mathbf{A} f_1, f_2) = (2\pi)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i<\xi'-\xi,x>} \mathbf{a}(x,\xi,\xi') \hat{f}_1(\xi') \overline{\hat{f}_2(\xi)} dx d\xi d\xi'. \tag{2.6}$$

We suppose that the amplitude $\mathbf{a}(x,\xi,\xi')$ belongs to the class $\mathcal{S}^m_{\rho,\delta,\delta}$ defined quite similarly to $\mathcal{S}^m_{\rho,\delta}$ with ξ replaced by (ξ,ξ') in (2.2) and that $\mathbf{a}(x,\xi,\xi')$ is compactly supported in ξ and ξ'. Again we denote $\mathcal{S}^m = \mathcal{S}^m_{1,0,0}$.

Let us consider (cf. (2.3)) a special class $\mathcal{C}^m(\Theta)$ of amplitudes admitting the representation

$$\mathbf{a}(x,\xi,\xi') = e^{i\Theta(x,\xi,\xi')} \mathbf{b}(x,\xi,\xi'), \quad \Theta \in \mathcal{S}^r, \quad r \in [0,1), \quad \mathbf{b} \in \mathcal{S}^m. \tag{2.7}$$

Such operators reduce to PDO with oscillating symbols.

Proposition 2.8 *Let $\mathbf{A}$ de defined by* (2.6) *with amplitude* (2.7). *Then $\mathbf{A} = A$ where A is a PDO* (2.1) *with symbol* (2.3) *and*

$$\begin{aligned} \Phi(x,\xi) &= \Theta(x,\xi,\xi) + \Theta_1(x,\xi), \quad \Theta_1 \in \mathcal{S}^{2r-1}, \\ b(x,\xi) &= \mathbf{b}(x,\xi,\xi) + b_1(x,\xi), \quad b_1 \in \mathcal{S}^{m+r-1}. \end{aligned}$$

In particular, if $\mathbf{b}(x,\xi,\xi) = 0$, then A belongs to the class $\mathcal{C}^{m+r-1}(\Phi)$.

This result (its proof can be found in [21]) allows us to extend Propositions 2.2 and 2.6 to operators $\mathbf{A}$ defined by (2.6).

Proposition 2.9 *Let $\mathbf{A} \in \mathcal{C}^m(\Theta)$. Then the operator $\mathbf{A}$ is bounded in the space $L_2(\mathbb{R}^d)$ if $m = 0$, and it is compact if $m < 0$. Moreover, the operator $\mathbf{A}\langle x\rangle^{-m}$ is also bounded.*

2.2. The spectral representation of $H_0 = -\Delta$ can be constructed in terms of the operator $\Gamma_0(\lambda)$ defined by (1.11). Let $\mathfrak{N} = L_2(\mathbb{S}^{d-1})$ and $(\mathcal{U}f)(\lambda) = \Gamma_0(\lambda)f,\ f \in \mathcal{S}(\mathbb{R}^d),\ \lambda > 0$. Then the operator $\mathcal{U}$ extends by continuity to a unitary operator $\mathcal{U} : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}_+;\mathfrak{N})$, and the operator H_0 is diagonalized by $\mathcal{U}$, that is $(\mathcal{U}H_0f)(\lambda) = \lambda(\mathcal{U}f)(\lambda)$ for $f \in \mathcal{D}(H_0)$.

The operator $\Gamma_0(\lambda)$ is directly related to the spectral family $E_0(\lambda)$ of the operator H_0 and to its resolvent $R_0(z) = (H_0 - z)^{-1}$. Put

$$\delta_\varepsilon(H_0 - \lambda) = (2\pi i)^{-1}(R_0(\lambda + i\varepsilon) - R_0(\lambda - i\varepsilon)).$$

For any $f_1, f_2 \in \mathcal{S}$ the limit and the derivative below exist and satisfy (see, e.g., [19])

$$\lim_{\varepsilon\to 0} (\delta_\varepsilon(H_0 - \lambda)f_1, f_2) = d\,(E_0(\lambda)f_1, f_2)\,/d\lambda = (\Gamma_0(\lambda)f_1, \Gamma_0(\lambda)f_2)_{\mathfrak{N}}\,.$$

We recall the Sobolev trace theorem.

Proposition 2.10 *For $p > 1/2$ the operator-function $\Gamma_0(\lambda)\langle x\rangle^{-p} : \mathcal{H} \to \mathfrak{N}$ is compact and Hölder continuous in norm with respect to the parameter $\lambda > 0$.*

For construction of the SM, we have to give a precise definition of the operator $\mathbf{A}^\flat(\lambda) := \Gamma_0(\lambda)\mathbf{A}\Gamma_0^*(\lambda)$ appearing in (1.10) which imposes rather stringent conditions on $\mathbf{A}$. Practically we consider only PDO $\mathbf{A}$ defined by (2.6), although the definition below is of abstract nature. Note that, for a PDO $\mathbf{A}$, the operator $\mathbf{A}^\flat(\lambda)$ is also defined as a PDO on the unit sphere $\mathbb{S}^{d-1}$. It is convenient to define the operator $\mathbf{A}^\flat(\lambda)$ in terms of its sesquilinear form $(\mathbf{A}^\flat(\lambda)g_1, g_2)$.

Definition 2.11 *Let $g_1, g_2 \in C^\infty(\mathbb{S}^{d-1})$, and let $\psi_1, \psi_2 \in C_0^\infty(\mathbb{R}_+)$ be such that $\psi_1(k) = \psi_2(k) = 1$. Set $\hat{f}_j(\xi) = g_j(\hat{\xi})\psi_j(|\xi|),\ j = 1, 2$. Then*

$$(\mathbf{A}^\flat(\lambda)g_1, g_2)_{\mathfrak{N}} := 2k^{2-d} \lim_{\varepsilon\to 0} (\mathbf{A}\delta_\varepsilon(H_0 - \lambda)f_1, \delta_\varepsilon(H_0 - \lambda)f_2)\,, \tag{2.8}$$

provided this limit exists for all $g_1, g_2 \in C^\infty(\mathbb{S}^{d-1})$.

By definition (1.11), if $g \in C^\infty(\mathbb{S}^{d-1}), \psi \in C_0^\infty(\mathbb{R}_+)$ and $\hat{f}(\xi) = g(\hat{\xi})\psi(|\xi|)$, then

$$(\Gamma_0(\lambda)f)\,(\omega) = 2^{-1/2}k^{(d-2)/2}\psi(k)g(\omega), \quad k = \lambda^{1/2}.$$

The operator $\Gamma_0^*(\lambda)$ formally adjoint to $\Gamma_0(\lambda)$ is defined, e.g., for $g \in C^\infty(\mathbb{S}^{d-1})$ by the equality

$$(\Gamma_0^*(\lambda)g)(x) = 2^{-1/2}k^{(d-2)/2}(2\pi)^{-d/2}\int_{\mathbb{S}^{d-1}} e^{ik<\omega,x>}g(\omega)d\omega, \quad k = \lambda^{1/2} > 0. \tag{2.9}$$

Therefore Definition 2.11 gives an exact meaning to the formal expression $\Gamma_0(\lambda)\mathbf{A}\Gamma_0^*(\lambda)$.

If $B = \mathcal{F}\mathbf{A}\mathcal{F}^*$ is an integral operator with kernel $B(\xi, \xi')$ which is continuous near the surface $|\xi| = |\xi'| = k$, then, by (1.11) and (2.9), $\mathbf{A}^\flat(\lambda)$ is also an integral operator with kernel

$$b(\omega, \omega'; \lambda) = 2^{-1}k^{d-2}B(k\omega, k\omega').$$

If $\mathbf{A}$ is a PDO (2.6) with amplitude $\mathbf{a} \in \mathcal{S}^m_{\rho,\delta,\delta}$, then the kernel

$$B(\xi,\xi') = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i<\xi'-\xi,x>} \mathbf{a}(x,\xi,\xi') dx$$

(understood as an oscillatory integral) of the operator $B = \mathcal{F}\mathbf{A}\mathcal{F}^*$ is well defined and infinitely differentiable off the diagonal $\xi = \xi'$.

Thus, we have the following obvious assertion.

Proposition 2.12 *Let $\mathbf{a} \in \mathcal{S}^m_{\rho,\delta,\delta}$ for some $\rho > 0$, $\delta < 1$, and let χ_1 and χ_2 be multiplication operators by $C^\infty(\mathbb{S}^{d-1})$-functions with disjoint supports. Then $(\chi_1 \mathbf{A} \chi_2)^\flat(\lambda)$ is an integral operator with C^∞-kernel of variables ω, ω' (and $\lambda > 0$).*

However due to a possible strong singularity of the function $B(\xi,\xi')$ at $\xi = \xi'$, the operator $\mathbf{A}^\flat(\lambda)$ exists only under special assumptions on $\mathbf{a}(x,\xi,\xi')$. In the case $\mathbf{a} \in \mathcal{C}^m(\Theta)$, $m < -1$, the diagonal singularity of $B(\xi,\xi')$ is weak, so no additional restrictions are required.

Proposition 2.13 *Let $\mathbf{A}$ belong to the class $\mathcal{C}^m(\Theta)$ where $m < -1$. Then $\mathbf{A}^\flat(\lambda)$ is well-defined for all $\lambda > 0$ and is a compact operator in $\mathfrak{N}$. Moreover, $\mathbf{A}^\flat(\lambda)$ depends norm continuously on λ.*

Proof. – Indeed,

$$\mathbf{A}^\flat(\lambda) = \Big(\Gamma_0(\lambda) \langle x \rangle^{-p}\Big) \Big(\langle x \rangle^p \mathbf{A} \langle x \rangle^p\Big) \Big(\langle x \rangle^{-p} \Gamma_0^*(\lambda)\Big), \quad p = m/2.$$

Since $p > 1/2$, Proposition 2.10 applies to the first and third factors. The second factor is bounded in $\mathcal{H}$ by Proposition 2.9. □

Let us now consider the general case $\mathbf{a} \in \mathcal{S}^m_{\rho,\delta,\delta}$ where m is arbitrary. It turns out (see [8, 21]) that for $m \geq -1$ the operator $\mathbf{A}^\flat(\lambda)$ exists if $\mathbf{a}(x,\xi,\xi') = 0$ for (x,ξ) belonging to a neighbourhood of the conormal bundle of the sphere $|\xi| = k$ and ξ' close to ξ. In view of Proposition 2.12, we may suppose that ξ and ξ' are close to each other or, to put it differently, we can take in Definition 2.11 functions g_1 and g_2 supported in a neighbourhood of the same point $\omega_0 \in \mathbb{S}^{d-1}$. Let Π_{ω_0} be the hyperplane orthogonal to ω_0, and let $\Omega = \Omega(\omega_0,\gamma) \subset \mathbb{S}^{d-1}$ be determined by the condition $\langle \omega, \omega_0 \rangle > \gamma > 0$. The number γ can be chosen arbitrary close to 1. We denote by $\zeta = \varkappa(\omega)$ the orthogonal projection of ω on Π_{ω_0}; in particular, we assume that $\varkappa(\omega_0) = 0$. We denote by Σ the orthogonal projection of Ω on the hyperplane Π_{ω_0} and identify points $\omega \in \Omega$ and $\zeta = \varkappa(\omega)$. The hyperplane Π_{ω_0} can be identified with $\mathbb{R}^{d-1}$. Let us also consider the unitary mapping $Z = Z_\varkappa : L_2(\Omega) \to L_2(\Sigma)$ defined by

$$(Zu)(\zeta) = (1 - |\zeta|^2)^{-1/4} u(\omega), \quad \zeta = \varkappa(\omega). \tag{2.10}$$

The next result (borrowed from [21]) gives conditions of the existence of the operator $\mathbf{A}^\flat(\lambda)$.

Proposition 2.14 *Let $\mathbf{a} \in \mathcal{S}^m_{\rho,\delta,\delta}$ for some $\rho > 0$, $\delta < 1$ with $\rho + \delta \leq 1$. Suppose that there exists $\varepsilon > 0$ such that $\mathbf{a}(x, \xi, \xi') = 0$ if the following two estimates*

$$|\xi - \xi'| < \varepsilon, \ | < \hat{x}, \hat{\xi} > | > 1 - \varepsilon \tag{2.11}$$

are satisfied. Then the operator $\mathbf{A}^\flat(\lambda)$ is well-defined in the sense of Definition 2.11 for all $\lambda > 0$. Moreover, the operator $\mathbf{A}^\flat_\varkappa(\lambda) = Z_\varkappa \mathbf{A}^\flat(\lambda) Z^*_\varkappa$ is the PDO with sesquilinear form

$$(\mathbf{A}^\flat_\varkappa(\lambda) u_1, u_2) = (2\pi)^{-d+1} \int_{\mathbb{R}^{d-1}} \int_\Sigma \int_\Sigma e^{i<\zeta - \zeta', y>} \mathbf{a}^\flat_\varkappa(y, \zeta, \zeta'; \lambda) u_1(\zeta') \overline{u_2(\zeta)} d\zeta d\zeta' dy \tag{2.12}$$

for all $u_1, u_2 \in \mathcal{C}^\infty_0(\Sigma)$. The amplitude $\mathbf{a}^\flat_\varkappa \in \mathcal{S}^{m+1}_{\rho,\delta,\delta}(\mathbb{R}^{d-1} \times \Sigma \times \Sigma)$ is given by the formula

$$\mathbf{a}^\flat_\varkappa(y, \zeta, \zeta'; \lambda) = (2\pi k)^{-1} \alpha(\zeta, \zeta') \int_{-\infty}^{\infty} \mathbf{a}\left((\omega + \omega') z - y/k, k\omega, k\omega'\right) dz \tag{2.13}$$

(this integral is taken over a finite interval due to condition (2.11)), where $k = \lambda^{1/2}$, $\zeta = \varkappa(\omega)$, $\zeta' = \varkappa(\omega')$ and

$$\alpha(\zeta, \zeta') = 2^{-1} \left(\left(\frac{1 - |\zeta|^2}{1 - |\zeta'|^2} \right)^{1/4} + \left(\frac{1 - |\zeta'|^2}{1 - |\zeta|^2} \right)^{1/4} \right). \tag{2.14}$$

Consequently, (2.12) is a smooth function of $\lambda > 0$ for all $u_1, u_2 \in \mathcal{C}^\infty_0(\Sigma)$.

Definition 2.15 *The function $\mathbf{a}^\flat_\varkappa(y, \zeta, \zeta'; \lambda)$ is called the amplitude of the operator $\mathbf{A}^\flat(\lambda)$ in the chart coordinates $(\omega_0, \varkappa)$.*

We emphasize that, from the point of view of PDO theory, in the formula (2.12) ζ plays the role of the space variable and y is the dual one.

A spectral information on PDO with oscillating amplitudes is given in the following assertion (see [21]).

Proposition 2.16 *Let $\mathbf{A} : C^\infty_0(\Sigma) \to C^\infty(\Sigma)$, $\Sigma \subset \mathbb{R}^{d-1}$, be a PDO with amplitude*

$$\mathbf{a}(y, \zeta, \zeta') = e^{i\Theta(y,\zeta,\zeta')} \mathbf{b}(y, \zeta, \zeta'), \tag{2.15}$$

where $\Theta \in \mathcal{S}^r$, $r \in (0, 1)$, $\mathbf{b} \in \mathcal{S}^0$. Suppose that for some points $y_0 \neq 0, \zeta_0 \in \Sigma$

$$|\Theta(\tau y_0, \zeta_0, \zeta_0)| \geq c\tau^r, \quad |\nabla_y \Theta(\tau y_0, \zeta_0, \zeta_0)| \geq c\tau^{r-1}, \quad c > 0, \tag{2.16}$$

for sufficiently large $\tau > 0$ and that

$$\lim_{\tau \to \infty} \mathbf{b}(\tau y_0, \zeta_0, \zeta_0) = 1. \tag{2.17}$$

Then for each $\mu \in \mathbb{T}$ and arbitrary small neighbourhood Σ_0 of ζ_0 there exists a (Weyl) sequence of functions $u_n \in C^\infty_0(\Sigma_0)$ such that

$$\|u_n\| = 1, \quad w - \lim_{n \to \infty} u_n = 0, \quad \lim_{n \to \infty} \|\mathbf{A} u_n - \mu u_n\| = 0.$$

In particular, the essential spectrum of the operator $\mathbf{A}$ in the space $L_2(\Sigma)$ covers the unit circle $\mathbb{T}$.

Remark 2.17 A simple version of Proposition 2.16 was proven and used in [20]. Namely, let $\mathbf{a} \in \mathcal{S}^0_{\rho,\delta,\delta}$ with $\rho > 1/2 > \delta$, and let representation (2.15) hold. Then the conclusion of Proposition 2.16 remains true if, for some $y_0 \neq 0, \zeta_0 \in \Sigma$, the condition (2.17) is satisfied and

$$\lim_{\tau\to\infty} \sup \Theta(\tau y_0, \zeta_0, \zeta_0) = \infty \quad \text{or} \quad \lim_{\tau\to\infty} \inf \Theta(\tau y_0, \zeta_0, \zeta_0) = -\infty. \tag{2.18}$$

3. LIMITING ABSORPTION PRINCIPLE AND RADIATIONS ESTIMATES

3.1. Let us recall some basic notions of the theory of smooth perturbations (see, e.g., [15, 19]).

Definition 3.1 *Let H be a self-adjoint operator in a Hilbert space $\mathcal{H}$, $R(z) = (H - z)^{-1}$ be the resolvent of H and K be an H-bounded operator. The operator K is called H-smooth (in the sense of Kato) if one of the two equivalent conditions is satisfied*

$$\sup_{f\in\mathcal{D}(\mathcal{H}),\|f\|=1} \int_{-\infty}^{\infty} \|Ke^{-itH} f\|^2 dt < \infty, \tag{3.1}$$

$$\sup_{\lambda\in\mathbb{R},\varepsilon>0} \|K(R(\lambda + i\varepsilon) - R(\lambda - i\varepsilon))K^*\| < \infty. \tag{3.2}$$

The following definition is essentially more flexible.

Definition 3.2 *Let $E(\cdot)$ be the spectral measure of H. Suppose that, for an H-bounded operator K, and intervals Λ_n such that $\mathbb{R} \setminus \bigcup_{n=1}^{\infty} \Lambda_n$ has the Lebesgue measure zero, the operators $KE(\Lambda_n)$ are H-smooth. Then we say that K is locally H-smooth.*

Note that condition (3.2), satisfied for $\lambda \in \Lambda$ only, implies that the operator $KE(\Lambda)$ is H-smooth.

The theory of H-smooth perturbations gives a sufficient condition for the existence of wave operators (which can easily be obtained from (3.1)).

Proposition 3.3 *Let H_0, H be a couple of self-adjoint operators, and let J be a bounded operator. Suppose that $T = HJ - JH_0$ admits the factorization*

$$T = \sum_{j=1}^{N} K_j^* K_{0,j},$$

where all operators $K_{0,j}$ are locally H_0-smooth and K_j are locally H-smooth. Denote by P_0 and P the orthogonal projections on the absolutely continuous subspaces of the operators H_0 and H, respectively. Then the wave operators

$$W_\pm(H, H_0; J) = s - \lim_{t\to\pm\infty} e^{itH} J e^{-itH_0} P_0$$

and

$$W_\pm(H_0, H; J^*) = s - \lim_{t\to\pm\infty} e^{iH_0 t} J^* e^{-iHt} P$$

exist.

Note, finally, that for an arbitrary self-adjoint operator H and a compact operator K

$$s-\lim_{|t|\to\infty} Ke^{-iHt}P = 0. \tag{3.3}$$

3.2. Let us return to the Schrödinger operator (1.1). Recall first of all that, although it is really unimportant for our results, under assumption (1.2) the operator H does not have positive eigenvalues (see [5, 18] where the proof of [15] was extended to magnetic potentials). The following well-known [1, 10] result (the limiting absorption principle) can be easily obtained by the Mourre method.

Proposition 3.4 *Let assumption* (1.2) *hold, and let* $p > 1/2$. *Then the operator-function* $\langle x\rangle^{-p} R(z) \langle x\rangle^{-p}$ *is continuous in norm with respect to the parameter* z *in the closed complex plane cut along* $[0,\infty)$ *with exception of the point* 0. *In particular, the operator* $\langle x\rangle^{-p}E(\Lambda)$ *is* H*-smooth for any compact interval* $\Lambda \subset (0,\infty)$. *The positive spectrum of* H *is absolutely continuous.*

The following resolvent estimates borrowed from [22] are called radiation estimates.

Proposition 3.5 *Let assumption* (1.2) *hold, and let the operators* $\nabla_j^{\perp}$ *be defined by* (1.7). *Set* $G_j = \langle x\rangle^{-1/2}\nabla_j^{\perp}$. *Then for all* $j = 1,\dots,d$ *the operators* $G_jE(\Lambda)$ *are* H*-smooth for any compact interval* $\Lambda \subset (0,\infty)$.

Propositions 3.4 and 3.5 are sufficient for the proof of existence and completeness of wave operators. However, our study of the SM requires additional resolvent estimates called usually propagation estimates.

Proposition 3.6 *Let assumption* (1.2) *be fulfilled. Let* $Q_\pm$ *be the PDO with symbol* $q_\pm(x,\xi) \in \mathcal{C}^m(\Phi_\pm)$ *for some* $\Phi_\pm \in \mathcal{S}^r$ *with* $r \in [0,1)$. *Suppose that* $q_\pm(x,\xi) = 0$ *for* $|\xi| \le \varepsilon$, $|x| \le \varepsilon$ *and that the support of* $q_\pm(x,\xi)$ *is contained in the cone*

$$\mp < \hat{\xi}, \hat{x} > \ge \varepsilon, \quad \hat{\xi} = \xi/|\xi|,\ \hat{x} = x/|x|,$$

for some $\varepsilon \in (0,1)$. *Then the operator-functions*

$$\langle x\rangle^{p-\sigma} Q_+^* R(z) \langle x\rangle^{-p}, \quad \langle x\rangle^{-p} R(z) Q_- \langle x\rangle^{p-\sigma}, \quad p > 1/2,\ \sigma > m+1,$$

are bounded and continuous in norm with respect to the parameter z *in the region* $\operatorname{Re} z \in (\lambda_0,\infty), \lambda_0 > 0,\ \operatorname{Im} z \ge 0$.

Proposition 3.7 *Under the assumptions of Proposition* 3.6 *the operator-function*

$$\langle x\rangle^{l} Q_+^* R(z) Q_- \langle x\rangle^{l}, \quad \forall l,$$

is bounded and continuous in norm with respect to the parameter z *in the region* $\operatorname{Re} z \in (\lambda_0,\infty), \lambda_0 > 0,\ \operatorname{Im} z \ge 0$.

These two assertions were proven in [6, 7, 11] for PDO $Q_\pm \in \mathcal{S}^m$. In the case $Q_\pm \in \mathcal{C}^m(\Phi_\pm)$ the proofs are quite similar. In particular, they rely on the Mourre estimate [10].

4. THE EIKONAL EQUATION

In this section our goal is to construct approximate eigenfunctions of the Schrödinger operator with long-range electric and magnetic potentials. It is natural to seek these eigenfunctions in the form $\Psi = e^{i\varphi}$ with a real function φ. A simple calculation shows that

$$\begin{aligned} &((i\nabla + A)^2 + V - |\xi|^2)\Psi \\ &= e^{i\varphi}(|\nabla\varphi|^2 - 2 < A, \nabla\varphi > -i\Delta\varphi + V + |A|^2 + i\mathrm{div}A - |\xi|^2), \quad \nabla = \nabla_x. \end{aligned} \tag{4.1}$$

Neglecting here the imaginary terms, which turn out to be short-range, we see that the Schrödinger equation reduces to the eikonal equation (1.4) for $\varphi = \varphi_\pm$. Considering this equation, we always remove a conical neighbourhood of the direction $\hat{x} = \mp\hat{\xi}$ and require that its right-hand side be short-range (not necessarily zero). We assume that $\varphi_\pm$ has the form (1.5) where $\Phi_\pm = o(|x|)$ as $|x| \to \infty$. Then equation (1.4) for $\varphi_\pm$ yields the equation

$$2 < \xi, \nabla\Phi_\pm(x,\xi) > + |\nabla\Phi_\pm(x,\xi)|^2 - 2 < A(x), \xi + \nabla\Phi_\pm(x,\xi) > + V(x) + |A(x)|^2 = q_\pm(x,\xi) \tag{4.2}$$

for $\Phi_\pm$ with a short-range term $q_\pm$. Thus, according to (4.1)

$$((i\nabla + A)^2 + V - |\xi|^2)(e^{i\varphi_\pm}) = e^{i\varphi_\pm}\mathrm{q}_\pm, \tag{4.3}$$

where

$$\mathrm{q}_\pm = q_\pm + i\mathrm{div}A - i\Delta\Phi_\pm \tag{4.4}$$

is again a short-range term.

Equation (4.2) can be easily solved by iterations. Let us first consider an auxiliary linear equation

$$< \xi, \nabla\phi(x,\xi) > + F(x,\xi) = 0. \tag{4.5}$$

All estimates below are uniform in ξ if $\lambda_0 \leq |\xi| \leq \lambda_1$ for some $0 < \lambda_0 < \lambda_1 < \infty$.

Lemma 4.1 *Suppose that $F = F_\pm$ satisfies for all multi-indices α, β the estimates*

$$|\partial_x^\alpha \partial_\xi^\beta F_\pm(x,\xi)| \leq C_{\alpha,\beta}(\kappa)\,(1+|x|)^{-\rho-|\alpha|}, \quad \pm < \hat{x}, \hat{\xi} > \geq \kappa,$$

for some $\rho \in (0,1)$ and any $\kappa > -1$. Then the function

$$\phi_\pm(x,\xi) = \pm\int_0^\infty \Big(F_\pm(x \pm t\xi, \xi) - F_\pm(\pm t\xi, \xi)\Big)dt \tag{4.6}$$

is a solution of equation (4.5) which satisfies for all α, β and $\kappa > -1$ the estimates

$$|\partial_x^\alpha \partial_\xi^\beta \phi_\pm(x,\xi)| \leq C_{\alpha,\beta}(\kappa)\,(1+|x|)^{1-\rho-|\alpha|}, \quad \pm < \hat{x}, \hat{\xi} > \geq \kappa.$$

The proof can be obtained by a direct differentiation of (4.6), see [20] for details.

Let us seek $\Phi_\pm = \Phi_\pm^{(N)}$ in the form

$$\Phi_\pm(x,\xi) = \sum_{n=1}^{N} \phi_\pm^{(n)}(x,\xi), \tag{4.7}$$

where the functions $\phi_\pm^{(n)}$ are solutions of equations

$$\begin{aligned} 2 < \xi, \nabla\phi_\pm^{(1)} > + V - 2 < A, \xi > &= 0, & (4.8)\\ 2 < \xi, \nabla\phi_\pm^{(2)} > + |\nabla\phi_\pm^{(1)}|^2 - 2 < A, \nabla\phi_\pm^{(1)} > + |A|^2 &= 0, & (4.9)\\ 2 < \xi, \nabla\phi_\pm^{(n)} > + \sum_{m+p=n} < \nabla\phi_\pm^{(m)}, \nabla\phi_\pm^{(p)} > -2 < A, \nabla\phi_\pm^{(n-1)} > &= 0, \ n \geq 3. & (4.10) \end{aligned}$$

Then the remainder $q_\pm(x,\xi) = q_\pm^{(N)}(x,\xi)$ defined by (4.2) satisfies the equations

$$q_\pm^{(1)} = |\nabla\phi_\pm^{(1)}|^2 - 2 < A, \nabla\phi_\pm^{(1)} > + |A|^2, \tag{4.11}$$

$$q_\pm^{(N)} = \sum_{m+p \geq N+1} < \nabla\phi_\pm^{(m)}, \nabla\phi_\pm^{(p)} > -2 < A, \nabla\phi_\pm^{(N)} >, \quad N \geq 2.$$

Using Lemma 4.1, we derive inductively estimates for functions $\phi_\pm^{(n)}$, which gives also estimates for $\Phi_\pm$ and $q_\pm$.

Proposition 4.2 *Let assumption* (1.2) *hold for some* $\rho \in (0,1)$ *such that* ρ^{-1} *is not integer and set* $N = [\rho^{-1}]$. *Then for all multi-indices* α, β *we have in the region* $\pm < \hat{x}, \hat{\xi} > \geq \kappa$ *(for any* $\kappa > -1$*)*

$$\begin{aligned} |\partial_x^\alpha \partial_\xi^\beta \phi_\pm^{(n)}(x,\xi)| &\leq C_{\alpha,\beta}(\kappa)\,(1+|x|)^{1-n\rho-|\alpha|} & (4.12)\\ |\partial_x^\alpha \partial_\xi^\beta \Phi_\pm(x,\xi)| &\leq C_{\alpha,\beta}(\kappa)\,(1+|x|)^{1-\rho-|\alpha|} & (4.13)\\ |\partial_x^\alpha \partial_\xi^\beta q_\pm(x,\xi)| &\leq C_{\alpha,\beta}(\kappa)\,(1+|x|)^{-1-\epsilon-|\alpha|}, \quad \epsilon > 0. & (4.14) \end{aligned}$$

Equality (4.4) and estimates (4.13) imply that the function $\mathrm{q}_\pm$ in the right-hand side of (4.3) obeys the same bound (4.14) as $q_\pm$.

We note that all assertions of Proposition 4.2 remain true for $\rho = 1$, except that in this case the function $\Phi_\pm(x,\xi) = \phi_\pm^{(1)}(x,\xi)$ may have a logarithmic growth as $|x| \to \infty$. Similarly, if ρ^{-1} is integer, then $N = \rho^{-1}$ and the function $\phi_\pm^{(N)}(x,\xi)$ has a logarithmic growth. The assumption that ρ^{-1} is not integer does not of course reduce the generality.

Let us finally give an explicit expression for $\Phi_\pm = \phi_\pm^{(1)}$ in the case $\rho > 1/2$

$$\Phi_\pm(x,\xi) = \pm 2^{-1} \int_0^\infty \Big(V(x \pm t\xi) - V(\pm t\xi) - 2 < A(x \pm t\xi) - A(\pm t\xi), \xi > \Big) dt. \tag{4.15}$$

5. WAVE OPERATORS

5.1. Here we consider wave operators for the pair of Hamiltonians $H_0 = -\Delta, H = (i\nabla + A)^2 + V$, acting in the space $\mathcal{H} = L_2(\mathbb{R}^d)$. We always assume condition (1.2) on A and V. Below we fix a spectral interval $\Lambda = [\lambda_0, \lambda_1] \subset \mathbb{R}_+ = (0, \infty)$. Let $\Phi_\pm$ be the function constructed in Proposition 4.2. We define the identification $J_\pm$ by the formula

$$(J_\pm f)(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i<x,\xi>+i\Phi_\pm(x,\xi)} \zeta_\pm(x,\xi) \hat{f}(\xi) d\xi, \tag{5.1}$$

where the cut-off function

$$\zeta_\pm(x,\xi) = \sigma_\pm(<\hat{\xi},\hat{x}>)\eta(x)\psi(|\xi|^2), \quad \hat{\xi} = \xi/|\xi|, \ \hat{x} = x/|x|. \tag{5.2}$$

Here $\sigma_\pm \in C^\infty$ is such that $\sigma_\pm(\tau) = 1$ near ± 1 and $\sigma_\pm(\tau) = 0$ near ∓ 1, so that it "kills" a conical neibourhood of the bad direction $\hat{x} = \mp\hat{\xi}$ where estimates (4.13) are violated. The function $\eta \in C^\infty(\mathbb{R}^d)$, such that $\eta(x) = 0$ near zero and $\eta(x) = 1$ for large $|x|$, is introduced only to get rid of the singularity of the function $\hat{x}$ at the point $x = 0$. Finally, $\psi \in C_0^\infty(\mathbb{R}_+)$ and $\psi(\lambda) = 1$ for $\lambda \in \Lambda$. Thus, our considerations are localized on a bounded disjoint from zero energy interval. Since the function $\Phi_\pm$ satisfies estimates (4.13) on the support of $\zeta_\pm$, the operator $J_\pm$ fits into the framework of PDO with oscillating symbols (see subsection 2.1). In particular, by Proposition 2.2, $J_\pm$ is a bounded operator on $\mathcal{H}$. Abusing somewhat notation, we write $J_\pm \in \mathcal{C}^0(\Phi_\pm)$.

Our first goal in this section is to show that the pair H_0, H, with identifications J_+ and J_-, fits into the framework of the theory of smooth perturbations, so the wave operators $W_\pm(H, H_0; J_\gamma)$ and $W_\pm(H_0, H; J_\gamma^*)$ exist both for $\gamma =$ "$+$" and $\gamma =$ "$-$". The structure of the operators $T_\pm = HJ_\pm - J_\pm H_0$ is described in the following

Lemma 5.1 *The operator $T_\pm$ admits the decomposition $T_\pm = T_\pm^{(s)} + T_\pm^{(r)}$ where $T_\pm^{(s)}, T_\pm^{(r)}$ are, respectively, PDO with symbols*

$$\begin{aligned} t_\pm^{(s)}(x,\xi) &= -2i\eta(x)e^{i\Phi_\pm(x,\xi)} <\xi, \nabla\sigma_\pm(<\hat{\xi},\hat{x}>)> \psi(|\xi|^2) \\ &= -2i\eta(x)|x|^{-1}e^{i\Phi_\pm(x,\xi)}(1- <\hat{\xi},\hat{x}>^2)\sigma'_\pm(<\hat{\xi},\hat{x}>)|\xi|\psi(|\xi|^2) \in \mathcal{C}^{-1}(\Phi_\pm) \end{aligned} \tag{5.3}$$

and $t_\pm^{(r)} \in \mathcal{C}^{-1-\epsilon}(\Phi_\pm)$ for some $\epsilon > 0$.

Proof. – According to (4.3) and (5.1), we have that

$$(T_\pm f)(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i<x,\xi>+i\Phi_\pm(x,\xi)} \tau_\pm(x,\xi) \hat{f}(\xi) d\xi,$$

where

$$\tau_\pm(x,\xi) = \mathrm{q}_\pm(x,\xi)\zeta_\pm(x,\xi) - 2i <\xi + \nabla\Phi_\pm(x,\xi) - A(x), \nabla\zeta_\pm(x,\xi)> -\Delta\zeta_\pm(x,\xi). \tag{5.4}$$

We single out here the term

$$\tau_\pm^{(s)}(x,\xi) = -2i\eta(x) <\xi, \nabla\sigma_\pm(<\hat{\xi},\hat{x}>)> \psi(|\xi|^2)$$

decaying as $|x|^{-1}$. According to (1.2), (4.13) and (4.14) all other terms belong to the class $\mathcal{S}^{-1-\epsilon}$. Thus, we define $t_\pm^{(s)}$ and $t_\pm^{(r)}$ by the equalities

$$t_\pm^{(s)}(x,\xi) = e^{i\Phi_\pm(x,\xi)}\tau_\pm^{(s)}(x,\xi),\ \ \tau_\pm^{(r)}(x,\xi) = \tau_\pm(x,\xi) - \tau_\pm^{(s)}(x,\xi),\ \ t_\pm^{(r)}(x,\xi) = e^{i\Phi_\pm(x,\xi)}\tau_\pm^{(r)}(x,\xi).$$

Then $t_\pm = t_\pm^{(s)} + t_\pm^{(r)}$. $\square$

Our study of the singular part $T_\pm^{(s)}$ relies on

Lemma 5.2 *Let T be a PDO with symbol*

$$t(x,\xi) = g(x,\xi) w(< \hat{x}, \hat{\xi} >) \eta(x) \psi(|\xi|^2),$$

where $g \in \mathcal{C}^{-1}(\Phi), \Phi \in \mathcal{S}^{1-\rho},\ \rho > 0,\ w \in C^\infty$ and $w(\pm 1) = 0$. Let $G_j, j = 1, \dots, d$, be the operators defined in Proposition 3.5. *Then T admits the representation* (1.8), *where $p = (1+\rho)/2$ and the operators $B^{(s)}, B^{(r)}$ are bounded.*

Proof. – Let $B^{(s)}$ and $T_j, j = 1, \dots, d$, be the PDO with symbols

$$\begin{aligned} b^{(s)}(x,\xi) &= \langle x \rangle\ |\xi|^{-2} (1 - < \hat{\xi}, \hat{x} >^2)^{-1} t(x,\xi), \\ t_j(x,\xi) &= \langle x \rangle^{-1} (\xi_j - |x|^{-2} x_j < \xi, x >)^2 b^{(s)}(x,\xi). \end{aligned}$$

The function $(1-\tau^2)^{-1} w(\tau)$ is C^∞, so that $b^{(s)} \in \mathcal{C}^0(\Phi)$ and hence, by Proposition 2.2, the operator $B^{(s)}$ is bounded. Since G_j is the PDO with symbol

$$g_j(x,\xi) = i \langle x \rangle^{-1/2} (\xi_j - |x|^{-2} x_j < \xi, x >)$$

and $t_j = |g_j|^2 b^{(s)}$, we get, by Proposition 2.7 where $m = 0$, $n = -1/2$, $r = 1 - \rho$, that the operator

$$B^{(r)} = \sum_{j=1}^{d} \langle x \rangle^p \left(G_j^* B^{(s)} G_j - T_j \right) \langle x \rangle^p$$

is bounded for $2p = \rho + 1$. It remains to remark that $t = \sum_{j=1}^d t_j$. $\square$

Now it is easy to obtain representation (1.8).

Proposition 5.3 *The operators $T_\pm$ admit representation* (1.8) *for $p > 1/2$ and some bounded operators $B_\pm^{(r)}, B_\pm^{(s)}$.*

Proof. – Let us proceed from Lemma 5.1. Since $t_\pm^{(s)} \in \mathcal{C}^{-1}(\Phi_\pm)$ and $\sigma_\pm'(\tau) = 0$ in neighbourhoods of the points -1 and 1, Lemma 5.2 can be directly applied to the operators $T_\pm^{(s)}$ for $p_s = (1+\rho)/2$. Since $t_\pm^{(r)} \in \mathcal{C}^{-1-\epsilon}(\Phi_\pm)$, the operator $\langle x \rangle^p T_\pm^{(r)} \langle x \rangle^p$ is bounded for $p_r = (1+\epsilon)/2$ by Proposition 2.6. This leads to (1.8) with $p = \min\{p_s, p_r\} > 1/2$. $\square$

It follows from Propositions 3.4 and 3.5 that the triples $\{H_0, H, J_+\}$ and $\{H_0, H, J_-\}$ satisfy the assumptions of Proposition 3.3. This yields

Theorem 5.4 *Suppose that A and V satisfy* (1.2). *Let $J_\pm = J_\pm(\zeta_\pm)$ be defined by* (5.1). *Then all wave operators*

$$W_\pm(H, H_0; J_\pm), \quad W_\pm(H_0, H; J^*_\pm) \tag{5.5}$$

and

$$W_\pm(H, H_0; J_\mp), \quad W_\pm(H_0, H; J^*_\mp) \tag{5.6}$$

exist. Operators (5.5) *as well as* (5.6) *are adjoint to each other.*

Corollary 5.5 *The wave operators satisfy the intertwining property, that is*

$$W_\pm(H, H_0; J_\gamma)E_0(X) = E(X)W_\pm(H, H_0; J_\gamma)$$

for all Borel sets $X \subset \mathbb{R}$ and both signs of γ. In particular,

$$\mathrm{Ran}\,(W_\pm(H, H_0; J_\gamma)E_0(X)) \subset E(X)\mathcal{H}. \tag{5.7}$$

5.2. Now we are able to prove the isometricity and completeness of the wave operators $W_\pm(H, H_0; J_\pm)$.

Lemma 5.6 *Let A be a PDO with symbol $a \in \mathcal{S}^0$ such that $a(x,\xi) = 0$ in some conical neighbourhood of the direction $\hat{x} = \pm\hat{\xi}$ at least for sufficiently large $|x|$. Then*

$$s - \lim_{t\to\pm\infty} Ae^{-iH_0 t} = 0.$$

Proof. – The stationary point $\xi = x/(2t)$ of the integral

$$(Ae^{-iH_0 t}f)(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i<\xi,x>-i|\xi|^2 t} a(x,\xi)\psi^2(|\xi|^2))\hat{f}(\xi)d\xi \tag{5.8}$$

does not belong to the support of the function $a(x,\xi)$ if $t \to \pm\infty$. Here we have taken into account that according to (3.3) values of $a(x,\xi)$ for bounded x can be neglected. Therefore supposing that $\hat{f} \in C_0^\infty(\mathbb{R}^d)$ and integrating by parts, we estimate integral (5.8) by $C_N(1 + |x| + |t|)^{-N}$ for arbitrary N. □

Lemma 5.7 *The following relations hold*

$$s - \lim_{t\to\pm\infty} (J^*_\pm J_\pm - \psi^2(H_0))e^{-iH_0 t} = 0, \tag{5.9}$$

$$s - \lim_{t\to\pm\infty} J^*_\mp J_\mp e^{-iH_0 t} = 0. \tag{5.10}$$

Proof. – By Proposition 2.3, up to compact terms (which are negligible according to (3.3)), $A_1 = J^*_\pm J_\pm - \psi^2(H_0)$ and $A_2 = J^*_\mp J_\mp$ are PDO with symbols $\zeta^2_\pm(x,\xi) - \psi^2(|\xi|^2)$ and $\zeta^2_\mp(x,\xi)$, respectively. Therefore Lemma 5.6 can be applied to both operators A_1 and A_2. □

Proposition 5.8 *Under the assumptions of Theorem* 5.4*, the operators* $W_\pm(H,H_0;J_\pm)$ *are isometric on the subspace* $E_0(\Lambda)\mathcal{H}$ *and*

$$W_\pm(H,H_0;J_\mp)=0, \quad W_\pm(H_0,H;J_\mp^*)=0. \tag{5.11}$$

Proof. – The results on the operators $W_\pm(H,H_0;J_\pm)$ and $W_\pm(H,H_0;J_\mp)$ are immediate consequences of (5.9) and (5.10), respectively. The second equality (5.11) is a consequence of the first because $W_\pm(H_0,H;J_\mp^*)=W_\pm^*(H,H_0;J_\mp)$. □

Proposition 5.9 *The operators* $W_\pm(H,H_0;J_\pm(\zeta_\pm))$ *do not depend on the choice of functions* $\sigma_\pm$ *and* η *in the definition* (5.2) *provided they satisfy the assumptions formulated at the beginning of this section.*

Proof. – Let $\sigma_\pm,\eta$ and $\tilde\sigma_\pm,\tilde\eta$ be a couple of such functions, and let $\zeta_\pm,\tilde\zeta$ be defined by (5.2). Set $L_\pm=J_\pm(\zeta_\pm)-J_\pm(\tilde\zeta_\pm)$. It suffices to check that $L_\pm e^{-H_0t}\to 0$ strongly as $t\to\pm\infty$ or that

$$s-\lim_{t\to\pm\infty} L_\pm^* L_\pm e^{-iH_0t}=0.$$

By Proposition 2.3, up to a compact term, $A_\pm = L_\pm^* L_\pm$ is the PDO with symbol $(\zeta_\pm(x,\xi)-\tilde\zeta_\pm(x,\xi))^2$, and hence Lemma 5.6 can be applied to both operators $A_\pm$. □

Our main result in this section is the following

Theorem 5.10 *Under the assumptions of Theorem* 5.4*, the asymptotic completeness holds*:

$$\mathrm{Ran}\,(W_\pm(H,H_0;J_\pm)E_0(\Lambda))=E(\Lambda)\mathcal{H}.$$

Proof. – By virtue of (5.7) for $X=\Lambda$, it suffices to check that for any $f\in E(\Lambda)\mathcal{H}$ there exists $f_0^{(\pm)}\in E_0(\Lambda)\mathcal{H}$ such that $f=W_\pm(H,H_0;J_\pm)f_0^{(\pm)}$ or, to put it differently, that

$$\lim_{t\to\pm\infty}||e^{-iHt}f-J_\pm e^{-iH_0t}f_0^{(\pm)}||=0. \tag{5.12}$$

Let us set $f_0^{(\pm)}=W_\pm(H_0,H;J_\pm^*)f$. Then, by the definition of this wave operator,

$$\lim_{t\to\pm\infty}||J_\pm^* e^{-iHt}f-e^{-iH_0t}f_0^{(\pm)}||=0$$

and hence

$$\lim_{t\to\pm\infty}||J_\pm J_\pm^* e^{-iHt}f-J_\pm e^{-iH_0t}f_0^{(\pm)}||=0. \tag{5.13}$$

The second equality (5.11) implies that $\lim_{t\to\pm\infty}||J_\mp^* e^{-iHt}f||=0$ and, consequently,

$$\lim_{t\to\pm\infty}||J_\mp J_\mp^* e^{-iHt}f||=0. \tag{5.14}$$

By Proposition 2.3, up to a compact term, $J_\pm J_\pm^*+J_\mp J_\mp^*$ is the PDO with symbol $\zeta_\pm^2(x,\xi)+\zeta_\mp^2(x,\xi)$. According to Proposition 5.9, we may assume that the functions $\sigma_\pm$ in (5.2) satisfy the condition $\sigma_+^2(\tau)+\sigma_-^2(\tau)=1$. Then it follows from (5.13) and (5.14) that

$$\lim_{t\to\pm\infty}||\psi^2(H_0)e^{-iHt}f-J_\pm e^{-iH_0t}f_0^{(\pm)}||=0.$$

Since the operator $\psi^2(H) - \psi^2(H_0)$ is compact and $\psi^2(H)f = f$, this yields (5.12). $\Box$

5.3. Here we find the asymptotics of the function

$$(J_\pm e^{-iH_0t}f)(x) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{i<x,\xi>+i\Phi_\pm(x,\xi)-i|\xi|^2t}\zeta_\pm(x,\xi)\hat{f}(\xi)d\xi, \quad f \in \mathcal{S}(\mathbb{R}^d), \quad (5.15)$$

as $t \to \pm\infty$. This will allow us to show that $W_\pm(H, H_0; J_\pm)$ coincide with wave operators defined (see [22]) in terms of a time-dependent modification of the free dynamics e^{-iH_0t}.

Stationary points $\omega(x,t)$ of the phase function in (5.15) are determined by the equation

$$x + (\nabla_\xi\Phi_\pm)(x, \omega(x,t)) - 2\omega(x,t)t = 0, \quad \pm t > 0. \quad (5.16)$$

Due to the function $\zeta_\pm(x,\xi)$ we are interested only in points $\omega = \omega(x,t)$ such that $|\omega|^2 \in \operatorname{supp}\psi$ and $\pm < \hat{\omega}, \hat{x} >\geq \kappa$ for some $\kappa > -1$. Using estimate (4.13) on $\nabla_\xi\Phi_\pm$, we see that for large $|t|$ equation (5.16) has a unique solution $\omega(x,t)$ and

$$\omega(x,t) = (2t)^{-1}x + O(|t|^{-\rho}) \quad (5.17)$$

uniformly in x such that

$$c_1|t| \leq |x| \leq c_2|t|, \quad 0 < c_1 < c_2 < \infty. \quad (5.18)$$

Let us set

$$\Xi(x,t) =< \omega(x,t), x > +\Phi_\pm(x, \omega(x,t)) - |\omega(x,t)|^2t, \quad \pm t > 0. \quad (5.19)$$

Applying the stationary phase method to integral (5.15) and taking into account the equality $\sigma_\pm(< \hat{\omega}(x,t), \hat{x} >) = 1$ for sufficiently large $\pm t$, we find that

$$(J_\pm e^{-iH_0t}f)(x) = e^{\mp\pi id/4}e^{i\Xi(x,t)}(2|t|)^{-d/2}\psi(|\omega(x,t)|^2)\hat{f}(\omega(x,t)) + \mathrm{r}_\pm(x,t), \quad (5.20)$$

where $\mathrm{r}_\pm(x,t)$ tends to zero in $L_2(\mathbb{R}^d)$ as $t \to \pm\infty$. Using equation (5.16), we can rewrite expression (5.19) as

$$\Xi(x,t) = |x|^2/(4t) + \Omega(x,t), \quad (5.21)$$

where

$$\Omega(x,t) = \Phi_\pm(x, \omega(x,t)) - (4t)^{-1}|(\nabla_\xi\Phi_\pm)(x, \omega(x,t))|^2, \quad \pm t > 0. \quad (5.22)$$

It follows from (5.17) that in equation (5.20) $\omega(x,t)$ can be replaced by $x/(2t)$. This gives us the following

Lemma 5.11 *Suppose that $|t|$ is sufficiently large. Let $\omega(x,t)$ satisfy equation* (5.16), *and let $\Xi(x,t)$ be defined by formulas* (5.21), (5.22). *Define a family of unitary operators by the equality*

$$(U_0(t)f)(x) = e^{\mp\pi di/4}e^{i\Xi(x,t)}(2|t|)^{-d/2}\hat{f}((2t)^{-1}x), \quad \pm t > 0. \quad (5.23)$$

Then for any function $\zeta_\pm$ defined by (5.2)

$$\lim_{t\to\pm\infty} ||J_\pm(\zeta_\pm)e^{-iH_0t}f - U_0(t)\psi(H_0)f|| = 0, \quad \forall f \in L_2(\mathbb{R}^d).$$

Lemma 5.11 allows us to reformulate the results of Theorems 5.4 and 5.10 as

Proposition 5.12 *Suppose that A and V satisfy* (1.2). *Define $U_0(t)$ by equality* (5.23) *for sufficiently large $|t|$. Then the wave operators*

$$\mathbf{W}_\pm = s - \lim_{t\to\pm\infty} e^{iHt} U_0(t) \tag{5.24}$$

exist, are isometric and Ran $\mathbf{W}_\pm = P\mathcal{H}$. *Furthermore, for any function* (5.2) *satisfying the assumptions formulated at the beginning of this section*

$$\mathbf{W}_\pm \psi(H_0) = W_\pm(H, H_0; J_\pm(\zeta_\pm)).$$

Proposition 5.12 shows again that the operators $W_\pm(H, H_0; J_\pm(\zeta_\pm))$ do not depend on the choice of functions $\sigma_\pm$ and η in the definition (5.2).

Remark 5.13 If $\Xi(x,t) = |x|^2/(4t)$ in (5.23), then

$$\lim_{t\to\pm\infty} ||e^{-iH_0t} f - U_0(t) f|| = 0, \quad \forall f \in L_2(\mathbb{R}^d),$$

and hence the wave operators (5.24) coincide with

$$W_\pm(H, H_0) = s - \lim_{t\to\pm\infty} e^{iHt} e^{-iH_0t}.$$

If $\rho > 1/2$ then the function (5.22) can be replaced by a simpler expression. Recall first of all that in this case $\Phi_\pm$ is defined by formula (4.15). It follows from (4.13) and (5.17) that in the region (5.18) the functions

$$(4t)^{-1} |\nabla_\xi \Phi_\pm(x, \omega(x,t))|^2 = O\left(|t|^{1-2\rho}\right), \quad \Phi_\pm(x, \omega(x,t)) - \Phi_\pm(x, x/(2t)) = O\left(|t|^{1-2\rho}\right)$$

tend to zero as $|t| \to \infty$. Thus, in place of (5.22) we can set $\Omega(x,t) = \Phi_\pm(x, x/(2t))$ which, according to (4.15), yields the expression

$$\Omega(x,t) = -t \int_0^1 V(sx) ds + \int_0^1 < A(sx), x > ds.$$

This allows us to simplify considerably the expression for the operators $J_\pm$ in the case $V = 0$.

Proposition 5.14 *Suppose that $V = 0$ and A satisfies assumption* (1.2) *with $\rho > 1/2$. Set*

$$(Jf)(x) = \exp\Big(i \int_0^1 < A(sx), x > ds\Big) f(x).$$

Then the wave operators $W_\pm(H, H_0; J)$ exist and coincide (*up to the factor $\psi(H_0)$*) *with operators* (1.3). *If additionally the transversal gauge condition* (1.9) *is satisfied, then $J = I$, so that the same conclusions hold for the usual wave operators $W_\pm(H, H_0)$.*

Remark 5.15 All the results of this section remain true if $V = V_0 + V_1$ and $A = A_0 + A_1$ where V_0 and A_0 satisfy assumption (1.2) and

$$\left.\begin{array}{lcl} |V_1(x)| & \leq & C(1+|x|)^{-\rho_1}, \\ |A_1(x)| + |\mathrm{div}\, A_1(x)| & \leq & C(1+|x|)^{-\rho_1} \end{array}\right\} \tag{5.25}$$

for some $\rho_1 > 1$. In this case the identifications $J_\pm$ can be constructed in terms of the functions V_0 and A_0 only.

Indeed, by multiplication theorem for wave operators, it suffices to check that the usual wave operators for the pair $H(V_0, A_0)$, $H(V, A)$ exist and are complete. Under condition (5.25) this follows from the limiting absorption principle (Proposition 3.4) satisfied by both operators $H(V_0, A_0)$ and $H(V, A)$.

5.4. The direct definition of wave operators (5.24) in [22] used the function (5.21) with $\Omega(x, t)$ such that

$$\begin{aligned} \Omega_t(x,t) + t^{-1} < x, (\nabla\Omega)(x,t) > + |(\nabla\Omega)(x,t)|^2 \\ - < A(x), x/t + 2\nabla\Omega(x,t) > + V(x) + |A(x)|^2 = O(|t|^{-1-\varepsilon}), \quad \varepsilon > 0, \end{aligned} \tag{5.26}$$

and

$$\partial_x^\alpha \Omega(x,t) = O(|t|^{1-\varepsilon-|\alpha|}), \quad |\alpha| = 1, 2, \tag{5.27}$$

in the region (5.18). These conditions allow to verify with the help of the Cook method the existence of wave operators (5.24). Of course, it is natural to expect that the functions $\Omega(x, t)$ defined in the previous subsection and in [22] coincide. Let us check it by a direct calculation.

Proposition 5.16 *Let $\omega(x, t)$ satisfy equation* (5.16) *and let $\Omega(x, t)$ be defined by formula* (5.22). *Then $\Omega(x, t)$ satisfies the conditions* (5.26) *and* (5.27).

Proof.– To simplify notation, we omit variables (x, t) and indices "$\pm$". Differentiating (5.22), we find that

$$\Omega_t = (2t)^{-2} |\nabla_\xi \Phi|^2 + \sum_j \Phi_{\xi_j} \Big(\partial\omega_j/\partial t - (2t)^{-1} \sum_k \Phi_{\xi_j \xi_k} \partial\omega_k/\partial t \Big), \tag{5.28}$$

$$\Omega_{x_i} = \Phi_{x_i} + \sum_j \Phi_{\xi_j} \Big(\partial\omega_j/\partial x_i - (2t)^{-1} (\Phi_{\xi_j x_i} + \sum_k \Phi_{\xi_j \xi_k} \partial\omega_k/\partial x_i) \Big). \tag{5.29}$$

Differentiating equation (5.16), we see that

$$\partial\omega_j/\partial t - (2t)^{-1} \sum_k \Phi_{\xi_j \xi_k} \partial\omega_k/\partial t = -2^{-1} t^{-2} x_j - 2^{-1} t^{-2} \Phi_{\xi_j} \tag{5.30}$$

and

$$\partial\omega_j/\partial x_i - (2t)^{-1} (\Phi_{\xi_j x_i} + \sum_k \Phi_{\xi_j \xi_k} \partial\omega_k/\partial x_i) = (2t)^{-1} \delta_{ij}, \tag{5.31}$$

where $\delta_{ii} = 1$ and $\delta_{ij} = 0$ if $i \neq j$. Comparing (5.28) and (5.30) we obtain that

$$\Omega_t = -2^{-1} t^{-2} < x, \nabla_\xi \Phi > -(2t)^{-2} |\nabla_\xi \Phi|^2.$$

Similarly, comparing (5.29) and (5.31), we obtain that

$$\nabla_x \Omega = \nabla_x \Phi + (2t)^{-1} \nabla_\xi \Phi. \tag{5.32}$$

These equalities imply that

$$\begin{aligned} \Omega_t + t^{-1} < x, \nabla_x \Omega > +|\nabla_x \Omega|^2 &= t^{-1} < x + \nabla_\xi \Phi, \nabla_x \Phi > +|\nabla_x \Phi|^2 \\ &= 2 < \omega, \nabla_x \Phi > +|\nabla_x \Phi|^2, \end{aligned}$$

where at the last step we have again used equation (5.16). Comparing (5.16) and (5.32), we see also that

$$(2t)^{-1} x + \nabla_x \Omega = \omega + \nabla_x \Phi.$$

Thus, the sum (5.26) can be written as

$$2 < \omega, \nabla_x \Phi > +|\nabla_x \Phi|^2 - 2 < A, \omega + \nabla_x \Phi > +V + |A|^2.$$

It follows from equation (4.2) that this expression equals $q(x, \omega(x,t))$ which, by estimate (4.14), is of order $|t|^{-1-\epsilon}$ in the region considered. The condition (5.27) is a consequence of estimates (4.13) and (5.17). $\square$

6. THE SCATTERING MATRIX

6.1. It follows from Theorems 5.4, 5.10 and Proposition 5.8 that the scattering operator

$$\mathbf{S} = W_+^*(H, H_0; J_+) W_-(H, H_0; J_-)$$

commutes with H_0 and is unitary on the subspace $E_0(\Lambda)\mathcal{H}$. Thus, in the diagonal representation of H_0 (discussed in subsection 2.2) $\mathbf{S}$ reduces to the operator of multiplication by the operator-function $S(\lambda) : \mathfrak{N} \to \mathfrak{N}$ called the SM. It is defined for almost all $\lambda \in (0, \infty)$ and is unitary for almost all $\lambda \in \Lambda$. We suppose that $\lambda \in \Lambda$.

We need a stationary formula for the SM in the case when identifications J_+ and J_- for $t \to +\infty$ and $t \to -\infty$ are different. Below we always assume that the functions $\sigma_\pm$ in definition (5.1) of the operators $J_\pm$ obey the conditions

$$\sigma_+(\tau) = 1 \text{ for } \tau \in [-\varepsilon, 1], \quad \sigma_+(\tau) = 0 \text{ for } \tau \in [-1, -2\varepsilon] \quad \text{and} \quad \sigma_-(\tau) = \sigma_+(-\tau) \tag{6.1}$$

for some $\varepsilon \in (0, 1/2)$. Formally, the SM admits (see, e.g., [22]) the representation (1.10) with $T_\pm = HJ_\pm - J_\pm H_0$. According to Lemma 5.1, $T_\pm = T_\pm^{(s)} + T_\pm^{(r)}$, where $T_\pm^{(s)}$ has symbol $t_\pm^{(s)}$ defined by (5.3) and $T_\pm^{(r)} \in \mathcal{C}^{-1-\epsilon}(\Phi_\pm)$, $\epsilon > 0$. It follows from (5.10) that in our case $W_+(H_0, H_0; J_+^* J_-) = 0$ and hence $\mathcal{W}(\lambda) = 0$. We shall show that other terms in (1.10) satisfy Definition 2.11. Let us set

$$S_0(\lambda) = -2\pi i \Gamma_0(\lambda) J_+^* T_-^{(s)} \Gamma_0^*(\lambda), \tag{6.2}$$

$$S_1(\lambda) = -2\pi i \Gamma_0(\lambda) J_+^* T_-^{(r)} \Gamma_0^*(\lambda),$$
$$S_2(\lambda) = 2\pi i \Gamma_0(\lambda) T_+^* R(\lambda + i0) T_- \Gamma_0^*(\lambda).$$

Then $S(\lambda) = S_0(\lambda) + S_1(\lambda) + S_2(\lambda)$. We emphasize that these operators depend on the choice of the cut-off functions $\sigma_\pm$ and η in the definition of $J_\pm$, but their sum does not depend on it (see Proposition 5.9).

For the study of the operator $S_2(\lambda)$, we have to consider previously the operator-function

$$B(z) = \langle x \rangle^l T_+^* R(z) T_- \langle x \rangle^l, \quad l < (1+\epsilon)/2.$$

Lemma 6.1 *The operator-function $B(z) : \mathcal{H} \to \mathcal{H}$ is bounded and continuous in norm with respect to z in the region $\operatorname{Re} z \in (\lambda_0, \infty), \lambda_0 > 0$, $\operatorname{Im} z \geq 0$.*

Proof. – Set $p = 1 + \epsilon - l$. Then $p > l$ and $p > 1/2$. Clearly, $B = B_1 + B_2 + B_3$, where

$$B_1(z) = \left(\langle x \rangle^p T_+^{(r)} \langle x \rangle^l\right)^* \left(\langle x \rangle^{-p} R(z) \langle x \rangle^{-p}\right) \left(\langle x \rangle^p T_-^{(r)} \langle x \rangle^l\right), \tag{6.3}$$

$$\begin{aligned} B_2(z) &= \left(\langle x \rangle^p T_+^{(r)} \langle x \rangle^l\right)^* \left(\langle x \rangle^{-p} R(z) T_-^{(s)} \langle x \rangle^l\right) \\ &+ \left(\langle x \rangle^l T_+^{(s)*} R(z) \langle x \rangle^{-p}\right) \left(\langle x \rangle^p T_-^{(r)} \langle x \rangle^l\right), \end{aligned}$$

$$B_3(z) = \langle x \rangle^l T_+^{(s)*} R(z) T_-^{(s)} \langle x \rangle^l.$$

Since $T_\pm^{(r)} \in \mathcal{C}^{-1-\epsilon}(\Phi_\pm)$, it follows from Proposition 2.6 that the operators $\langle x \rangle^p T_\pm^{(r)} \langle x \rangle^l$ are bounded. The second factor in (6.3) satisfies the assumptions of Proposition 3.4. According to (5.3) and (6.1), the operator $T_\pm^{(s)} \in \mathcal{C}^{-1}(\Phi_\pm)$ and its symbol is contained in the cone $\mp < \hat{x}, \hat{\xi} > \geq \varepsilon > 0$. Therefore we can apply Proposition 3.6 to the operators

$$\langle x \rangle^{-p} R(z) T_-^{(s)} \langle x \rangle^l, \quad \langle x \rangle^l T_+^{(s)*} R(z) \langle x \rangle^{-p}$$

and Proposition 3.7 to the operator $B_3(z)$. $\square$

Now it is easy to check

Proposition 6.2 *The operators $S_1(\lambda)$ and $S_2(\lambda)$ are compact and norm-continuous in $\lambda > 0$.*

Proof. – Let $l \in (1/2, (1+\epsilon)/2)$. The operator $S_1(\lambda)$ factorizes as

$$S_1(\lambda) = 2\pi i \Big(\Gamma_0(\lambda) \langle x \rangle^{-l}\Big) \left(\langle x \rangle^{-l} J_+ \langle x \rangle^l\right)^* \left(\langle x \rangle^l T_-^{(r)} \langle x \rangle^l\right) \Big(\Gamma_0(\lambda) \langle x \rangle^{-l}\Big)^*,$$

where the operators $\langle x \rangle^{-l} J_+ \langle x \rangle^l$ and $\langle x \rangle^l T_-^{(r)} \langle x \rangle^l$ are bounded according to Proposition 2.6 and the operators $\Gamma_0(\lambda) \langle x \rangle^{-l}$ are compact and norm-continuous according to Proposition 2.10. The operator

$$S_2(\lambda) = 2\pi i \Big(\Gamma_0(\lambda) \langle x \rangle^{-l}\Big) B(\lambda + i0) \Big(\Gamma_0(\lambda) \langle x \rangle^{-l}\Big)^*$$

is well defined, compact and norm-continuous according to Proposition 2.10 and Lemma 6.1. $\square$

6.2. Let us now consider the singular term (6.2). The PDO $\mathbf{A}_0 = -2\pi i J_+^* T_-^{(s)}$ is defined by formula (2.6) with amplitude

$$\mathbf{a}_0(x,\xi,\xi') = -2\pi i \overline{j_+(x,\xi)} t_-^{(s)}(x,\xi').$$

Therefore, by equations (5.1) – (5.3),

$$\mathbf{a}_0(x,\xi,\xi') = -4\pi e^{i\Theta(x,\xi,\xi')}\sigma_+(<\hat{x},\hat{\xi}>) <\xi', \nabla\sigma_-(<\hat{x},\hat{\xi}'>)> \eta^2(x)\psi(|\xi|^2)\psi(|\xi'|^2), \quad (6.4)$$

where

$$\Theta(x,\xi,\xi') = \Phi_-(x,\xi') - \Phi_+(x,\xi) \quad (6.5)$$

and the functions $\Phi_\pm$ are defined in Proposition 4.2. Estimates (4.13) imply

Lemma 6.3 *The function $\Theta(x,\xi,\xi')$ satisfies the estimates*

$$|\partial_x^\alpha \partial_{\xi,\xi'}^\beta \Theta(x,\xi,\xi')| \leq C_{\alpha,\beta}(\kappa)(1+|x|)^{1-\rho-|\alpha|} \quad (6.6)$$

in the region $<\hat{x},\hat{\xi}>\geq \kappa$, $-<\hat{x},\hat{\xi}'>\geq \kappa$ for any $\kappa > -1$ and all α,β.

Since Θ satisfies the estimates of the class $\mathcal{S}^{1-\rho}$ on the support of $\mathbf{a}_0$, (6.4) has the form (2.7) and hence the operator $\mathbf{A}_0$ fits into the framework of PDO with oscillating amplitudes (see subsection 2.1). Abusing somewhat notation, we say that $\mathbf{A}_0$ belongs to the class $\mathcal{C}^{-1}(\Theta)$. Moreover, due to the function $\nabla\sigma_-(<\hat{x},\hat{\xi}'>)$, the operator $\mathbf{A}_0$ satisfies the assumptions of Proposition 2.14. Below we use the notation and the chart coordinates $(\omega_0,\varkappa)$ introduced in subsection 2.2. Thus, we have the following

Proposition 6.4 *The operator $S_0(\lambda) = \mathbf{A}_0^\flat(\lambda)$ is well-defined in the sense of Definition 2.11 for all $\lambda > 0$. Moreover, in the chart coordinates $(\omega_0,\varkappa)$, $\mathbf{A}_0^\flat(\lambda)$ is the PDO with amplitude*

$$\mathbf{a}_0^\flat(y,\zeta,\zeta';\lambda) = (2\pi k)^{-1}\alpha(\zeta,\zeta')\int_{-\infty}^{\infty} \mathbf{a}_0\,(x,k\omega,k\omega')\,dz,$$

where α is the function (2.14), $\zeta = \varkappa(\omega)$, $\zeta' = \varkappa(\omega')$ and

$$x = (\omega+\omega')z - y/k, \quad <y,\omega_0>=0. \quad (6.7)$$

It follows from Propositions 2.12 and 2.14 that $(S_0(\lambda)g_1,g_2)$ is a smooth function of λ for $g_j \in C^\infty(\mathbb{S}^{d-1})$. Below we shall see that the operator $S_0(\lambda)$ is bounded in the space $\mathfrak{N}$.

Our next goal is to find a simple expression for a genuinely non-compact part of the operator $S_0(\lambda)$. First, we transform expression (6.4) for $\mathbf{a}_0$ to a more convenient form.

Proposition 6.5 *Let $\mathbf{A}_1$ be the PDO with amplitude*

$$\mathbf{a}_1(x,\xi,\xi') = -2\pi e^{i\Theta(x,\xi,\xi')}\sigma_+(<\hat{x},\hat{\xi}>) <\xi+\xi', \nabla\sigma_-(<\hat{x},\hat{\xi}'>)> \eta^2(x)\psi(|\xi|^2)\psi(|\xi'|^2) \quad (6.8)$$

where Θ is given by (6.5). Then the operator $\mathbf{A}_1^\flat(\lambda)$ exists and the operator $S_0(\lambda)-\mathbf{A}_1^\flat(\lambda)$ is compact on $\mathfrak{N}$. Moreover, in the chart coordinates $(\omega_0,\varkappa)$, $\mathbf{A}_1^\flat(\lambda)$ is the PDO with amplitude

$$\mathbf{a}_1^\flat(y,\zeta,\zeta';\lambda) = -\alpha(\zeta,\zeta')\int_0^\infty e^{i\Theta(x,k\omega,k\omega')} <\omega+\omega', \nabla\sigma_-(<\hat{x},\omega'>)> \eta^2(x)dz, \quad (6.9)$$

where x and y,z are related by formula (6.7).

Proof. – Comparing (6.4) and (6.8), we see that the difference $\mathbf{B} = \mathbf{A}_0 - \mathbf{A}_1$ is a PDO with amplitude

$$\mathbf{b}(x,\xi,\xi') = -2\pi e^{i\Theta(x,\xi,\xi')}\sigma_+(<\hat{x},\hat{\xi}>) < \xi'-\xi, \nabla\sigma_-(<\hat{x},\hat{\xi'}>) > \eta^2(x)\psi(|\xi|^2)\psi(|\xi'|^2).$$

Since $\mathbf{b}(x,\xi,\xi) = 0$, it follows from Proposition 2.8 that the operator $\mathbf{B}$ belongs to the class $\mathcal{C}^{-1-\rho}(\tilde{\Theta})$ for some $\tilde{\Theta} \in \mathcal{S}^{1-\rho}$. Thus, by Proposition 2.13, $\mathbf{B}^\flat(\lambda)$ is a compact operator on $\mathfrak{N}$.

It follows from conditions (6.1) that for ω' sufficiently close to ω the function $\sigma_+(<\hat{x}, \omega>) = 1$ on the support of $\nabla\sigma_-(<\hat{x},\omega'>)$. Hence σ_+ may be omitted in (6.8). We also take into account that, due to $\nabla\sigma_-(<\hat{x},\omega'>)$, the function $\mathbf{a}_1((\omega+\omega')z-y/k, k\omega, k\omega')$ is supported in the region $z \geq 0$ and $\psi(\lambda) = 1$ for $\lambda \in \Lambda$. Thus, applying Proposition 2.14 to the PDO $\mathbf{A}_1$, we obtain formula (6.9). □

An obvious drawback of this assertion is that the amplitude $\mathbf{a}_1^\flat$ contains the cut-off function σ_- although the SM does not depend on it.

6.3. Let us show that, up to a compact term, the operator $\mathbf{A}_1^\flat(\lambda)$ does not actually depend on σ_-. Below we always work in fixed chart coordinates $(\omega_0, \varkappa)$ and denote by χ a function from the class $C_0(\Omega)$. Recall, that for fixed $\omega, \omega' \in \Omega(\omega_0,\gamma)$, we always use in $\mathbb{R}^d$ the coordinates $(y,z) \in \mathbb{R}^{d-1}\times\mathbb{R}$ defined by (6.7). Then

$$k < \omega'-\omega, x > = < \omega-\omega', y > = < \zeta-\zeta', y >, \tag{6.10}$$

$$dx = k^{-d+1} < \omega+\omega', \omega_0 > dzdy \tag{6.11}$$

and

$$< \omega+\omega', \nabla f(x) > = \partial_z f(x)$$

for any differentiable function f.

Integrating by parts in (6.9) and taking into account that $\sigma_-(<\hat{x},\omega'>)$ varies from 1 to 0 as z varies from 0 to ∞, we get that

$$\mathbf{a}_1^\flat(y,\zeta,\zeta';\lambda) = \alpha(\zeta,\zeta')e^{i\Theta(-y/k,k\omega,k\omega')}\eta^2(-y/k) + \mathbf{b}(y,\zeta,\zeta';\lambda), \tag{6.12}$$

where

$$\mathbf{b}(y,\zeta,\zeta';\lambda) = \alpha(\zeta,\zeta')\int_0^\infty e^{i\Theta(x,k\omega,k\omega')}\Big(i\partial_z\Theta(x,k\omega,k\omega')\eta^2(x) + \partial_z\eta^2(x)\Big)\sigma_-(<\hat{x},\omega'>)dz. \tag{6.13}$$

By Lemma 6.3 and Proposition 2.9, the first term in the right-hand side of (6.12) is the amplitude of a bounded operator in $L_2(\mathbb{S}^{d-1})$. We shall show that $\mathbf{b}$ is the amplitude of a compact operator $\mathbf{B}$. This is obvious for the part of $\mathbf{b}$ which contains $\partial_z\eta^2$ since it is compactly supported in y. To consider the part of $\mathbf{b}$ which contains $\partial_z\Theta$, we use the following

Lemma 6.6 *The function* (6.5) *satisfies the equality*

$$< \xi+\xi', \nabla\Theta(x,\xi,\xi') > = < \nabla G(x,\xi,\xi'), F(x,\xi,\xi') > + q_0(x,\xi,\xi'), \tag{6.14}$$

where the functions G, F and q_0 are defined by

$$\begin{aligned} G(x,\xi,\xi') &= <\xi'-\xi, x> + \Theta(x,\xi,\xi'), \\ F(x,\xi,\xi') &= 2A(x) - \nabla\left(\Phi_-(x,\xi') + \Phi_+(x,\xi)\right), \\ q_0(x,\xi,\xi') &= q_-(x,\xi') - q_+(x,\xi) \end{aligned} \tag{6.15}$$

and $q_\pm$ are functions (4.2). *The functions F and q_0 satisfy in the region $<\hat{x},\hat{\xi}>\geq \kappa$, $-<\hat{x},\hat{\xi}'>\geq\kappa$, for any $\kappa>-1$ and all α,β, the estimates*

$$|\partial_x^\alpha \partial_{\xi,\xi'}^\beta F(x,\xi,\xi')| \leq C_{\alpha,\beta}(\kappa)(1+|x|)^{-\rho-|\alpha|} \tag{6.16}$$

$$|\partial_x^\alpha \partial_{\xi,\xi'}^\beta q_0(x,\xi,\xi')| \leq C_{\alpha,\beta}(\kappa)(1+|x|)^{-1-\epsilon-|\alpha|}, \quad \epsilon>0. \tag{6.17}$$

Proof. – One proceeds from eikonal equation (4.2) for $\Phi_+(x,\xi)$ and $\Phi_-(x,\xi')$ and takes their difference. This yields the equality (6.14). Estimates (6.16) and (6.17) are direct consequences of (4.13) and (4.14), respectively. □

Let us plug expression (6.14) into (6.13) and consider the amplitude

$$\mathbf{b}_0(y,\zeta,\zeta';\lambda) = i\alpha(\zeta,\zeta')k^{-1}\int_0^\infty e^{i\Theta(x,k\omega,k\omega')} <\nabla G(x,k\omega,k\omega'), F(x,k\omega,k\omega')> \eta^2(x)dz \tag{6.18}$$

corresponding to the first term in the right-side of (6.14).

Lemma 6.7 *The PDO $\mathbf{B}_0$ with amplitude* (6.18) *admits the representation $\mathbf{B}_0=\mathbf{B}_1+\mathbf{B}_2$ where $\mathbf{B}_1$, $\mathbf{B}_2$ are PDO with amplitudes*

$$\mathbf{b}_1(y,\zeta,\zeta';\lambda) = -k^{-1}\alpha_1(\zeta,\zeta')e^{i\Theta(-y/k,k\omega,k\omega')} <\omega_0, F(-y/k,k\omega,k\omega')> \eta^2(-y/k), \tag{6.19}$$

$$\mathbf{b}_2(y,\zeta,\zeta';\lambda) = \alpha(\zeta,\zeta')\int_0^\infty e^{i\Theta(x,k\omega,k\omega')} q(x,k\omega,k\omega')dz. \tag{6.20}$$

Here

$$\alpha_1(\zeta,\zeta') = <\omega+\omega',\omega_0>^{-1}\alpha(\zeta,\zeta')$$

and

$$q(x,\xi,\xi') = -k^{-1}\mathrm{div}_x\left(F(x,\xi,\xi')\sigma_-(<\hat{x},\hat{\xi}'>)\eta^2(x)\right). \tag{6.21}$$

Proof. – According to (6.10), (6.11), the PDO $\mathbf{B}_0$ has integral kernel

$$\begin{aligned} \mathrm{p}_0(\zeta,\zeta';\lambda) &= (2\pi)^{-d+1}\int_{\mathbb{R}^{d-1}} e^{i<\zeta-\zeta',y>}\mathbf{b}_0(y,\zeta,\zeta';\lambda)dy \\ &= i(2\pi)^{-d+1}k^{d-2}\alpha_1(\zeta,\zeta')\int_{<x,\omega_0>\geq 0}\exp\Big(iG(x,k\omega,k\omega')\Big) \\ &\times\ <\nabla G(x,k\omega,k\omega'), F(x,k\omega,k\omega')> \sigma_-(<\hat{x},\omega'>)\eta^2(x)dx. \end{aligned}$$

Integrating here by parts, we find that $\mathrm{p}_0=\mathrm{p}_1+\mathrm{p}_2$ where

$$\begin{aligned} \mathrm{p}_1(\zeta,\zeta';\lambda) = -\alpha_1(\zeta,\zeta')(2\pi)^{-d+1}k^{-1}\int_{\mathbb{R}^{d-1}} e^{iG(-y/k,k\omega,k\omega')} \\ \times <\omega_0, F(-y/k,k\omega,k\omega')> \eta^2(-y/k)dy, \end{aligned}$$

$$\mathrm{p}_2(\zeta,\zeta';\lambda) = \alpha_1(\zeta,\zeta')(2\pi)^{-d+1}k^{d-1}\int_{<x,\omega_0>\geq 0} e^{iG(x,k\omega,k\omega')}q(x,k\omega,k\omega')dx.$$

The amplitudes of PDO with kernels $\mathrm{p}_1,\mathrm{p}_2$ equal $\mathbf{b}_1,\mathbf{b}_2$, respectively. □

Let us consider the operators $\mathbf{B}_1$ and $\mathbf{B}_2$ separately.

Lemma 6.8 *The PDO* $\mathbf{B}_1$ *with amplitude* (6.19) *belongs to the class* $\mathcal{C}^{-\rho}(\Theta_0)$*, where*

$$\Theta_0(y,\zeta,\zeta';\lambda)=\Theta_0(-k^{-1}y,k\varkappa(\zeta),k\varkappa(\zeta'))$$

is given by (6.5)*. In particular,* $\chi\mathbf{B}_1\chi$ *is a compact operator in* $L_2(\Sigma)$*.*

Proof. – Since $<y,\omega_0>=0$, we have that $|<\hat{y},\omega>|\leq\varepsilon<1$ and $|<\hat{y},\omega'>|\leq\varepsilon<1$ for $\omega,\omega'\in\Omega$. According to (6.6), (6.16) this implies that functions Θ and F in (6.19) belong to the classes $\mathcal{S}^{1-\rho}(\mathbb{R}^{d-1})$ and $\mathcal{S}^{-\rho}(\mathbb{R}^{d-1})$, respectively. Thus, $\mathbf{b}_1$ belongs to the class $\mathcal{C}^{-\rho}(\Theta_0)$, and hence $\mathbf{B}_1$ is a compact operator by Proposition 2.9. □

Lemma 6.9 *Suppose that a function* $q(x,\xi,\xi')$ *equals zero in a conical neighbourhood of the set* $\hat{x}=\hat{\xi}'$ *and* $q(x,\xi,\xi')$ *satisfies the estimates* (6.17) *for* $<\hat{x},\hat{\xi}>\geq\kappa>-1$ *and all* α,β*. Let* $\mathbf{B}_2$ *be PDO with amplitude* (6.20)*. Then the operator* $\chi\mathbf{B}_2\chi$ *is compact in* $L_2(\Sigma)$*.*

Proof. – Let $\vartheta_0\in C^\infty(\mathbb{R})$ be such that $\vartheta_0(z)=0$ for $z\leq 1/2$ and $\vartheta_0(z)=1$ for $z\geq 1$ and $\vartheta_1=1-\vartheta_0$. We first consider the operator $\mathbf{B}_{2,0}$ with amplitude

$$\mathbf{b}_{2,0}(y,\zeta,\zeta';\lambda)=\alpha(\zeta,\zeta')\int_{-\infty}^{\infty}e^{i\Theta(x,k\omega,k\omega')}q(x,k\omega,k\omega')\vartheta_0(<\omega+\omega',\omega_0>z)dz. \quad (6.22)$$

Let $f\in C^\infty(\mathbb{R}^d)$, $f(\xi)=1$ for $\xi\in\Omega$ and $f(\xi)=0$ off a small neighbourhood of Ω. We introduce an auxiliary PDO $\mathbf{G}$ with amplitude

$$\mathbf{g}(x,\xi,\xi')=f(\xi)f(\xi')\exp(i\Theta(x,\xi,\xi'))q(x,\xi,\xi')\vartheta_0(<x,\omega_0>).$$

The function $q\vartheta_0$ equals zero in conical neighbourhoods of the sets $\hat{x}=\hat{\xi}'$ and $\hat{x}=-\hat{\xi}$, so that, by Lemma 6.3, Θ satisfies estimates (6.6) on the support of $q\vartheta_0$. Thus, $\mathbf{g}\in C^{-1-\epsilon}(\Theta)$. Comparing (2.13) and (6.22), we see that $\mathbf{B}_{2,0}(\lambda)=2\pi k\mathbf{G}^\flat(\lambda)$. Therefore the operator $\mathbf{B}_{2,0}$ is compact by Proposition 2.13.

Next we consider the PDO $\mathbf{B}_{2,1}$ with amplitude

$$\mathbf{b}_{2,1}(y,\zeta,\zeta';\lambda)=\alpha(\zeta,\zeta')e^{i\Theta(-y/k,k\omega,k\omega')}$$
$$\times\int_0^\infty e^{i(\Theta(x,k\omega,k\omega')-\Theta(-y/k,k\omega,k\omega'))}q(x,k\omega,k\omega')\vartheta_1(<\omega+\omega',\omega_0>z)dz,$$

where, due to the function ϑ_1, the integral is actually taken over a finite interval. Therefore it follows from Lemma 6.3 that for $\omega,\omega'\in\Omega$ and sufficiently large $|y|$

$$\left|\partial_y^\alpha\partial_{\zeta,\zeta'}^\beta e^{i(\Theta(x,k\omega,k\omega')-\Theta(-y/k,k\omega,k\omega'))}\right|=O\left(\langle y\rangle^{-\rho-|\alpha|}\right).$$

Taking also into account that $q\vartheta_1$ satisfies estimates (6.17), we obtain that $\mathbf{b}_{2,1}\in\mathcal{C}^{-1-\rho-\epsilon}(\Theta_0)$. Hence the operator $\chi\mathbf{B}_{2,1}\chi$ is also compact. □

Since the function (6.21) satisfies the assumptions of this lemma, it applies directly to the operator with amplitude (6.20). Moreover, the same conclusion is true if the role of $q(x,\xi,\xi')$ is played by the function $q_0(x,\xi,\xi')\sigma_-(<\hat{x},\omega'>)$, where q_0 is defined by (6.15). This yields

Proposition 6.10 *The PDO* $\mathbf{B}$ *with amplitude* (6.13) *is compact in* $L_2(\Sigma)$.

Thus, we have verified that all terms in the right-hand side of (1.10) are well defined as bounded operators in the space $\mathfrak{N}$. As shown in [22], this justifies the representation (1.10) for the SM $S(\lambda)$. Since $(S(\lambda)g_1, g_2)$ is a smooth function of λ for $g_j \in C^\infty(\mathbb{S}^{d-1})$, it follows from unitarity of $S(\lambda)$ that actually $S(\lambda)$ is a strongly continuous operator-function of $\lambda > 0$. Moreover, comparing Propositions 6.5 and 6.10, we see that, modulo compact operators, the SM $S(\lambda)$ can be considered as a PDO with amplitude given by the first term in the right-hand side of (6.12). Finally, neglecting again compact operators, we can omit in this term $\alpha(\zeta, \zeta')$ because $\alpha(\zeta, \zeta) = 1$ and $\eta^2(-y/k)$ because it is 1 for sufficiently large $|y|$.

Let us summarize the results obtained in the following theorem.

Theorem 6.11 *Let condition* (1.2) *hold and let* $J_\pm$ *be defined by* (5.1) *where the function* (5.2) *satisfies the assumption* (6.1). *Then the SM* $S(\lambda)$ *for the pair* H_0, H *with identifications* $J_\pm$ *admits representation* (1.10) *where* $\mathcal{W}(\lambda) = 0$. *The function* $S(\lambda) : \mathfrak{N} \to \mathfrak{N}$ *is strongly continuous in* $\lambda > 0$. *If* χ_1 *and* χ_2 *are multiplications by* $C^\infty(\mathbb{S}^{d-1})$*-functions with disjoint supports, then the operator* $\chi_1 S(\lambda)\chi_2$ *is compact. Finally, up to a compact term,* $S(\lambda)$ *is the PDO on* $\mathbb{S}^{d-1}$ *with amplitude given in arbitrary chart coordinates* $(\omega_0, \varkappa)$ *by the formula*

$$\mathbf{s}(y, \zeta, \zeta'; \lambda) = \exp\left(i\Theta(-\lambda^{-1/2}y, \lambda^{1/2}\omega, \lambda^{1/2}\omega')\right). \tag{6.23}$$

6.4. Using Theorem 6.11, we can now describe the spectrum of the SM. Recall that the function $\Theta(x, \xi, \xi')$ is defined by formula (6.5) where $\Phi_\pm$ are constructed in Proposition 4.2. Let us consider the first approximation $\mathbf{V}(x, \xi) = \phi_-^{(1)}(x, \xi) - \phi_+^{(1)}(x, \xi)$ to the function $\Theta(x, \xi, \xi)$. According to (4.15)

$$\mathbf{V}(x, \xi) = 2^{-1} \int_{-\infty}^{\infty} \Big(V(t\xi) - V(x + t\xi) + 2 < A(x + t\xi) - A(t\xi), \xi >\Big) dt. \tag{6.24}$$

Note that the function $\mathbf{V}$ does not depend on the projection of x on the direction of ξ. Typically, for asymptotically homogeneous functions $V(x)$ and $A(x)$ of order $-\rho$, the function $\mathbf{V}(x, \xi)$ is asymptotically homogeneous function of x as $|x| \to \infty$ of order $1 - \rho$ if $\rho \in (0, 1)$ and it has a logarithmic growth if $\rho = 1$.

Theorem 6.12 *Let condition* (1.2) *hold and suppose that for some* $x_0 \neq 0$, $\omega_0 \in \mathbb{S}^{d-1}$, $< x_0, \omega_0 >= 0$, *and* $k > 0$ *the function* (6.24) *satisfies the conditions*

$$\limsup_{\tau \to \infty} \mathbf{V}(\tau x_0, k\omega_0) = \infty \quad \text{or} \quad \liminf_{\tau \to \infty} \mathbf{V}(\tau x_0, k\omega_0) = -\infty \tag{6.25}$$

if $\rho > 1/2$ *or*

$$|\mathbf{V}(\tau x_0, k\omega_0))| \geq c\tau^{1-\rho}, \quad |\nabla \mathbf{V}(\tau x_0, k\omega_0))| \geq c\tau^{-\rho}, \quad c > 0, \tag{6.26}$$

for sufficiently large $\tau > 0$ *if* $\rho \in (0, 1/2]$. *Then the spectrum of the SM* $S(\lambda)$ *coincides for* $\lambda = k^2$ *with the unit circle* $\mathbb{T}$.

Proof. – It follows from definition (4.7) and estimates (4.12) for $n \geq 2$ that

$$|\partial_x^\alpha (\Theta(x,\xi,\xi) - \mathbf{V}(x,\xi))| \leq C_\alpha (1+|x|)^{1-2\rho-|\alpha|}, \quad < x, \xi >= 0, \quad |\alpha| = 0, 1.$$

Thus, the function $\Theta(\tau x_0, k\omega_0, k\omega_0)$ where $< x_0, \omega_0 >= 0$ also satisfies conditions (6.25) or (6.26). Let us consider the chart diffeomormism $(\omega_0, \varkappa)$ and the PDO $S_0^\varkappa(\lambda)$ with amplitude (6.23). We apply Proposition 2.16 to the operator $\chi_0^\varkappa S_0^\varkappa(\lambda)\chi_0^\varkappa$ where $\chi_0^\varkappa \in C_0^\infty(\Sigma)$ and $\chi_0^\varkappa(\zeta) = 1$ in a neighbourhood of the point $\zeta_0 = 0$. Conditions (2.16) (or (2.18)) for $\zeta_0 = 0$ and $y_0 = -x_0$ are satisfied according to (6.26) (or (6.25)). Therefore for each $\mu \in \mathbb{T}$ there exists a Weyl sequence u_n such that $||u_n|| = 1$, $u_n \to 0$ weakly and

$$\chi_0^\varkappa S_0^\varkappa(\lambda)\chi_0^\varkappa u_n - \mu u_n \to 0$$

strongly as $n \to \infty$. Moreover, we may assume that supports of u_n are contained in an arbitrary neighbourhood of the point $0 \in \Pi_{\omega_0}$. Let the operator Z be defined by formula (2.10) and $\chi_0(\omega) = \chi_0^\varkappa(\zeta)$. It follows from Theorem 6.11 that $f_n = Z^* u_n$ is a Weyl sequence for the operator $\chi_0 S(\lambda)\chi_0$. Let $\chi_1 \in C^\infty(\mathbb{S}^{d-1})$ be supported on the set where $\chi_0(\omega) = 1$. Then supp $(1-\chi_0) \cap$ supp $\chi_1 = \emptyset$ and hence, by Theorem 6.11, the operator $(1-\chi_0)S(\lambda)\chi_1$ is compact. Without loosing generality, we assume that $\chi_0 f_n = \chi_1 f_n = f_n$. Since both terms in the right-hand side of the estimate

$$||S(\lambda)f_n - \mu f_n|| \leq ||\chi_0 S(\lambda)\chi_0 f_n - \mu f_n|| + ||(1-\chi_0)S(\lambda)\chi_1 f_n||$$

tend to zero as $n \to \infty$, the spectrum of the SM $S(\lambda)$ covers the unit circle. Finally, we take into account that $S(\lambda)$ is unitary, so that its spectrum actually coincides with $\mathbb{T}$. □

We emphasize that, at least in the cases $V = 0$ or $A = 0$, the conditions (6.25) and (6.26) are fulfilled for all points k at the same time.

According to Proposition 5.14 and Theorem 6.12, the following result is true for the SM defined in terms of the usual wave operators.

Proposition 6.13 *Let $V = 0$ and let A satisfy estimates* (1.2) *for $\rho \in (1/2, 1)$ and the transversal condition* (1.9) (*at least for large $|x|$*). *Suppose that the corresponding function*

$$\mathbf{V}(x,\xi) = \int_{-\infty}^{\infty} < A(x+t\xi), \xi > dt$$

satisfies condition (6.25) *for some point (x_0, ω_0), $\omega_0 \in \mathbb{S}^{d-1}$, $< x_0, \omega_0 >= 0$. Then the spectrum of the SM $S(\lambda)$ covers the unit circle $\mathbb{T}$ for all $\lambda > 0$.*

As an example of a potential satisfying the assumptions of Proposition 6.13 we note (for $d = 2$) the potential $A(x) = \gamma |x|^{-1-\rho}(-x_2, x_1)$, $|x| \geq R$, $\rho \in (1/2, 1), \gamma \neq 0$. In this case for any $(x,\xi) \in \mathbb{R}^{2d}$, $< x, \xi >= 0$, $|x| \geq R$,

$$\mathbf{V}(x,\xi) = \pm \gamma v_0 |x|^{1-\rho}, \quad v_0 = \int_{-\infty}^{\infty} \langle u \rangle^{-1-\rho} du > 0,$$

if $\hat{\xi}$ is obtained from $\hat{x}$ by rotation at the angle $\pm\pi/2$. Thus, condition (6.25) is satisfied.

Of course, Proposition 6.13 remains true for all $\rho > 0$, but, if $\rho \leq 1/2$, then the SM is defined in terms of modified wave operators and condition (6.25) should be replaced by (6.26). On the contrary, if $\rho = 1$, then under transversal condition (1.9) $\mathbf{V}(x,\xi)$ has typically the finite limits as $|x| \to \infty$, so that the essential spectrum of $S(\lambda)$ might be a subset of $\mathbb{T}$. This is the case for the Aharonov-Bohm potential $A(x) = \gamma|x|^{-2}(-x_2, x_1)$; for more general potentials of this type, see [16].

Proposition 6.13 contradicts a preceding result of F. Nicoleau. It is claimed in [12] that under the assumptions of Proposition 6.13 the operator $S(\lambda) - I$ is compact. This is incompatible with the assertion that the essential spectrum of $S(\lambda)$ covers the unit circle.

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