ON POVZNER–WIENHOLTZ-TYPE SELF-ADJOINTNESS RESULTS FOR MATRIX-VALUED STURM–LIOUVILLE OPERATORS

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Abstract. We derive Povzner–Wienholtz-type self-adjointness results for $m\times m$ matrix-valued Sturm–Liouville operators $T = R^{-1}\big[-\frac{d}{dx}P\frac{d}{dx}+Q\big]$ in $L^2((a,b);Rdx)^m$, $m\in\mathbb{N}$, for (a,b) a half-line or $\mathbb{R}$.

1. Introduction

In two recent papers concerned with certain aspects of inverse spectral theory for matrix-valued one dimensional Schrödinger operators (cf. [1], [7]), it became necessary to infer the limit point case at $\pm\infty$ from nonoscillatory properties of the Schrödinger differential expression at $\pm\infty$. Put differently, it was necessary to conclude that boundedness from below implies (essential) self-adjointness for matrix-valued Schrödinger operators under minimal regularity assumptions on the potential coefficient. While the problem of self-adjointness of matrix-valued (and more generally, operator-valued) differential operators has been studied in the literature (see, e.g., [15] and [28]), apparently, no result assuming lower semiboundedness of the associated minimal differential operator alone, appears to be known. (Reference [28], e.g., assumes additional local boundedness of the operator-valued potential in the Schrödinger operator in question.) In this note we close this gap by proving Povzner–Wienholtz-type self-adjointness results for one-dimensional scalar and matrix-valued Sturm–Liouville operators under optimal conditions on the potential coefficient.

Hartman [9] and Rellich [21] were first to show that boundedness below of the scalar Sturm–Liouville expression $\ell = r^{-1}(-\frac{d}{dx}p\frac{d}{dx}+q)$ and integral conditions on the functions r and p, which are necessarily satisfied when $p, r = 1$, are together sufficient to imply essential self-adjointness of the minimal operator for ℓ (cf. Section 2). Independently, Povzner [18] and subsequently Wienholtz [29] were first to show that semi-boundedness for the operator $L_0 = (-\Delta+q) \restriction C_0^\infty(\mathbb{R}^n)$ implies its essential self-adjointness when q is real-valued and continuous. This result has since been shown for more general q as may be seen by the work and references found in [11], [25], [26], and [27]. Closely related self-adjointness results based on the concept of a finite rate of propagation were discussed by Berezanskii [2, Sect. VI.1.7] and further devloped, for instance, in [3], [4, Sect. 13.8], [17], [22]. These works typically address the scalar case for ordinary or partial differential expressions, and

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not the one-dimensional matrix setting that is our interest. After completing our work we became aware of a recent very detailed study of essential self-adjointness of general Schrödinger operators of the type D^*D+V, where D is a first-order elliptic differential operator acting on the space of sections of a hermitian vector bundle over a manifold, by Braverman, Milatovic, and Shubin [5]. This paper appears to contain the state of the art in the multi-dimensional context.

In Section 2, we define Sturm–Liouville operators for the scalar case on the whole or half-line and recall in Theorem 2.3 the equivalence between the boundedness below of our operators and the nonoscillation of solutions of the associated homogeneous equation near endpoints of the given interval. The fundamental results of Hartman [9] and of Rellich [21] are recalled in Theorem 2.4 for the scalar Sturm–Liouville case. In Section 3, we obtain in Theorem 3.1 a result for the scalar Sturm–Liouville operator, and in Theorem 3.7 a result for the matrix-valued Sturm–Liouville operator, like that of Hartman and Rellich but independent of both as seen in Examples 3.3 and 3.4.

Hartman proved his result using the concept of principal and nonprincipal solutions while Rellich obtained a proof of this result by focusing upon the boundedness below of the associated minimal operator. Each of these approaches appears not to extend to the matrix setting. Instead, our results are obtained using the approach introduced by Wienholtz [29] to prove the result first due to Povzner [18].

2. Self-adjointness for the scalar Sturm–Liouville case

Beginning with the scalar case, we introduce the following basic assumptions.

Hypothesis 2.1.
(i) Let $c < b \leq \infty$ and suppose that p, q, r are (Lebesgue) measurable on $[c, b)$, and that

$$p > 0, r > 0 \text{ a.e. on } [c, b),\ q \text{ real-valued}, \tag{2.1a}$$

$$1/p, q, r \in L^1([c, d]; dx) \text{ for all } d \in [c, b). \tag{2.1b}$$

(ii) Let $-\infty \leq a < b \leq \infty$ and suppose that p, q, r are (Lebesgue) measurable on (a, b), and that

$$p > 0, r > 0 \text{ a.e. on } (a, b),\ q \text{ real-valued}, \tag{2.2a}$$

$$1/p, q, r \in L^1_{\text{loc}}((a, b); dx). \tag{2.2b}$$

The case $-\infty \leq a < c$ and p, q, r (Lebesgue) measurable on $(a, c]$, $p > 0, r > 0$ a.e. on $(a, c]$, q real-valued, and $1/p, q, r \in L^1([d, c]; dx)$, for all $d \in (a, c]$, follows upon reflection from case (i) in Hypothesis 2.1 and hence will not be separately discussed in the following.

Given Hypothesis 2.1, we consider the differential expression

$$\ell = \frac{1}{r}\bigg(-\frac{d}{dx}p\frac{d}{dx} + q\bigg) \tag{2.3}$$

on $[c,b)$ and (a,b), respectively. Next, we introduce for $c \in (a,b)$,

$$\begin{aligned}
\mathcal{D}_{\rm max,c,b} = \Big\{ u \in L^2([c,b);rdx) \,\Big|\, & u, pu' \in AC([c,d]) \text{ for all } d \in [c,b); \\
& u(c) = 0;\ \ell u \in L^2([c,b);rdx)\Big\}, \qquad (2.4)
\end{aligned}$$

$$\mathcal{D}_{\rm min,c,b} = \{u \in \mathcal{D}_{\rm max,c,b} \,|\, \mathrm{supp}(u) \subset [c,b) \text{ compact}\}, \tag{2.5}$$

$$\mathcal{D}_{\rm max} = \{u \in L^2((a,b);rdx) \,|\, u, pu' \in AC_{\rm loc}((a,b)); \tag{2.6}$$

$$\ell u \in L^2((a,b);rdx)\}, \tag{2.7}$$

$$\mathcal{D}_{\rm min} = \{u \in \mathcal{D}_{\rm max} \,|\, \mathrm{supp}(u) \subset (a,b) \text{ compact}\}, \tag{2.8}$$

where $AC_{\rm loc}((a,b))$ denotes the set of locally absolutely continuous functions on (a,b) and $AC([c,d])$ represents the set of absolutely continuous functions on $[c,d]$.

Minimal operators $T_{\rm min,c,b}$ and $T_{\rm min}$ and maximal operators $T_{\rm max,c,b}$ and $T_{\rm max}$ in $L^2((c,b);rdx)$ and $L^2((a,b);rdx)$, respectively, associated with ℓ, are then defined by

$$T_{\rm min,c,b} u = \ell u, \quad u \in \mathrm{dom}(T_{\rm min,c,b}) = \mathcal{D}_{\rm min,c,b}, \tag{2.9}$$

$$T_{\rm max,c,b} u = \ell u, \quad u \in \mathrm{dom}(T_{\rm max,c,b}) = \mathcal{D}_{\rm max,c,b}, \tag{2.10}$$

$$T_{\rm min} u = \ell u, \quad u \in \mathrm{dom}(T_{\rm min}) = \mathcal{D}_{\rm min}, \tag{2.11}$$

$$T_{\rm max} u = \ell u, \quad u \in \mathrm{dom}(T_{\rm max}) = \mathcal{D}_{\rm max}, \tag{2.12}$$

respectively. Then $T_{\rm min}$ and $T_{\rm min,c,b}$ are densely defined and (cf. [16, p. 64, 88])

$$T_{\rm min,c,b}{}^* = T_{\rm max,c,b}, \tag{2.13}$$

$$T_{\rm min}{}^* = T_{\rm max}. \tag{2.14}$$

In the following we will frequently refer to solutions $u(\cdot,z)$ of $\ell u = zu$ for some $z \in \mathbb{C}$. Such solutions are always assumed to be distributional solutions, that is, we tacitly assume

$$u(\cdot,z),\, pu'(\cdot,z) \in AC([c,d]),\ d \in [c,b) \ \text{(resp., } AC_{\rm loc}((a,b)), \text{etc.)} \tag{2.15}$$

in such a case.

Next we recall the standard notion of (non)oscillatory differential expressions.

Definition 2.2. Assume Hypothesis 2.1 (ii) and fix $c \in (a,b)$. Then ℓ is said to be *nonoscillatory near* a *(or* b*)* for some $\lambda \in \mathbb{R}$ if and only if every real-valued solution u of $\ell u = \lambda u$ has finitely many zeros in (a,c) (resp., (c,b)). Otherwise, ℓ is called *oscillatory near* a *(resp.,* b*)*.

The following is a key result in this connection.

Theorem 2.3 ([10], [12], [21], [23], [24])**.**
Assume Hypothesis 2.1 (i)*. Then the following assertions are equivalent:*
(i) $T_{\min,c,b}$ *(and hence any symmetric extension of* $T_{\min,c,b}$*) is bounded from below.*
(ii) *There exists a* $\lambda_0 \in \mathbb{R}$ *such that* ℓ *is nonoscillatory near* b *for all* $\lambda < \lambda_0$*.*
Assume Hypothesis 2.1 (ii)*. Then the following assertions are equivalent:*
(iii) $T_{\rm min}$ *(and hence any symmetric extension of* $T_{\rm min}$*) is bounded from below.*
(iv) *There exists a* $\lambda_0 \in \mathbb{R}$ *such that* ℓ *is nonoscillatory near* a *and* b *for all* $\lambda < \lambda_0$*.*

Next, we mention a fundamental result which links the nonoscillatory behavior at one end point with the limit point property at that endpoint.

Theorem 2.4 (Hartman [9] (cf. also [6], [21])). *Suppose Hypothesis* 2.1 (*i*) *and assume that for some* $\lambda_0 \in \mathbb{R}$, $\ell - \lambda_0$ *is nonoscillatory near* b, *or equivalently, that* $T_{\min,c,b}$ *is bounded from below. Then, if*

$$\left| \int^b dx \left(\frac{r(x)}{p(x)} \right)^{1/2} \right| = \infty, \tag{2.16}$$

ℓ *is in the limit point case at* b.

Originally, Hartman proved the special case $r = 1$ by an elegant application of the concept of (non)principal solutions (the proof easily extends to the general situation $r \neq 1$ described in Theorem 2.4). Theorem 2.4 was subsequently derived by Rellich [21] by focusing on operators $T_{\rm min,c,b}$ bounded from below. (Actually, Hartman and Rellich assume $p, p' \in C((a,b))$ and q, r piecewise continuous in (a,b) in addition to Hypothesis 2.1 (i) but this is easily seen to be unnecessary.)

For an extension of Theorem 2.4 to polynomials of ℓ we refer to Read [19].

For essential self-adjointness results using conditions of the type (2.16) but with semiboundedness from below of the corresponding minimal differential operators replaced by alternative conditions, we refer to the recent work by Lesch and Malamud [14]. The latter treats the case of matrix-valued Schrödinger operators as well as the case of general canonical systems.

Finally, for extensions of Theorem 2.4 to the multi-dimensional case we refer to our account in the introduction.

3. Povzner–Wienholtz-type self-adjointness results

Next we formulate a result that resembles one by Povzner [18] in the context of partial differential operators. We will adapt a method of proof due to Wienholtz [29], who independently proved Povzner's result. (Wienholtz's proof is also reproduced in Glazman [8, Chapter1, Theorem 35].)

Theorem 3.1. *Assume Hypothesis* 2.1 (*i*) *with* $b = \infty$ *and* $p \in AC([c, c+\rho])$ *for all* $\rho > 0$ *and* $r^{-1}(p')^2, r^{-1} \in L^1_{\rm loc}((c,\infty))$. *Moreover, suppose that* $T_{\min,c,\infty}$ *is bounded from below. Then, if*

$$\|p/r\|_{L^\infty((\rho/2,\rho))} \underset{\rho\uparrow\infty}{=} O(\rho^2), \tag{3.1}$$

ℓ *is in the limit point case at* ∞.

Proof. By von Neumann's theory of self-adjoint extensions (cf. [20, p. 137]), to prove that ℓ is in the limit point case at ∞, or equivalently, that $T_{\rm min,c,\infty}$ is essentially self-adjoint and hence its closure $T_{\rm max,c,\infty}$ is self-adjoint (since $\overline{T_{\rm min,c,\infty}} = T^*_{\rm min,c,\infty} = T_{\rm max,c,\infty}$), we need to show that

$$\ker(T^*_{\rm min,c,\infty} - C) = \{0\} \tag{3.2}$$

for $C > 0$ sufficiently large. Next, we assume, without loss of generality, that for $u \in \mathrm{dom}(T_{\rm min,c,\infty})$

$$\int_c^\infty r(x)dx\, \overline{u(x)}(\ell u)(x) > \int_c^\infty r(x)dx\, |u(x)|^2, \tag{3.3}$$

so we may take $C = 0$ in (3.2). In addition, we choose $c = 0$ for simplicity.

To start the proof we assume that ℓ is not in the limit point case at ∞. Then, by Weyl's alternative, all solutions u of

$$\ell u = 0 \tag{3.4}$$

are necessarily in $L^2([0,\infty); r dx)$, and hence there exists a nontrivial real-valued solution $\hat{u} \in L^2([0,\infty); r dx)$ of (3.4) that satisfies the Dirichlet boundary condition $\hat{u}(0) = 0$. To complete the proof, it is sufficient to show that (3.3) rules out the existence of such a solution $\hat{u} \in L^2([0,\infty); r dx)$ of (3.4).

Let

$$u_\rho(x) = \theta_\rho(x)\hat{u}(x)\,, \quad 0 \le x < \infty, \tag{3.5}$$

where

$$\theta_\rho(x) = \theta(x/\rho), \quad \theta \in \mathbb{C}^\infty([0,\infty)), \quad \theta(x) = \begin{cases} 1 & 0 \le x \le 1/2, \\ 0 & x > 1. \end{cases} \tag{3.6}$$

As defined, $u_\rho \in \operatorname{dom}(T_{\min,0,\infty})$. Indeed, since $p \in AC([0,\rho])$ for all $\rho > 0$ by hypothesis, one verifies that

$$\begin{aligned} \ell u_\rho &= \ell(\theta_\rho \hat{u}) = \theta_\rho(\ell \hat{u}) - r^{-1}(p\theta_\rho')'\hat{u} - 2r^{-1}\theta_\rho'(p\hat{u}') \\ &= -r^{-1}p'\theta_\rho'\hat{u} - r^{-1}p\theta_\rho''\hat{u} - 2r^{-1}\theta_\rho'(p\hat{u}'). \end{aligned} \tag{3.7}$$

Since

$$\int_{\rho/2}^{\rho} dx\, r^{-1}(p')^2(\theta_\rho')^2\hat{u}^2 \le \|(\theta_\rho')^2\hat{u}^2\|_{L^\infty((\rho/2,\rho))} \int_{\rho/2}^{\rho} dx\, r^{-1}(p')^2 < \infty, \tag{3.8}$$

$$\begin{aligned} \int_{\rho/2}^{\rho} dx\, (r^{-1}p)\big[p(\theta_\rho'')^2\hat{u}^2\big] &\le (\rho/2)\|p/r\|_{L^\infty((\rho/2,\rho))}\|p\|_{L^\infty((\rho/2,\rho))} \\ &\quad \times \|(\theta_\rho'')^2\hat{u}^2\|_{L^\infty((\rho/2,\rho))} < \infty, \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} \int_{\rho/2}^{\rho} dx\, r^{-1}(\theta_\rho')^2(p\hat{u}')^2 &\le \|(\theta_\rho')^2\|_{L^\infty((\rho/2,\rho))}\|(p\hat{u}')^2\|_{L^\infty((\rho/2,\rho))} \\ &\quad \times \int_{\rho/2}^{\rho} dx\, r^{-1} < \infty, \end{aligned} \tag{3.10}$$

one obtains $u_\rho \in \operatorname{dom}(T_{\min,0,\infty})$. Hence u_ρ satisfies (3.3), that is,

$$\int_0^\infty r(x)dx\, \theta_\rho(x)^2\hat{u}(x)^2 < \int_0^\infty r(x)dx\, u_\rho(x)(\ell u_\rho)(x). \tag{3.11}$$

Since

$$r u_\rho(\ell u_\rho) = -\theta_\rho(p\theta_\rho')'\hat{u}^2 - 2\theta_\rho\theta_\rho'\hat{u}(p\hat{u}') = (\theta_\rho')^2 p\hat{u}^2 - \big[\theta_\rho\theta_\rho' p\hat{u}^2\big]' \tag{3.12}$$

and $\theta_\rho(\rho) = \theta'_\rho(0) = 0$, one obtains

$$\begin{aligned}
&\int_0^{\rho/2} r(x)dx\, \hat{u}(x)^2 < \int_0^\infty r(x)dx\, \theta_\rho(x)^2 \hat{u}(x)^2 \\
&< \int_0^\infty r(x)dx\, u_\rho(x)(\ell u_\rho)(x) = \int_0^\infty dx\, \theta'_\rho(x)^2 p(x)\hat{u}(x)^2 \\
&= \int_0^\infty r(x)dx\, \theta'_\rho(x)^2[p(x)/r(x)]\hat{u}(x)^2 \\
&\leq \frac{C^2}{\rho^2}\|p/r\|_{L^\infty((\rho/2,\rho))} \int_{\rho/2}^\rho r(x)dx\, \hat{u}(x)^2 \\
&\underset{\rho\uparrow\infty}{=} O(1) \int_{\rho/2}^\rho r(x)dx\, \hat{u}(x)^2 \underset{\rho\uparrow\infty}{=} o(1)
\end{aligned} \tag{3.13}$$

for some constant $C > 0$, implying $\hat{u} = 0$ a.e. on $[0,\infty)$. $\square$

Remark 3.2. Even though Theorems 2.4 (for $b = \infty$) and 3.1 show some resemblance, their proofs follow quite different strategies. In particular, the proof of Theorem 3.1 seems to work only for half-lines (i.e., $b = \infty$). Moreover, as the following examples show, neither theorem implies the other.

Example 3.3. Consider $c = 0$, $b = \infty$, $q = 0$, $p, r \in C^\infty([0,\infty))$, and

$$\frac{r(x)}{p(x)} = \frac{1}{x^2 \ln(x)} \text{ for } x \geq e. \tag{3.14}$$

Then one easily verifies that condition (2.16) is satisfied while (3.1) is not.

Example 3.4. Consider $c = 0$, $b = \infty$ $q = 0$, $p, r \in C([0,\infty))$, and

$$\frac{r(x)}{p(x)} = \begin{cases} \frac{1}{x^2}, & x \in [n, n+(1/n)],\ n \in \mathbb{N}, \\ 0, & \text{otherwise}. \end{cases} \tag{3.15}$$

Then one easily verifies that condition (3.1) is satisfied while (2.16) is not.

We note that $q = 0$ implies the nonoscillatory behavior of ℓ since solving $\ell u = 0$ readily yields

$$u(x) = C\int^x dx'\, p(x')^{-1} + D, \quad C, D \in \mathbb{C}, \tag{3.16}$$

and clearly u in (3.16) has at most one zero. Moreover, if r is not integrable at ∞, there exists a particular solution $u_0 = D \notin L^2([0,\infty); rdx)$ (choosing $D \neq 0$ in (3.16)) of $\ell u = 0$ and hence ℓ is also in the limit point case in this situation.

Remark 3.5. Examining the proof of Theorem 3.1, it is possible to relax the assumption $p \in AC_{\text{loc}}(\mathbb{R})$ by an appropriate modification of condition (3.1). In fact, modifying the cutoff function θ_ρ in (3.6) as follows,

$$\theta_\rho \in C^\infty([0,\infty)), \quad \theta_\rho(x) = \begin{cases} 1, & 0 \leq x \leq \rho/2, \\ f_\rho\big(\rho^{-1}\int_0^x dx'\, p(x')^{-1}\big), & \rho/2 \leq x \leq \rho, \\ 0, & x \geq \rho, \end{cases} \tag{3.17}$$

$$f_\rho\Big(\rho^{-1}\int_0^{\rho/2} dx\, p(x)^{-1}\Big) = 1, \quad f_\rho\Big(\rho^{-1}\int_0^{\rho} dx\, p(x)^{-1}\Big) = 0 \tag{3.18}$$

for an appropriate smooth function f_ρ, and assuming

$$\left\| (pr)^{-1}\Big[f_\rho'\Big(\rho^{-1}\int_0^{\cdot} dx'\, p(x')^{-1}\Big)\Big]^2\right\|_{L^\infty((\rho/2,\rho))} \underset{\rho\uparrow\infty}{=} O(\rho^2), \tag{3.19}$$

instead of condition (3.1), one verifies that

$$\begin{aligned} \ell u_\rho &= \ell(\theta_\rho \hat u) = \theta_\rho(\ell \hat u) - r^{-1}(p\theta_\rho')'\hat u - 2r^{-1}\theta_\rho'(p\hat u') \\ &= -r^{-1}(p\theta_\rho')'\hat u - 2r^{-1}\theta_\rho'(p\hat u') \\ &= -\rho^{-2}(pr)^{-1} f_\rho''\Big(\rho^{-1}\int_0^x dx'\, p(x')^{-1}\Big)\hat u \\ &\quad - 2\rho^{-1}(pr)^{-1} f_\rho'\Big(\rho^{-1}\int_0^x dx'\, p(x')^{-1}\Big)(p\hat u') \end{aligned} \tag{3.20}$$

is well-defined. Following the proof of Theorem 3.1 step by step then results in

$$\begin{aligned} &\int_0^\infty dx\, \theta_\rho'(x)^2 p(x)\hat u(x)^2 = \int_0^\infty r(x)dx\, \theta_\rho'(x)^2 p(x) r(x)^{-1}\hat u(x)^2 \\ &\le \frac{C^2}{\rho^2}\left\| (pr)^{-1}\Big[f_\rho'\Big(\rho^{-1}\int_0^{\cdot} dx'\, p(x')^{-1}\Big)\Big]^2\right\|_{L^\infty((\rho/2,\rho))} \int_{\rho/2}^{\rho} r(x)dx\, \hat u(x)^2 \end{aligned} \tag{3.21}$$

for some constant $C > 0$ and hence again in $\hat u = 0$ a.e. on $[0,\infty)$ as in (3.13).

When trying to extend Theorem 2.4 to the matrix-valued context, one observes that the use of (matrix-valued) principal solutions can indeed exclude the limit circle case at b under the assumption of $T_{\min,c,b}$ being bounded from below. However, it is not clear how to exclude all intermediate cases between those of limit circle and limit point that we may then infer the limit point case at b. The strategy of proof of Theorem 3.1, on the other hand, will now be shown to extend to the matrix-valued case in due course.

Throughout the rest of this note all matrices will be considered over the field of complex numbers $\mathbb{C}$, and the corresponding linear space of $k\times\ell$ matrices will be denoted by $\mathbb{C}^{k\times\ell}$, $k,\ell\in\mathbb{N}$. Moreover, I_k denotes the identity matrix in $\mathbb{C}^{k\times k}$ and M^* the adjoint (i.e., complex conjugate transpose) of a matrix M. A positive definite matrix P is denoted by $P > 0$, a nonnegative matrix Q by $Q \ge 0$.

We start by introducing the following hypothesis.

Hypothesis 3.6.
Let $c < b \le \infty$ and suppose that $P, Q, R \in \mathbb{C}^{m\times m}$ are (Lebesgue) measurable on $[c, b)$, and that

$$P > 0, R > 0 \text{ a.e. on } [c,b),\ Q = Q^* \text{ self-adjoint}, \tag{3.22a}$$

$$P^{-1}, Q, R \in L^1([c,d];dx)^{m\times m} \text{ for all } d\in[c,b). \tag{3.22b}$$

As in the scalar context, the analogous case of the half-line $(a, c]$ follows upon reflection from the case $[c, b)$ in Hypothesis 3.6 and hence is not separately discussed in the following.

Given Hypothesis 3.6, we consider the $m\times m$ matrix-valued differential expression

$$L = R^{-1}\Big(-\frac{d}{dx}P\frac{d}{dx} + Q\Big) \tag{3.23}$$

on $[c,b)$. Next, we introduce

$$\mathcal{D}_{\max,c,b}=\Big\{u\in L^2([c,b);Rdx)^m\,\Big|\,u,Pu'\in AC([c,d])^m \text{ for all } d\in[c,b); \\ u(c)=0;\ \ell u\in L^2([c,b);Rdx)^m\Big\}, \tag{3.24}$$

$$\mathcal{D}_{\min,c,b}=\{u\in\mathcal{D}_{\max,c,b}\,|\,\operatorname{supp}(u)\subset[c,b)\text{ compact}\}, \tag{3.25}$$

where $AC([c,d])^m$ denotes the set of absolutely continuous vectors in $\mathbb{C}^m$ on $[c,d]$. Moreover, $L^2(\Omega;Rdx)^m$ denotes the Hilbert space of $\mathbb{C}^m$-valued (Lebesgue) measurable vectors u,v on $\Omega\subseteq\mathbb{R}$ corresponding to the $m\times m$ matrix-valued measure Rdx with scalar product $(u,v)_{L^2(\Omega;Rdx)^m}$, given by

$$(u,v)_{L^2(\Omega;Rdx)^m}=\int_\Omega dx\,(u(x),R(x)v(x))_{\mathbb{C}^m}\,, \tag{3.26}$$

where $(f,g)_{\mathbb{C}^m}=\sum_{k=1}^m\overline{f}_kg_k,\ f,g\in\mathbb{C}^m$. For notational simplicity we will abbreviate the scalar product in $L^2(\Omega;Rdx)^m$ by (u,v) only.

Minimal operators $T_{\min,c,b}$ and maximal operators $T_{\max,c,b}$ in $L^2((c,b);Rdx)^m$ associated with L are then defined as in the scalar context by

$$T_{\min,c,b}u=Lu,\quad u\in\operatorname{dom}(T_{\min,c,b})=\mathcal{D}_{\min,c,b}, \tag{3.27}$$

$$T_{\max,c,b}u=Lu,\quad u\in\operatorname{dom}(T_{\max,c,b})=\mathcal{D}_{\max,c,b}, \tag{3.28}$$

respectively. Then $T_{\min}$ and $T_{\min,c,b}$ are densely defined and (2.13)–(2.14) continue to hold in the matrix-valued context.

After these preparations we are ready for the matrix-valued extension of Theorem 3.1, the principal objective of this note. (We denote by $\|M\|$ the operator norm of an $m\times m$ matrix M.)

Theorem 3.7. *Assume Hypothesis* 3.6 *with* $b=\infty$ *and* $P\in AC([c,c+\rho])^{m\times m}$ *for all* $\rho>0$ *and* $\|P'R^{-1}P'\|,\|R^{-1}\|\in L^1_{\text{loc}}((c,\infty))$. *Moreover, suppose that* $T_{\min,c,\infty}$ *is bounded from below. Then, if*

$$\operatorname{ess\,sup}_{x\in[\rho/2,\rho]}\big\|R(x)^{-1/2}P(x)R(x)^{-1/2}\big\|\underset{\rho\uparrow\infty}{=}O(\rho^2), \tag{3.29}$$

L is in the limit point case at ∞.

Proof. The strategy of proof is completely analogous to the scalar context, Theorem 3.1. As before we need to show that

$$\ker(T^*_{\min,c,\infty}-C)=\{0\} \tag{3.30}$$

for $C>0$ sufficiently large. Again we assume, without loss of generality, that for $u\in\operatorname{dom}(T_{\min,c,\infty})$

$$\int_c^\infty dx\,(u(x),R(x)(Lu)(x))_{\mathbb{C}^m}>\int_c^\infty dx\,(u(x),R(x)u(x))_{\mathbb{C}^m}\,, \tag{3.31}$$

so we may take $C=0$ in (3.30). Moreover, we choose $c=0$ as before.

To start the proof we assume that L is not in the limit point case at ∞. Then, by the matrix analog of Weyl's alternative (cf., e.g., [13], [14]), $m+1$ solutions u of

$$Lu=0 \tag{3.32}$$

are necessarily in $L^2([0,\infty);Rdx)^m$, and hence there exists a nontrivial real-valued solution $\hat u\in L^2([0,\infty);Rdx)$ of (3.4) that satisfies the Dirichlet boundary condition

$\hat u(0) = 0$. To complete the proof, it is sufficient to show that (3.31) rules out the existence of such a solution $\hat u$.

Let

$$u_\rho(x) = \theta_\rho(x)\hat u(x)\,, \quad 0 \le x < \infty, \tag{3.33}$$

where

$$\theta_\rho(x) = \theta(x/\rho), \quad \theta \in \mathbb{C}^\infty([0,\infty)), \quad \theta(x) = \begin{cases} 1 & 0 \le x \le 1/2, \\ 0 & x > 1. \end{cases} \tag{3.34}$$

As defined, $u_\rho \in \mathrm{dom}(T_{\min,0,\infty})$. Indeed, since $P \in AC([0,\rho])$ for all $\rho > 0$ by hypothesis, one verifies that

$$\begin{aligned} Lu_\rho = L(\theta_\rho \hat u) &= \theta_\rho(L\hat u) - R^{-1}(P\theta_\rho')'\hat u - 2R^{-1}\theta_\rho'(P\hat u') \\ &= -R^{-1}P'\theta_\rho'\hat u - R^{-1}P\theta_\rho''\hat u - 2R^{-1}\theta_\rho'(P\hat u'). \end{aligned} \tag{3.35}$$

Since

$$\begin{aligned} \int_{\rho/2}^{\rho} dx\, \big(\theta_\rho'\hat u, P'R^{-1}P'\theta_\rho'\hat u\big)_{\mathbb{C}^m} &\le \operatorname{ess\,sup}_{x\in[\rho/2,\rho]} \|\theta_\rho'(x)\hat u(x)\|^2_{\mathbb{C}^m} \\ &\quad \times \int_{\rho/2}^{\rho} dx\, \big\|P'R^{-1}P'\big\| < \infty, \end{aligned} \tag{3.36}$$

$$\begin{aligned} &\int_{\rho/2}^{\rho} dx\, \big(P\theta_\rho''\hat u, R^{-1}P\theta_\rho''\hat u\big)_{\mathbb{C}^m} = \int_{\rho/2}^{\rho} dx\, \big(P^{1/2}\theta_\rho''\hat u, P^{1/2}R^{-1}P^{1/2}P^{1/2}\theta_\rho''\hat u\big)_{\mathbb{C}^m} \\ &\le (\rho/2)\operatorname{ess\,sup}_{x\in[\rho/2,\rho]}\big\|P(x)^{1/2}\big\|^2 \operatorname{ess\,sup}_{x\in[\rho/2,\rho]}\|\theta_\rho''(x)\hat u(x)\|^2_{\mathbb{C}^m} \\ &\quad \times \operatorname{ess\,sup}_{x\in[\rho/2,\rho]}\big\|P(x)^{1/2}R(x)^{-1}P(x)^{1/2}\big\| \\ &= (\rho/2)\operatorname{ess\,sup}_{x\in[\rho/2,\rho]}\big\|P(x)^{1/2}\big\|^2 \operatorname{ess\,sup}_{x\in[\rho/2,\rho]}\|\theta_\rho''(x)\hat u(x)\|^2_{\mathbb{C}^m} \\ &\quad \times \operatorname{ess\,sup}_{x\in[(\rho/2,\rho)]}\big\|R(x)^{-1/2}P(x)R(x)^{-1/2}\big\| < \infty, \end{aligned} \tag{3.37}$$

and

$$\begin{aligned} &\int_{\rho/2}^{\rho} \big(R^{-1}\theta_\rho'(P\hat u'), RR^{-1}\theta_\rho'(P\hat u')\big)_{\mathbb{C}^m} \\ &\le \big\|(\theta_\rho')^2\big\|_{L^\infty([\rho/2,\rho])} \operatorname{ess\,sup}_{x\in[\rho/2,\rho]}\big\|P(x)\hat u'(x)\big\|^2_{\mathbb{C}^m} \int_{\rho/2}^{\rho} dx\, \big\|R^{-1}\big\| < \infty, \end{aligned} \tag{3.38}$$

and thus $u_\rho \in \mathrm{dom}(T_{\min,0,\infty})$. Hence u_ρ satisfies (3.31), that is,

$$\int_0^\infty dx\, \theta_\rho(x)^2(\hat u(x), R(x)\hat u(x))_{\mathbb{C}^m} < \int_0^\infty dx\, (u_\rho(x), R(x)(Lu_\rho)(x))_{\mathbb{C}^m}\,. \tag{3.39}$$

Since

$$\big[(u\rho, R(Lu_\rho))_{\mathbb{C}^m} + (R(Lu_\rho), u_\rho)_{\mathbb{C}^m}\big]/2 = \big(\theta_\rho'\big)^2(\hat u, P\hat u)_{\mathbb{C}^m} - \big[\theta_\rho\theta_\rho'(\hat u, P\hat u)_{\mathbb{C}^m}\big]' \tag{3.40}$$

and $\theta_\rho(\rho) = \theta'_\rho(0) = 0$, one obtains

$$\begin{aligned}
&\int_0^{\rho/2} dx\, (\hat u(x), R(x)\hat u(x))_{\mathbb{C}^m} < \int_0^\infty dx\, \theta_\rho(x)^2 (\hat u(x), R(x)\hat u(x))_{\mathbb{C}^m} \\
&< \int_0^\infty (u_\rho(x), R(x)(Lu_\rho)(x))_{\mathbb{C}^m} \\
&= (1/2)\int_0^\infty (u_\rho(x), R(x)(Lu_\rho)(x))_{\mathbb{C}^m} + (1/2)\int_0^\infty (R(x)(Lu_\rho)(x), u_\rho(x))_{\mathbb{C}^m} \\
&= \int_0^\infty \theta'_\rho(x)^2 (\hat u(x), P(x)\hat u(x))_{\mathbb{C}^m} \\
&= \int_0^\infty dx\, \theta'_\rho(x)^2 \big(R(x)^{1/2}\hat u(x), R(x)^{-1/2}P(x)R(x)^{-1/2}R(x)^{1/2}\hat u(x)\big)_{\mathbb{C}^m} \\
&\le \frac{C^2}{\rho^2} \operatorname{ess\,sup}_{x\in[\rho/2,\rho]} \big\|R(x)^{-1/2}P(x)R(x)^{-1/2}\big\| \int_{\rho/2}^\rho dx\, (\hat u(x), R(x)\hat u(x))_{\mathbb{C}^m} \\
&\underset{\rho\uparrow\infty}{=} O(1)\int_{\rho/2}^\rho dx\, (\hat u(x), R(x)\hat u(x))_{\mathbb{C}^m} \underset{\rho\uparrow\infty}{=} o(1)
\end{aligned} \tag{3.41}$$

for some constant $C > 0$, implying $\hat u = 0$ a.e. on $[0,\infty)$. □

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