GOOD LOCAL BOUNDS FOR SIMPLE RANDOM WALKS

CHRISTINE RITZMANN

Abstract. We give a local central limit theorem for simple random walks on $\mathbb{Z}^d$, including Gaussian error estimates. The detailed proof combines standard large deviation techniques with Cramér-Edgeworth expansions for lattice distributions.

A simple random walk on $\mathbb{Z}^d$ is a sequence of independent identically distributed random variables (steps) on $\mathbb{Z}^d$. We here suppose that the steplength is bounded, that is, the measure describing a single step has bounded support, and our goal is to give local approximations for the sum of the first n steps of the walk. Of course the Central Limit Theorem applies, and there also exist local CLTs, see for example Bhattacharya/Rao, [1].

In our work on local approximations for polymer measures (see the joint paper with E. Bolthausen [2]), we considered weakly self-avoiding walks as perturbations of simple walks. In order to obtain good bounds for the perturbations we thus needed good bounds for the simple walks as well. In particular, we needed Gaussian tail estimates, which were not provided by [1]. Therefore we proved a version of a local CLT, which was tailored for our application there.

It turns out that such a LCLT is useful not just in the treatment of polymer measures, but in all cases where similar perturbation methods are used and hence good local bounds for simple random walks are needed.

Therefore we present a local central limit theorem with Gaussian tail estimates for walks with bounded steplength. This restriction may exclude some possible applications, but the setting is sufficient for many purposes and it allows a precise control on the error terms. A major advantage over [1] is that we make the dependence of the constants explicit.

In order to formulate the theorem, we need some definitions and notations: Let G be the single step distribution of a random walk on $\mathbb{Z}^d$. The walk has *bounded steplength* ℓ if $G(x) = 0$ for all $|x| > \ell$. The walk is called *maximal*, if the support of G is not contained in any affine hyperplane of $\mathbb{Z}^d$. The walk is called *aperiodic*, if the greatest common divisor of all $n \in \mathbb{N}$ with $G^{*n}(0) > 0$ is one.

In the whole paper, φ_Γ denotes the density of the centered normal distribution on $\mathbb{R}^d$ with covariance matrix Γ, that is,

$$\varphi_\Gamma(x) = (2\pi)^{-d/2}(\det\Gamma)^{-1/2}\exp[-x\cdot(\Gamma^{-1}\cdot x)/2].$$

Furthermore we abbreviate $\varphi_{(\eta\cdot\mathrm{Id}_d)}$ by φ_η, that is

$$\varphi_\eta(x) = (2\pi\eta)^{-d/2}\exp\left[-x^2/(2\eta)\right].$$

Note that for $x \in \mathbb{R}^d$ we abbreviate $|x|^k$ by x^k.

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Angewandte Mathematik, Universität Zürich, Winterthurer Strasse 190, CH-8057 Zürich, Switzerland. Tel. xx41/1/635 58 54, `chritz@amath.unizh.ch`.

Local Central Limit Theorem. *Let G be the single step distribution of a maximal and aperiodic random walk on $\mathbb{Z}^d$ with mean zero and bounded steplength $\ell \in \mathbb{N}$. Let Γ denote the covariance matrix of G. Then there exist polynomials P_3 and P_6 of degree three and six, respectively, and a positive constant C such that for all x in $\mathbb{Z}^d$, $\alpha \in (0, 1/2)$ and for all natural n,*

$$\left|G^{*n}(x) - \left[1 + n^{-1/2} P_3(x/\sqrt{n}) + n^{-1} P_6(x/\sqrt{n})\right] \varphi_{n\Gamma}(x)\right| \leq C n^{-1-\alpha} \varphi_{n2d\ell^2}(x).$$

The coefficients of the polynomials depend (polynomially) on Γ^{-1} and on the moments of G up to order four, whereas C can be chosen independently of the specific law of G, depending only on d, ℓ, α and on a lower bound for the smallest eigenvalue of Γ.

Before proving the theorem, we make some comments on the assumptions. The fact that the measure has bounded support is essential for the proof and can not be given up easily. The other assumptions are only technical: By shifting the measure we can arrange zero expectation and by restricting to a subspace we achieve maximality. For periodic measures we have to adapt the inversion formula for lattice measures (10). The effect of the period is that for fixed n we have non zero measure of G^{*n} only on a sublattice of $\mathbb{Z}^d$, which leads to an additional factor in the approximating term but leaves the rest of the proof unchanged.

The proof of the LCLT is split in two parts. The first part contains the large deviation arguments. In the second part we prove the Cramér-Edgeworth expansions, which are used in the first part to approximate the measures obtained by tilting the original distribution.

Proof of the LCLT. Let

$$Z(t) \stackrel{\text{def}}{=} \sum_{x \in \mathbb{Z}^d} \exp[t \cdot x] G(x)$$

$$\text{and} \qquad I(\xi) \stackrel{\text{def}}{=} \sup_{t \in \mathbb{R}^d} \{t \cdot \xi - \log Z(t)\}.$$

Standard large deviation theory (see for example [3]) yields a large deviation principle with entropy function I for the laws of $G^{*n}(x/n)$. Let S_G denote the convex closure of the set of points with nonzero G measure. Then I is convex on $\mathbb{R}^d$ and even strictly convex on $\operatorname{int} S_G$, that is, the interior of S_G. Outside S_G, I equals $+\infty$.

The function $t \mapsto D \log Z(t)$ is an analytic diffeomorphism from $\mathbb{R}^d$ onto $\operatorname{int} S_G$ (for a proof see [3], page 261). Therefore, for any $\xi \in \operatorname{int} S_G$, there exists a unique $t_\xi \in \mathbb{R}^d$ with $D \log Z(t_\xi) = \xi$. Clearly $D \log Z(0) = 0$ and $D^2 \log Z(0) = \Gamma$. For $\xi \in \operatorname{int} S_G$, we have $I(\xi) = t_\xi \cdot \xi - \log Z(t_\xi)$. Now for $t \in \mathbb{R}^d$ denote by G_t the tilted measure

$$G_t(x) \stackrel{\text{def}}{=} \frac{G(x) \exp[t \cdot x]}{Z(t)}.$$

Using this, we see that for $\xi \stackrel{\text{def}}{=} x/n \in \operatorname{int} S_G$, we can write

$$G^{*n}(x) = \exp[-nI(\xi)] \cdot G_{t_\xi}^{*n}(x). \tag{1}$$

Case $|x| \leq n^{(5-\alpha)/9}$:

Since G is centered and maximal, the boundary of S_G is bounded away from zero. The covariance matrix Γ_ξ of G_{t_ξ} depends analytically on ξ with $\Gamma_0 = \Gamma$. Let

γ be the smallest eigenvalue of Γ. We can find a constant $\delta > 0$ such that for all $\xi \in \mathbb{R}^d$ with $|\xi| \leq \delta$ we have

- $\xi \in \operatorname{int} S_G$ and
- the smallest eigenvalue of Γ_ξ is greater or equal $\gamma/2$.

We will now argue that δ can be chosen depending only on ℓ and γ, but not on the specific measure G. There is only a finite number of possible supports for a maximal and centered measure H with steplength ℓ on $\mathbb{Z}^d$, and the convex hulls of these supports all contain zero in their interior. Therefore we find a constant δ' only depending on ℓ, such that for all $|\xi| \leq \delta'$ we have $\xi \in \operatorname{int} S_H$ for all such H.

On the other hand we consider the set $\mathcal{H}$ of all measures on $\mathbb{Z}^d$ with steplength ℓ, whose covariance matrices have smallest eigenvalue greater or equal γ. $\mathcal{H}$ is obviously compact, and the measures in $\mathcal{H}$ are all maximal (the covariance matrix of a non maximal measure has zero as an eigenvalue). The function that maps (H, ξ) to the smallest eigenvalue of the covariance matrix of H_{t_ξ} is continuous in both arguments. Hence we find a constant $\delta \leq \delta'$ such that the smallest eigenvalue of the covariance matrix of H_{t_ξ} is greater or equal $\gamma/2$ for all $H \in \mathcal{H}$ and $|\xi| \leq \delta$. This shows the claimed independence.

Now we come back to the proof of the LCLT. We know $|\xi| = |x/n| \leq n^{-(4+\alpha)/9}$ for a fixed constant $\alpha \in (0, 1/2)$. Thus for almost all n we have $|\xi| \leq \delta$. It is sufficient to prove the estimate for these n, since there is only a finite number of n and x with $\delta n < |x| \leq n^{(5-\alpha)/9}$. We can cover these cases by choosing C large enough. Moreover, using the compactness of $\mathcal{H}$ again, we see that we find C depending on d, ℓ, γ and α only.

So let $\xi \in S_G$ with $|\xi| \leq \delta \wedge n^{-(4+\alpha)/9}$. To estimate the first factor in (1), we use Taylor expansion for I at zero.

Remember $I(\xi) = t_\xi \cdot \xi - \log Z(t_\xi)$. We have $DI(\xi) = t_\xi$, and a simple computation yields $D^2 I(\xi) = \Gamma_\xi^{-1}$. In general $D^k I(\zeta)(\xi, \ldots, \xi)$ is a homogeneous polynomial of degree k, whose coefficients depend (polynomially) on Γ_ζ^{-1} and on the first to kth moments of G_ζ.

Now let $T^{(k)}$ denote a homogeneous polynomial of degree k. Using $I(0) = 0$ and $DI(0) = 0$, we obtain

$$I(\xi) = \xi \cdot (\Gamma^{-1} \cdot \xi)/2 + T^{(3)}(\xi) + T^{(4)}(\xi) + R_5(I, 0, \xi).$$

To deal with the error term R_5 observe that, since the steplength of the tilted measures is also bounded by ℓ, we can find upper bounds (depending only on d and ℓ) for their moments. Furthermore we have $|\Gamma^{-1} \cdot \xi| \leq |\xi|/\gamma$. Thus we know

$$|R_5(I, 0, \xi)| \leq \operatorname{const}(d, \ell, \gamma)|\xi|^5,$$

and hence

$$\begin{aligned} \exp[-nI(\xi)] &= \exp[-n\,\xi \cdot (\Gamma^{-1} \cdot \xi)/2] \times \exp[-nT^{(3)}(\xi) - nT^{(4)}(\xi) - nO((\xi)^5))] \\ &= \exp[-n\,\xi \cdot (\Gamma^{-1} \cdot \xi)/2] \times \big[1 - nT^{(3)}(\xi) - nT^{(4)}(\xi) + n^2 T^{(6)}(\xi) \\ &\qquad + \underbrace{O(n|\xi|^5) + O(n^2|\xi|^7) + O(n^3|\xi|^9)\exp[O(n|\xi|^3)]}_{=O(n^{-1-\alpha}),\text{ since } |\xi| \leq n^{-(4+\alpha)/9}}\big] \\ &= \exp[-x \cdot (\Gamma^{-1} \cdot x)/(2n)] \\ &\qquad \times \big[1 - n^{-1/2} T^{(3)}(x/\sqrt{n}) - n^{-1}(T^{(4)} + T^{(6)})(x/\sqrt{n}) + O(n^{-1-\alpha})\big]. \end{aligned} \tag{2}$$

The coefficients of the polynomials $T^{(k)}$ are depending polynomially on Γ^{-1} and the moments of G up to order four, and $|O(n^{-1-\alpha})| \leq \mathrm{const}(d, \ell, \gamma, \alpha)\, n^{-1-\alpha}$.

As a next step we approximate the second factor in (1), $G^{*n}_{t_\xi}(x)$. This is done in the second part of the paper, using Cramér-Edgeworth expansions for lattice distributions. Applying the corollary on page 8 on the measures G_{t_ξ}, we obtain

$$\left|G^{*n}_{t_\xi}(x) - (2\pi n)^{-d/2}(\det \Gamma_\xi)^{-1/2}\left[1 + n^{-1}L(\xi)\right]\right| \leq C\, n^{-(d+3)/2},$$

where $L(\xi)$ depends polynomially on the moments of G_{t_ξ} up to order four and on the matrix Γ_ξ^{-1}. Since $\gamma/2$ is a lower bound for the smallest eigenvalue of Γ_ξ, the constant C can be chosen depending only on d, ℓ and γ.

We know that Γ_ξ, Γ_ξ^{-1} and the other moments of G_{t_ξ} are analytic functions of ξ. Taylor expansion around $\xi = 0$ thus yields

$$\begin{aligned} G^{*n}_{t_\xi}(x) &= (2\pi n)^{-d/2}(\det \Gamma)^{-1/2} \\ &\qquad \times \left[1 + T^{(1)}(\xi) + T^{(2)}(\xi) + O(\xi^3) + n^{-1}[T^{(0)} + O(\xi)] + O(n^{-3/2})\right] \\ &= (2\pi n)^{-d/2}(\det \Gamma)^{-1/2} \\ &\qquad \times \left[1 + n^{-1/2}T^{(1)}(x/\sqrt{n}) + n^{-1}(T^{(0)} + T^{(2)})(x/\sqrt{n}) + O(n^{-1-\alpha})\right], \end{aligned} \tag{3}$$

with the same dependencies as in (2). Inserting (2) and (3) in (1) yields

$$G^{*n}(x) = \varphi_{n\Gamma}(x) \times \left[1 - n^{-1/2}P^{(3)}(x/\sqrt{n}) + n^{-1}P^{(6)}(x/\sqrt{n}) + O(n^{-1-\alpha})\right],$$

where $P^{(k)}$ denotes a polynomial of order k. Here $P^{(3)}$ contains only odd order terms and $P^{(6)}$ only even order terms.

We still have to estimate $\varphi_{n\Gamma}$. Since the coefficients of the covariance matrix Γ are bounded by ℓ^2, we can bound the maximal eigenvalue of Γ by $d\ell^2$. Therefore

$$\begin{aligned} \varphi_{n\Gamma}(x) &= (2\pi n)^{-d/2}(\det \Gamma)^{-1/2}\, \exp[-(x \cdot \Gamma^{-1}x)/(2n)] \\ &\leq (2\pi n\gamma)^{-d/2}\, \exp[-x^2/(2nd\ell^2)] \\ &\leq (d\ell^2/\gamma)^{d/2}\, \varphi_{nd\ell^2}(x) \\ &\leq (2d\ell^2/\gamma)^{d/2}\, \varphi_{n2d\ell^2}(x). \end{aligned}$$

Choosing C large enough yields the claim for $|x| \leq n^{(5-\alpha)/9}$.

Case $|x| > n^{(5-\alpha)/9}$*:*

For 'big' x we estimate $G^{*n}(x)$ and $\left[1 + n^{-1/2}P_3(x/\sqrt{n}) + n^{-1}P_6(x/\sqrt{n})\right]\varphi_{n\Gamma}(x)$ separately. For fixed natural k we have $|x|^k \exp[-x^2] \leq \mathrm{const}(k) \exp[-x^2/\sqrt{2}]$ for all $x \in \mathbb{Z}^d$. Therefore $\left|1 + n^{-1/2}P_3(x/\sqrt{n}) + n^{-1}P_6(x/\sqrt{n})\right|\varphi_{n\Gamma}(x)$ is bounded by $\mathrm{const}(d, \ell, \gamma)\, \varphi_{\sqrt{2}nd\ell^2}(x)$. Now we use the fact that there exists a natural number k such that $n^{1+\alpha} \leq (x/\sqrt{n})^k$ (k depending on α). Thus we find $C = \mathrm{const}(d, \ell, \gamma, \alpha)$ such that for all n

$$\left|1 + n^{-1/2}P_3(x/\sqrt{n}) + n^{-1}P_6(x/\sqrt{n})\right|\varphi_{n\Gamma}(x) \leq C\, n^{-1-\alpha}\varphi_{2nd\ell^2}(x). \tag{4}$$

Now consider $G^{*n}(x)$. If $\xi \notin S_G$, $G^{*n}(x)$ equals zero. If $\xi \in \mathrm{int}\, S_G$, we use Taylor expansion of $I(\xi)$ to obtain

$$I(\xi) = \int_0^1 (1-s)D^2 I(s\xi)(\xi, \xi)ds = \int_0^1 (1-s)(\xi \cdot \Gamma_{s\xi}^{-1} \cdot \xi)ds \geq \xi^2/(2d\ell^2). \tag{5}$$

In the last inequality we used that the maximal eigenvalue of $\Gamma_{s\xi}$ is majorized by $d\ell^2$ for all ξ and s. Inserting (5) into (1) yields $G^{*n}(x) \leq \exp[-x^2/(2d\ell^2 n)]$. Now the same arguments that lead to (4) yield the desired estimate for $\xi \in \operatorname{int} S_G$.

Finally we consider $\xi \in \partial S_G$. Approximating ξ with a sequence $(\xi_k)_{k\in\mathbb{N}}$ of points in $\operatorname{int} S_G$, the corresponding sequence of measures $G_{t_{\xi_k}}$ converges to a measure G'_ξ with support on a subset of the support of G, and $I(\xi_k) \to I(\xi)$. Using (1) and (5) we therefore obtain

$$G^{*n}(x) = \exp[-nI(\xi)] \cdot G'^{*n}_{\xi}(x) \leq \exp[-x^2/(2d\ell^2 n)].$$

The rest of the argument proceeds as before. □

Cramér-Edgeworth expansions for lattice distributions

This second part yields the bounds we use for the tilted measures in the LCLT. The proof follows the arguments in Bhattacharya/Rao, [1]. The statement here differs from their result in two points: On one hand, we only want to approximate the walk at time n up to order $n^{-(d+3)/2}$, and we suppose bounded range of the measure. Thus we only look at a special case of their result. On the other hand, we need more precise control of the error terms. This means that most of the required lemmas from [1] would have to be adjusted. Rather than doing that, we prove everything we need directly.

The object of interest here is H, the single step distribution of an aperiodic and maximal random walk on $\mathbb{Z}^d$ with bounded steplength $\ell \in \mathbb{N}$. Let E denote the expectation and Γ the covariance matrix of H. The aim is to give a good local bound for $H^{*n}(nE)$, which we use in the proof of the LCLT.

First we introduce some notations and basic estimates (for more details see [1], chapter 6). We denote by $\widehat{H}$ the Fourier transform of H, that is

$$\widehat{H}(t) \stackrel{\text{def}}{=} \sum_{x\in\mathbb{Z}^d} e^{it\cdot x}\, H(x).$$

Since H has bounded steplength, all moments of H exist. We denote by μ_ν the νth moment of H, where ν is a d-dimensional integral vector, that is

$$\mu_\nu \stackrel{\text{def}}{=} \sum_{x\in\mathbb{Z}^d} x^\nu H(x), \quad \text{and we have} \quad i^{|\nu|}\mu_\nu = (D^\nu \widehat{H})(0).$$

The so called cumulants χ_ν are given by

$$i^{|\nu|}\chi_\nu = (D^\nu \log \widehat{H})(0)$$

and can be expressed in terms of moments by equating coefficients of t^ν on both sides of the formal identity

$$\sum_{|\nu|\geq 1} \chi_\nu \frac{(it)^\nu}{\nu!} = \log \widehat{H}(t) = \sum_{s=1}^{\infty} (-1)^{s+1}\, \frac{1}{s} \Big(\sum_{|\nu|\geq 1} \mu_\nu \frac{(it)^\nu}{\nu!} \Big)^s. \tag{6}$$

For $r \in \mathbb{N}$ we define the polynomial $\chi_r(z)$ in $z \in \mathbb{C}^d$ by

$$\chi_r(z) = \sum_{|\nu|=r} \frac{r!}{\nu!}\, \chi_\nu\, z^\nu.$$

We have $\chi_1(z) = E \cdot z$ and $\chi_2(z) = z \cdot (\Gamma \cdot z)$. For $r \in \mathbb{N}$ we can bound

$$\begin{aligned}
\Big|\frac{\chi_r(z)}{r!}\Big| &= \Big|\sum_{|\nu|=r} \chi_\nu \frac{z^\nu}{\nu!}\Big| = \Big|\sum_{k=1}^{r}(-1)^{k+1}\frac{1}{k} \sum_{\substack{|\nu^{(1)}|,\dots,|\nu^{(k)}|\geq 1 \\ \sum_{j=1}^{k}|\nu^{(j)}|=r}} \mu_{\nu^{(1)}}\dots\mu_{\nu^{(k)}} \frac{z^{\sum_{j=1}^{k}\nu^{(j)}}}{\prod_{j=1}^{k}\nu^{(j)}!}\Big| \\
&\leq |z|^r\,\ell^r \sum_{k=1}^{r}\frac{1}{k}\Big(\underbrace{\sum_{|\nu|\geq 1}\frac{1}{\nu!}}_{=e^d-1}\Big)^k \leq |z|^r\, r\,[\ell(e^d-1)]^r \\
&\leq r(K|z|)^r, \qquad (7)
\end{aligned}$$

where $K \overset{\text{def}}{=} \ell(e^d-1)$. Thus for $|t| < 1/K$ the Taylor series $\sum_{r=1}^{\infty}\frac{\chi_r(it)}{r!}$ is convergent, that is,

$$\log \widehat{H}(t) = E \cdot it - \frac{t \cdot (\Gamma \cdot t)}{2} + \sum_{r=3}^{\infty}\frac{\chi_r(it)}{r!}. \qquad (8)$$

Lemma. *There exists a constant C such that for all x in $\mathbb{Z}^d$, $\alpha \in (0,1/2)$ and for all natural n,*

$$\begin{aligned}
&\Big|H^{*n}(x) - n^{-d/2}\Big[1 - \frac{\chi_3(D)}{6\sqrt{n}} + \frac{\chi_4(D)}{24\,n} + \frac{\chi_3^{*2}(D)}{72\,n}\Big]\,\varphi_\Gamma\Big(\frac{x-nE}{\sqrt{n}}\Big)\Big| \\
&\leq C\, n^{-(d+3)/2}, \qquad (9)
\end{aligned}$$

where the formal polynomials $\chi_r(D)$ are given by

$$\chi_r(D)f(x) \overset{\text{def}}{=} \sum_{|\nu|=r}\frac{r!}{\nu!}\chi_\nu\, D^\nu f(x) \quad \textit{and} \quad \chi_r^{*2}(D)f(x) \overset{\text{def}}{=} (\chi_r(D)f * \chi_r(D)f)(x).$$

The constant C depends only on the dimension d, the steplength ℓ of H and on a lower bound for the smallest eigenvalue of Γ.

Proof. First we use the Fourier inversion formula for aperiodic lattice measures (see for example [1] (21.28)) to express H. We have

$$\begin{aligned}
H^{*n}(x) &= (2\pi)^{-d}\int_{[-\pi,\pi]^d}\exp[-it\cdot x]\,\widehat{H}^n(t)\,dt \\
&= (2\pi\sqrt{n})^{-d}\int_{[-\sqrt{n}\pi,\sqrt{n}\pi]^d}\exp[-it\cdot\frac{x}{\sqrt{n}} + n\log\widehat{H}(\frac{t}{\sqrt{n}})]\,dt. \qquad (10)
\end{aligned}$$

With the usual inversion formula for continuous measures we obtain

$$\begin{aligned}
&n^{-d/2}\Big[1 - \frac{\chi_3(D)}{6\sqrt{n}} + \frac{\chi_4(D)}{24\,n} + \frac{\chi_3^{*2}(D)}{72\,n}\Big]\,\varphi_\Gamma\Big(\frac{x-nE}{\sqrt{n}}\Big) \qquad (11) \\
&= (2\pi\sqrt{n})^{-d}\int_{\mathbb{R}^d}\exp\Big[-it\cdot\frac{x-nE}{\sqrt{n}} - \frac{t\cdot(\Gamma\cdot t)}{2}\Big]\Big[1 + \frac{\chi_3(it)}{6\sqrt{n}} + \frac{\chi_4(it)}{24\,n} + \frac{\chi_3^2(it)}{72\,n}\Big]\,dt,
\end{aligned}$$

where we used

$$\widehat{D^\nu\varphi_\Gamma}(t) = (-it)^\nu\,\widehat{\varphi_\Gamma}(t) = (-it)^\nu\exp\Big[-\frac{t\cdot(\Gamma\cdot t)}{2}\Big].$$

Now let $\mathcal{E}$ be a small ball around 0 with radius $\varepsilon > 0$ (precise conditions on $\varepsilon = \varepsilon(d, \ell, \gamma)$ will be determined later). Using (10) and (11) we can bound the left hand side of (9) by the sum of three integrals I, II and III with

$$I = (2\pi\sqrt{n})^{-d} \int_{\sqrt{n}\mathcal{E}} \Big| \exp\big[n \log \widehat{H}(\frac{t}{\sqrt{n}})\big] - \exp\big[it \cdot \frac{nE}{\sqrt{n}} - \frac{t\cdot(\Gamma \cdot t)}{2}\big] \times \Big[1 + \frac{\chi_3(it)}{6\sqrt{n}} + \frac{\chi_4(it)}{24\,n} + \frac{\chi_3^2(it)}{72\,n}\Big]\Big|\, dt,$$

and

$$II = (2\pi\sqrt{n})^{-d} \int_{[-\sqrt{n}\pi, \sqrt{n}\pi]^d \setminus \sqrt{n}\mathcal{E}} \Big| \widehat{H}^n(\frac{t}{\sqrt{n}}) \Big|\, dt,$$

and

$$III = (2\pi\sqrt{n})^{-d} \int_{\mathbb{R}^d \setminus \sqrt{n}\mathcal{E}} \exp\big[-\frac{t\cdot(\Gamma\cdot t)}{2}\big] \Big| 1 + \frac{\chi_3(it)}{6\sqrt{n}} + \frac{\chi_4(it)}{24\,n} + \frac{\chi_3^2(it)}{72\,n}\Big|\, dt.$$

We start with the essential term, the integral I. As a first condition we let $\varepsilon \le 1/(2K)$. Then we have $K|t|/\sqrt{n} \le 1/2$ for all $t \in \sqrt{n}\mathcal{E}$. Denote by γ (a lower bound for) the smallest eigenvalue of Γ. We use (8) and (7) to obtain

$$\begin{aligned}
I &\le (2\pi\sqrt{n})^{-d} \int_{\sqrt{n}\mathcal{E}} \exp\big[-\frac{t\cdot(\Gamma\cdot t)}{2}\big] \\
&\qquad \times \Big| \exp\big[\sum_{r=3}^{\infty} n^{-\frac{r-2}{2}} \frac{\chi_r(it)}{r!}\big] - \Big[1 + \frac{\chi_3(it)}{6\sqrt{n}} + \frac{\chi_4(it)}{24\,n} + \frac{\chi_3^2(it)}{72\,n}\Big]\Big|\, dt \\
&\le (2\pi\sqrt{n})^{-d} \int_{\sqrt{n}\mathcal{E}} \exp\big[-\frac{\gamma}{2}t^2\big] \times \Big[\sum_{r=5}^{\infty} n^{-\frac{r-2}{2}} \frac{|\chi_r(it)|}{r!} \\
&\qquad + \frac{1}{2} \sum_{\substack{r,s\ge 3\\ r+s\ge 7}} n^{-\frac{r+s-4}{2}} \frac{|\chi_r(it)||\chi_s(it)|}{r!s!} + \sum_{s=3}^{\infty} \Big(\sum_{r=3}^{\infty} n^{-\frac{r-2}{2}} \frac{|\chi_r(it)|}{r!}\Big)^s / s!\Big]\, dt \\
&\le (2\pi\sqrt{n})^{-d} \int_{\sqrt{n}\mathcal{E}} \exp\big[-\frac{\gamma}{2}t^2\big] \Big[n^{-3/2} \sum_{r=5}^{\infty} r(Kt)^r + \frac{1}{2} n^{-3/2} \sum_{\substack{r,s\ge 3\\ r+s\ge 7}} rs(Kt)^{r+s} \\
&\qquad + \sum_{s=3}^{\infty} n^{-s/2}(2Kt)^{3s}/s!\Big]\, dt \\
&\le n^{-(d+3)/2}\, (2\pi)^{-d} \int_{\mathbb{R}^d} \exp[-\frac{\gamma}{2}t^2] \operatorname{const}(d,\ell) \big[t^5 + t^7 + t^9 \exp[\varepsilon(2K)^3)t^2]\big]\, dt \\
&\le n^{-(d+3)/2}\, (2\pi)^{-d} \int_{\mathbb{R}^d} \operatorname{const}(d,\ell,\gamma)\, \exp[-\frac{\gamma}{4}t^2]\, dt,
\end{aligned}$$

if we choose $\varepsilon = \varepsilon(d, \ell, \gamma)$ small enough. Therefore we have

$$I \le \operatorname{const}(d,\ell,\gamma)\, n^{-(d+3)/2}.$$

Now we come to the error term II. Since $|\widehat{H}| = 1$ in zero and $|\widehat{H}| < 1$ away from zero, there exists a constant $K' < 1$ with $|\widehat{H}(t)| \le K'$ on $[-\pi,\pi]^d \setminus \mathcal{E}$. As in the proof of the LCLT we use the fact, that the set of all measures on $\mathbb{Z}^d$ with steplength ℓ such that the smallest eigenvalue of the covariance matrix is greater or equal γ, is compact. Since the mapping $(H,t) \mapsto \widehat{H}(t)$ is continuous in both arguments, the constant K' can be chosen depending on d, ℓ and γ only. Therefore we have

$$II = (2\pi)^{-d} \int_{[-\pi,\pi]^d \setminus \mathcal{E}} \left|\widehat{H}^n(t)\right| dt \le K'^n \le \mathrm{const}(d,\ell,\gamma)\, n^{-(d+3)/2}.$$

III is not much more complicated. Using (7) to majorize the χ_r, we can bound

$$\begin{aligned} III &= (2\pi\sqrt{n})^{-d} \int_{\mathbb{R}^d \setminus \sqrt{n}\mathcal{E}} \exp\Big[-\frac{t\cdot(\Gamma\cdot t)}{2}\Big] \Big|1 + \frac{\chi_3(it)}{6\sqrt{n}} + \frac{\chi_4(it)}{24\,n} + \frac{\chi_3^2(it)}{72\,n}\Big|\, dt \\ &\le (2\pi\sqrt{n})^{-d} \int_{\mathbb{R}^d \setminus \sqrt{n}\mathcal{E}} \mathrm{const}(d,\ell) \exp\Big[-\frac{\gamma t^2}{3}\Big]\, dt \\ &\le \mathrm{const}(d,\ell,\gamma)\, n^{-d/2} \exp\Big[-\frac{\gamma\varepsilon^2 n}{4}\Big] \le \mathrm{const}(d,\ell,\gamma)\, n^{-(d+3)/2}. \end{aligned}$$

The combination of these three estimates completes the proof of the Lemma. ☐

For the proof of the Local CLT we only need an approximation of $H^{*n}(x)$ at the point $x = nE$. In this case the approximating term

$$n^{-d/2}\Big[1 - \frac{\chi_3(D)}{6\sqrt{n}} + \frac{\chi_4(D)}{24\,n} + \frac{\chi_3^{*2}(D)}{72\,n}\Big]\, \varphi_\Gamma\Big(\frac{x-nE}{\sqrt{n}}\Big)$$

simplifies to

$$n^{-d/2}\big[1 + n^{-1}\big(\chi_4(D)/24 + \chi_3^{*2}(D)/72\big)\big]\, \varphi_\Gamma(0),$$

since the odd derivatives of a centered normal density vanish at zero. Therefore we obtain the following corollary as a special case of the lemma.

Corollary. *There exist constants C and L such that for all natural n,*

$$\Big|H^{*n}(nE) - (2\pi n)^{-d/2}(\det\Gamma)^{-1/2}\Big[1 + n^{-1}L\Big]\Big| \le C\, n^{-(d+3)/2}.$$

Here L is a constant depending polynomially on the moments of H up to order four (coming from the χ terms) and on the matrix Γ^{-1} (coming from the derivatives of φ_Γ). The constant C depends only on the dimension d, the steplength ℓ of H and on a lower bound for the smallest eigenvalue of Γ.

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