Non-universality in Ising models with four spin interaction.

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ABSTRACT. *We consider two bidimensional classical Ising models, coupled by a weak interaction bilinear in the energy densities of the two systems; the model contains, as limiting cases, the Ashkin-Teller and the Eight-vertex models for certain values of their parameters. We write the energy-energy correlations and the specific heat as Grassman integrals formally describing Dirac 1+1 dimensional interacting massive fermions on a lattice, and an expansion based on Renormalization Group is written for them, convergent up to temperatures very close to the critical temperature and for weak coupling. The critical behaviour is determined by critical indices which are continuous functions of the coupling.*

KEY WORDS *Coupled Ising models, non-universality, Renormalization Group, fermions, critical indices.*

1. Introduction and main results

1.1 *Motivations.* It is well known that the partition function and many correlations of the $d = 2$ classical *Ising model* can be written as Grassman integrals formally describing a system of $d = 1{+}1$ *free* fermions, see [H],[SML],[K],[MW],[S],[ID]. This mapping of the Ising model in terms of a fermionic theory is quite useful; as the corresponding action is quadratic in the Grassman variables, it can be diagonalized by a Bogolubov transformation so that the partition function and many observables can be exactly computed. However, even a slight perturbation of the Ising model makes the corresponding fermionic action *non-quadratic*. The relationship between spin Ising-like models and interacting fermionic models is deeply investigated in the literature; it is for instance claimed that models of coupled Ising models with an interaction quartic in the spin, like the *Eight-vertex* model, are in the same *class of universality* of models of *interacting* $d = 1 + 1$ fermions in the continuum, like the *massive Thirring* model or the *massive Luttinger* model, see for instance [N], [LP], [PB]. This relationship is however found under many approximations and it was not possible up to now to use it to prove rigorous results about such spin models. In this paper we consider two Ising models coupled by a general interaction bilinear in the spin densities; the model contains as particular cases the Ashkin-Teller and Eight-vertex model for certain values of their parameters. We clarify the relation between this general spin model and systems of interacting fermions by writing the partition function and some correlations as Grassman integrals with a formal action resembling but not coinciding with the action of the massive Thirring or Luttinger model; the main difference (in addition to the fact that the formal action describes fermions in a lattice, not in the continuum) is that it does not verify some special symmetries (like invariance under local Gauge transformations) which are generally verified in models of interacting fermions and which play an important role in their analysis. The main interest in this representation in terms of Grassman integrals is that we can apply the Renormalization Group techniques developed for interacting fermions (see for instance [BG] or [GM]) for writing a convergent perturbative expansion for the partition function and some correlations of the spin model we are considering. A naive power series expansion in the coupling is not expected to be convergent close to the critical temperature (the free and interacting model are not analytically close) and one needs a *resummed* or *renormalized* expansion which is provided by Renormalization Group.

1.2 *Spin systems with quartic interactions.* We consider two Ising models coupled via a four spin interaction bilinear in the energy densities of the two sublattices. Given $\Lambda_M \in Z^2$ a square lattice with side M and periodic boundary condition, we call $\mathbf{x} = (x, x_0)$ a site of Λ_M. If $\sigma^{(1)}_{\mathbf{x}} = \pm 1$ and $\sigma^{(2)}_{\mathbf{x}} = \pm 1$, we write the following Hamiltonian

$$H_\Lambda(\sigma^{(1)}, \sigma^{(2)}) = H_I(\sigma^{(1)}) + H_I(\sigma^{(2)}) + V(\sigma^{(1)}, \sigma^{(2)}) \tag{1.1}$$

where, if $\alpha = 1,2$ and $\beta = 1,2$

$$H_I(\sigma^{(\alpha)}) = -\sum_{x,x_0=1}^{M} [J_1^{(\alpha)}\sigma^{(\alpha)}_{x,x_0}\sigma^{(\alpha)}_{x+1,x_0} + J_2^{(\alpha)}\sigma^{(\alpha)}_{x,x_0}\sigma^{(\alpha)}_{x,x_0+1}] \tag{1.2}$$

$$\begin{aligned} V(\sigma^{(1)},\sigma^{(2)}) = &-\lambda a[\sigma^{(1)}_{x,x_0}\sigma^{(1)}_{x+1,x_0}\sigma^{(2)}_{x,x_0}\sigma^{(2)}_{x+1,x_0} + \sigma^{(1)}_{x,x_0}\sigma^{(1)}_{x,x_0+1}\sigma^{(2)}_{x,x_0}\sigma^{(2)}_{x,x_0+1}] \\ &-\lambda b[\sigma^{(1)}_{x,x_0}\sigma^{(1)}_{x+1,x_0}\sigma^{(2)}_{x,x_0}\sigma^{(2)}_{x,x_0+1} + \sigma^{(1)}_{x,x_0}\sigma^{(1)}_{x,x_0+1}\sigma^{(2)}_{x-1,x_0+1}\sigma^{(2)}_{x,x_0+1}] \\ &-\lambda c\sum_{\alpha=1,2}[\sigma^{(\alpha)}_{x,x_0}\sigma^{(\alpha)}_{x+1,x_0}\sigma^{(\alpha)}_{x,x_0}\sigma^{(\alpha)}_{x,x_0+1} + \sigma^{(\alpha)}_{x,x_0}\sigma^{(\alpha)}_{x+1,x_0}\sigma^{(\alpha)}_{x+1,x_0-1}\sigma^{(\alpha)}_{x+1,x_0}] \end{aligned} \tag{1.3}$$

The above Hamiltonian can be a model for a number of physical problems; for instance two magnetic spin planes coupled by an interplanar interaction quartic in the spins (in this interpretation $\sigma^{(1)}$ and $\sigma^{(2)}$ are the spin of the first or second plane). Moreover it contains, as limiting cases, two well known models, the *Ashkin-Teller* and the *Eight-vertex* model.

1.3 *The Ashkin-Teller model.* Ashkin and Teller introduced their model as a generalization of the Ising model to a four state system. Each site $\mathbf{x}$ is occupied by one of the four kinds of atom, said A,B,C,D, and two neighboring atoms interact with an energy ε_0 for AA, BB, CC, DD; ε_1 for AB, CD, ε_2 for AC, BD, ε_3 for AD, BC. It is quite easy to express this model in terms of Ising spins, see [F]. One associates to each site of the lattice two spins $\sigma^{(1)}_{\mathbf{x}}$ and $\sigma^{(2)}_{\mathbf{x}}$, see Fig.1; then $\sigma^{(1)}_{\mathbf{x}}, \sigma^{(2)}_{\mathbf{x}} = (+,+)$ if there is an A atom at site $\mathbf{x}$, $(+,-)$ if there is an atom B, $(-,+)$ if there is an atom C and $(-,-)$ if there is an atom D. Then the hamiltonian is given by

$$\begin{aligned} H_\Lambda(\sigma^{(1)},\sigma^{(2)}) &= H_I(\sigma^{(1)}) + H_I(\sigma^{(2)}) \\ -J''\sum_{x,x_0=1}^{M}[\sigma^{(2)}_{x,x_0}\sigma^{(2)}_{x,x_0+1}\sigma^{(1)}_{x,x_0}\sigma^{(1)}_{x,x_0+1} &+ \sigma^{(2)}_{x,x_0}\sigma^{(2)}_{x+1,x_0}\sigma^{(1)}_{x,x_0}\sigma^{(1)}_{x+1,x_0}] - J_0 \end{aligned} \tag{1.4}$$

where $H_I(\sigma^{(\alpha)})$ is given by (1.2) with $J_1^{(\alpha)} = J_2^{(\alpha)} = J^{(\alpha)}$ and

$$\begin{aligned} -J^{(1)} &= (\varepsilon_0+\varepsilon_1-\varepsilon_2-\varepsilon_3)/4 \quad -J^{(2)} = (\varepsilon_0+\varepsilon_2-\varepsilon_3-\varepsilon_1)/4 \\ -J'' &= (\varepsilon_0+\varepsilon_3-\varepsilon_1-\varepsilon_2)/4 \quad -J_0 = (\varepsilon_0+\varepsilon_1+\varepsilon_2+\varepsilon_3)/4 \end{aligned} \tag{1.5}$$

The model is not soluble, except in some particular case, and we refer to [Ba] chapt.12 and references therein for a review of the main results about it; it is of the form (1.1) with $b = c = 0$.

1.4 *The Eight-vertex model.* In such model one associates to each site of a square lattice a vertex with four arrows with different orientations, see [Ba] chapt. 10.

Among the sixteen possibilities, only eight kind of vertices are allowed, and an energy ε_i, $i = 1, .., 8$, is associated to each of them. It was solved in [Ba] if $\varepsilon_1 = \varepsilon_2$, $\varepsilon_3 = \varepsilon_4$, $\varepsilon_5 = \varepsilon_6$, $\varepsilon_7 = \varepsilon_8$ and for suitable values of the ε_i it reduces to the *ice-model*, solved in [Li]. Such models are called *vertex models* and were introduced to describe crystal with idrogen bounding. As explained in [Ba] or [W] they can be written in terms of the following Hamiltonian, see Fig. 2

$$H_\Lambda(\sigma^{(1)},\sigma^{(2)}) = -\sum_{x,x_0}[J\sigma_{x,x_0+1}\sigma_{x+1,x_0} + J'\sigma_{x,x_0}\sigma_{x+1,x_0+1}$$

$$+J''\sigma_{x,x_0}\sigma_{x+1,x_0+1}\sigma_{x,x_0+1}\sigma_{x+1,x_0}] \tag{1.6}$$

with

$$\varepsilon_1 = \varepsilon_2 = -J - J' - J'' \quad \varepsilon_3 = \varepsilon_4 = J + J' - J'' \tag{1.7}$$

$$\varepsilon_5 = \varepsilon_6 = J' - J + J'' \quad \varepsilon_7 = \varepsilon_8 = -J' + J + J''$$

If $J'' = 0$ the above hamiltonian decouples in the hamiltonian of two Ising models, involving spins located in two sublattices, one with $x + x_0$ equal to an even number and the other to an odd number. Each site of the sublattices is in the center of the unit square of the other one. Calling $\sigma^{(1)}$ and $\sigma^{(2)}$ the spin located in the first or the second sublattice, relabeling the spins and performing a rotation of $\frac{\pi}{4}$ the hamiltonian can be written as, see Fig.3

$$H_\Lambda(\sigma^{(1)},\sigma^{(2)}) = H_I(\sigma^{(1)}) + H_I(\sigma^{(2)})$$

$$-J''\sum_{x,x_0=1}^{M}[\sigma^{(2)}_{x,x_0}\sigma^{(2)}_{x,x_0+1}\sigma^{(1)}_{x,x_0}\sigma^{(1)}_{x+1,x_0} + \sigma^{(1)}_{x,x_0}\sigma^{(1)}_{x,x_0+1}\sigma^{(2)}_{x-1,x_0+1}\sigma^{(2)}_{x,x_0+1}] \tag{1.8}$$

where

$$H_I(\sigma^{(\alpha)}) = -\sum_{x,x_0=1}^{M} J[\sigma^{(\alpha)}_{x,x_0}\sigma^{(\alpha)}_{x+1,x_0} + \sigma^{(\alpha)}_{x,x_0}\sigma^{(\alpha)}_{x,x_0+1}]$$

which is again of the form (1.1) with $a = c = 0$ (of course this identification is exact if boundary conditions in the Eight-vertex model are chosen properly).

1.5 *Main results.* We will consider in this paper the case

$$J_1^{(2)} = J_2^{(1)} = J_1^{(2)} = J_2^{(2)} = J \tag{1.9}$$

If $\lambda = 0$ the problem reduces to the Ising model whose critical temperature is given by $\tanh\beta_c J_0 = \sqrt{2} - 1 \equiv t_c$ where $J \equiv \beta J_0$ and β^{-1} is the temperature. The interaction changes in general the value of the critical temperature by terms $O(\lambda)$, and it is instead technically convenient to fix its value still in correspondence of $t_c = \sqrt{2} - 1$, by choosing properly the molecular energy parameter J as a function of λ; in this way the critical temperature of the system with $\lambda = 0$ and $\lambda \neq 0$ is the

same. We consider then the model (1.1) with J_r replacing J, and we will choose $J_r = J + O(\lambda)$ so that the critical temperature is in correspondence $t_c = \sqrt{2} - 1$.
We define

$$< O(\mathbf{x})O(\mathbf{y}) >_\Lambda = \frac{1}{Z_\Lambda} \sum_{\substack{\sigma^{(1)}_{\mathbf{x}}, \sigma^{(2)}_{\mathbf{x}} = \pm 1 \\ \mathbf{x} \in \Lambda_M}} O(\mathbf{x})O(\mathbf{y}) e^{-H_\Lambda(\sigma^{(1)}, \sigma^{(2)})} \tag{1.10}$$

where $Z_\Lambda = \sum_{\substack{\sigma^{(1)}_{\mathbf{x}}, \sigma^{(2)}_{\mathbf{x}} = \pm 1 \\ \mathbf{x} \in \Lambda_M}} e^{-H_\Lambda(\sigma^{(1)}, \sigma^{(2)})}$ is the *Partition function.* The specific heat is $C_v = \frac{\partial^2}{\partial J^2} \frac{1}{|\Lambda|} \log Z_\Lambda$ and the correlation function of the observable $O(\mathbf{x})$

$$< O(\mathbf{x})O(\mathbf{y}) >_{\Lambda,T} = < O(\mathbf{x})O(\mathbf{y}) >_\Lambda - < O(\mathbf{x}) >_\Lambda < O(\mathbf{y}) >_\Lambda . \tag{1.11}$$

We will prove the following theorem, where $f(\lambda) \simeq f_1(\lambda)$ means $|f(\lambda) - f_1(\lambda)| \leq \frac{|f_1(\lambda)|}{2}$.

Theorem 1. *Consider the hamiltonian (1.1), with J_r replacing J, and assume that $|a+b| > 0$ and (1.9). There exist $\varepsilon > 0, a_3 > 0$ constants and a function $\nu(\lambda) = \tanh J_r - \tanh J$ such that, for $|\lambda| \leq \varepsilon$ and $|t - t_c| \geq e^{-\frac{1}{a_3 \lambda^2}}$, the energy-energy correlation verifies*

$$\lim_{|\Lambda| \to \infty} < \sigma^{(\alpha)}_{\mathbf{x}} \sigma^{(\alpha)}_{\mathbf{x}'} \sigma^{(\alpha)}_{\mathbf{y}} \sigma^{(\alpha)}_{\mathbf{y}'} >_{\Lambda,T} = \Omega^{(\alpha),a}(\mathbf{x},\mathbf{y}) + \Omega^{(\alpha),b}(\mathbf{x},\mathbf{y}) + \Omega^{(\alpha),c}(\mathbf{x},\mathbf{y}) \tag{1.12}$$

where $\mathbf{x}, \mathbf{x}'$ and $\mathbf{y}, \mathbf{y}'$ are nearest neighbor pairs respectively and, for any integer positive N

$$\begin{aligned} |\Omega^{(\alpha)a}(\mathbf{x},\mathbf{y})| &\leq \frac{1}{|\mathbf{x}-\mathbf{y}|^{2+\eta_1}} \frac{C_N}{1 + (\Delta|\mathbf{x}-\mathbf{y}|)^N} \\ |\Omega^{(\alpha),b}(\mathbf{x},\mathbf{y})| &\leq \frac{1}{|\mathbf{x}-\mathbf{y}|^{2+\eta_3}} \frac{C_N}{1 + (\Delta|\mathbf{x}-\mathbf{y}|)^N} \\ |\Omega^{(\alpha),c}(\mathbf{x},\mathbf{y})| &\leq \frac{1}{|\mathbf{x}-\mathbf{y}|^{2+\tau}} \frac{C_N}{1 + (\Delta|\mathbf{x}-\mathbf{y}|)^N} \end{aligned} \tag{1.13}$$

where C_N, τ are positive constants, $\Delta = |t - t_c|^{1+\eta_2}$, and

$$\eta_1(\lambda) \simeq -a_1 \lambda \quad \eta_2(\lambda) \simeq a_2 \lambda \quad \eta_3(\lambda) \simeq a_1 \lambda \tag{1.14}$$

with $a_1 > 0, a_2 > 0$ constants. If $1 \leq |\mathbf{x}| \leq \Delta^{-1}$

$$\Omega^{(\alpha)a}(\mathbf{x},\mathbf{y}) = \frac{1 + f^{(\alpha),a}_\lambda(\mathbf{x},\mathbf{y})}{\tilde{Z}_0^2 (x_0^2 + x^2)^{1+\eta_1}} \quad \Omega^{(\alpha)b}(\mathbf{x},\mathbf{y}) = \frac{1 + f^{(\alpha),b}_\lambda(\mathbf{x},\mathbf{y})}{\tilde{Z}_0^2 (x_0^2 + x^2)^{1+\eta_3}} \tag{1.15}$$

and $|\Omega^{(\alpha),c}(\mathbf{x},\mathbf{y})| \leq \frac{C}{|\mathbf{x}-\mathbf{y}|^{2+\tau}}$, where $\tilde{Z}_0 > 0$ is a constant and $f_\lambda(\mathbf{x},\mathbf{y})$ are smooth bounded $O(\lambda)$ functions. Finally the specific heat verifies

$$C_v = \tilde{f}_{1,\lambda} \frac{1}{\eta_1}[1 - |\Delta|^{2\eta_1}] + \tilde{f}_{2,\lambda} \frac{1}{\eta_3}[1 - |\Delta|^{2\eta_3}] + \tilde{f}_{3,\lambda} \tag{1.16}$$

where $C_i \leq |\tilde{f}_{i,\lambda}| \leq \tilde{C}_i$ *with* $C_i, \tilde{C}_i$ *constants and* $i = 1, 2, 3$.

1.6 *Remarks.* The above theorem will be proved by writing the partition function of the model (1.1) as a Grassman integral, formally describing four kinds of interacting fermions. The presence of V in (1.1) has the effect that the fermionic action contains terms which are quartic in the fermions, so the corresponding fermionic theory is an *interacting* one. One can perform a change of variables and in the new fermionic variables the partition function becomes identical to the one of a system of four kinds of *Majorana fermions*, two massive and the other two with a mass which is vanishing at $t = t_c$. We can *integrate out* the heavy fermions, and we arrive to a Grassman integral involving only two kinds of Majorana fermions with a small mass (vanishing at the critical temperature). By performing a Bogolubov transformation we can write the Grassman integral of the partition function in terms of the lattice version of a system of Dirac fermions on a lattice, coupled by non local interactions. We can integrate such fermions using Renormalization Group methods, by writing the partition function as a series in a set of running coupling constants; due to special symmetries there is only one *relevant* running coupling constant, corresponding to the effective mass at a certain scale; the other running couplings are dimensionally *marginal*. Finally also the correlation functions and the specific heat can be written as Grassman integrals, and one has to introduce other running coupling constants; the critical indices are related to the *anomalous flow* under Renormalization Group of such running coupling constants and of the effective mass. If $|a + b| > 0$ the system is not in the same "universality class" of the Ising model; in this case one expects $C_1 \log(|t - t_c|^{-1}) \leq |C_v| \leq C_2 \log(|t - t_c|^{-1})$, for some positive λ-independent constants C_1, C_2, a property which is clearly contradicted by (1.16). In the particular case $a = c = 0$ our model reduces to the Eight-vertex model, which is exactly soluble [Ba]; our results agree with the informations obtained by the exact solution, in which non universal critical behaviour is found. In the case $a = b = 0$ our model reduces to a model of two *non interacting* Ising models with nearest and next to nearest neighbor interaction, which was studied in [PS] (which is indeed the first paper in which fermionic RG methods were applied to classical Ising-like models, and it is a major source of inspiration for the present work). In this case $C_1 \log(|t - t_c|^{-1}) \leq |C_v| \leq C_2 \log(|t - t_c|^{-1})$ (up to $t = t_c$) as a consequence of the fact that the interaction is in this case *irrelevant* (while if $|a + b| > 0$ is *marginal* in the Renormalization Group sense).

We remark finally that our fermionic techniques should be allow us to consider the case in which (1.9) does not hold, or the case $t = t_c$. In particular the condition $|t - t_c| \geq e^{-\frac{1}{a_3 \lambda^2}}$ is used to control the flow of the effective interaction, but it could possibly be removed by exploiting suitable cancellations at every order of the expansion, as in the theory of $d = 1$ interacting Fermi systems, see for instance [BGPS] and [BM]. We hope to return on this in a future publication.

2. Fermionic representation

2.1 *Grassman integrals.* If $\lambda = 0$ the hamiltonian (1.1) is given by the sum of two independent Ising model hamiltonians, and the partition function is given by $Z_I^{(1)} Z_I^{(2)}$ where

$$Z_I^{(\alpha)} = \sum_{\substack{\sigma_{\mathbf{x}}^{(\alpha)}=\pm 1\\ \mathbf{x}\in\Lambda_M}} e^{-H_I(\sigma^{(\alpha)})}. \tag{2.1}$$

It is well known that such partion function can be written in terms of Grassman integrals, so we recall first their definition. Grassman variables η_α, $\alpha = 1, 2, .., 2n$, ,n even, are anticommuting variables satisfying

$$\{\eta_\alpha, \eta_{\alpha'}\} = 0 \qquad \eta_\alpha^2 = 0 \tag{2.2}$$

The Grassman integration $\int d\eta_\alpha$ is a linear operation defined as

$$\int d\eta_\alpha = 0 \qquad \int d\eta_\alpha \eta_\alpha = 1 \tag{2.3}$$

It holds that

$$\int \prod_\alpha d\eta_\alpha e^{\frac{1}{2}\sum_{\alpha,\beta}\eta_\alpha A_{\alpha,\beta}\eta_\beta} = \mathrm{Pf}A \tag{2.4}$$

where A is an even antisymmetric $2n$-matrix and PfA denotes the *Pfaffian.* It holds

$$\int d\eta_1 ... d\eta_{2n} \exp \frac{1}{2} \sum_{\alpha,\beta} \eta_\alpha A_{\alpha,\beta}\eta_\beta = \tag{2.5}$$

$$\int d\eta_{2n}...d\eta_1 \prod_{\alpha<\beta}(1 + A_{\alpha,\beta}\eta_\alpha\eta_\beta) = \frac{1}{2^n n!}\sum_p (-1)^p A_{p_1,p_2}A_{p_3,p_4}...A_{p_{2n-1},p_{2n}} \equiv \mathrm{Pf}A$$

where the sum is over all the permutations. We can consider another set of Grassman variables η^+_α, $\alpha = 1, .., 2n$, and

$$\int \prod_\alpha d\eta_\alpha \prod_\alpha d\eta_\alpha^+ \exp[\sum_{\alpha,\beta} \eta_\alpha B_{\alpha,\beta}\eta_\beta^+] = \mathrm{Det}B \tag{2.6}$$

The well known relation $(\mathrm{Pf}A)^2 = \det A$ can be quite easily deduced by the above Grassman integrals; it can be written as

$$\int \prod_\alpha d\eta_\alpha d\eta_\alpha^+ e^{\eta_\alpha A_{\alpha,\beta}\eta_\beta^+} = \int \prod_\alpha d\eta_\alpha^{(1)} e^{\frac{1}{2}\eta_\alpha^{(1)} A_{\alpha,\beta}\eta_\beta^{(1)}} \int \prod_\alpha d\eta_\alpha^{(2)} e^{\frac{1}{2}\eta_\alpha^{(2)} A_{\alpha,\beta}\eta_\beta^{(2)}} \tag{2.7}$$

which can be proved by the change of variables

$$\eta_\alpha^+ = \frac{1}{\sqrt{2}}(\eta_\alpha^{(1)} + i\eta_\alpha^{(2)}) \quad \eta_\alpha = \frac{1}{\sqrt{2}}(\eta_\alpha^{(1)} - i\eta_\alpha^{(2)}) \tag{2.8}$$

in $\int d\eta_\alpha d\eta_\alpha^+ e^{\eta_\alpha A_{\alpha,\beta}\eta_\beta^+}$. Then $d\eta_\alpha d\eta_\alpha^+ = id\eta_\alpha^{(1)} d\eta_\alpha^{(2)}$ and

$$\eta_\alpha A_{\alpha,\beta}\eta_\beta^+ = \frac{1}{2}\eta_\alpha^{(1)} A_{\alpha,\beta}\eta_\beta^{(1)} + \frac{1}{2}\eta_\alpha^{(2)} A_{\alpha,\beta}\eta_\beta^{(2)} \tag{2.9}$$

as

$$\eta_\alpha^{(1)} A_{\alpha,\beta}\eta_\beta^{(2)} - \eta_\alpha^{(2)} A_{\alpha,\beta}\eta_\beta^{(1)} = \eta_\alpha^{(1)} A_{\alpha,\beta}\eta_\beta^{(2)} - \eta_\beta^{(1)} A_{\beta,\alpha}\eta_\alpha^{(2)} = 0 \tag{2.10}$$

2.2 *The fermionic representation of the Ising model.* It is well known that the partition function can be written as a Grassman integral. It is proved in [K],[H],[MW],[S] that

$$Z_I^{(\alpha)} = (\cosh J_r)^B 2^S \frac{1}{2} \int \prod_{\mathbf{x}\in\Lambda_M} dH_{\mathbf{x}}^{(\alpha)} d\bar{H}_{\mathbf{x}}^{(\alpha)} dV_{\mathbf{x}}^{(\alpha)} d\bar{V}_{\mathbf{x}}^{(\alpha)} [-e^{S_{+,+}} + e^{S_{+,-}} + e^{S_{-,+}} + e^{S_{-,-}}] \tag{2.11}$$

where B is the total number of bonds and S is the total number of sites,

$$S_{\varepsilon,\varepsilon'}^{(\alpha)} = \sum_{\mathbf{x}\in\Lambda_M} \tanh J_r [\bar{H}_{x,x_0}^{(\alpha)} H_{x+1,x_0}^{(\alpha)} + \bar{V}_{x,x_0}^{(\alpha)} V_{x,x_0+1}^{(\alpha)}] +$$

$$\sum_{\mathbf{x}\in\Lambda_M} [\bar{H}_{x,x_0}^{(\alpha)} H_{x,x_0}^{(\alpha)} + \bar{V}_{x,x_0}^{(\alpha)} V_{x,x_0}^{(\alpha)} + \bar{V}_{x,x_0}^{(\alpha)} \bar{H}_{x,x_0}^{(\alpha)} + V_{x,x_0}^{(\alpha)} \bar{H}_{x,x_0}^{(\alpha)} + H_{x,x_0}^{(\alpha)} \bar{V}_{x,x_0}^{(\alpha)} + V_{x,x_0}^{(\alpha)} H_{x,x_0}^{(\alpha)}] \tag{2.12}$$

and $H_{\mathbf{x}}^{(\alpha)}, \bar{H}_{\mathbf{x}}^{(\alpha)}, V_{\mathbf{x}}^{(\alpha)}, \bar{V}_{\mathbf{x}}^{(\alpha)}$ are *Grassman variables* such that

$$\bar{H}_{x,x_0+M}^{(\alpha)} = \varepsilon \bar{H}_{x,x_0}^{(\alpha)} \qquad \bar{H}_{x+M,x_0}^{(\alpha)} = \varepsilon' \bar{H}_{x,x_0}^{(\alpha)}$$

$$H_{x,x_0+M}^{(\alpha)} = \varepsilon H_{x,x_0}^{(\alpha)} \quad H_{x+M,x_0}^{(\alpha)} = \varepsilon' H_{x,x_0}^{(\alpha)} \tag{2.13}$$

and identical relations hold for the variables $V^{(\alpha)}, \bar{V}^{(\alpha)}$. The Grassman integration $\int \prod_x dH_x^{(\alpha)} d\bar{H}_x^{(\alpha)}$ is defined as the linear functional on the Grassmanian algebra, such that, given a monomial $Q(H^{(\alpha)}, \bar{H}^{(\alpha)})$ in the variables $H_{\mathbf{x}}^{(\alpha)}, \bar{H}_{\mathbf{x}}^{(\alpha)}$, $\mathbf{x} \in \Lambda_M$, its value is 0, except in the case $Q(H^{(\alpha)}, \bar{H}^{(\alpha)}) = \prod_{\mathbf{x}} H_{\mathbf{x}}^{(\alpha)} \bar{H}_{\mathbf{x}}^{(\alpha)}$, up to a permutation of the variables. In this case the value of the functional is determined, by using the anticommuting properties of the variables, by the condition

$$\int \left[\prod_{\mathbf{x}\in\Lambda_M} d\bar{H}_{\mathbf{x}}^{(\alpha)} dH_{\mathbf{x}}^{(\alpha)}\right] \prod_{\mathbf{x}\in\Lambda_M} H_{\mathbf{x}}^{(\alpha)} \bar{H}_{\mathbf{x}}^{(\alpha)} = 1 \; . \tag{2.14}$$

In a similar way is defined the Grassman integration for $V^{(\alpha)}, \bar{V}^{(\alpha)}$, just exchanging $H, \bar{H}$ with $V, \bar{V}$. In order to prove (2.11) the starting point is the high temperature expansion

$$Z_I^{(\alpha)} = (\cosh J_r)^B 2^S \sum (\tanh J_r)^l \tag{2.15}$$

where the sum is over all the closed polygons which may have points but not sides in common, see Fig.4. One then replaces the Ising lattice with another lattice, called *second lattice*, in which each site is replaced by four surroundings sites or *terminals*, see Fig.5. If we associate $\bar{V}_{\mathbf{x}}$ with $N_{\mathbf{x}}$, $V_{\mathbf{x}}$ with $S_{\mathbf{x}}$, $\bar{H}_{\mathbf{x}}$ with $E_{\mathbf{x}}$ and $H_{\mathbf{x}}$ with $W_{\mathbf{x}}$, see Fig.5, it is easy to verify, by the rules of Grassman integrations, that $\int e^{S_{\varepsilon,\varepsilon'}}$ can be represented as a sum over polygon configurations over the second lattice. Such polygon configurations are defined such that each terminal must be coupled to another terminal, $E_{\mathbf{x}}$ may be coupled to W_{x+1,x_0}; $N_{x_0,x}$ may be coupled S_{x,x_0+1},; $E_{\mathbf{x}}$ may be coupled to $N_{\mathbf{x}}, W_{\mathbf{x}}, S_{\mathbf{x}}$; $N_{\mathbf{x}}$ may be coupled to $W_{\mathbf{x}}$ or $S_{\mathbf{x}}$; $W_{\mathbf{x}}$ may be coupled to $S_{\mathbf{x}}$, and no other types of couplings are allowed. An example of polygon configuration in the second lattice is in Fig.5. By a local identification of Ising configurations and configurations over the second lattice (see Fig.6), there is a correspondence between polygon configurations on the Ising lattice and polygon configurations on the second lattice. The correspondence is not one to one, as the configuration with zero lines in the Ising model corresponds to three configurations in the second lattice. However the signs add so that the sum over polygon configurations in the Ising lattice in (2.15) is exactly equal to the sum over polygon configurations in (2.11); the proof is based on a subtle combinatorial analysis first done in [K]. In order to ensure periodic boundary conditions, one has to sum over four terms as in (2.11), as explained in [MW]. We write $e^{S^{(\alpha)}_{\varepsilon,\varepsilon'}} = e^{S^{(\alpha),0}_{\varepsilon,\varepsilon'}} e^{S^{(\alpha),\nu}_{\varepsilon,\varepsilon'}}$ where $S^{(\alpha),0}_{\varepsilon,\varepsilon'}$ is given by (2.12) with J replacing J_r and

$$S^{(\alpha),\nu}_{\varepsilon,\varepsilon'} = \nu \sum_{\mathbf{x}\in\Lambda_M} [\bar{H}^{(\alpha)}_{x,x_0} H^{(\alpha)}_{x+1,x_0} + \bar{V}^{(\alpha)}_{x,x_0} V^{(\alpha)}_{x,x_0+1}] \tag{2.16}$$

If J is not constant but it depends on the bounds one obtains a similar formula in which $S^{(\alpha),0}_{\varepsilon,\varepsilon'}$ is given by

$$\begin{aligned} S^{(\alpha),0}_{\varepsilon,\varepsilon'} = & \sum_{\mathbf{x}} \tanh J^{(\alpha)}_{1;x,x_0;x+1,x_0} \bar{H}^{(\alpha)}_{x,x_0} H^{(\alpha)}_{x+1,x_0} + \tanh J^{(\alpha)}_{2;x,x_0;x,x_0+1} \bar{V}^{(\alpha)}_{x,x_0} V^{(\alpha)}_{x,x_0+1}]+ \\ & \sum_{\mathbf{x}} [\bar{H}^{(\alpha)}_{x,x_0} H^{(\alpha)}_{x,x_0} + \bar{V}^{(\alpha)}_{x,x_0} V^{(\alpha)}_{x,x_0} + \bar{V}^{(\alpha)}_{x,x_0} \bar{H}^{(\alpha)}_{x,x_0} + V^{(\alpha)}_{x,x_0} \bar{H}^{(\alpha)}_{x,x_0} + H^{(\alpha)}_{x,x_0} \bar{V}^{(\alpha)}_{x,x_0} + V^{(\alpha)}_{x,x_0} H^{(\alpha)}_{x,x_0}] \end{aligned} \tag{2.17}$$

and the factor $(\cosh J)^B$ is replaced by $\prod_b \cosh J_b$, where the product is over all the possible nearest neighbor bounds. We will call $Z^{(\alpha)}_I(J_{\mathbf{x},\mathbf{x}'})$ the Ising model partition function with non constant J.

2.3 *The fermionic representation of coupled Ising models.* The partition function of (1.1) is

$$Z_{2I} = \sum_{\substack{\sigma^{(1)}_{\mathbf{x}}=\pm 1\\ \mathbf{x}\in\Lambda_M}} \sum_{\substack{\sigma^{(2)}_{\mathbf{x}}=\pm 1\\ \mathbf{x}=\Lambda_M}} e^{-H_I(\sigma^{(1)})} e^{-H_I(\sigma^{(2)})} e^{-V(\sigma^{(1)},\sigma^{(2)})} \tag{2.18}$$

Let us consider first the essentially equivalent expression

$$\hat{Z}_{2I} = \sum_{\substack{\sigma^{(1)}=\pm 1\\ \mathbf{x}\in\Lambda_M}} \sum_{\substack{\sigma^{(2)}=\pm 1\\ \mathbf{x}\in\Lambda_M}} e^{-H_I(\sigma^{(1)})} e^{-H_I(\sigma^{(2)})} \times$$

$$\prod_{\mathbf{x}}[1+\lambda a\sigma^{(1)}_{x,x_0}\sigma^{(1)}_{x+1,x_0}\sigma^{(2)}_{x,x_0}\sigma^{(2)}_{x+1,x_0}]\prod_{\mathbf{x}}[1+\lambda a\sigma^{(1)}_{x,x_0}\sigma^{(1)}_{x,x_0+1}\sigma^{(2)}_{x,x_0}\sigma^{(2)}_{x,x_0+1}]$$

$$\prod_{\mathbf{x}}[1+\lambda b\sigma^{(1)}_{x,x_0}\sigma^{(1)}_{x+1,x_0}\sigma^{(2)}_{x,x_0}\sigma^{(2)}_{x,x_0+1}]\prod_{\mathbf{x}}[1+\lambda b\sigma^{(1)}_{x,x_0}\sigma^{(1)}_{x,x_0+1}\sigma^{(2)}_{x-1,x_0+1}\sigma^{(2)}_{x,x_0+1}] \quad (2.19)$$

$$\prod_{\alpha}\prod_{\mathbf{x}}[1+\lambda c\sigma^{(\alpha)}_{x,x_0}\sigma^{(\alpha)}_{x+1,x_0}\sigma^{(\alpha)}_{x,x_0}\sigma^{(\alpha)}_{x,x_0+1}]\prod_{\alpha}\prod_{\mathbf{x}}[1+\lambda c\sigma^{(\alpha)}_{x,x_0}\sigma^{(\alpha)}_{x+1,x_0}\sigma^{(\alpha)}_{x+1,x_0-1}\sigma^{(\alpha)}_{x+1,x_0}))]$$

Noting that

$$\sigma^{(\alpha)}_{x,x_0}\sigma^{(\alpha)}_{x+1,x_0}e^{-H_I(\sigma^{(\alpha)})} = \frac{\partial}{\partial J^{(\alpha)}_{1;x,x_0;x+1,x_0}} Z^{(\alpha)}_I(J^{(\alpha)}_{\mathbf{x},\mathbf{x}'})|_{\{J^{(\alpha)}_{\mathbf{x},\mathbf{x}'}\}=\{J^{(\alpha)}\}} \qquad (2.20)$$

and from (2.17) this derivative gives an extra factor $\tanh J^{(\alpha)} + \mathrm{sech}^2 J^{(\alpha)} \bar{H}^{(\alpha)}_{x,x_0} H^{(\alpha)}_{x+1,x_0}$ in (2.11). In the same way

$$\sigma^{(\alpha)}_{x,x_0}\sigma^{(\alpha)}_{x,x_0+1}e^{-H_I(\sigma^{(\alpha)})} = \frac{\partial}{\partial J^{(\alpha)}_{2;x,x_0;x,x_0+1}} Z^{(\alpha)}_I(J^{(\alpha)}_{\mathbf{x},\mathbf{x}'})|_{\{J^{(\alpha)}_{\mathbf{x},\mathbf{x}'}\}=\{J^{(\alpha)}\}} \qquad (2.21)$$

and this derivative gives a factor $\tanh J^{(\alpha)} + \mathrm{sech}^2 J^{(\alpha)} \bar{V}^{(\alpha)}_{x,x_0} V^{(\alpha)}_{x,x_0+1}$. We can write than, if $\delta_{+,+} = 1$ and $\delta_{+,-} = \delta_{-,+} = \delta_{-,-} = 2$

$$\hat{Z}_{2I} = \sum_{\varepsilon^{(1)},\varepsilon'^{(1)}} (-1)^{\delta_{\varepsilon^{(1)},\varepsilon'^{(1)}}} \sum_{\varepsilon^{(2)},\varepsilon'^{(2)}} (-1)^{\delta_{\varepsilon^{(2)},\varepsilon'^{(2)}}} \hat{Z}^{\varepsilon^{(1)},\varepsilon'^{(1)},\varepsilon^{(2)},\varepsilon'^{(2)}}_{2I} \qquad (2.22)$$

where

$$\hat{Z}^{\varepsilon^{(1)},\varepsilon'^{(1)},\varepsilon^{(2)},\varepsilon'^{(2)}}_{2I} = (\cosh J)^{2B} 2^{2S} \frac{1}{4} \qquad (2.23)$$

$$\int \prod_{\alpha=1}^{2} [\prod_{\mathbf{x}} dH^{(\alpha)}_{\mathbf{x}} d\bar{H}^{(\alpha)}_{\mathbf{x}} dV^{(\alpha)}_{\mathbf{x}} d\bar{V}^{(\alpha)}_{\mathbf{x}}] e^{S^{(1)}_{\varepsilon^{(1)},\varepsilon'^{(1)}}} e^{S^{(2)}_{\varepsilon^{(2)},\varepsilon'^{(2)}}}$$

$$\prod_{\mathbf{x}}[1+\lambda a(\tanh J + \mathrm{sech}^2 J \bar{H}^{(1)}_{x,x_0} H^{(1)}_{x+1,x_0})(\tanh J + \mathrm{sech}^2 J \bar{H}^{(2)}_{x,x_0} H^{(2)}_{x+1,x_0})]$$

$$\prod_{\mathbf{x}}[1+\lambda a(\tanh J + \mathrm{sech}^2 J \bar{V}^{(1)}_{x,x_0} V^{(1)}_{x,x_0+1})(\tanh J + \mathrm{sech}^2 J \bar{V}^{(2)}_{x,x_0} V^{(2)}_{x,x_0+1})]$$

$$\prod_{\mathbf{x}}[1+\lambda b(\tanh J + \mathrm{sech}^2 J \bar{H}^{(1)}_{x,x_0} H^{(1)}_{x+1,x_0})(\tanh J + \mathrm{sech}^2 J \bar{V}^{(2)}_{x,x_0} V^{(2)}_{x,x_0+1})]$$

$$\prod_{\mathbf{x}}[1+\lambda b(\tanh J+\mathrm{sech}^2 J\bar V^{(1)}_{x,x_0}V^{(1)}_{x,x_0+1})(\tanh J+\mathrm{sech}^2 J\bar H^{(2)}_{x-1,x_0+1}H^{(2)}_{x,x_0+1})]$$

$$\prod_{\alpha}\prod_{\mathbf{x}}[1+\lambda c(\tanh J+\mathrm{sech}^2 J\bar H^{(\alpha)}_{x,x_0}H^{(\alpha)}_{x+1,x_0})(\tanh J+\mathrm{sech}^2 J\bar V^{(\alpha)}_{x,x_0}V^{(\alpha)}_{x,x_0+1})]$$

$$\prod_{\alpha}\prod_{\mathbf{x}}[1+\lambda c(\tanh J+\mathrm{sech}^2 J\bar H^{(\alpha)}_{x,x_0}H^{(\alpha)}_{x+1,x_0})(\tanh J+\mathrm{sech}^2 J\bar V^{(\alpha)}_{x+1,x_0-1}V^{(\alpha)}_{x+1,x_0})]$$

The above expression can be rewritten as

$$\hat Z_{2I}^{\varepsilon^{(1)},\varepsilon'^{(1)},\varepsilon^{(2)},\varepsilon'^{(2)}}=(\cosh J)^{2B}2^{2S}\frac{1}{4}\int[\prod_{\alpha=1}^{2}\prod_{\mathbf{x}}dH^{(\alpha)}_{\mathbf{x}}d\bar H^{(\alpha)}_{\mathbf{x}}dV^{(\alpha)}_{\mathbf{x}}d\bar V^{(\alpha)}_{\mathbf{x}}e^{S^{(\alpha)}_{\varepsilon^{(\alpha)},\varepsilon'^{(\alpha)}}}]e^{\mathcal{V}} \tag{2.24}$$

with

$$\mathcal{V}=\mathcal{V}_a+\mathcal{V}_b+\mathcal{V}_c \tag{2.25}$$

and, if $f_i=log(1+\lambda[i]\tanh^2 J)$ and $[i]=a,b,c$

$$\mathcal{V}_a=\sum_{\mathbf{x}}[f_a+\tilde\lambda_a[\bar H^{(1)}_{x,x_0}H^{(1)}_{x+1,x_0}+\bar H^{(2)}_{x,x_0}H^{(2)}_{x+1,x_0}]+\lambda_a\bar H^{(1)}_{x,x_0}H^{(1)}_{x+1,x_0}\bar H^{(2)}_{\mathbf{x}}H^{(2)}_{x+1,x_0}]$$

$$+\sum_{\mathbf{x}}[f_a+\tilde\lambda_a[\bar V^{(1)}_{x,x_0}V^{(1)}_{x,x_0+1}+\bar V^{(2)}_{x,x_0}V^{(2)}_{x,x_0+1}]+\lambda_a\bar V^{(1)}_{\mathbf{x}}V^{(1)}_{x,x_0+1}\bar V^{(2)}_{x,x_0}V^{(2)}_{x,x_0+1}] \tag{2.26}$$

$$\mathcal{V}_b=\sum_{\mathbf{x}}[f_b+\tilde\lambda_b[\bar H^{(1)}_{x,x_0}H^{(1)}_{x+1,x_0}+\bar V^{(2)}_{x,x_0}V^{(2)}_{x,x_0+1}]+\lambda_b\bar H^{(1)}_{x,x_0}H^{(1)}_{x+1,x_0}\bar V^{(2)}_{\mathbf{x}}V^{(2)}_{x,x_0+1}]$$

$$+\sum_{\mathbf{x}}[f_b+\tilde\lambda_b[\bar V^{(1)}_{x,x_0}V^{(1)}_{x,x_0+1}+\bar H^{(2)}_{x-1,x_0+1}H^{(2)}_{x,x_0+1}]+\lambda_b\bar V^{(1)}_{x,x_0}V^{(1)}_{x,x_0+1}\bar H^{(2)}_{x-1,x_0+1}H^{(2)}_{x,x_0+1}]$$

$$\mathcal{V}_c=\sum_{\mathbf{x}}\sum_{\alpha}[f_c+\tilde\lambda_c[\bar H^{(\alpha)}_{x,x_0}H^{(\alpha)}_{x+1,x_0}+\bar V^{(\alpha)}_{x,x_0}V^{(\alpha)}_{x,x_0+1}]+\lambda_c\bar H^{(\alpha)}_{x,x_0}H^{(\alpha)}_{x+1,x_0}\bar V^{(\alpha)}_{\mathbf{x}}V^{(\alpha)}_{x,x_0+1}]$$

$$+\sum_{\mathbf{x}}\sum_{\alpha}[f_c+\tilde\lambda_c[\bar V^{(\alpha)}_{x+1,x_0-1}V^{(\alpha)}_{x+1,x_0}+\bar H^{(\alpha)}_{x,x_0}H^{(\alpha)}_{x+1,x_0}]+\lambda_c\bar V^{(\alpha)}_{x+1,x_0-1}V^{(\alpha)}_{x+1,x_0}\bar H^{(\alpha)}_{\mathbf{x}}H^{(\alpha)}_{x+1,x_0}]$$

It is easy in fact to verify that

$$e^{f_i+\tilde\lambda_i[\bar H^{(\alpha)}_{\mathbf{x}}H^{(\alpha)}_{x+1,x_0}+\bar V^{(\alpha)}_{\mathbf{x}}V^{(\alpha)}_{x,x_0+1}]+\lambda_i\bar H^{(\alpha)}_{\mathbf{x}}H^{(\alpha)}_{x+1,x_0}\bar V^{(\beta)}_{\mathbf{x}}V^{(\beta)}_{x,x_0+1}}=\{(1+\lambda[i]\tanh^2 J)$$
$$[1+\tilde\lambda_i[\bar H^{(\alpha)}_{\mathbf{x}}H^{(\alpha)}_{x+1,x_0}+\bar V^{(\beta)}_{\mathbf{x}}V^{(\beta)}_{x,x_0+1}]+(\lambda_i+(\tilde\lambda_i)^2)\bar H^{(\alpha)}_{\mathbf{x}}H^{(\alpha)}_{x+1,x_0}\bar V^{(\beta)}_{\mathbf{x}}V^{(\beta)}_{x,x_0+1}]\} \tag{2.27}$$

hence the equality between (2.23) and (2.24) holds with the identification

$$\tilde\lambda_i(1+\lambda[i])\tanh^2 J)=\lambda[i]\mathrm{sech}^2 J\tanh J$$

$$(1+\lambda[i]\tanh^2 J)(\lambda_i+(\tilde\lambda_i)^2)=\lambda[i]\mathrm{sech}^4 J \tag{2.28}$$

An expression identical to (2.22),(2.23) holds for Z_{2I}, the only difference being that the relation with respect $\tilde\lambda_i,\lambda_i$ is slightly more complicated than (2.28), but for small λ again

$$\tilde\lambda_i=[i]\lambda(\tanh J\mathrm{sech}^2 J+O(\lambda))\quad \lambda_i=[i]\lambda(\mathrm{sech}^4 J+O(\lambda)) \tag{2.29}$$

2.4 *Massive and massless fermions.* We define

$$P^{(\alpha)}_{\varepsilon^{(\alpha)},\varepsilon'^{(\alpha)}}(dH^\alpha,dV^{(\alpha)})=\prod_{\mathbf{x}} dH^{(\alpha)}_{\mathbf{x}}d\bar H^{(\alpha)}_{\mathbf{x}}dV^{(\alpha)}_{\mathbf{x}}d\bar V^{(\alpha)}_{\mathbf{x}}e^{S^{(\alpha),0}_{\varepsilon^{(\alpha)},\varepsilon'^{(\alpha)}}} \tag{2.30}$$

It is convenient to perform the following change of variables [ID], [SP]

$$\bar H^{(\alpha)}_{\mathbf{x}}+iH^{(\alpha)}_{\mathbf{x}}=e^{i\frac{\pi}{4}}\psi^{(\alpha)}_{\mathbf{x}}-e^{i\frac{\pi}{4}}\chi^{(\alpha)}_{\mathbf{x}}\qquad \bar H^{(\alpha)}_{\mathbf{x}}-iH^{(\alpha)}_{\mathbf{x}}=e^{-i\frac{\pi}{4}}\bar\psi^{(\alpha)}_{\mathbf{x}}-e^{-i\frac{\pi}{4}}\bar\chi^{(\alpha)}_{\mathbf{x}}$$

$$\bar V^{(\alpha)}_{\mathbf{x}}+iV^{(\alpha)}_{\mathbf{x}}=\psi^{(\alpha)}_{\mathbf{x}}+\chi^{(\alpha)}_{\mathbf{x}}\qquad \bar V^{(\alpha)}_{\mathbf{x}}-iV^{(\alpha)}_{\mathbf{x}}=\bar\psi^{(\alpha)}_{\mathbf{x}}+\bar\chi^{(\alpha)}_{\mathbf{x}} \tag{2.31}$$

It holds that

$$P^{(\alpha)}_{\varepsilon^{(\alpha)},\varepsilon'^{(\alpha)}}(dH^\alpha,dV^{(\alpha)})=P^{(\alpha)}_{\varepsilon^{(\alpha)},\varepsilon'^{(\alpha)}}(d\psi^\alpha)P^{(\alpha)}_{\varepsilon^{(\alpha)},\varepsilon'^{(\alpha)}}(d\chi^\alpha)e^{Q(\chi^{(\alpha)},\psi^{(\alpha)})} \tag{2.32}$$

where, if $t=\tanh J$

$$P^\alpha_{\varepsilon,\varepsilon'}(d\psi^{(\alpha)})=[\prod_{\mathbf{x}\in\Lambda_M}d\psi^{(\alpha)}_{\mathbf{x}}d\bar\psi^{(\alpha)}_{\mathbf{x}}]\exp\{\frac{t}{4}\sum_{\mathbf{x}\in\Lambda_M}\psi^{(\alpha)}_{\mathbf{x}}(\partial_1-i\partial_0)\psi^{(\alpha)}_{\mathbf{x}}+\bar\psi^{(\alpha)}_{\mathbf{x}}(\partial_1+i\partial_0)\bar\psi^{(\alpha)}_{\mathbf{x}})$$

$$+\frac{t}{4}\sum_{\mathbf{x}\in\Lambda_M}[-i\bar\psi^{(\alpha)}_{\mathbf{x}}(\partial_1\psi^{(\alpha)}_{x}+\partial_0\psi^{(\alpha)}_{\mathbf{x}})+i\psi^{(\alpha)}_{\mathbf{x}}(\partial_1\bar\psi^{(\alpha)}_{\mathbf{x}}+\partial_0\bar\psi^{(\alpha)}_{\mathbf{x}})]$$

$$+\sum_{\mathbf{x}\in\Lambda_M}i(\sqrt{2}-1-t)\bar\psi^{(\alpha)}_{\mathbf{x}}\psi^{(\alpha)}_{\mathbf{x}}\} \tag{2.33}$$

where

$$\partial_1\psi^{(\alpha)}_{\mathbf{x}}=\psi^{(\alpha)}_{x+1,x_0}-\psi^{(\alpha)}_{\mathbf{x}}\qquad \partial_0\psi^{(\alpha)}_{x}=\psi^{(\alpha)}_{x,x_0+1}-\psi^{(\alpha)}_{\mathbf{x}} \tag{2.34}$$

Moreover

$$P^{(\alpha)}_{\varepsilon,\varepsilon'}(d\chi^{(\alpha)})=[\prod_{\mathbf{x}\in\Lambda_M}d\chi^{(\alpha)}_{\mathbf{x}}d\bar\chi^{(\alpha)}_{\mathbf{x}}]\exp\{\frac{t}{4}\sum_{\mathbf{x}\in\Lambda_M}\chi^{(\alpha)}_{\mathbf{x}}(\partial_1-i\partial_0)\chi^{(\alpha)}_{\mathbf{x}}+\bar\chi^{(\alpha)}_{\mathbf{x}}(\partial_1+i\partial_0)\bar\chi^{(\alpha)}_{\mathbf{x}}+$$

$$\frac{t}{4}\sum_{\mathbf{x}\in\Lambda_M}[-i\bar\chi^{(\alpha)}_{\mathbf{x}}(\partial_1\chi^{(\alpha)}_{x}+\partial_0\chi^{(\alpha)}_{\mathbf{x}})+i\chi^{(\alpha)}_{\mathbf{x}}(\partial_1\bar\chi^{(\alpha)}_{\mathbf{x}}+\partial_0\bar\chi^{(\alpha)}_{\mathbf{x}})]-\sum_{\mathbf{x}\in\Lambda_M}i(\sqrt{2}+1+t)\bar\chi^{(\alpha)}_{\mathbf{x}}\chi^{(\alpha)}_{\mathbf{x}}\} \tag{2.35}$$

and finally

$$Q^{(\alpha)}(\psi^{(\alpha)},\chi^{(\alpha)})=\sum_{\mathbf{x}}\frac{t}{4}\{-\psi^{(\alpha)}_{\mathbf{x}}(\partial_1\chi^{(\alpha)}_{\mathbf{x}}+i\partial_0\chi^{(\alpha)}_{\mathbf{x}})-\bar\psi^{(\alpha)}_{\mathbf{x}}(\partial_1\bar\chi^{(\alpha)}_{\mathbf{x}}-i\partial_0\bar\chi^{(\alpha)}_{\mathbf{x}})-$$

$$\chi^{(\alpha)}_{\mathbf{x}}(\partial_1\psi^{(\alpha)}_{\mathbf{x}}+i\partial_0\psi^{(\alpha)}_{\mathbf{x}})-\bar\chi^{(\alpha)}_{\mathbf{x}}(\partial_1\bar\psi^{(\alpha)}_{\mathbf{x}}-i\partial_0\bar\psi^{(\alpha)}_{\mathbf{x}})+i\bar\psi^{(\alpha)}_{\mathbf{x}}(\partial_1\chi^{(\alpha)}_{\mathbf{x}}-\partial_0\chi^{(\alpha)}_{\mathbf{x}}) \quad (2.36)$$
$$+\bar\psi^{(\alpha)}_{\mathbf{x}}(-\partial_1\chi^{(\alpha)}_{\mathbf{x}}-\partial_0\chi^{(\alpha)}_{\mathbf{x}})+\bar\chi^{(\alpha)}_{\mathbf{x}}(-\partial_1\psi^{(\alpha)}_{\mathbf{x}}-\partial_0\psi^{(\alpha)}_{\mathbf{x}})+\bar\chi^{(\alpha)}_{\mathbf{x}}(-\partial_1\psi^{(\alpha)}_{\mathbf{x}}+\partial_0\psi^{(\alpha)}_{\mathbf{x}})\}$$

We find convenient to rewrite the Grassman variables in momentum space. We call $D_{\varepsilon,\varepsilon'}$ the set of $\mathbf{k}$ such that

$$k=\frac{2\pi n_1}{M}+\frac{(\varepsilon-1)\pi}{2M}\qquad k_0=\frac{2\pi n_0}{M}+\frac{(\varepsilon'-1)\pi}{2M} \tag{2.37}$$

and $-[M/2]\le n_0\le[(M-1)/2]$, $-[M/2]\le n_1\le[(M-1)/2]$, $n_0,n_1\in Z$. We can write if $\mathbf{k}=(k_0,k)$

$$\psi^{(\alpha)}_{\mathbf{x}}=\frac{1}{M^2}\sum_{\mathbf{k}\in D_{\varepsilon,\varepsilon'}}\psi^{(\alpha)}_{\mathbf{k}}e^{-i\mathbf{k}\mathbf{x}}\qquad \bar\psi^{(\alpha)}_{\mathbf{x}}=\frac{1}{M^2}\sum_{\mathbf{k}\in D_{\varepsilon,\varepsilon'}}\bar\psi^{(\alpha)}_{\mathbf{k}}e^{-i\mathbf{k}\mathbf{x}} \tag{2.38}$$

Hence

$$P^{(\alpha)}_{\varepsilon,\varepsilon'}(d\psi)=[\prod_{\mathbf{k}\in D_{\varepsilon,\varepsilon'}}d\bar\psi^{(\alpha)}_{\mathbf{k}}d\psi^{(\alpha)}_{\mathbf{k}}]\exp[\frac{t}{4M^2}\sum_{\mathbf{k}\in D_{\varepsilon,\varepsilon'}}[\psi^{(\alpha)}_{\mathbf{k}}\psi^{(\alpha)}_{-\mathbf{k}}(i\sin k+\sin k_0)$$

$$+\bar\psi^{(\alpha)}_{\mathbf{k}}\bar\psi^{(\alpha)}_{-\mathbf{k}}(i\sin k-\sin k_0)+i2m_\psi(\mathbf{k})\bar\psi^{(\alpha)}_{\mathbf{k}}\psi^{(\alpha)}_{-\mathbf{k}}]] \tag{2.39}$$

where $m_\psi(\mathbf{k})=\frac{2}{t}(\sqrt{2}-1-t)+(2\cos k_0+2\cos k-4)=2\frac{(t-t_c)}{t}+O(\mathbf{k}^2)$. In deriving the above formula we have used that

$$\sum_{\mathbf{x}}\psi^{(\alpha)}_{\mathbf{x}}\partial_0\psi^{(\alpha)}_{\mathbf{x}}=\sum_{\mathbf{k}\in D_{\varepsilon,\varepsilon'}}(e^{ik_0}-1)\psi^{(\alpha)}_{\mathbf{k}}\psi^{(\alpha)}_{-\mathbf{k}}=\sum_{\mathbf{k}\in D_{\varepsilon,\varepsilon'}}i\sin k_0\psi^{(\alpha)}_{\mathbf{k}}\psi^{(\alpha)}_{-\mathbf{k}} \tag{2.40}$$

as

$$\sum_{\mathbf{k}\in D_{\varepsilon,\varepsilon'}}(\cos k_0-1)\psi^{(\alpha)}_{\mathbf{k}}\psi^{(\alpha)}_{-\mathbf{k}}=-\sum_{\mathbf{k}\in D_{\varepsilon,\varepsilon'}}(\cos k_0-1)\psi^{(\alpha)}_{-\mathbf{k}}\psi^{(\alpha)}_{\mathbf{k}}$$

$$=-\sum_{\mathbf{k}\in D_{\varepsilon,\varepsilon'}}(\cos k_0-1)\psi^{(\alpha)}_{\mathbf{k}}\psi^{(\alpha)}_{-\mathbf{k}}=0 \tag{2.41}$$

2.5 *Majorana and Dirac fermions.* We write $\psi^{(\alpha)}_{\mathbf{x}}$ in (2.38) as

$$\psi^{(\alpha)}_{\mathbf{x}}=e^{-i\mathbf{p}_{\varepsilon,\varepsilon'}\mathbf{x}}\frac{1}{M^2}\sum_{\mathbf{k}'\in D_{-,-}}\psi^{(\alpha)}_{\mathbf{k}'+\mathbf{p}_{\varepsilon,\varepsilon'}}e^{-i\mathbf{k}'\mathbf{x}}=e^{-i\mathbf{p}_{\varepsilon,\varepsilon'}\mathbf{x}}\psi'^{(\alpha)}_{\mathbf{x}} \tag{2.42}$$

where $\mathbf{p}_\varepsilon = (\frac{\pi(\varepsilon+1)}{2M}, \frac{\pi(\varepsilon'+1)}{2M})$, and $\psi^{'(\alpha)}_{\mathbf{k}'} = \psi^{(\alpha)}_{\mathbf{k}'+\mathbf{p}_{\varepsilon,\varepsilon'}}$; moreover

$$P^{(\alpha)}_{\varepsilon,\varepsilon'}(d\psi) = P^{(\alpha)}(d\psi')e^{\tilde{Q}^{(\alpha)}_{\psi,\varepsilon,\varepsilon'}} \tag{2.43}$$

where

$$P^{(\alpha)}(d\psi') = [\prod_{\mathbf{k}'\in D_{-,-}} d\bar{\psi}^{'(\alpha)}_{\mathbf{k}'} d\psi^{'(\alpha)}_{\mathbf{k}'}] \exp[\frac{t}{4M^2} \sum_{\mathbf{k}'\in D_{-,-}} [\psi^{'(\alpha)}_{\mathbf{k}'} \psi^{'(\alpha)}_{-\mathbf{k}'} (i \sin k' + \sin k'_0)$$
$$+ \bar{\psi}^{'(\alpha)}_{\mathbf{k}'} \bar{\psi}^{'(\alpha)}_{-\mathbf{k}'} (i \sin k' - \sin k'_0) + i2m_\psi(\mathbf{k}') \bar{\psi}^{'(\alpha)}_{\mathbf{k}} \psi^{'(\alpha)}_{-\mathbf{k}'}]] \tag{2.44}$$

and $\tilde{Q}^{\alpha}_{\psi,\varepsilon,\varepsilon'}$ is defined by (2.43) and is formally vanishing in the limit $M \to \infty$. Proceeding in the same way for $P(d\chi)$ and Q we find

$$\int [\prod_{\alpha=1}^{2} P^{(\alpha)}_{\varepsilon^{(\alpha)},\varepsilon'^{(\alpha)}}(d\psi^\alpha) P^{(\alpha)}_{\varepsilon^{(\alpha)},\varepsilon'^{(\alpha)}}(d\chi^\alpha)] e^{Q(\chi,\psi)} e^{\mathcal{V}(\psi,\chi)} = \tag{2.45}$$

$$\int [\prod_{\alpha=1}^{2} P^{(\alpha)}(d\psi') P^{(\alpha)}(d\chi')] e^{\tilde{Q}_\varepsilon(\psi',\chi')} e^{Q(\psi',\chi')} e^{\mathcal{V}(\psi',\chi')}$$

where

$$\tilde{Q}_\varepsilon(\psi',\chi') = \sum_{\alpha=1}^{2} [\tilde{Q}^{(\alpha)}_{\psi,\varepsilon^{(\alpha)},\varepsilon'^{(\alpha)}} + \tilde{Q}^{(\alpha)}_{\chi,\varepsilon^{(\alpha)},\varepsilon'^{(\alpha)}} + \tilde{Q}^{(\alpha)}_{\psi\chi,\varepsilon^{(\alpha)},\varepsilon'^{(\alpha)}}] \qquad (2.46);$$

of course $\tilde{Q}_{-,-,-,-}(\psi',\chi') = 0$. If

$$D^{+}_{-,-} = \{\mathbf{k}' \in D_{+,-} : k' > 0\} \tag{2.47}$$

we can write $P^{(\alpha)}(d\psi')$ in the following way

$$P^{(\alpha)}(d\psi^{'(\alpha)}) = \prod_{\mathbf{k}'\in D^{+}_{-,-}} d\psi^{'(\alpha)}_{\mathbf{k}'} d\psi^{'(\alpha)}_{-\mathbf{k}'} d\bar{\psi}^{'(\alpha)}_{\mathbf{k}'} d\bar{\psi}^{('\alpha)}_{-\mathbf{k}'} \exp[\frac{t}{4M^2} \sum_{\mathbf{k}'\in D^{+}_{-,-}} \xi^{(\alpha)T}_{\mathbf{k}'} A(\mathbf{k}') \xi^{(\alpha)}_{\mathbf{k}'}] \tag{2.48}$$

where

$$\xi^{(\alpha)T}_{\mathbf{k}'} = (\psi^{'(\alpha)}_{\mathbf{k}'}, \psi^{'(\alpha)}_{-\mathbf{k}'}, \bar{\psi}^{'(\alpha)}_{\mathbf{k}'}, \bar{\psi}^{'(\alpha)}_{-\mathbf{k}'}) \tag{2.49}$$

and $A(\mathbf{k}')$ is

$$\begin{pmatrix} 0 & \sin k'_0 + i \sin k' & 0 & 0 \\ -\sin k'_0 - i \sin k' & 0 & -i m_\psi(\mathbf{k}') & 0 \\ 0 & i m_\psi(\mathbf{k}') & 0 & -\sin k'_0 + i \sin k' \\ 0 & 0 & \sin k'_0 - i \sin k' & 0 \end{pmatrix} \tag{2.50}$$

Hence if we perform the change of variables

$$\psi^-_{1,\mathbf{k}'} = \frac{1}{\sqrt{2}}(\psi^{'(1)}_{\mathbf{k}'} + i\psi^{'(2)}_{\mathbf{k}'}) \quad \psi^+_{1,\mathbf{k}'} = \frac{1}{\sqrt{2}}(\psi^{'(1)}_{\mathbf{k}'} - i\psi^{'(2)}_{\mathbf{k}'}) \tag{2.51}$$

$$\psi^-_{-1,\mathbf{k}'} = \frac{1}{\sqrt{2}}(\bar\psi^{'(1)}_{\mathbf{k}'} + i\bar\psi^{'(2)}_{\mathbf{k}'}) \quad \psi^+_{-1,\mathbf{k}'} = \frac{1}{\sqrt{2}}(\bar\psi^{'(1)}_{\mathbf{k}'} - i\bar\psi^{'(2)}_{\mathbf{k}'})$$

we find by (2.7)

$$P^{(1)}(d\psi')P^{(2)}(d\psi') = \tag{2.52}$$

$$\prod_{\mathbf{k}'\in D^+_{-,-}} \prod_{\omega=\pm1} d\psi^+_{\mathbf{k}',\omega} d\psi^-_{\mathbf{k}',\omega} d\psi^+_{-\mathbf{k}',\omega} d\psi^-_{-\mathbf{k}',\omega} \exp[\frac{t}{4M^2} \sum_{\mathbf{k}'\in D^+_{-,-}} \tilde\xi^{\mathbf{T}}_{\mathbf{k}'} \mathbf{A}(\mathbf{k}')\tilde\xi^{(+)}_{\mathbf{k}'}]$$

where

$$\tilde\xi^{\mathbf{T}}_{\mathbf{k}} = (\psi^-_{\mathbf{k},1}, \psi^-_{-\mathbf{k},1}, \psi^-_{\mathbf{k},-1}, \psi^-_{-\mathbf{k},-1}) \tag{2.53}$$

$$\tilde\xi^{+,\mathbf{T}}_{\mathbf{k}} = (\psi^+_{\mathbf{k},1}, \psi^+_{-\mathbf{k},1}, \psi^+_{\mathbf{k},-1}, \psi^+_{-\mathbf{k},-1})$$

We can make another change of variables $\psi^+_{\mathbf{k},\omega} \to \psi^+_{-\mathbf{k},\omega}$ so that at the end, if $P^{(1)}(d\psi)\ P^{(2)}(d\psi) = P(d\psi)$

$$P(d\psi) = \tag{2.54}$$

$$\prod_{\mathbf{k}\in D^+_{-,-}} \prod_{\omega=\pm1} d\psi^+_{-\mathbf{k},\omega} d\psi^-_{\mathbf{k},\omega} d\psi^+_{\mathbf{k},\omega} d\psi^-_{-\mathbf{k},\omega} \exp[\frac{t}{2M^2} \sum_{\mathbf{k}\in D^+_{-,-}} \hat\xi^{\mathbf{T}}_{\mathbf{k}} \mathbf{A}(\mathbf{k})\hat\xi^{(+)}_{\mathbf{k}}]$$

where

$$\hat\xi^{\mathbf{T}}_{\mathbf{k}} = (\psi^-_{\mathbf{k},1}, \psi^-_{-\mathbf{k},1}, \psi^-_{\mathbf{k},-1}, \psi^-_{-\mathbf{k},-1})$$

$$\hat\xi^{+,\mathbf{T}}_{\mathbf{k}} = (\psi^+_{-\mathbf{k},1}, \psi^+_{\mathbf{k},1}, \psi^+_{-\mathbf{k},-1}, \psi^+_{\mathbf{k},-1}) \tag{2.55}$$

and in terms of the original variables

$$\psi^-_{1,\mathbf{k}'} = \frac{1}{\sqrt{2}}(\psi^{'(1)}_{\mathbf{k}'} + i\psi^{'(2)}_{\mathbf{k}'}) \quad \psi^+_{1,-\mathbf{k}'} = \frac{1}{\sqrt{2}}(\psi^{'(1)}_{\mathbf{k}'} - i\psi^{'(2)}_{\mathbf{k}'}) \tag{2.56}$$

$$\psi^-_{-1,\mathbf{k}} = \frac{1}{\sqrt{2}}(\bar\psi^{'(1)}_{\mathbf{k}'} + i\bar\psi^{'(2)}_{\mathbf{k}'}) \quad \psi^+_{-1,-\mathbf{k}} = \frac{1}{\sqrt{2}}(\bar\psi^{'(1)}_{\mathbf{k}'} - i\bar\psi^{'(2)}_{\mathbf{k}'})$$

In the following we will call $\mathbf{k}'$ simply $\mathbf{k}$. If we write

$$\psi^\pm_{\mathbf{x}} = \frac{1}{M^2} \sum_{\mathbf{k}\in D_{-,-}} e^{\pm i\mathbf{k}\mathbf{x}} \psi^\pm_{\mathbf{k}} \tag{2.57}$$

we can rewrite (2.56) in the coordinate space as

$$\psi^-_{1,\mathbf{x}} = \frac{1}{\sqrt{2}}(\psi^{'(1)}_{\mathbf{x}} + i\psi^{'(2)}_{\mathbf{x}}) \quad \psi^+_{1,\mathbf{x}} = \frac{1}{\sqrt{2}}(\psi^{'(1)}_{\mathbf{x}} - i\psi^{'(2)}_{\mathbf{x}}) \tag{2.58}$$

$$\psi^-_{-1,\mathbf{x}} = \frac{1}{\sqrt{2}}(\bar\psi'^{(1)}_{\mathbf{x}} + i\bar\psi'^{(2)}_{\mathbf{x}}) \quad \psi^+_{-1,\mathbf{x}} = \frac{1}{\sqrt{2}}(\bar\psi'^{(1)}_{\mathbf{x}} - i\bar\psi'^{(2)}_{\mathbf{x}})$$

or in terms of the original variables

$$\begin{aligned}
\psi^-_{1,\mathbf{x}} &= \frac{1}{\sqrt{2}}(e^{i\mathbf{p}_{\varepsilon(1),\varepsilon'(1)}\mathbf{x}}\psi^{(1)}_{\mathbf{x}} + ie^{i\mathbf{p}_{\varepsilon(2),\varepsilon'(2)}\mathbf{x}}\psi^{(2)}_{\mathbf{x}})\\
\psi^+_{1,\mathbf{x}} &= \frac{1}{\sqrt{2}}(e^{i\mathbf{p}_{\varepsilon(1),\varepsilon'(1)}\mathbf{x}}\psi^{(1)}_{\mathbf{x}} - ie^{i\mathbf{p}_{\varepsilon(2),\varepsilon'(2)}\mathbf{x}}\psi^{(2)}_{\mathbf{x}}) \qquad (2.59)\\
\psi^-_{-1,\mathbf{x}} &= \frac{1}{\sqrt{2}}(e^{i\mathbf{p}_{\varepsilon(1),\varepsilon'(1)}\mathbf{x}}\bar\psi^{(1)}_{\mathbf{x}} + ie^{i\mathbf{p}_{\varepsilon(2),\varepsilon'(2)}\mathbf{x}}\bar\psi^{(2)}_{\mathbf{x}})\\
\psi^+_{-1,\mathbf{x}} &= \frac{1}{\sqrt{2}}(e^{i\mathbf{p}_{\varepsilon(1),\varepsilon'(1)}\mathbf{x}}\bar\psi^{(1)}_{\mathbf{x}} - ie^{i\mathbf{p}_{\varepsilon(2),\varepsilon'(2)}\mathbf{x}}\bar\psi^{(2)}_{\mathbf{x}})
\end{aligned}$$

If $\mathbf{k} \in D^+_{-,-}$ then $-\mathbf{k} \in D^-_{-,-}$ with $D^+_{-,-} \bigcup D^-_{-,-} = D_{-,-}$ so that

$$P(d\psi) = \qquad (2.60)$$

$$\prod_{\mathbf{k}\in D_{-,-}}\prod_{\omega=\pm1} d\psi^+_{\mathbf{k},\omega} d\psi^-_{\mathbf{k},\omega} \exp[\frac{t}{2M^2}\sum_{\mathbf{k}\in D_{-,-}} \tilde\xi^{\mathbf{T}}_{\mathbf{k}}\tilde{\mathbf{A}}(\mathbf{k})\tilde\xi^{(+)}_{\mathbf{k}}]$$

$$\tilde A(\mathbf{k}) = \begin{pmatrix} i\sin k + \sin k_0 & im_\psi(\mathbf{k}) \\ -im_\psi(\mathbf{k}) & i\sin k - \sin k_0 \end{pmatrix}$$

$$\tilde\xi^{\mathbf{T}}_{\mathbf{k}} = (\psi^-_{\mathbf{k},1}, \psi^-_{\mathbf{k},-1}) \quad \tilde\xi^{+,\mathbf{T}}_{\mathbf{k}} = (\psi^+_{\mathbf{k},1}, \psi^+_{\mathbf{k},-1}) \qquad (2.61)$$

Remark. In the physical language, the change of variables (2.59) means that one is describing the system in terms of *Dirac fermions* instead in terms of *Majorana fermions.*

A similar computation can be done for $P(d\chi)$; proceeding exactly as above we find

$$P^{(1)}(d\chi^{(1)})P^{(2)}(d\chi^{(2)}) = P(d\chi) \qquad (2.62)$$

where

$$P(d\chi) = \prod_{\mathbf{k}\in D_{-,-}}\prod_{\omega=\pm1} d\chi^+_{\mathbf{k},\omega} d\chi^-_{\mathbf{k},\omega} \exp[\frac{t}{2M^2}\sum_{\mathbf{k}\in D_{-,-}} \bar\eta^{\mathbf{T}}_{\mathbf{k}}\tilde{\mathbf{B}}(\mathbf{k})\bar\eta^{(+)}_{\mathbf{k}}] \qquad (2.63)$$

$$\tilde B(\mathbf{k}) = \begin{pmatrix} i\sin k + \sin k_0 & im_\chi(k) \\ -im_\chi(k) & i\sin k - \sin k_0 \end{pmatrix}$$

$$\bar\eta^{\mathbf{T}}_{\mathbf{k}} = (\chi^-_{\mathbf{k},1}, \chi^-_{\mathbf{k},-1}) \quad \bar\eta^{+,\mathbf{T}}_{\mathbf{k}} = (\chi^+_{\mathbf{k},1}, \chi^+_{\mathbf{k},-1}) \qquad (2.64)$$

Note that $tm_\chi(\mathbf{k}) = 2(\sqrt{2}+1+t) + 2t(2-\cos k_0 - \cos k_1)$, and the mass of the χ fields never vanishes. It holds that

$$\int P(d\chi)\,\chi^-_{\mathbf{x},\omega}\chi^+_{\mathbf{y},\omega'} = g^{(\chi)}_{\omega,\omega'}(\mathbf{x}-\mathbf{y})\;, \tag{2.65}$$

where

$$g^{(\chi)}_{\omega,\omega'}(\mathbf{x}-\mathbf{y}) = \frac{1}{2tM^2}\sum_{\mathbf{k}} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}[\tilde{B}^{-1}(\mathbf{k})]_{\omega,\omega'}\;, \tag{2.66}$$

and $\tilde{B}^{-1}(\mathbf{k})$ is the inverse of the $\tilde{B}(\mathbf{k})$ defined in (2.63).

If we set

$$A(\mathbf{k}) = \det\tilde{B}(\mathbf{k}) = -\sin^2 k_0 - \sin^2 k - [m^\chi(\mathbf{k})]^2\;, \tag{2.67}$$

then

$$B^{-1}(\mathbf{k}) = \frac{1}{A(\mathbf{k})}\begin{pmatrix} -\sin k_0 + i\sin k & -im^\chi(\mathbf{k}) \\ im^\chi(\mathbf{k}) & \sin k_0 + i\sin k\end{pmatrix}\;. \tag{2.68}$$

Similar formulas hold for $g^{(\psi)}(\mathbf{x}-\mathbf{y})$ and, if $i=\psi,\chi$, the following bounds holds, for any $N>1$

$$|g^{(i)}_{\omega,\omega}(\mathbf{x}-\mathbf{y})| \le \frac{1}{1+|\mathbf{d}_M(\mathbf{x}-\mathbf{y})|}\frac{C_N}{1+|m_i\mathbf{d}_M(\mathbf{x}-\mathbf{y})|^N} \tag{2.69}$$

$$|g^{(i)}_{\omega,-\omega}(\mathbf{x}-\mathbf{y})| \le \frac{|m_i|C_N}{1+|m_i\mathbf{d}_M(\mathbf{x}-\mathbf{y})|^N} \tag{2.70}$$

where

$$\mathbf{d}_M(\mathbf{x}-\mathbf{y}) = (\frac{M}{\pi}\sin(\frac{\pi(x-y)}{M}), \frac{M}{\pi}\sin(\frac{\pi(x_0-y_0)}{M}) \tag{2.71}$$

The same transformations are done for the term $e^{S^\nu_{\varepsilon,\varepsilon'}}$ which can be written as the product of three terms similar to (2.33), (2.35) and (2.36) with t replaced by ν and $\sqrt{2}-1$ in (2.33) and $\sqrt{2}+1$ in (2.35) replaced by 0; then (2.33),(2.35) can be written in terms of Dirac fermions as (2.61) and (2.64) with t replaced by ν and $\sqrt{2}-1$ or $\sqrt{2}+1$ replaced by 0.

We have then written the partition function as Grassman integral over the χ-fields, which are massive with a mass $O(1)$, and the ψ-fields with a small mass $O(t-t_c)$ close to the critical temperature; we integrate out first the *heavy fermions* χ to get an effective theory in terms of the *light fermions* ψ only.

3. Integration of heavy fermions

3.1 *Local interactions.* By the change of variables in the preceding section we can write (2.23) as

$$Z^{\varepsilon^{(1)},\varepsilon'^{(1)},\varepsilon^{(2)},\varepsilon'^{(2)}}_{2I} = e^{M^2\mathcal{N}}\int P(d\psi)\int P(d\chi)e^{\tilde{Q}_\varepsilon(\chi,\psi)}e^{Q(\chi,\psi)}e^{\mathcal{V}(\psi,\chi)} \tag{3.1}$$

where $\mathcal{N}$ is a normalization constant, $P(\psi)$ and $P(d\chi)$ are given by (2.60), (2.63), $\mathcal{V}$, Q, $\tilde{Q}_\varepsilon$ are obtained respectively from (2.36) and (2.46) by the change of variables (2.59). We write

$$\mathcal{V}(\psi,\chi)=\mathcal{V}_2(\psi)+\mathcal{V}_4(\psi)+\mathcal{V}_\chi(\psi,\chi) \tag{3.2}$$

where $\mathcal{V}_2(\psi)$ is a sum of monomials bilinear in the ψ,ψ^+ variables, $\mathcal{V}_4(\psi)$ is a sum of monomials quartic in the ψ,ψ^+ and $\mathcal{V}_\chi(\psi,\chi)$ is sum of monomials bilinear or quartic each one containing at least a χ_ω field. It holds that, from (2.31),

$$\bar{H}^{(\alpha)}_{\mathbf{x}}H^{(\alpha)}_{\mathbf{x}}=\frac{i}{2}\psi^{(\alpha)}_{\mathbf{x}}\bar{\psi}^{(\alpha)}_{\mathbf{x}}+R^{(\alpha)}_1\quad \bar{V}^{(\alpha)}_{\mathbf{x}}V^{(\alpha)}_{\mathbf{x}}=\frac{i}{2}\psi^{(\alpha)}_{\mathbf{x}}\bar{\psi}^{(\alpha)}_{\mathbf{x}}+R^{(\alpha)}_2 \tag{3.3}$$

where $R^{(\alpha)}_i$ is sum of monomials bilinear in the fields $\chi,\bar{\chi},\psi,\bar{\psi}$ and containing at least one field $\bar{\chi}$ or χ. Moreover, from (2.58)

$$\psi^{(1)}_{\mathbf{x}}\bar{\psi}^{(1)}_{\mathbf{x}}=\frac{1}{2}[\psi_{1,\mathbf{x}}\psi_{-1,\mathbf{x}}+\psi_{1,\mathbf{x}}\psi^+_{-1,\mathbf{x}}+\psi^+_{1,\mathbf{x}}\psi_{-1,\mathbf{x}}+\psi^+_{1,\mathbf{x}}\psi^+_{-1,\mathbf{x}}] \tag{3.4}$$

$$\psi^{(2)}_{\mathbf{x}}\bar{\psi}^{(2)}_{\mathbf{x}}=\frac{1}{2}[-\psi_{1,\mathbf{x}}\psi_{-1,\mathbf{x}}-\psi^+_{1,\mathbf{x}}\psi^+_{-1,\mathbf{x}}+\psi^+_{1,\mathbf{x}}\psi_{-1,\mathbf{x}}+\psi_{1,\mathbf{x}}\psi^+_{-1,\mathbf{x}}]$$

hence

$$\psi^{(\alpha)}_{\mathbf{x}}\bar{\psi}^{(\alpha)}_{\mathbf{x}}\psi^{(\alpha)}_{\mathbf{x}}\bar{\psi}^{(\alpha)}_{\mathbf{x}}=0$$

$$\psi^{(1)}_{\mathbf{x}}\bar{\psi}^{(1)}_{\mathbf{x}}\psi^{(2)}_{\mathbf{x}}\bar{\psi}^{(2)}_{\mathbf{x}}=e^{-2i(\mathbf{p}_{\varepsilon(1),\varepsilon(1)}+\mathbf{p}_{\varepsilon(1),\varepsilon(1)})\mathbf{x}}\psi^+_{1,\mathbf{x}}\psi^+_{-1,\mathbf{x}}\psi^-_{-1,\mathbf{x}}\psi^-_{1,\mathbf{x}} \tag{3.5}$$

and from (2.26)

$$\mathcal{V}_4(\psi)=-2e^{-2i(\mathbf{p}_{\varepsilon(1),\varepsilon(1)}+\mathbf{p}_{\varepsilon(1),\varepsilon(1)})\mathbf{x}}(\tilde{\lambda}_a+\tilde{\lambda}_b)\sum_{\mathbf{x}}\psi^+_{1,\mathbf{x}}\psi^+_{-1,\mathbf{x}}\psi^-_{-1,\mathbf{x}}\psi^-_{1,\mathbf{x}}+\mathcal{V}^R_4(\psi) \tag{3.6}$$

and $\mathcal{V}^R_4$ is a sum of quartic monomials with coupling $O(\lambda)$ in which at least a $\partial\psi$ field. In the same way it is easy to check that

$$\mathcal{V}_2=\frac{1}{2M^2}\sum_{\mathbf{k},\omega}\{[\nu+f_1(\lambda)][i\omega+\cos k_0+\cos k-2]\psi_{\mathbf{k},\omega}\psi^+_{\mathbf{k},-\omega}+$$

$$[\nu+f_2(\lambda)][\omega\sin k_0+i\sin k]\psi_{\mathbf{k},\omega}\psi^+_{\mathbf{k},\omega}\} \tag{3.7}$$

where $|f_1|,|f_2|\le C|\lambda|$.

3.2 *Integration of the heavy fields* We now integrate the heavy χ fields

$$\int\bar{P}(d\psi)e^{M^2\mathcal{N}^{(1)}+\mathcal{V}^{(1)}(\psi)}=\int P(d\psi)\int P(d\chi)e^{Q(\chi,\psi)}e^{\tilde{Q}_\varepsilon(\chi,\psi)}e^{\mathcal{V}(\psi,\chi)} \tag{3.8}$$

where $\mathcal{N}^{(1)}$ is a constant. Let us consider a set of coordinates $\mathbf{x}_1, ..., \mathbf{x}_{2n}$ which are not all different one to the other and we will denote by $\sum^*_{\mathbf{x}_1,...,\mathbf{x}_{2n}}$ the sum over all the distinct $\leq 2n$ variables. We prove the following result.

Theorem 2. *There exists an ε such that, for $|\lambda|, |\nu| \leq \varepsilon$*

$$\mathcal{V}^{(0)} = \sum_{n\geq 1} \sum_{\{\varepsilon,\alpha,\omega\}} \sum^*_{\mathbf{x}_1,..,\mathbf{x}_{2n}} W_n(\mathbf{x}_1, .., \mathbf{x}_{2n}) \partial^{\alpha_1} \psi^{\varepsilon_1}_{\mathbf{x}_1,\omega_1} ... \partial^{\alpha_{2n}} \psi^{\varepsilon_{2n}}_{\mathbf{x}_{2n},\omega_{2n}} \tag{3.9}$$

and, for $n > 2$

$$\sum^*_{\mathbf{x}_1,...\mathbf{x}_{2n}} |W_n(\mathbf{x}_1, ...\mathbf{x}_n)| \leq M^2 C^n |\lambda|^{n/2} \tag{3.10}$$

For $n = 2$

$$\sum^*_{\mathbf{x}_1,..,\mathbf{x}_4} W_2(\mathbf{x}_1, .., \mathbf{x}_{2n}) \partial^{\alpha_1} \psi^{\varepsilon_1}_{\mathbf{x}_1,\omega_1} ... \partial^{\alpha_{2n}} \psi^{\varepsilon_{2n}}_{\mathbf{x}_{2n},\omega_{2n}} =$$

$$\mathcal{V}_4(\psi) + \sum^*_{\mathbf{x}_1,..,\mathbf{x}_4} \sum_{\{\varepsilon,\alpha,\omega\}} W_2(\mathbf{x}_1, .., \mathbf{x}_4) \partial^{\alpha_1} \psi^{\varepsilon_1}_{\mathbf{x}_1,\omega_1} ... \partial^{\alpha_{2n}} \psi^{\varepsilon_{2n}}_{\mathbf{x}_{2n},\omega_{2n}} \tag{3.11}$$

with $\sum^*_{\mathbf{x}_1,..,\mathbf{x}_4} |W_2(\mathbf{x}_1, .., \mathbf{x}_4)| \leq M^2 C|\lambda|$ and $\hat{W}_2(\mathbf{0}, ..., \mathbf{0}) = O(\lambda^2)$; for $n = 1$

$$\sum^*_{\mathbf{x}_1,\mathbf{x}_2} \sum_{\{\varepsilon,\alpha,\omega\}} W_1(\mathbf{x}_1, \mathbf{x}_2) \partial^{\alpha_1} \psi^{\varepsilon_1,\omega_1}_{\mathbf{x}_1} \partial^{\alpha_2} \psi^{\varepsilon_2}_{\mathbf{x}_2,\omega_2} =$$

$$\mathcal{V}_2(\psi) + \sum^*_{\mathbf{x}_1,\mathbf{x}_2} \sum_{\{\varepsilon,\alpha,\omega\}} W_1(\mathbf{x}_1, \mathbf{x}_2) \partial^{\alpha_1} \psi^{\varepsilon_1}_{\mathbf{x}_1,\omega_1} \partial^{\alpha_2} \psi^{\varepsilon_2}_{\mathbf{x}_{2n},\omega_{2n}} \tag{3.12}$$

with $\sum^*_{\mathbf{x}_1,\mathbf{x}_2} |W_1(\mathbf{x}_1, \mathbf{x}_2)| \leq M^2 C|\lambda|$; $\mathcal{V}_2$ and $\mathcal{V}_4$ are given by (3.6) and (3.7). Finally

$$\bar{P}(d\psi) = \prod_{\mathbf{k}\in D_{-,-}} \prod_{\omega=\pm 1} d\psi^+_{\mathbf{k},\omega} d\psi^-_{\mathbf{k},\omega} \exp[-\frac{t}{M^2} \sum_{\mathbf{k}\in D_{-,-}} \psi^+_{\mathbf{k},\omega} \tilde{A}_{\omega,\omega'}(\mathbf{k}) \psi^-_{\mathbf{k},\omega'}] \tag{3.13}$$

with

$$\tilde{A}(\mathbf{k}) = \frac{1}{C_0 + f_{0,0}(\mathbf{k})} \begin{pmatrix} Z_1(i \sin k + \sin k_0) + f_{1,1}(\mathbf{k}) & -i(t - t_c)C_0 + f_{1,2}(\mathbf{k}) \\ i(t - t_c)C_0 + f_{2,1}(\mathbf{k}) & Z_1(i \sin k - \sin k_0) + f_{2,2}(\mathbf{k}) \end{pmatrix}$$

with $C_0 = (t + 1 + \sqrt{2})^2$ and $Z_1 = \frac{t}{2}[(2t + 2\sqrt{2}t) + (2\sqrt{2} + 3 + t^2)]$ and $f_{0,0}(\mathbf{k})$, $f_{i,j}(\mathbf{k})$, $i, j = 1, 2$ analytic $O(\mathbf{k}^2)$ functions.

Remark 1. The fact that $\hat{W}_2(\mathbf{0}, ..., \mathbf{0}) = O(\lambda^2)$ can be checked by an explicit computations of all the contributions with coupling $O(\lambda)$ to W_2, noting that they can be only obtained contracting a terms quartic in the χ fields with one of the addend of (2.36); each of such terms carries a derivative in the coordinate space, hence the Fourier transform of such terms is vanishing at zero momentum.

Remark 2. The fermionic integration $\bar{P}(d\psi)$ reduces, neglecting the functions $f_{0,0}(\mathbf{k})$, $f_{i,j}(\mathbf{k})$ which are $O(\mathbf{k}^2)$, to the integration of a system of Dirac fermions on a lattice with mass $O(t-t_c)$. Hence, apart from the functions $f_{0,0}(\mathbf{k})$, $f_{i,j}(\mathbf{k})$, it coincides with the *free* action of the Thirring model. Note however that the interaction in the Thirring model is quartic and invariant under the transformation $\psi^\varepsilon_{\mathbf{x},\omega} \to e^{i\varepsilon\alpha_\omega}\psi^\varepsilon_{\mathbf{x},\omega}$ in the massless case; both this properties are not true for (3.9).

Proof. We start from the definition of truncated expectation:

$$\mathcal{E}^T_\chi(X;n) = \frac{\partial^n}{\partial\lambda^n}\log\int P(d\chi)e^{\lambda X(\chi)}|_{\lambda=0} \tag{3.14}$$

so that, calling

$$\bar{\mathcal{V}}(\chi,\psi) = Q(\chi,\psi) + \tilde{Q}_\varepsilon(\chi,\psi) + \mathcal{V}(\chi,\psi) \tag{3.15}$$

we obtain

$$M^2\mathcal{N}^{(1)} + \mathcal{V}^{(1)}(\psi) = \log\int P(d\chi)e^{-\bar{\mathcal{V}}(\chi,\psi)} = \sum_{n=0}^\infty \frac{(-1)^n}{n!}\mathcal{E}^T_\chi(V(.,\psi;n)) \tag{3.16}$$

We label each one of the monomials (whose number will be called C_0) in $\bar{\mathcal{V}}$ by an index v_i, so that each monomial can be written as

$$\sum_{\mathbf{x}_{v_i}} v(\mathbf{x}_{v_i}) \prod_{f\in\tilde{P}_{v_i}} \partial^{\alpha(f)}\psi^{\varepsilon(f)}_{\omega(f),\mathbf{x}(f)} \prod_{f\in P_{v_i}} \partial^{\alpha(f)}\chi^{\varepsilon(f)}_{\omega(f),\mathbf{x}(f)} \tag{3.17}$$

where $\mathbf{x}_{v_i}$ is the total set of coordinates associated to v_i and P_{v_i} and $\tilde{P}_{v_i}$ are set of indices labeling the χ or ψ-fields. We can write

$$\mathcal{V}^{(1)}(\psi) = \sum_{\tilde{P}_{v_0}\neq 0} \mathcal{V}^{(1)}(\tilde{P}_{v_0})\ , \tag{3.18}$$

$$\mathcal{V}^{(1)}(\tilde{P}_{v_0}) = \sum_{\mathbf{x}_{v_0}}[\prod_{f\in\tilde{P}_{v_0}} \partial^{\alpha(f)}\psi^{\varepsilon(f)}_{\omega(f),\mathbf{x}(f)}]K^{(0)}_{\tilde{P}_{v_0}}(\mathbf{x}_{v_0})\ , \tag{3.19}$$

$$K^{(0)}_{\tilde{P}_{v_0}}(\mathbf{x}_{v_0}) = \sum_{n=1}^\infty \frac{1}{n!}\sum_{v_1,..,v_n} \mathcal{E}^T_0[\tilde{\chi}(P_{v_1}),\ldots,\tilde{\chi}(P_{v_n})]\prod_{i=1}^n v_i(\mathbf{x}_{v_i})\ , \tag{3.20}$$

where $\sum_{v_1,\ldots v_n} \leq C_0^n$, $\tilde{P}_{v_0} = \bigcup_i \tilde{P}_{v_i}$ and $\mathbf{x}_{v_0} = \bigcup_i \mathbf{x}_{v_i}$. We use now the well known expression for $\mathcal{E}^T$ (see for instance [Le])

$$\mathcal{E}^T_\chi(\tilde{\chi}(P_1),...,\tilde{\chi}(P_s)) = \sum_T\prod_{l\in T} g^\chi_{\omega^-,\omega^+}(\mathbf{x}_l - \mathbf{y}_l)\int dP_T(\mathbf{t})\det G^T(\mathbf{t}) \tag{3.21}$$

where:
-)P is a set of indices, and

$$\tilde{\chi}(P) = \prod_{f\in P} \partial^{\alpha(f)} \chi^{\varepsilon(f)}_{\mathbf{x}(f),\omega(f)} \tag{3.22}$$

-)T is a set of lines forming an *anchored tree* between the cluster of poins $P_1, .., P_s$ *i.e.* T is a set of lines which becomes a tree if one identifies all the points in the same clusters.
-)$\mathbf{t} = \{t_{i,i'} \in [0,1], 1 \le i, i' \le s\}$, $dP_T(\mathbf{t})$ is a probability measure with support on a set of $\mathbf{t}$ such that $t_{i,i'} = \mathbf{u}_i \cdot \mathbf{u}_{i'}$ for some family of vectors $\mathbf{u}_i \in \mathbb{R}^s$ of unit norm.
-)$G^T(\mathbf{t})$ is a $(n-s+1)\times(n-s+1)$ matrix, whose elements are given by $G^T_{ij,i'j'} = t_{i,i'} \tilde{g}_{\omega^-,\omega^+}(\mathbf{x}_{ij} - \mathbf{y}_{i'j'})$ with $(f^-_{ij}, f^+_{i'j'})$ not belonging to T.
If $s = 1$ the sum over T is empty, but we can still use the above equation by iterpreting the r.h.s. as 1 if P_1 is empty, and $detG(P_1)$ otherwise.
We bound the determinant using the well known *Gram-Hadamard inequality*, stating that, if M is a square matrix with elements M_{ij} of the form $M_{ij} =< A_i, B_j >$, where A_i, B_j are vectors in a Hilbert space with scalar product $< \cdot, \cdot >$, then

$$|\det M| \le \prod_i ||A_i|| \cdot ||B_i|| \; . \tag{3.23}$$

where $||\cdot||$ is the norm induced by the scalar product.
Let $\mathcal{H} = \mathbb{R}^s \otimes \mathcal{H}_0$, where $\mathcal{H}_0$ is the Hilbert space of complex four dimensional vectors $F(\mathbf{k}) = (F_1(\mathbf{k}), \dots, F_4(\mathbf{k}))$, $F_i(\mathbf{k})$ being a function on the set $\mathcal{D}_{-,-}$, with scalar product

$$< F, G >= \sum_{i=1}^{4} \frac{1}{M^2} \sum_{\mathbf{k}} F^*_i(\mathbf{k}) G_i(\mathbf{k}) \; . \tag{3.24}$$

and it is easy to veriy that

$$G^T_{ij,i'j'} = t_{i,i'} g^{(\chi)}_{\omega^-_l,\omega^+_l}(\mathbf{x}_{ij} - \mathbf{y}_{i'j'}) =< \mathbf{u}_i \otimes A_{\mathbf{x}(f^-_{ij}),\omega(f^-_{ij})}, \mathbf{u}_{i'} \otimes B_{\mathbf{x}(f^+_{i'j'}),\omega(f^+_{i'j'})} > , \tag{3.25}$$

where $\mathbf{u}_i \in \mathbb{R}^s$, $i = 1, \dots, s$, are the vectors such that $t_{i,i'} = \mathbf{u}_i \cdot \mathbf{u}_{i'}$, and

$$\begin{aligned} A_{\mathbf{x},\omega}(\mathbf{k}) &= e^{i\mathbf{k}'\mathbf{x}} \frac{1}{\sqrt{-A(\mathbf{k})}} \cdot \begin{cases} (-\sin k_0 + i \sin k, 0, -im_\chi(\mathbf{k}), 0), & \text{if } \omega = +1, \\ (0, im_\chi(\mathbf{k}), 0, m_\chi(\mathbf{k})), & \text{if } \omega = -1, \end{cases} \\ B_{\mathbf{x},\omega} &= e^{i\mathbf{k}'\mathbf{y}} \frac{1}{\sqrt{-A(\mathbf{k})}} \cdot \begin{cases} (1, 1, 0, 0), & \text{if } \omega = +1, \\ (0, 0, 1, (\sin k_0 + i \sin k)/m_\chi(\mathbf{k})), & \text{if } \omega = -1. \end{cases} \end{aligned} \tag{3.26}$$

Hence from (3.23) we immediately find

$$|G^T_{ij,i'j'}| \le C_1^n \tag{3.27}$$

Note that C_1 is an $O(1)$ constant; this follows from the fact that the Grassman variable χ has an $O(1)$ mass. Finally we get

$$\sum_{\mathbf{x}_{v_0}} |K^{(1)}_{\tilde{P}_{v_0}}(\mathbf{x}_{v_0})| \leq \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{v_1,..,v_n} \sum_{\mathbf{x}_{v_1},...,\mathbf{x}_{v_n}} C_1^n \sum_T [\prod_{l\in T} |\tilde{g}_\chi(\mathbf{x}_l - \mathbf{y}_l)|] \prod_{i=1}^n |v_i(\mathbf{x}_{v_i})| \tag{3.28}$$

where we have used that $\int dP_T(\mathbf{t}) = 1$. Regarding the sum over T, it is empty if $n = 1$. If $n > 1$ the number of anchored trees with d_i lines branching from the vertex v_i can be bounded, by using Caley's formula, by

$$\frac{(s_v - 2)!}{(d_1 - 1)!...(d_{s_v} - 1)!} |P_{v_1}|^{d_1} ... |P_{v_n}|^{d_n} \ ; \tag{3.29}$$

hence the number of addenda in $\sum_T$ is bounded by $n! C_2^n$. Finally T and the $\bigcup_i \mathbf{x}_{v_i}$ form a tree connecting all points, so that using that the propagator is massive and that the interactions are short ranged $\sum_{\mathbf{x}_{v_1},...\mathbf{x}_{v_n}} \sum_T [\prod_{l\in T} |\tilde{g}_\chi(\mathbf{x}_l - \mathbf{y}_l)|] \prod_{i=1}^n |v_i(\mathbf{x}_{v_i})| \leq C_3^n |\lambda|^{\tilde{n}} M^2$, where $\tilde{n}$ is the number of coupling $O(\lambda)$.

Let us consider the case $|P_{v_0}| \geq 4$. Note that if to v_i are associated only terms from $\mathcal{V}(\psi, \chi)$, then $\tilde{n} = n$. The same bound holds for M large enough, if there are v_i associated with $\tilde{Q}_\varepsilon$; in fact such terms are vanishing as $M \to \infty$ (one has in the bounds an extra M^{-1} for any of such v_i, for dimensional reasons). Let us consider now the case in which there are end-points associated to $Q(\psi, \chi)$, which have $O(1)$ coupling; there are at most $|\tilde{P}_{v_0}|$ end-points associated with $Q(\psi, \chi)$. In fact in $Q(\psi, \chi)$ there are only terms of the form $\psi\chi$, so at most the number of them is equal to the number of ψ fields. If we call $n_\lambda \leq \tilde{n}$ the number of vertices quartic in the fields it is clear that $n_\lambda \geq \max\{1, |\tilde{P}_{v_0}|/2 - 1\}$; hence

$$\sum_{\mathbf{x}_{v_0}} |K^{(1)}_{\tilde{P}_{v_0}}(\mathbf{x}_{v_0})| \leq M^2 \sum_{\tilde{n}=1}^{\infty} C^{\tilde{n}+|\tilde{P}_{v_0}|} |\lambda|^{\frac{\tilde{n}}{2}} |\lambda|^{\max\{\frac{1}{2}, |\tilde{P}_{v_0}|/4 - 1/2\}} \tag{3.30}$$

and (3.10) holds for $|\tilde{P}_{v_0}| \geq 4$ (in the r.h.s. of (3.30) the sum over the number of v_i to which are associated quadratic monomials with coupling $O(\lambda)$ is already done).

Consider now the case $|\tilde{P}_{v_0}| = 2$; in this case there are terms λ independent, obtained when to all the v_i are associated with elements of $Q(\psi, \chi)$. It is convenient to include all such terms in the free measure, as they cannot be considered as perturbations (they are not $O(\lambda)$). Instead of computing all such terms, we can proceed in a more rapid way by noting that

$$\int \bar{P}(d\psi) = \int P(d\psi) \int P(d\chi) e^{Q(\psi,\chi)} \tag{3.31}$$

and if, if $< X >_0 = \int \bar{P}^0(d\psi) X$, it holds

$$< \psi^-_{\mathbf{x},1} \psi^+_{\mathbf{y},1} >_0 = \frac{1}{2} < (\psi^{(1)}_{\mathbf{x}} + i\psi^{(2)}_{\mathbf{x}})(\psi^{(1)}_{\mathbf{y}} - i\psi^{(2)}_{\mathbf{y}}) >_0 = < \psi^{(1)}_{\mathbf{x}} \psi^{(1)}_{\mathbf{y}} >_0 \tag{3.32}$$

as $<\psi^{(1)}\psi^{(2)}>_0=0$ if $\lambda=0$ (the two systems are decoupled). In the same way

$$<\psi^-_{\mathbf{x},-1}\psi^+_{\mathbf{y},-1}>_0=\frac{1}{2}<(\bar\psi^{(1)}_{\mathbf{x}}+i\bar\psi^{(2)}_{\mathbf{x}})(\bar\psi^{(1)}_{\mathbf{y}}-i\bar\psi^{(2)}_{\mathbf{y}})>_0=<\bar\psi^{(1)}_{\mathbf{x}}\bar\psi^{(1)}_{\mathbf{y}}>_0 \quad (3.33)$$

and finally

$$<\psi^-_{\mathbf{x},1}\psi^+_{\mathbf{y},-1}>_0=\frac{1}{2}<(\psi^{(1)}_{\mathbf{x}}+i\psi^{(2)}_{\mathbf{x}})(\bar\psi^{(1)}_{\mathbf{y}}-i\bar\psi^{(2)}_{\mathbf{y}})>_0=<\psi^{(1)}_{\mathbf{x}}\bar\psi^{(1)}_{\mathbf{y}}>_0 \quad (3.34)$$

Note that $<\psi^\alpha_{\mathbf{x}}\psi^\alpha_{\mathbf{y}}>_0$ are the Ising correlation computed in [MPW], [SP] where is found that

$$<\psi^{(\alpha)}_{\mathbf{x}}\psi^{(\alpha)}_{\mathbf{y}}>_0=\frac{1}{M^2}\sum_{\mathbf{k}}\frac{e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}}{\Delta}c_{1,1}(\mathbf{k}) \quad (3.35)$$

$$<\bar\psi^{(\alpha)}_{\mathbf{x}}\bar\psi^{(\alpha)}_{\mathbf{y}}>_0=\frac{1}{M^2}\sum_{\mathbf{k}}\frac{e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}}{\Delta}c_{2,2}(\mathbf{k}) \quad (3.36)$$

$$<\psi^{(\alpha)}_{\mathbf{x}}\bar\psi^{(\alpha)}_{\mathbf{y}}>_0=\frac{1}{M^2}\sum_{\mathbf{k}}\frac{e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}}{\Delta}c_{1,2}(\mathbf{k}) \quad (3.37)$$

where

$$\Delta=2t(1-t^2)(2-\cos k_0-\cos k)+(t+1+\sqrt{2})^2(t-\sqrt{2}+1)^2 \quad (3.38)$$

and

$$c_{1,1}(\mathbf{k})=\frac{t}{2}[(2t+2\sqrt{2}t)(\sin k_0\cos k-i\sin k\cos k_0)+(2\sqrt{2}+3+t^2)(\sin k_0-i\sin k)] \quad (3.39)$$

$$c_{2,2}(\mathbf{k})=\frac{t}{2}[(2t+2\sqrt{2}t)(\sin k_0\cos k-i\sin k\cos k_0)+(2\sqrt{2}+3+t^2)(-\sin k_0-i\sin k)] \quad (3.40)$$

$$c_{1,2}(\mathbf{k})=-c_{1,2}(\mathbf{k})=\frac{-i}{2}[(2\sqrt{2}t^2+4t^2)\cos k\cos k_0+$$
$$(t^3+2t\sqrt{2}+t)(\cos k+\cos k_0)-2-2\sqrt{2}+2t^2] \quad (3.41)$$

Note that

$$\frac{c_{1,1}(\mathbf{k})c_{2,2}(\mathbf{k})-c_{1,2}(\mathbf{k})c_{2,1}(\mathbf{k})}{\Delta(\mathbf{k})}-(t+1+\sqrt{2})^2=f(\mathbf{k})$$

where $f(\mathbf{k})=O(\mathbf{k}^2)$ and with bounded derivatives.

4. Renormalization Group for light fermions

4.1 *Multiscale analysis.* We start considering $Z_{2I}^{-,-,-,-}$ in the sum (2.22), written as in (3.8). We introduce a *scaling parameter* $\gamma > 1$ and a positive function $\chi(\mathbf{k}) \in C^\infty$ such that

$$\chi(\mathbf{k}) = \chi(-\mathbf{k}) = \begin{cases} 1 & \text{if } |\mathbf{k}| < t_0 \equiv a_0/\gamma \ , \\ 0 & \text{if } |\mathbf{k}| > a_0 \ , \end{cases} \tag{4.1}$$

where

$$|\mathbf{k}| = \sqrt{\sin k_0^2 + \sin k^2} \ . \tag{4.2}$$

We define also, for any integer $h \le 0$,

$$f_h(\mathbf{k}) = \chi(\gamma^{-h}\mathbf{k}) - \chi(\gamma^{-h+1}\mathbf{k}) \ ; \tag{4.3}$$

we have, for any $h_M < 0$,

$$\chi(\mathbf{k}) = \sum_{h=h_M+1}^{0} f_h(\mathbf{k}) + \chi(\gamma^{-h_M}\mathbf{k}) \ . \tag{4.4}$$

Note that, if $h \le 0$, $f_h(\mathbf{k}) = 0$ for $|\mathbf{k}| < t_0\gamma^{h-1}$ or $|\mathbf{k}| > t_0\gamma^{h+1}$, and $f_h(\mathbf{k}) = 1$, if $|\mathbf{k}| = t_0\gamma^h$. Therefore

$$f_h(\mathbf{k}) = 0 \quad \forall h < h_M = \min\{h : t_0\gamma^{h+1} > \sqrt{2}(\pi M^{-1})^2\} \ , \tag{4.5}$$

and

$$1 = \sum_{h=h_M}^{1} f_h(\mathbf{k}) \qquad f_1 = 1 - \chi(\mathbf{k}) \ . \tag{4.6}$$

We define a sequence of *effective potentials* $\mathcal{V}^{(h)}(\psi)$ defined iteratively in the following way; assuming that we have integrated the scales $h = 1, 0, -1, -2, ..., h+1$

$$e^{-M^2 E_M} = \int P_{m_h,C_h}(d\psi^{(\le h)})\, e^{-\mathcal{V}^{(h)}(\psi^{(\le h)}) - M^2 E_h} \ , \quad \mathcal{V}^{(h)}(0) = 0 \ , \tag{4.7}$$

where

$$\begin{aligned} P_{m_h,C_h}(d\psi^{(\le h)}) = &\prod_{\mathbf{k}:C_h^{-1}(\mathbf{k})>0} \prod_{\omega=\pm1} \frac{d\hat\psi^{(\le h)+}_{\mathbf{k},\omega} d\hat\psi^{(\le h)-}_{\mathbf{k}',\omega}}{\mathcal{N}(\mathbf{k})} \cdot \\ &\exp\left\{ -\frac{1}{M^2} \sum_{\mathbf{k}:C_h^{-1}(\mathbf{k})>0} C_h(\mathbf{k}) \sum_{\omega,\omega'=\pm1} \hat\psi^{(\le h)+}_{\mathbf{k},\omega} T^{(h+1)}_{\omega,\omega'} \hat\psi^{(\le h)-}_{\mathbf{k},\omega'} \right\} \ , \end{aligned} \tag{4.8}$$

$$C_h(\mathbf{k})^{-1} = \sum_{j=h_M}^{h} f_j(\mathbf{k}') \ , \tag{4.9}$$

and the 2×2 matrix $T_h(\mathbf{k}')$ is given by

$$T_h(\mathbf{k}) = \frac{1}{C_0 + f_{0,0}(\mathbf{k})} \begin{pmatrix} Z_1(i\sin k + \sin k_0) + f_{1,1}(\mathbf{k}) & -im_{h-1}(\mathbf{k}) + f_{1,2}(\mathbf{k}) \\ im_{h-1}(\mathbf{k}) + f_{2,1}(\mathbf{k}) & Z_1(i\sin k - \sin k_0) + f_{2,2}(\mathbf{k}) \end{pmatrix} \tag{4.10}$$

with $m_1 = C_0(t - t_c)$ and C_0, Z_1 defined in Theorem 2.

The *single scale* propagator is

$$\int P_{m_h, f_h^{-1}}(d\psi^{(h)})\, \psi^{(h)-}_{\mathbf{x},\omega}\, \psi^{(h)+}_{\mathbf{y},\omega'} = g^{(h)}_{\omega,\omega'}(\mathbf{x}-\mathbf{y}) \ , \tag{4.11}$$

where

$$g^{(h)}_{\omega,\omega'}(\mathbf{x}-\mathbf{y}) = \frac{1}{M^2}\sum_{\mathbf{k}} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})} f_h(\mathbf{k}) [T_h^{-1}(\mathbf{k})]_{\omega,\omega'} \ , \tag{4.12}$$

and $T_h^{-1}(\mathbf{k})$ is the inverse of the $T_h(\mathbf{k})$ defined in (4.10). The large distance behaviour of $g^{(h)}_{\omega,\omega'}(\mathbf{x}-\mathbf{y})$ is given by, given the positive integers N, n_0, n_1 and putting $n = n_0 + n_1$, and the constant $C_{N,n}$

$$|\partial_{x_0}^{n_0}\partial_x^{n_1} g^{(h)}_{\omega,\omega}(\mathbf{x}-\mathbf{y})| \le C_{N,n}\frac{\gamma^{h+n}}{1 + (\gamma^h|\mathbf{d}(\mathbf{x}-\mathbf{y}))|^N} \tag{4.13}$$

$$|\partial_{x_0}^{n_0}\partial_x^{n_1} g^{(h)}_{\omega,-\omega}(\mathbf{x}-\mathbf{y})| \le C_{N,n}|\frac{m_h}{\gamma^h}|\frac{\gamma^{h+n}}{1 + (\gamma^h|\mathbf{d}(\mathbf{x}-\mathbf{y})|)^N} \ . \tag{4.14}$$

where ∂_x denotes the discrete derivative. It will be useful to write

$$g^{(h)}_{\omega,\omega}(\mathbf{x}-\mathbf{y}) = g^{(h)}_{L;\omega,\omega}(\mathbf{x}-\mathbf{y}) + \tilde g^{(h)}_{\omega,\omega}(\mathbf{x}-\mathbf{y}) + \hat g^{(h)}_{\omega,\omega}(\mathbf{x}-\mathbf{y}) \tag{4.15}$$

with

$$g^{(h)}_{L;\omega,\omega}(\mathbf{x}-\mathbf{y}) = \frac{1}{M^2}\sum_{\mathbf{k}} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}\frac{f_h(\mathbf{k})}{-Z_1\omega\sin k_0 + iZ_1\sin k} \ , \tag{4.16}$$

obeying to the bound (4.13) while

$$|\partial_{x_0}^{n_0}\partial_x^{n_1} \tilde g^{(h)}_{\omega,\omega}(\mathbf{x}-\mathbf{y})| \le C_{N,n}\frac{\gamma^{2h+n}}{1 + (\gamma^h|\mathbf{d}(\mathbf{x}-\mathbf{y}))|^N} \tag{4.17}.$$

$$|\partial_{x_0}^{n_0}\partial_x^{n_1} \hat g^{(h)}_{\omega,\omega}(\mathbf{x}-\mathbf{y})| \le C_{N,n}|\frac{m_h}{\gamma^h}|^2\frac{\gamma^{h+n}}{1 + (\gamma^h|\mathbf{d}(\mathbf{x}-\mathbf{y})|)^N} \ . \tag{4.18}$$

Moreover

$$g^{(h)}_{\omega,-\omega}(\mathbf{x}-\mathbf{y}) = \hat g^{(h)}_{\omega,-\omega}(\mathbf{x}-\mathbf{y}) + \tilde g^{(h)}_{\omega,-\omega}(\mathbf{x}-\mathbf{y}) \tag{4.19}$$

with

$$\hat{g}^{(h)}_{\omega,-\omega}(\mathbf{x}-\mathbf{y}) = \frac{1}{M^2}\sum_{\mathbf{k}} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})} f_h(\mathbf{k}) \frac{-im_h(\mathbf{k})}{Z_1^2 \sin^2 k_0 + Z_1^2 \sin k^2 + m_h^2(\mathbf{k})} \tag{4.20}$$

verifying (4.14) and $\tilde{g}^{(h)}_{\omega,-\omega}(\mathbf{x}-\mathbf{y})$ verifying (4.17).
Finally $\mathcal{V}^{(h)}$ is given by

$$\mathcal{V}^{(h)}(\psi^{(\le h)}) = \sum_{n=1}^{\infty} \sum^{*}_{\substack{\mathbf{x}_1,\ldots,\mathbf{x}_{2n},\\ \underline{\sigma},\underline{\omega},\alpha}} \prod_{i=1}^{2n} \partial^{\alpha_i} \psi^{(\le h)\sigma_i}_{\mathbf{x}_i,\omega_i} W^{(h)}_{2n,\underline{\sigma},\underline{\omega}}(\mathbf{x}_1,...,\mathbf{x}_{2n}) \tag{4.21}$$

4.2 *The localization operator.* We define an $\mathcal{L}$ operation, for $h \le 0$, in the following way:

1) If $2n = 4$, then

$$\mathcal{L}\hat{W}^{(h)}_{4,\underline{\sigma},\underline{\omega}}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = \hat{W}^{(h)}_{4,\underline{\sigma},\underline{\omega}}(\bar{\mathbf{k}}_{++},\bar{\mathbf{k}}_{++},\bar{\mathbf{k}}_{++})\ , \tag{4.22}$$

where

$$\bar{\mathbf{k}}_{\eta\eta'} = \left(\eta\frac{\pi}{M},\eta'\frac{\pi}{M}\right)\ . \tag{4.23}$$

2) If $2n = 2$ then

$$\mathcal{L}\hat{W}^{(h)}_{2,\underline{\sigma},\underline{\omega}}(\mathbf{k}) = \frac{1}{4}\sum_{\eta,\eta'=\pm 1}[\hat{W}^{(h)}_{2,\underline{\sigma},\underline{\omega}}(\bar{\mathbf{k}}_{\eta\eta'})\ \cdot$$

$$\cdot\ + \hat{W}^{(h)}_{2,\underline{\sigma},\underline{\omega}}(\bar{\mathbf{k}}_{\eta\eta'})(\eta\frac{L}{\pi}\sin k + \eta'\frac{\beta}{\pi}\sin k_0)] \tag{4.24}$$

3) In all the other cases

$$\mathcal{L}\hat{W}^{h}_{2n,\underline{\sigma},\underline{\omega}}(\mathbf{k}_1,\ldots,\mathbf{k}_{2n-1}) = 0\ . \tag{4.25}$$

By (4.22) the operator $\mathcal{L}$ satisfies the relation $\mathcal{R}\mathcal{L} = 0$.

Remark. First note that in the limit $M \to \infty$ (4.24) becomes simply

$$\mathcal{L}\hat{W}^{(h)}_{2,\underline{\sigma},\underline{\omega}}(\mathbf{k}) = [\hat{W}^{(h)}_{2,\underline{\sigma},\underline{\omega}}(\mathbf{0}) + k_0\partial_{k_0}\hat{W}^{(h)}_{2,\underline{\sigma},\underline{\omega}}(\mathbf{0}) + k\partial_k \hat{W}^{a(h)}_{2,\underline{\sigma},\underline{\omega}}(\mathbf{0})], \tag{4.26}$$

hence $\mathcal{L}\hat{W}^{(h)}_{2,\underline{\sigma},\underline{\omega}}(\mathbf{k})$ has to be understood as a discrete version of the Taylor expansion up to order 1. Moreover the localization operator acting on the effective potential in the x-space representation can be seen as an operator on monomials of grassman variables; for instance (4.22) implies in the x-space representation, in the $M\to\infty$ limit

$$\mathcal{L}\sum^{*}_{\mathbf{x}_1,...,\mathbf{x}_4}\prod_{i=1}^4\psi^{(\le h)\varepsilon_i}_{\mathbf{x}_i,\omega_i}W_4^{(h)}(\mathbf{x}_1,...,\mathbf{x}_4)=\hat W_4(\mathbf{0},\mathbf{0},\mathbf{0})\prod_{i=1}^4\psi^{(\le h)\varepsilon_i}_{\mathbf{x}_1,\omega_i}\tag{4.27}$$

where the r.h.s. of the above equation is always vanishing unless $\prod_{i=1}^4\psi^{(\le h)\varepsilon_i}_{\mathbf{x}_1,\omega_i}$ is a permutation of $\psi^{(\le h)+}_{\mathbf{x}_1,+}\psi^{(\le h)-}_{\mathbf{x}_1,+}\psi^{(\le h)+}_{\mathbf{x}_1,-}\psi^{(\le h)-}_{\mathbf{x}_1,-}$; hence $\mathcal{L}$ acts on a quartic monomial producing a *local* expression. Analogous considerations can be done for $n=1$.

4.3 We have, before continuing, to exploit the consequences of some symmetries. There are no local terms of the form $\psi^+_{\mathbf{x},1}\psi^-_{\mathbf{x},1}$; such local terms can be written as $\psi^{(1)}_{\mathbf{x}}\psi^{(2)}_{\mathbf{x}}$, but the model is invariant under the transformation

$$\psi^{(1)},\bar\psi^{(1)},\chi^{(1)},\bar\chi^{(1)}\to-\psi^{(1)},-\bar\psi^{(1)},-\chi^{(1)},-\bar\chi^{(1)}$$

$$\psi^{(2)},\bar\psi^{(2)},\chi^{(2)},\bar\chi^{(2)}\to\psi^{(2)},\bar\psi^{(2)},\chi^{(2)},\bar\chi^{(2)},\tag{4.28}$$

hence such terms cannot be present as they violate such symmetry.

There are no local terms of the form $\psi_{1,\mathbf{x}}\psi_{-1,\mathbf{x}}$ or $\psi^+_{1,\mathbf{x}}\psi^+_{-1,\mathbf{x}}$; in fact,

$$\psi_{1,\mathbf{x}}\psi_{-1,\mathbf{x}}=\frac{1}{2}[\psi^{(1)}_{\mathbf{x}}\bar\psi^{(1)}_{\mathbf{x}}-\psi^{(2)}_{\mathbf{x}}\bar\psi^{(2)}_{\mathbf{x}}+i\psi^{(1)}_{\mathbf{x}}\bar\psi^{(2)}_{\mathbf{x}}+i\psi^{(2)}_{\mathbf{x}}\bar\psi^{(1)}_{\mathbf{x}}]\tag{4.29}$$

and the last two terms violates the symmetry (4.28); moreover the first two terms are *odd* in the exchange $(1),(2)\to(2),(1)$ and the model is invariant in the exchange $(1),(2)\to(2),(1)$. Moreover the model is invariant under complex coniugation and the exchange

$$\psi^{(\alpha)}_{\mathbf{x}},\bar\psi^{(\alpha)}_{\mathbf{x}}\to\bar\psi^{(\alpha)}_{\mathbf{x}},\psi^{(\alpha)}_{\mathbf{x}}\quad\chi^{(\alpha)}_{\mathbf{x}},\bar\chi^{(\alpha)}_{\mathbf{x}}\to\bar\chi^{(\alpha)}_{\mathbf{x}},\chi^{(\alpha)}_{\mathbf{x}};\tag{4.30}$$

this follows from the fact that, from (2.31), $\bar H^{(\alpha)},H^{(\alpha)},\bar V^{(\alpha)},V^{(\alpha)}$, written in terms of $\bar\psi^{(\alpha)},\psi^{(\alpha)},\bar\chi^{(\alpha)},\chi^{(\alpha)}$, are invariant under such transformation. Hence the coefficient of the local part of the quartic (non vanishing) terms is real; in fact $\hat w(0,0,0)\psi^+_{1,\mathbf{x}}\psi_{1,\mathbf{x}}\psi^+_{-1,\mathbf{x}}\psi_{-1,\mathbf{x}}=\hat w(0,0,0)\psi^{(1)}_{\mathbf{x}}\bar\psi^{(1)}_{\mathbf{x}}\psi^{(2)}_{\mathbf{x}}\bar\psi^{(1)}_{\mathbf{x}}$ must be equal, by the above invariance, to $\hat w^*(0,0,0)\bar\psi^{(1)}_{\mathbf{x}}\psi^{(1)}_{\mathbf{x}}\bar\psi^{(2)}_{\mathbf{x}}\psi^{(1)}_{\mathbf{x}}$, hence $\hat w(0,0,0)=\hat w^*(0,0,0)$. Finally the combination of local terms $\psi^+_{\mathbf{x},1}\psi^-_{\mathbf{x},-1}+\psi^+_{\mathbf{x},-1}\psi^-_{\mathbf{x},1}$ is equal to $\frac{1}{2}[\psi^{(1)}_{\mathbf{x}}\bar\psi^{(2)}_{\mathbf{x}}-\psi^{(2)}_{\mathbf{x}}\bar\psi^{(1)}_{\mathbf{x}}]$ so it cannot be present as it violates the symmetry (4.28). On the other hand $\psi^+_{\mathbf{x},1}\psi^-_{\mathbf{x},-1}-\psi^+_{\mathbf{x},-1}\psi^-_{\mathbf{x},1}$ is equal to $\frac{1}{2}[\psi^{(1)}_{\mathbf{x}}\bar\psi^{(1)}_{\mathbf{x}}+\psi^{(2)}_{\mathbf{x}}\bar\psi^{(2)}_{\mathbf{x}}]$; hence the coefficient of the local part is imaginary; in fact $\hat w(0)[\psi^{(1)}_{\mathbf{x}}\bar\psi^{(1)}_{\mathbf{x}}+\psi^{(2)}_{\mathbf{x}}\bar\psi^{(2)}_{\mathbf{x}}]$ must be equal to $\hat{w}^*(0)[\bar{\psi}^{(1)}_{\mathbf{x}}\psi^{(1)}_{\mathbf{x}}+\bar{\psi}^{(2)}_{\mathbf{x}}\bar{\psi}^{(2)}_{\mathbf{x}}]$, by the invariance under complex conjugation and (4.30), hence $\hat{w}(0)=-\hat{w}^*(0)$.

4.4 By (4.22),(4.24),(4.25) and the symmetry relations in §4.3, we can write $\mathcal{L}\mathcal{V}^{(h)}$ in the following way:

$$\mathcal{L}\mathcal{V}^{(h)}(\psi^{(\le h)})=(s_h+\gamma^h n_h)F_\sigma^{(\le h)}-\lambda_h F_\lambda^{(\le h)}+\sum_{i=1}^2\sum_\omega z_{i,\omega,h}F_i^{(\le h)}+\sum_{i=1}^2\sum_\omega \tilde{z}_{i,\omega,h}\tilde{F}_i^{(\le h)}\;, \tag{4.31}$$

where, if $|\lambda|,|\nu|\le\varepsilon$, $z_1^i,\tilde{z}_1^i=O(\varepsilon)$, $s_1=O(\varepsilon)$, $\lambda_1=2\lambda\mathrm{sech}^2 J(a+b)+O(\varepsilon^2)$, $\nu_1=\nu+O(\varepsilon)$ and

$$\begin{aligned}
F_m^{(\le h)}&=\sum_{\omega=\pm1}\frac{i\omega}{M^2}\sum_{\mathbf{k}\in\mathcal{D}_M}\hat{\psi}^{(\le h)+}_{\mathbf{k},\omega}\hat{\psi}^{(\le h)-}_{\mathbf{k},-\omega}\;,\\
F_\lambda^{(\le h)}&=\frac{1}{(M)^4}\sum_{\mathbf{k}_1,\dots,\mathbf{k}_4\in\mathcal{D}_M}\hat{\psi}^{(\le h)+}_{\mathbf{k}_1,+1}\hat{\psi}^{(\le h)-}_{\mathbf{k}_2,-1}\hat{\psi}^{(\le h)+}_{\mathbf{k}_3,-1}\hat{\psi}^{(\le h)-}_{\mathbf{k}_4,+1}\delta(\mathbf{k}_1-\mathbf{k}_2+\mathbf{k}_3-\mathbf{k}_4)\\
F_i^{(\le h)}&=\frac{1}{M}\sum_{\mathbf{k}\in\mathcal{D}_M}f_i(\mathbf{k})\hat{\psi}^{(\le h)+}_{\mathbf{k},\omega}\hat{\psi}^{(\le h)-}_{\mathbf{k},\omega}\\
\tilde{F}_i^{(\le h)}&=\frac{1}{M}\sum_{\mathbf{k}\in\mathcal{D}_M}f_i(\mathbf{k})\hat{\psi}^{(\le h)+}_{\mathbf{k},\omega}\hat{\psi}^{(\le h)-}_{\mathbf{k},-\omega}
\end{aligned} \tag{4.32}$$

with $f_1=\sin k$ and $f_2=\sin k_0$. The constants n_h,s_h,λ_h are real, and we will call $v_h=(\bigcup_{i,\omega}z_{i,\omega,h},\bigcup_{i,\omega}\tilde{z}_{i,\omega,h},\lambda_h)$. The set of (v_h,m_h,ν_h) are called *running coupling constants*. At the end of our iterative construction it will appear that $\hat{W}^{(h)}_{2n,\underline{\sigma},\underline{\omega}}$ is written as a function of (v_k,m_k,ν_k), with $1\ge k\ge h+1$. It is possible to write (see Remark 1 in §4.9)

$$\hat{W}^{(h)}_{2,\underline{\sigma},\underline{\omega}}=\hat{W}^{a(h)}_{2,\underline{\sigma},\underline{\omega}}+\hat{W}^{b(h)}_{2,\underline{\sigma},\underline{\omega}} \tag{4.33}$$

with $\hat{W}^{a(h)}_{2,\underline{\sigma},\underline{\omega}}$ is vanishing if at least one $m_k=0$, for $1\ge k\ge h+1$, and $\hat{W}^{b(h)}_{2,\underline{\sigma},\underline{\omega}}$ is the rest; we define

$$s_h=\delta_{\omega,-\omega}[\frac{1}{4}\sum_{\eta,\eta'=\pm1}\hat{W}^{a(h)}_{2,\underline{\sigma},\underline{\omega}}(\bar{\mathbf{k}}_{\eta\eta'})]\qquad \gamma^h n_h=\delta_{\omega,-\omega}[\frac{1}{4}\sum_{\eta,\eta'=\pm1}\hat{W}^{b(h)}_{2,\underline{\sigma},\underline{\omega}}(\bar{\mathbf{k}}_{\eta\eta'})]. \tag{4.34}$$

We *renormalize* the free integration $P_{m_h,C_h}(d\psi^{(\le h)})$ by adding to it part of the r.h.s. of (4.31). We get

$$\begin{aligned}
&\int P_{m_h,C_h}(d\psi^{(\le h)})\,e^{-\mathcal{V}^{(h)}(\psi^{(\le h)})}=\\
&\qquad=e^{-L\beta t_h}\int P_{m_{h-1},C_h}(d\psi^{(\le h)})\,e^{-\hat{\mathcal{V}}^{(h)}(\psi^{(\le h)})}\;,
\end{aligned} \tag{4.35}$$

where $P_{m_{h-1},C_h}(d\psi^{(\le h)})$ is obtained from $P_{m_h,C_h}(d\psi^{(\le h)})$ by substituting $m_h(\mathbf{k})$ with

$$m_{h-1}(\mathbf{k}) = [m_h(\mathbf{k}) + C_h^{-1}(\mathbf{k}')(C_0 + f_{0,0}(\mathbf{k}))s_h] \; ; \tag{4.36}$$

moreover $\hat{\mathcal{V}}^{(h)} = \mathcal{V}^{(h)} - s_h F_\sigma^{(\le h)}$. We will call $m_h(0) \equiv m_h$. The r.h.s of (4.35) can be written as

$$e^{-M^2 t_h} \int P_{m_{h-1},C_{h-1}}(d\psi^{(\le h-1)}) \int P_{m_{h-1},f_h^{-1}}(d\psi^{(h)})\, e^{-\hat{\mathcal{V}}^{(h)}(\psi^{(\le h)})} \; , \tag{4.37}$$

and the factor $\exp(-M^2 t_h)$ in (4.35) takes into account the different normalization of the two integrations. If we now define

$$e^{-\mathcal{V}^{(h-1)}(\psi^{(\le h-1)}) - M^2 \tilde{E}_h} = \int P_{m_{h-1},\tilde{f}_h^{-1}}(d\psi^{(h)})\, e^{-\hat{\mathcal{V}}^{(h)}(\psi^{(\le h)})} \; , \tag{4.38}$$

it is easy to see that $\mathcal{V}^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\le h-1)})$ is of the form (4.21) and that

$$E_{h-1} = E_h + t_h + \tilde{E}_h \; . \tag{4.39}$$

It is sufficient to use the well known identity

$$M^2 \tilde{E}_h + \mathcal{V}^{(h-1)}(\psi^{(\le h-1)}) = \sum_{n=1}^{\infty} \frac{1}{n!}(-1)^{n+1} \mathcal{E}_h^{T,n}(\hat{\mathcal{V}}^{(h)}(\psi^{(\le h)})) \; , \tag{4.40}$$

where $\mathcal{E}_h^{T,n}$ denotes the *truncated expectation of order* n with propagator $g^{(h)}_{\omega,\omega'}$, see (4.11), and observe that $\psi^{(\le h)} = \psi^{(\le h-1)} + \psi^{(h)}$.

Let us define

$$h^* = \inf\{h : 0 \ge h \ge h_M, a_0 \gamma^{\bar{h}-1} \ge 4|m_{\bar{h}}|, \forall \bar{h} : 0 \ge \bar{h} \ge h\} \; . \tag{4.41}$$

Of course this definition is meaningful only for m_0 such that $|m_0| \le \frac{a_0}{4\gamma}$.

The integration of the scales from h^* to h_M will be performed "in a single step". This follows from thw following result

4.5 Lemma. *Assume that h^* is finite uniformly in M, so that $|m_{h^*-1}\gamma^{-h^*}| \ge \bar{\kappa}$, for a suitable constant $\bar{\kappa}$ and define*

$$\bar{g}^{(\le h^*)}_{\omega,\omega'}(\mathbf{x} - \mathbf{y}) \equiv \int P_{m_{h^*-1},C_{h^*}}(d\psi^{(\le h^*)}) \psi^{(\le h^*)-}_{\mathbf{x},\omega} \psi^{(\le h^*)+}_{\mathbf{y},\omega'} \; . \tag{4.42}$$

Then, given the positive integers N, n_0, n_1 and putting $n = n_0 + n_1$, there exist a constant $C_{N,n}$ such that

$$|\partial_{x_0}^{n_0} \partial_x^{n_1} g^{(\le h^*)}_{\omega,\omega'}(\mathbf{x};\mathbf{y})| \le C_{N,n} \frac{\gamma^{h^*+n}}{1 + (\gamma^{h^*}|\mathbf{d}(\mathbf{x}-\mathbf{y})|)^N} \; . \tag{4.43}$$

4.6 *Remark.* Let us now explain the main motivations of the integration procedure discussed above. In a Renormalization Group framework one has to identify the relevant, marginal and irrelevant effective interactions. By a power counting argument one sees that the terms bilinear in the fields are *relevant*, and the terms quartic in the fields are marginal. The Renormalization Group flow of the relevant terms can be controlled by the introduction of a *counterterm*; as we have at our disposal only one counterterm, it is important to have only one kind of relevant effective interactions; other *a priori* possible relevant terms are forbidden by symmetry considerations, see §4.3. The unique relevant effective interaction has the form $\psi^+_{\mathbf{x},1}\psi^-_{\mathbf{x},-1} - \psi^+_{\mathbf{x},-1}\psi^-_{\mathbf{x},1}$, which can be interpreted as a *mass term* in a fermionic quantum field theory like the Thirring model. There is however an important difference: generally in fermionic models if there is no mass term in the action then no mass terms are generated by Renormalization Group iterations; the reason is that such models are invariant under the local gauge transformation $\psi^\varepsilon_{\mathbf{x},\omega} \to e^{i\varepsilon\alpha_\omega}\psi^\varepsilon_{\mathbf{x},\omega}$ if there is no mass term in the action. In our model this is not true, as the interaction is not invariant under this symmetry; hence even if $t = t_c$ (or $m_1 = 0$) a mass term in the Renormalization Group integrations can be generated. Hence we collect all the relevant terms which are vanishing if $m_k = 0$, $1 \geq k \geq h+1$, in s_h, which we include in the fermionic iterations; the "mass" has a non trivial flow producing at the end the critical index of the correlation length. The remaining terms are left in the effective interaction; they are constituting the running coupling constant ν_h (in a gauge invariant theory $\nu_h = 0$) whose flow is controlled by the counterterm ν. The running coupling constant v_h are dimensionally marginal. Note that marginal effective interaction of the form $\psi^+_{\mathbf{x},\omega}\partial_x\psi^-_{\mathbf{x},-\omega}$ or $\psi^+_{\mathbf{x},\omega}\partial_{x_0}\psi^-_{\mathbf{x},-\omega}$ are generated in the Renormalization Group iterations, even if they are not present in the free fermionic integration; again this would be impossible in a fermionic relativistic theory. Finally, due to the mass gap, the propagator of the integration of all the scales between h^* and h_M has the same bound as the propagator of the integration of a single scale greater than h^*; this property is used to perform the integration of all the scales $\leq h^*$ in a single step.

4.7 THEOREM 3. *Let $h > h^* \geq 0$ and, for some constants c_1, if*

$$\max_{k\geq h}[|v_k| + |\nu_k|] \leq \varepsilon_h \qquad \sup_{h'>h}\left|\frac{m_{h'}}{m_{h'-1}}\right| \leq e^{c_1\varepsilon_h}, \tag{4.44}$$

there exists a constant $\bar\varepsilon$ (depending on c_1) such that, if $\varepsilon_h \leq \bar\varepsilon$, then, for a suitable constant c_0, independent of c_1, as well as of M, the kernels in (4.21) verify

$$\sum^*_{\mathbf{x}_1,\ldots,\mathbf{x}_{2n}} |W^{(h)}_n(\mathbf{x}_1, \ldots, \mathbf{x}_{2n})| \leq M^2\gamma^{-hD_k(P_{v_0})}(c_0\varepsilon_h)^{max(1,n-1)} \tag{4.45}$$

where

$$D_k(P_{v_0}) = -2 + n + k \ . \tag{4.46}$$

and $k = \sum_{i=1}^{2n} \alpha_i$.

4.8 *Proof.* We write $\mathcal{V}^{(h)}$ in terms of a *tree expansion*, similar to that described, for example, in [BM].

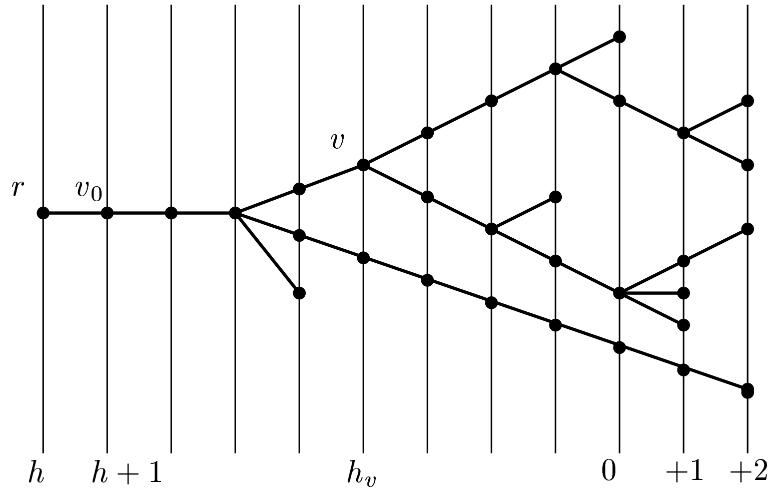

Fig. 1

We need some definitions and notations.

1) Let us consider the family of all trees which can be constructed by joining a point r, the *root*, with an ordered set of $n \geq 1$ points, the *endpoints* of the *unlabeled tree*, so that r is not a branching point. n will be called the *order* of the unlabeled tree and the branching points will be called the *non trivial vertices*. The unlabeled trees are partially ordered from the root to the endpoints in the natural way; we shall use the symbol $<$ to denote the partial order. Two unlabeled trees are identified if they can be superposed by a suitable continuous deformation, so that the endpoints with the same index coincide. It is then easy to see that the number of unlabeled trees with n end-points is bounded by 4^n. We shall consider also the *labeled trees* (to be called simply trees in the following); they are defined by associating some labels with the unlabeled trees, as explained in the following items.

2) We associate a label $h \leq 0$ with the root and we denote $\mathcal{T}_{h,n}$ the corresponding set of labeled trees with n endpoints. Moreover, we introduce a family of vertical lines, labeled by an an integer taking values in $[h, 2]$, and we represent any tree $\tau \in \mathcal{T}_{h,n}$ so that, if v is an endpoint or a non trivial vertex, it is contained in a vertical line with index $h_v > h$, to be called the *scale* of v, while the root is on the line with index h. There is the constraint that, if v is an endpoint, $h_v > h+1$; if there is only one end-point its scale must be equal to $h+2$, for $h \leq 0$.

The tree will intersect in general the vertical lines in set of points different from the root, the endpoints and the non trivial vertices; these points will be called *trivial vertices*. The set of the *vertices* of τ will be the union of the endpoints, the trivial vertices and the non trivial vertices. Note that, if v_1 and v_2 are two vertices and $v_1 < v_2$, then $h_{v_1} < h_{v_2}$.

Moreover, there is only one vertex immediately following the root, which will be denoted v_0 and can not be an endpoint; its scale is $h+1$.

3) With each endpoint v of scale $h_v = +2$ we associate one of the contributions to $\mathcal{V}^{(1)}$ given by (3.9); with each endpoint v of scale $h_v \le 1$ one of the terms in $\mathcal{L}V^{(h_v-1)}$ defined in (4.31). Moreover, we impose the constraint that, if v is an endpoint and $h_v \le 1$, $h_v = h_{v'}+1$, if v' is the non trivial vertex immediately preceding v.

4) If v is not an endpoint, the *cluster* L_v with frequency h_v is the set of endpoints following the vertex v; if v is an endpoint, it is itself a (*trivial*) cluster. The tree provides an organization of endpoints into a hierarchy of clusters.

5) We introduce a *field label* f to distinguish the field variables appearing in the terms associated with the endpoints as in item 3); the set of field labels associated with the endpoint v will be called I_v. Analogously, if v is not an endpoint, we shall call I_v the set of field labels associated with the endpoints following the vertex v; $\mathbf{x}(f)$, $\varepsilon(f)$ and $\omega(f)$ will denote the space-time point, the ε index and the ω index, respectively, of the field variable with label f.

If $h \le 0$, the effective potential can be written in the following way:

$$\mathcal{V}^{(h)}(\psi^{(\le h)}) + M^2\tilde{E}_{h+1} = \sum_{n=1}^{\infty}\sum_{\tau\in\mathcal{T}_{h,n}} \mathcal{V}^{(h)}(\tau,\psi^{(\le h)}) \ , \tag{4.47}$$

where, if v_0 is the first vertex of τ and $\tau_1,..,\tau_s$ $(s = s_{v_0})$ are the subtrees of τ with root v_0,
$\mathcal{V}^{(h)}(\tau,\psi^{(\le h)})$ is defined inductively by the relation

$$\begin{aligned}&\mathcal{V}^{(h)}(\tau,\psi^{(\le h)}) = \\ &\frac{(-1)^{s+1}}{s!}\mathcal{E}^T_{h+1}[\bar{\mathcal{V}}^{(h+1)}(\tau_1,\psi^{(\le h+1)});\ldots;\bar{\mathcal{V}}^{(h+1)}(\tau_s,\psi^{(\le h+1)})] \ ,\end{aligned} \tag{4.48}$$

and $\bar{\mathcal{V}}^{(h+1)}(\tau_i,\psi^{(\le h+1)})$

a) is equal to $\mathcal{R}\mathcal{V}^{(h+1)}(\tau_i,\psi^{(\le h+1)})$ if the subtree τ_i is not trivial;

b) if τ_i is trivial and $h \le -1$, it is equal to one of the terms in $\mathcal{L}\mathcal{V}^{(h+1)}(\psi^{(\le h+1)})$ or, if $h = 0$, to one of the terms contributing to $\mathcal{V}^{(1)}(\psi^{\le 1})$ (3.9).

In (4.48) $\mathcal{E}^T_{h+1}$ denotes the truncated expectation with respect to the measure $\prod_\omega P(d\psi^{(h+1)}_\omega)$. The r.h.s. of (4.48) can be written more explicitly in the following way. Given $\tau\in\mathcal{T}_{0,n}$, there are n endpoints of scale 2 and only another one vertex, v_0, of scale 1; let us call $v_1,\ldots,v_n$ the endpoints. We choose, in any set I_{v_i}, a subset Q_{v_i} and we define $P_{v_0} = \cup_i Q_{v_i}$. We have

$$\mathcal{V}^{(0)}(\tau,\psi^{(\le 0)}) = \sum_{P_{v_0}} \mathcal{V}^{(0)}(\tau,P_{v_0}) \ , \tag{4.49}$$

$$\mathcal{V}^{(0)}(\tau, P_{v_0}) = \int d\mathbf{x}_{v_0} \tilde\psi^{\le 0}(P_{v_0}) K^{(1)}_{\tau, P_{v_0}}(\mathbf{x}_{v_0}) \;, \tag{4.50}$$

$$K^{(1)}_{\tau,P_{v_0}}(\mathbf{x}_{v_0}) = \frac{1}{n!}\mathcal{E}^T_1[\tilde\psi^{(\le 1)}(P_{v_1}\backslash Q_{v_1}),\ldots,\tilde\psi^{(\le 1)}(P_{v_n}\backslash Q_{v_n})]\prod_{i=1}^n K^{(2)}_{v_i}(\mathbf{x}_{v_i}) \;, \tag{4.51}$$

where we use the definitions $\tilde\psi^{(\le h)}(P_v) = \prod_{f\in P_v} \partial^{m(f)} \psi^{(\le h)\varepsilon(f)}_{\mathbf{x}(f),\omega(f)}$, and $K^{(2)}_{v_i}(\mathbf{x}_{v_i})$ are the kernels in (3.9). We now write $\mathcal{V}^{(0)}$ as $\mathcal{L}\mathcal{V}^{(0)} + \mathcal{R}\mathcal{V}^{(0)}$, with $\mathcal{L}\mathcal{V}^{(0)}$ defined as in (4.31), and we write for $\mathcal{R}\mathcal{V}^{(0)}$ a decomposition similar to the previous one, with $\mathcal{R}\mathcal{V}^{(0)}(\tau, P_{v_0})$ in place of $\mathcal{V}^{(0)}(\tau, P_{v_0})$. By iterating the previous procedure, one gets for $\mathcal{V}^{(h)}(\tau, \psi^{(\le h)})$, for any $\tau \in \mathcal{T}_{h,n}$, the representation described below (see [GM] and [BM] for details).

We associate with any vertex v of the tree a subset P_v of I_v, the *external fields* of v. These subsets must satisfy various constraints. First of all, if v is not an endpoint and $v_1,\ldots,v_{s_v}$ are the vertices immediately following it, then $P_v \subset \cup_i P_{v_i}$; if v is an endpoint, $P_v = I_v$. We shall denote Q_{v_i} the intersection of P_v and P_{v_i}; this definition implies that $P_v = \cup_i Q_{v_i}$. The subsets $P_{v_i}\backslash Q_{v_i}$, whose union $\mathcal{I}_v$ will be made, by definition, of the *internal fields* of v, have to be non empty, if $s_v > 1$. Given $\tau \in \mathcal{T}_{h,n}$, there are many possible choices of the subsets P_v, $v \in \tau$, compatible with all the constraints; we shall denote $\mathcal{P}_\tau$ the family of all these choices and $\mathbf{P}$ the elements of $\mathcal{P}_\tau$. We can write

$$\mathcal{V}^{(h)}(\tau, \psi^{(\le h)}) = \sum_{\mathbf{P}\in\mathcal{P}_\tau} \mathcal{V}^{(h)}(\tau, \mathbf{P}) \;. \tag{4.52}$$

$\mathcal{V}^{(h)}(\tau, \mathbf{P})$ can be represented as

$$\mathcal{V}^{(h)}(\tau, \mathbf{P}) = \int d\mathbf{x}_{v_0} \tilde\psi^{(\le h)}(P_{v_0}) K^{(h+1)}_{\tau,\mathbf{P}}(\mathbf{x}_{v_0}) \;, \tag{4.53}$$

with $K^{(h+1)}_{\tau,\mathbf{P}}(\mathbf{x}_{v_0})$ defined inductively (recall that $h_{v_0} = h+1$) by the equation, valid for any $v \in \tau$ which is not an endpoint,

$$K^{(h_v)}_{\tau,\mathbf{P}}(\mathbf{x}_v) = \frac{1}{s_v!}\prod_{i=1}^{s_v}[K^{(h_v+1)}_{v_i}(\mathbf{x}_{v_i})] \;\; \mathcal{E}^T_{h_v}[\tilde\psi^{(h_v)}(P_{v_1}\backslash Q_{v_1}),\ldots,\tilde\psi^{(h_v)}(P_{v_{s_v}}\backslash Q_{v_{s_v}})] \;, \tag{4.54}$$

Moreover, if v is an endpoint, $K^{(2)}_v(\mathbf{x}_v)$ is defined as in (3.9) if $h_v = 2$, otherwise

$$K^{(h_v)}_v(\mathbf{x}_v) = \begin{cases} \lambda_{h_v-1}\delta(\mathbf{x}_1-\mathbf{x}_2)\delta(\mathbf{x}_2-\mathbf{x}_3)\delta(\mathbf{x}_3-\mathbf{x}_4) & \text{if } v \text{ is of type } \lambda, \\ \gamma^{h_v-1}\nu_{h_v-1}\delta(\mathbf{x}-\mathbf{y}) & \text{if } v \text{ is of type } \nu, \\ z_{h_v-1,\omega,i}\delta(\mathbf{x}-\mathbf{y}), \quad \tilde z_{h_v-1,\omega,i}\delta(\mathbf{x}-\mathbf{y}) & \text{if } v \text{ is of type } z, \tilde z \end{cases} \tag{4.55}$$

If v_i is not an endpoint,

$$K^{(h_v+1)}_{v_i}(\mathbf{x}_{v_i}) = \mathcal{R} K^{(h_v+1)}_{\tau_i,\mathbf{P}^{(i)}}(\mathbf{x}_{v_i}) \;, \tag{4.56}$$

where τ_i is the subtree of τ starting from v and passing through v_i (hence with root the vertex immediately preceding v), $\mathbf{P}^{(i)}$ are the restrictions to τ_i of $\mathbf{P}$.

(4.52) is not the final form of our expansion, since we further decompose $\mathcal{V}^{(h)}(\tau,\mathbf{P})$, by using the representation of the truncated expectation (3.21). If we apply the expansion (3.21) in each non trivial vertex of τ, we get an expression of the form

$$\mathcal{V}^{(h)}(\tau,\mathbf{P}) = \sum_{T\in\mathbf{T}} \int d\mathbf{x}_{v_0} \tilde\psi^{(\le h)}(P_{v_0}) W^{(h)}_{\tau,\mathbf{P},T}(\mathbf{x}_{v_0}) \;, \tag{4.57}$$

where $\mathbf{T}$ is a special family of graphs on the set of points $\mathbf{x}_{v_0}$, obtained by putting together an anchored tree graph T_v for each non trivial vertex v. Note that any graph $T\in\mathbf{T}$ becomes a tree graph on $\mathbf{x}_{v_0}$, if one identifies all the points in the sets x_v, for any vertex v which is also an endpoint.

Taking into account the effect of the $\mathcal{R}$ operation we obtain (see [BM], §3 for a detailed proof of a similar formula)

$$\begin{aligned} W_{\tau,\mathbf{P},T}(\mathbf{x}_{v_0}) = & \sum_{\alpha\in A_T} \left[\prod_{i=1}^n K^{h_i}_{v_i^*}\right] \Bigg\{ \prod_{\substack{v\\ \text{not e.p.}}} \frac{1}{s_v!} \int dP_{T_v}(\mathbf{t}_v) \det G^{h_v,T_v}_\alpha(\mathbf{t}_v)\cdot \\ & \cdot \Big[\prod_{l\in T_v} \partial^{q_\alpha(f_l^-)} \partial^{q_\alpha(f_l^+)} [(\mathbf{x}_l-\mathbf{y}_l)^{b_\alpha(l)} \partial^{m_l} \tilde g^{(h_v)}_{\omega_l^-,\omega_l^+}(\mathbf{x}_l-\mathbf{y}_l)]\Big]\Bigg\} \;, \end{aligned} \tag{4.58}$$

where "e.p." is an abbreviation of "endpoint" and, together with the definitions used before, we are using the following ones:

1) A_T is a set of indices which allows to distinguish the different terms produced by the non trivial $\mathcal{R}$ operations and the iterative decomposition of the zeros;
2) $v_1^*,\ldots,v_n^*$ are the endpoints of τ and $h_i = h_{v_i^*}$;
3) $b_\alpha(l)$, $q_\alpha(f_l^-)$ and $q_\alpha(f_l^+)$ are positive integers ≤ 3; $\hat\partial^0 = I$;
4) if v is a non trivial vertex (so that $s_v>1$),the elements $G^{h_v,T_v}_{\alpha,ij,i'j'}$ of $G^{h_v,T_v}_\alpha(\mathbf{t}_v)$ are of the form

$$G^{h_v,T_v}_{\alpha,ij,i'j'} = t_{i,i'} \hat\partial^{q_\alpha(f^-_{ij})} \hat\partial^{q_\alpha(f^+_{i'j'})} \partial^{m(f_l^-)} \partial^{m(f_l^-)} \tilde g^{(h_v)}_{\omega^-,\omega^+}(\mathbf{x}_{ij}-\mathbf{y}_{i'j'}) \;; \tag{4.59}$$

if v is trivial, T_v is empty and $\int dP_{T_v}(\mathbf{t}_v)\det G^{h_v,T_v}_\alpha(\mathbf{t}_v)$ has to be interpreted as 1, if $|\mathcal{I}_v|=0$ ($\mathcal{I}_v$ is the set of internal fields of v), otherwise it is the determinant of a matrix of the form (4.59) with $t_{i,i'}=1$.

It would be very difficult to give a precise description of the various contributions to the sum over A_T, but fortunately we only need to know some very general properties, in particular that $|A_T|\le C^n$ for some constant C and that for any $\alpha\in A_T$, the following inequality is satisfied

$$\Big[\prod_{f\in I_{v_0}} \gamma^{h_\alpha(f) q_\alpha(f)}\Big]\Big[\prod_{l\in T} \gamma^{-h_\alpha(l) b_\alpha(l)}\Big] \le \prod_{v \text{ not e.p.}} \gamma^{-z(P_v)} \;, \tag{4.60}$$

where $h_\alpha(f) = h_{v_0} - 1$ if $f \in P_{v_0}$, otherwise it is the scale of the vertex where the field with label f is contracted; $h_\alpha(l) = h_v$, if $l \in T_v$ and

$$z(P_v) = \begin{cases} 1 & \text{if } |P_v| = 4, \\ 2 & \text{if } |P_v| = 2, \\ 0 & \text{otherwise.} \end{cases} \tag{4.61}$$

By a standard computation (see for instance [BM], §3) by bounding the determinant by the Gram-Hadamard inequality (see (3.23)) we obtain

$$\begin{aligned} &\int d\mathbf{x}_{v_0} |W_{\tau,\mathbf{P},\mathbf{T}}(\mathbf{x}_{v_0})| \le C^n M^2 \varepsilon_h^n \gamma^{-hD_k(P_{v_0})} \cdot \\ &\cdot \prod_{v \text{ not e.p.}} \left\{ \frac{1}{s_v!} C^{\sum_{i=1}^{s_v} |P_{v_i}| - |P_v|} \gamma^{-[-2+\frac{|P_v|}{2}+z(P_v)]} \right\}, \end{aligned} \tag{4.62}$$

with $-2 + \frac{|P_v|}{2} + z(P_v) > 0$. In order to perform the sums note that the number of unlabeled trees is $\le 4^n$; fixed an unlabeled tree, the number of terms in the sum over the various labels of the tree is bounded by C^n, except the sums over the scale labels. In order to bound the sums over the scale labels and $\mathbf{P}$ we first use the inequality

$$\begin{aligned} &\prod_{v \text{ not e.p.}} \gamma^{-[-2+\frac{|P_v|}{2}+z(P_v)]} \le \\ &\le [\prod_{\tilde v} \gamma^{-2\alpha(h_{\tilde v} - h_{\tilde v'})}][\prod_{v \text{ not e.p.}} \gamma^{-2\alpha|P_v|}], \end{aligned} \tag{4.63}$$

where $\tilde v$ are the non trivial vertices, and $\tilde v'$ is the non trivial vertex immediately preceding $\tilde v$ or the root. The factors $\gamma^{-2\alpha(h_{\tilde v} - h_{\tilde v'})}$ in the r.h.s. of (4.63) allow to bound the sums over the scale labels by C^n; α is a suitable constant (one finds $\alpha = \frac{1}{40}$).

Finally the sum over $\mathbf{P}$ can be bounded by using the following combinatorial inequality, trivial for γ large enough. Let $\{p_v, v \in \tau\}$ a set of integers such that $p_v \le \sum_{i=1}^{s_v} p_{v_i}$ for all $v \in \tau$ which are not endpoints; then

$$\prod_{v \text{ not e.p.}} \sum_{p_v} \gamma^{-\frac{p_v}{40}} \le C^n . \tag{4.64}$$

It follows that

$$\sum_{\substack{\mathbf{P} \\ |P_{v_0}|=2m}} \prod_{v \text{ not e.p.}} \gamma^{-\frac{|P_v|}{40}} \le \prod_{v \text{ not e.p.}} \sum_{p_v} \gamma^{-\frac{p_v}{40}} \le C^n . \tag{4.65}$$

4.9 *Remark 1.* The decomposition in (4.33) respects the determinant structure of the truncated expectations, as we can decompose the propagators as in

(4.15),(4.19), obtaining, for any tree with n end-points, a family of C^n different contributions to $\hat W_2^{(h)}$, which can be bounded as before. Hence we can include in $\hat W^{(a)}$ the terms with at least a propagator $\hat g_{\omega,-\omega}$, defined in (4.20).

Remark 2. If the tree τ has an end-point on scale k, the bound (4.45) can be improved by a factor $\gamma^{\alpha(h-k)}$, as it follows immediately from (4.63). In particular if to an end-point is associated a term $\mathcal{R}V^{(1)}$ there is an extra factor $\gamma^{\alpha h}$; this is a *short memory* property.

4.10 The above results were proved for $Z_{2I}^{-,-,-,-}$; a similar analysis can be repeated for $Z_{2I}^{\varepsilon^{(1)},\varepsilon'^{(1)},\varepsilon^{(2)},\varepsilon'^{(2)}}$, for any value of $\varepsilon^{(1)},\varepsilon'^{(1)},\varepsilon^{(2)},\varepsilon'^{(2)}$. The only difference is that one has in addition the function $\tilde Q_\varepsilon$ in the interaction and the oscillating functions $e^{i\mathbf{p}_M\mathbf{x}}$. One can split $\mathcal{V}^1$ in a part identical to the one for $Z_{-,-,-,-}$, called $\bar{\mathcal{V}}^{(1)}$, and the rest; hence we repeat the multiscale analysis by writing $\mathcal{V}^{(h)}=\bar{\mathcal{V}}^{(h)}+\mathcal{V}'^{(h)}$, with $\mathcal{V}'^{(h)}$ given by a sum of trees with at least an end-point associated to $\mathcal{V}^{(1)}-\bar{\mathcal{V}}^{(1)}$; we define the localization operators acting non trivially only on $\bar{\mathcal{V}}^{(h)}$ (and defined as above). It is easy to see that the terms from trees with at least one end-point associated to $\mathcal{V}^{(1)}-\bar{\mathcal{V}}^{(1)}$ are vanishing in the limit $M\to\infty$; in fact the bounds for such terms is improved by the factor $\frac{\gamma^{-h^*}}{M}$, by simply dimensional considerations, and we will see in the following section that γ^{-h^*} is a finite number independent from M, hence such terms are vanishing in the limit $M\to\infty$.

5. The flow of the running coupling constants

5.1 By the analysis of the preceding section it follows that the running coupling constants v_k,ν_k,m_k, $1\ge k\ge h^*$, verify a set of recursive equations called *Beta function equations*, of the form

$$\begin{aligned}\nu_{h-1}&=\gamma\nu_h+\beta_\nu^h(v_h,\nu_h;...;v_1,\nu_1)\\ v_{h-1}&=v_h+\beta_v^h(v_h,\nu_h;...;v_1,\nu_1)\\ \frac{m_{h-1}}{m_h}&=1+\beta_m^h(v_h,\nu_h;...;v_1,\nu_1)\end{aligned}\tag{5.1}$$

By repeating the analysis for proving Theorem 3 to the functions β_h^i we have that, if (4.44) holds, then the β_h^i are expressed by convergent series. We want to show that there exists a positive constant a_3 such that, if $|t-t_c|\ge e^{-\frac{1}{a_3\lambda^2}}$, it is possible to find a function ν_1 so that (4.44) holds (if λ is small enough). Iterating the first of (5.1) we find

$$\nu_h=\gamma^{-h+1}\left[\nu_1+\sum_{k=h+1}^{1}\gamma^{k-2}\beta_\nu^k(\nu_k,\ldots,\nu_0)\right]\ ,\tag{5.2}$$

where now the functions β^k_ν are thought as functions of $\nu_k, \ldots, \nu_1$ only.
If we put $h = h^*$ in (5.2), we get the following identity:

$$\nu_{h^*} = \gamma^{-h^*+1}\left[\nu_1 + \sum_{k=h^*+1}^{1} \gamma^{k-2}\beta^k_\nu(\nu_k, \ldots, \nu_0)\right] , \tag{5.3}$$

and we look for a ν_1 verifying

$$\nu_1 = -\sum_{k=h^*+1}^{1} \gamma^{k-2}\beta^k_\nu(\nu_k, \ldots, \nu_1) . \tag{5.4}$$

so that

$$\nu_h = -\gamma^{-h}\sum_{k=h^*+1}^{h} \gamma^{k-1}\beta^k_\nu(\nu_k, \ldots, \nu_1) , \qquad h^* < h \le 1 . \tag{5.5}$$

Let be $\tilde\nu \equiv \{\tilde\nu_k, h^* \le h \le 1\}$ and $||\tilde\nu|| = \sup_{h^*\le h \le 1} |\tilde\nu_h|$; we call M the set of $\tilde\nu$ with bounded norm. We decouple the beta function equations (5.1) imagining that in the second and third equation of (5.1) ν_k is replaced by $\tilde\nu_k$ acting as a parameter; we call $m_h(\tilde\nu), v_k(\tilde\nu)$ the solution of the second and third of (5.1) as functions of the parameter $\tilde\nu$. We shall prove the following lemma.

5.2 LEMMA. *There exists ε and a_3 such that, for $|t-t_c| \ge e^{\frac{-1}{a_3\lambda_1^2}}$, $|\lambda| \le \varepsilon$ and any $\tilde\nu$ such that $||\tilde\nu||, ||\tilde\nu'|| \le C_\nu|\lambda|$, it holds, if $m_0 = |t-t_c|$ and C_ν, C are constants*

$$|\lambda_h(\tilde\nu) - \lambda_1| \le \frac{|\lambda_1|}{4} \qquad |\lambda_h(\tilde\nu) - \lambda_h(\tilde\nu')| \le C\lambda^2 \max_{j>h} |\tilde\nu_k - \tilde\nu'_k| \tag{5.6}$$

$$|\tilde z_{i,\omega,h}(\tilde\nu)| \le |\tilde z_{0,\omega,h}| + |\lambda| \qquad |z_{i,\omega,h}(\tilde\nu)| \le |z_{i,\omega,0}| + |\lambda|$$

$$|\tilde z_{i,\omega,h}(\tilde\nu) - \tilde z_{i,\omega,h}(\tilde\nu')| \le C\lambda^2 \max_{k>h} |\tilde\nu_k - \tilde\nu'_k| \qquad |z_{i,\omega,h}(\tilde\nu) - z_{i,\omega,h}(\tilde\nu')| \le C\lambda^2 \max_{k>h} |\tilde\nu_k - \tilde\nu'_k|$$

$$|m_0|\gamma^{c_1 a_2 \lambda_1 h} \le |m_h| \le |m_0|\gamma^{c_2 a_2 \lambda_1 h} \qquad |\frac{m_h(\tilde\nu)}{m_h(\tilde\nu')} - 1| \le \max_{j>h} |\tilde\nu_k - \tilde\nu'_k|$$

with c_1, c_2 constants.

Remark. An obvious corollary of the above statement is that there exits a finite (uniformly in M) h^* such that $m_{h^*-1}\gamma^{h^*-1} \ge \kappa$ for a suitable constant κ.

5.3 *Proof.* The proof is done by induction. By iterating the second equation of (5.1) we find

$$\lambda_{h-1}(\tilde\nu) - \lambda_1 = \sum_{k=h}^{1} \beta^k_\lambda(v_h(\tilde\nu), \tilde\nu_h; \ldots; v_1(\tilde\nu), \tilde\nu_1) \tag{5.7}$$

and by induction

$$|\beta^k_\lambda(v_h(\tilde\nu), \tilde\nu_h; \ldots; v_1(\tilde\nu), \tilde\nu_1)| \le C_1[|\lambda_1|^3 + \lambda_1^2 \frac{|m_h|}{\gamma^h} + \lambda_1^2\gamma^{\alpha h}]. \tag{5.8}$$

The first addend of the r.h.s. of (5.8) is a bound on the terms containing only propagators $g^{(h)}_{L,\omega}(\mathbf{x}-\mathbf{y})$, see (4.16), and trees with end-points v with $h_v \leq 1$, and the bound follows from the fact that the second order terms cancels out. The second addend is a bound on the terms containing at least a propagator $\hat g^{(h)}_{\omega_1,\omega_2}(\mathbf{x}-\mathbf{y})$ (4.20),(4.18) (the bound follows from the short memory property and (4.14), (4.18)). The third addend is a bound on the sum of terms with at least a propagator $\tilde g^{(h)}_{\omega_1,\omega_2}(\mathbf{x}-\mathbf{y})$ or from trees with at least an end-point v with $h_v = 2$, and we have used (4.17).

Inserting (5.8) in (5.7) we get

$$|\lambda_{h-1}(\tilde\nu)-\lambda_1| \leq C_1[|h||\lambda_1|^3 + \lambda_1^2 \sum_{k=h}^{0}[\frac{|m_k|}{\gamma^k}+\gamma^{\alpha k}]| \leq \frac{|\lambda_1|}{4} \tag{5.9}$$

for a_3 large enough and λ small enough, where we have used that $\sum_{k=h}^0 \frac{|m_k|}{\gamma^k}$ is bounded (by induction and the definition of h^*).

Moreover the expansion for $\lambda_{h-1}(\tilde\nu)-\lambda_{h-1}(\tilde\nu')$ is given by a sum of terms similar to the ones for $\lambda_{h-1}(\tilde\nu)$ in which a v_k is replaced by $v_k(\tilde\nu)-v_k(\tilde\nu')$ bounded by $C\lambda^2 \max_{j>k}|\tilde\nu_j-\tilde\nu'_j|$ or m_k is replaced by $m_k(\tilde\nu)-m_k(\tilde\nu')$ bounded by $\max_{j>k}|\tilde\nu_j-\tilde\nu'_j|$; one finds, for a_3 large and λ small enough

$$|\lambda_{h-1}(\tilde\nu)-\lambda_{h-1}(\tilde\nu')| \leq C_2[|h|C|\lambda|^4$$

$$+\lambda^2(1+C|\lambda|)\sum_{k=h}^{0}[\frac{|m_k|}{\gamma^k}|+\gamma^{\alpha k}]] \max_{j>h}|\tilde\nu_k-\tilde\nu'_k| \leq C\lambda^2 \max_{j>h}|\tilde\nu_k-\tilde\nu'_k|, \tag{5.10}$$

where the first term is a bound on the terms containing only propagators $g^{(h)}_{L,\omega}(\mathbf{x}-\mathbf{y})$, and trees with end-points v with $h_v \leq 1$, containing at least a $v_k(\tilde\nu)-v_k(\tilde\nu')$ or a $\tilde\nu_k-\tilde\nu'_k$ (note also that the first term with end-point ν is $\lambda_h^2\nu_h^2$).

The recursive equation for $\tilde z_{i,\omega,h}$ is

$$\tilde z_{i,\omega,h-1}(\tilde\nu) = \tilde z_{i,\omega,h}(\tilde\nu) + \tilde z_{i,\omega,h}(\lambda_h^2 a_{\tilde z} + \beta_z) + R^{(h)}_{\tilde z} \tag{5.11}$$

with $|\beta^h_z| \leq C_3|\lambda|^3$ and $|R^h_{\tilde z}| \leq C_4[\gamma^{\alpha h}+\frac{|m_h|}{\gamma^{\alpha h}}]\lambda^2$. In $\lambda_h^2(a_2+\beta_z)$ in (5.11) are the contributions from terms with only diagonal propagators $g^{(h)}_{L,\omega}(\mathbf{x})$ and from trees with endpoints v with $h_v \leq 1$; there is necessarily at least a vertex $\tilde z$. In fact assume that a contribution has n_λ end-points of type λ and n end-points of type ν; there are then $n_\lambda - \frac12 + \frac n2$ propagators $\omega = 1$ and $n_\lambda - \frac12 + \frac n2$ propagators $\omega = -1$, so that n must be *odd*; the total number of propagators $2n_\lambda - 1 + n$ is an *even* number; the derivated integrand is odd under the exchange $\mathbf{k} \to -\mathbf{k}$ and so it is vanishing. Moreover there is no contribution of order $\lambda_h z_h$ (they corresponds to *tadpole* graphs , whose derivative is zero). In $R^{(h)}_{\tilde z}$ are the trees with at least an end point v with $h_v = 2$, or with a propagator $\hat g^{(k)}_{\omega_1,\omega_2}$, or with a propagator $\tilde g^{(k)}_{\omega_1,\omega_2}$.

It holds by iterating, for a_3 large and λ small enough, $|\tilde z_{i,\omega,h}| \le |\tilde z_{0,\omega,h}| + |\lambda|$; an equation very similar holds for $z_{i,\omega,h}$. Moreover

$$|\tilde z_{i,\omega,h-1}(\tilde\nu) - \tilde z_{i,\omega,h-1}(\tilde\nu')| \le C_3 \max_{j>h} |\tilde\nu_k - \tilde\nu'_k| [C\lambda^4 |h| + \lambda^2 (1 + C|\lambda|) \sum_{k=h}^{0} [\frac{|m_k|}{\gamma^k} + \gamma^{\alpha k}]] \le$$

$$C\lambda^2 \max_{j>h} |\tilde\nu_k - \tilde\nu'_k| \tag{5.12}$$

By definition (see (4.33)) $m_{h-1}(\tilde\nu)$ is given by a sum over terms with at least a non diagonal propagator $\hat g^{(h)}_{\omega,-\omega}(\mathbf{x}-\mathbf{y})$, hence

$$\frac{m_{h-1}(\tilde\nu)}{m_h(\tilde\nu)} = 1 + \lambda_h(-a_2 + h_m) \tag{5.13}$$

where $a_2 > 0$ is a constant and $|h_m| \le C|\lambda_1|$; then (assuming $\lambda_1 > 0$; similar computations can be repeated for $\lambda_1 < 0$), for λ small enough

$$(1 - \frac{5}{4} a_2 \lambda_1) \le \frac{|m_{h-1}(\tilde\nu)|}{|m_h(\tilde\nu)|} \le (1 - \frac{3}{4} a_2 \lambda_1) \tag{5.14}$$

Then

$$|m_{h-1}(\tilde\nu)| \ge |m_0| \gamma^{2a_2\lambda_1(h-1)} \frac{(1 - \frac{5}{4}a_2\lambda_1)}{\gamma^{-2a_2\lambda_1}} \ge |m_0| \gamma^{2a_2\lambda_1(h-1)} \tag{5.15}$$

and

$$|m_{h-1}(\tilde\nu)| \le |m_0| \gamma^{\frac{a_2}{2}\lambda_1(h-1)} \frac{(1 - \frac{3}{4}a_2\lambda_1)}{\gamma^{-\lambda_1\frac{a_2}{2}}} \le |m_0| \gamma^{\frac{a_2}{2}\lambda_1(h-1)} \tag{5.16}$$

Finally, for a_3 large enough

$$|\frac{m_{h-1}(\tilde\nu)}{m_{h-1}(\tilde\nu')} - 1| = |\frac{\prod_{k=h}^1 (1 + \beta_m^k(\tilde\nu)) - \prod_{k=h}^1 (1 + \beta_m^k(\tilde\nu'))}{\prod_{k=h}^1 (1 + \beta_m^k(\tilde\nu'))}|$$

$$\le C_4 \log\beta \max_{0\le k\le h} |v_k(\tilde v) - v_k(\tilde v')| \le \max_{j>h} |\tilde\nu_k - \tilde\nu'_k| \tag{5.17}$$

5.4 LEMMA. *There exists* ε, a_3 *and* C_ν *such that, for* $|t - t_c| \ge e^{\frac{-1}{a_3\lambda^2}}$ *and* $|\lambda| \le \varepsilon$, *there exists a* $\nu_1(\lambda)$ *so that* $\max_{k\ge h^*} |\nu_k| \le C_\nu |\lambda|$.

Proof. It is sufficient to look for a fixed point for the operator $\mathbf{T} : M \to M$, and $\mathbf{T}$ is defined in the following way, if $\tilde\nu' = \mathbf{T}(\tilde\nu)$, see (5.5):

$$\tilde\nu'_h = -\gamma^{-h} \sum_{k=h^*+1}^{h} \gamma^{k-1} \beta_k^\nu (v_k(\tilde\nu), \tilde\nu_k, \ldots, \nu_0(\tilde\nu), \tilde\nu_0) \;, \qquad h^* < h \le 1 \;. \tag{5.18}$$

We want to prove that it is possible to choose the constant $C_\nu \geq 1$ in Lemma (5.2) so that, if $|\lambda|$ is small enough, the set $\mathcal{F} = \{\tilde{\nu} \in M : ||\tilde{\nu}|| \leq C_\nu |\lambda|\}$ is invariant under $\mathbf{T}$ and $\mathbf{T}$ is a contraction on it. This is sufficient to prove the lemma, since M is a Banach space, as one can easily show.

By (5.18) and Theorem 3

$$|\tilde{\nu}'_h| \leq \sum_{j=h^*+1}^{h} \gamma^{-h+j-1} \left[C_{1,\nu}|\lambda| + \sum_{n=2}^{\infty} c^n |\lambda|^n \right] , \tag{5.19}$$

where $C_{1,\nu}$ is a constant depending on the first order contribution (*i.e.* the *tadpole*). So, for a proper C_ν, $||\tilde{\nu}'_h|| \leq C_\nu |\lambda|$.

We then show that $\mathbf{T}$ is a contraction on $\mathcal{F}$. In fact, given $\tilde{\nu}_1, \tilde{\nu}_2 \in \mathcal{F}$, by using Theorem 3 and Lemma (5.2), we see that, for λ small enough

$$\tilde{\nu}_{1,h} - \tilde{\nu}_{2,h} = -\gamma^{-h} \sum_{k=h^*+1}^{h} \gamma^{k-1} \times$$

$$[\beta^\nu_k(v_k(\tilde{\nu}_1), \tilde{\nu}_{k,1}, \ldots, v_1(\tilde{\nu}_1), \tilde{\nu}_{1,1}) - \beta^\nu_k(v_k(\tilde{\nu}_2), \tilde{\nu}_{k,2}, \ldots, v_1(\tilde{\nu}_2), \tilde{\nu}_{1,2})] \tag{5.20}$$

and

$$|\tilde{\nu}_{1,h} - \tilde{\nu}_{2,h}| \leq C_1|\lambda|[\gamma^{-h} \sum_{k=h^*+1}^{h} \gamma^{k-1}][\sum_{j\geq h} \gamma^{\alpha(h-j)} \max_{i \geq j} |\tilde{\nu}_{1,i} - \tilde{\nu}_{2,i}|] \leq C_2|\lambda| ||\tilde{\nu}_1 - \tilde{\nu}_2||$$

with C_1, C_2 constants, so that $\mathbf{T}$ is a contraction.

5.5 If $a = b = 0$ one has the case treated in [SP], corresponding to two independent Ising models with a quartic interaction. In such a case the local part of the quartic terms is vanishing by Pauli principle, so that if $|P_v| = 4$ one can apply "freely" a first order renormalization obtaining an additional $\gamma^{-(h_v - h_{v'})}$, for any v such that $|P_v| = 4$, in the bounds. At each step one can include all the quadratic running coupling constants in the free integrations; their beta function is $\tilde{z}_{i,h-1} = \tilde{z}_{i,h} + O(\gamma^{\alpha h}\lambda)$ and $z_{i,h-1} = z_{i,h} + O(\gamma^{\alpha h}\lambda)$, as the beta function is sum over all the trees with end-points necessarily at scale $h_v = 2$ (contrary to our case, as the local part of the quartic terms is vanishing, and there are no end-points associated to the quadratic terms as they are included in the free integration). Hence all the quadratic couplings are $O(\lambda)$ and they do not change the scaling properties of the propagator on a single scale; the qualitative behaviour of C_v for $\lambda = 0$ or $\lambda \neq 0$ are then the same.

6. Correlation functions

6.1 *Flow of observables.* We consider the following functional integral

$$e^{\mathcal{S}(\phi)} = \int P(d\psi) e^{\mathcal{V}^{(1)}(\psi)+\mathcal{B}(\phi,\psi)} \tag{6.1}$$

where $\mathcal{V}^{(1)}$ given by (3.8) and $\mathcal{B} = \mathcal{B}^1 + \mathcal{B}^2$ with

$$\mathcal{B}^{(1)}(\psi,\phi) = \int d\mathbf{x}\phi^{(1)}(\mathbf{x})[\psi^+_{\mathbf{x},1}\psi^-_{\mathbf{x},-1} + \psi^-_{\mathbf{x},1}\psi^+_{\mathbf{x},-1}] \tag{6.2}$$

$$\mathcal{B}^{(2)}(\psi,\phi) = \int d\mathbf{x}\phi^{(2)}(\mathbf{x})[\psi^+_{\mathbf{x},1}\psi^+_{\mathbf{x},-1} + \psi^-_{\mathbf{x},1}\psi^-_{\mathbf{x},-1}]\ . \tag{6.3}$$

After integrating the fields $\psi^{(1)}, ...\psi^{(h+1)}$, $0 \geq h \geq h^*$, we find

$$e^{\mathcal{S}(\phi)} = e^{-L\beta E_h + S^{(h+1)}(\phi)} \int P_{m_h,C_h}(d\psi^{\leq h}) e^{-\mathcal{V}^{(h)}(\psi^{(\leq h)})+\mathcal{B}^{(h)}(\psi^{(\leq h)},\phi)}\ , \tag{6.4}$$

where $P_{m_h,C_h}(d\psi^{(\leq h)})$ and $\mathcal{V}^h$ are given by (4.8) and (4.21), respectively, while $S^{(h+1)}\ (\phi)$, which denotes the sum over all the terms dependent on ϕ but independent of the ψ field, and $\mathcal{B}^{(h)}(\psi^{(\leq h)},\phi)$, which denotes the sum over all the terms containing at least one ϕ field and two ψ fields, can be represented in the form

$$S^{(h+1)}(\phi) = \sum_{m=1}^{\infty} \int d\mathbf{x}_1 \cdots d\mathbf{x}_m S^{(h+1)}_m(\mathbf{x}_1,\ldots,\mathbf{x}_m)\Big[\prod_{i=1}^m \phi^{(\alpha_i)}(\mathbf{x}_i)\Big] \tag{6.5}$$

$$\begin{aligned}\mathcal{B}^{(h)}(\psi^{(\leq h)},\phi) = &\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\underline{\sigma},\underline{\omega}} \int d\mathbf{x}_1\cdots d\mathbf{x}_m d\mathbf{y}_1\cdots d\mathbf{y}_{2n}\ \cdot \\ &\cdot\ B^{(h)}_{m,2n,\underline{\sigma},\underline{\omega}}(\mathbf{x}_1,\ldots,\mathbf{x}_m;\mathbf{y}_1,\ldots,\mathbf{y}_{2n})\Big[\prod_{i=1}^m \phi^{(\alpha_i)}(\mathbf{x}_i)\Big]\Big[\prod_{i=1}^{2n}\psi^{(\leq h)\sigma_i}_{\mathbf{y}_i,\omega_i}\Big]\ .\end{aligned} \tag{6.6}$$

Since the field ϕ is equivalent, from the point of view of dimensional considerations, to two ψ fields, the only terms in the r.h.s. of (6.6) which are not irrelevant are those with $m=1$ and $n=1$, which are marginal. The localization $\mathcal{L}$ is defined equal to zero except in the following cases

$$\begin{aligned}&\mathcal{L}\int d\mathbf{x}d\mathbf{y}d\mathbf{z}W^h_{1,2}(\mathbf{x};\mathbf{y},\mathbf{z})\phi^{(1)}(\mathbf{x})\psi^\sigma_{\mathbf{y},1}\psi^\sigma_{\mathbf{z},-1} = \int d\mathbf{x}d\mathbf{y}d\mathbf{z}W^h_{1,2}(\mathbf{x};\mathbf{y},\mathbf{z})\phi^{(1)}(\mathbf{x})\psi^\sigma_{\mathbf{x},1}\psi^\sigma_{\mathbf{x},-1} \\ &\mathcal{L}\int d\mathbf{x}d\mathbf{y}d\mathbf{z}W^h_{1,2}(\mathbf{x};\mathbf{y},\mathbf{z})\phi^{(2)}(\mathbf{x})\psi^\sigma_{\mathbf{y},1}\psi^{-\sigma}_{\mathbf{z},-1} = \int d\mathbf{x}d\mathbf{y}d\mathbf{z}W^h_{1,2}(\mathbf{x};\mathbf{y},\mathbf{z})\phi^{(2)}(\mathbf{x})\psi^\sigma_{\mathbf{x},1}\psi^{-\sigma}_{\mathbf{x},-1}\end{aligned} \tag{6.7}$$

Hence

$$\mathcal{L}\mathcal{B}^{(h)}(\psi^{(\leq h)},\phi)=Z_h^{(1)}F_1^{(\leq h)}+Z_h^{(2)}F_2^{(\leq h)}\ , \tag{6.8}$$

where $Z_h^{(1)}$ and $Z_h^{(2)}$ are real numbers, such that $Z_1^{(1)}=Z_1^{(2)}=1$ and

$$F_1^{(\leq h)}=\int d\mathbf{x}\phi^{(1)}(\mathbf{x})[\psi^{(\leq h)+}_{\mathbf{x},1}\psi^{(\leq h)-}_{\mathbf{x},-1}+\psi^{(\leq h)-}_{\mathbf{x},1}\psi^{(\leq h)+}_{\mathbf{x},-1}]\ , \tag{6.9}$$

$$F_2^{(\leq h)}=\int d\mathbf{x}\phi^{(2)}(\mathbf{x})[\psi^{(\leq h)+}_{\mathbf{x},1}\psi^{(\leq h)+}_{\mathbf{x},-1}+\psi^{(\leq h)-}_{\mathbf{x},1}\psi^{(\leq h)-}_{\mathbf{x},-1}]\ . \tag{6.10}$$

By using the notation of the preceding section

$$\begin{aligned} &e^{-M^2t_h}\int P_{m_{h-1},C_h}(d\psi^{(\leq h)})e^{-\mathcal{V}^{(h)}\psi^{(\leq h)})+\mathcal{B}^{(h)}(\psi^{(\leq h)},\phi)}\ = \\ &=e^{-M^2t_h}\int P_{m_{h-1},C_{h-1}}(d\psi^{(\leq h-1)})\ \cdot \\ &\cdot\int P_{m_{h-1},f_h^{-1}}(d\psi^{(h)})e^{-\hat{\mathcal{V}}^{(h)}(\psi^{(\leq h)})+\mathcal{B}^{(h)}(\psi^{(\leq h)},\phi)}\ , \end{aligned} \tag{6.11}$$

where $\mathcal{B}^{(h-1)}(\psi^{(\leq h-1)},\phi)$ and $S^{(h)}(\phi)$ are then defined through the analogous of (4.38), that is

$$\begin{aligned} &e^{-\mathcal{V}^{(h-1)}(\psi^{(\leq h-1)})+\mathcal{B}^{(h-1)}(\psi^{(\leq h-1)},\phi)-L\beta\tilde{E}_h+\tilde{S}^{(h)}(\phi)}= \\ &=\int P_{m_{h-1},f_h^{-1}}(d\psi^{(h)})e^{-\hat{\mathcal{V}}^{(h)}(\psi^{(\leq h)})+\hat{\mathcal{B}}^{(h)}(\psi^{(\leq h)},\phi)}\ . \end{aligned} \tag{6.12}$$

The definitions (6.8) easily imply that

$$\frac{Z_{h-1}^{(1)}}{Z_h^{(1)}}=1+a_1\lambda_h+h_1 \qquad \frac{Z_{h-1}^{(2)}}{Z_h^{(2)}}=1-a_1\lambda_h+h_2 \tag{6.13}$$

with $|h_i|\leq C\lambda^2$, $i=1,2$; proceeding as in the proof of Lemma 5.2

$$\gamma^{-\lambda_1c_4h}<\frac{Z_h^{(1)}}{Z_0^1}<\gamma^{-\lambda_1c_3h}\quad \gamma^{\lambda_1c_1h}<\frac{Z_h^{(2)}}{Z_0^1}<\gamma^{\lambda_1c_2h}\ , \tag{6.14}$$

The fields of scale between h^* and $h_{L,\beta}$ are integrated in a single step, and it follows that

$$S(\phi)=-L\beta E_{L,\beta}+S^{(h)}(\phi)=-L\beta E_{L,\beta}+\sum_{h=h^*}^{1}\tilde{S}^{(h)}(\phi)\ ; \tag{6.15}$$

As a ϕ fields is dimensionally analogous to two external fields, we get, proceeding as in the proof of Theorem 3 (see also [BM] §5 for a detailed proof of similar bounds in a related case)

$$|\frac{\partial^2}{\partial\phi^{(1)}(\mathbf{x})\partial\phi^{(1)}(\mathbf{y})}\tilde{S}^{(h)}(\phi)||_{\phi=0}\leq\gamma^{2h}[Z_h^{(1)}]^2\frac{C_N}{1+(\gamma^h|d(\mathbf{x}-\mathbf{y})|)^N} \tag{6.16}$$

$$|\frac{\partial^2}{\partial\phi^{(2)}(\mathbf{x})\partial\phi^{(2)}(\mathbf{y})}\tilde S^{(h)}(\phi)||_{\phi=0}\le\gamma^{2h}[Z_h^{(2)}]^2\frac{C_N}{1+(\gamma^h|d(\mathbf{x}-\mathbf{y})|)^N} \tag{6.17}$$

and if $\mathbf{x}_\alpha=\mathbf{x},\mathbf{y}$ and $\phi_\alpha=\phi_1,\phi_2$

$$|\frac{\partial^3\tilde S^{(h)}(\phi)}{\partial\phi^{(\alpha_1)}(\mathbf{x}_1)\partial\phi^{(\alpha_2)}(\mathbf{x}_2)\partial\phi^{(\alpha_3)}(\mathbf{x}_3)}||_{\phi=0}\le\gamma^{-h}\gamma^{4h}[\prod_{i=1}^3 Z_h^{(\alpha_i)}]\frac{C_N}{1+(\gamma^h|d(\mathbf{x}-\mathbf{y})|)^N} \tag{6.18}$$

$$|\frac{\partial^4\tilde S^{(h)}(\phi)}{\partial\phi^{(\alpha_1)}(\mathbf{x}_1)\partial\phi^{(\alpha_2)}(\mathbf{x}_2)\partial\phi^{(\alpha_3)}(\mathbf{x}_3)\partial\phi_{\alpha_4}(\mathbf{x}_4)}||_{\phi=0}\le$$

$$\gamma^{-2h}\gamma^{6h}[\prod_{i=1}^4 Z_h^{(\alpha_i)}]\frac{C_N}{1+(\gamma^h|d(\mathbf{x}-\mathbf{y})|)^N} \tag{6.19}$$

6.2 *Correlation functions.* The correlation function $<\sigma_{\mathbf{x}}^{(\alpha)}\sigma_{\mathbf{x}'}^{(\alpha)}\sigma_{\mathbf{y}}^{(\alpha)}\sigma_{\mathbf{y}'}^{(\alpha)}>_T$, where $\mathbf{x}'=(x+1,x_0)$ or (x,x_0+1) is given by

$$<\sigma_{\mathbf{x}}^{(\alpha)}\sigma_{\mathbf{x}'}^{(\alpha)}\sigma_{\mathbf{y}}^{(\alpha)}\sigma_{\mathbf{y}'}^{(\alpha)}>_T=\frac{\partial}{\partial J_{\mathbf{x},\mathbf{x}'}^{(\alpha)}}\frac{\partial}{\partial J_{\mathbf{y},\mathbf{y}'}^{(\alpha)}}\log Z_{2I}(\{J_{\mathbf{x},\mathbf{x}'}^{(\alpha)}\})|_{\{J_{\mathbf{x},\mathbf{x}'}\}=\{J\}} \tag{6.20}$$

If $O^{(\alpha)}(\mathbf{x})=\sigma_{x,x_0}^{(\alpha)}\sigma_{x+1,x_0}^{(\alpha)}$ each derivative produces a factor

$$\mathrm{sech}^2 J\bar H_{\mathbf{x}}^{\alpha}H_{x+1,x_0}^{(\alpha)}+\tanh J+\frac{\partial}{\partial J_{\mathbf{x};x+1,x_0}}\mathcal{V} \tag{6.21}$$

with $\mathcal{V}$ given by (2.25). We define, using (2.30),(2.25)

$$<O^{(\alpha)}(\mathbf{x})O^{(\alpha)}(\mathbf{y})>_{\Lambda;\varepsilon^{(1)},\varepsilon'^{(1)},\varepsilon^{(2)},\varepsilon'^{(2)}}=$$

$$\frac{\int[\prod_{\alpha=1}^2 P_{\varepsilon^{(\alpha)},\varepsilon'^{(\alpha)}}^{(\alpha)}(dH^{(\alpha)},dV^{(\alpha)})]e^{\mathcal{V}}O^{(\alpha)}(\mathbf{x})O^{(\alpha)}(\mathbf{y})}{\int[\prod_{\alpha=1}^2 P_{\varepsilon^{(\alpha)},\varepsilon'^{(\alpha)}}^{(\alpha)}(dH^{(\alpha)},dV^{(\alpha)})]e^{\mathcal{V}}} \tag{6.22}$$

Moreover we call

$$<O^{(\alpha)}(\mathbf{x})O^{(\alpha)}(\mathbf{y})>_{\Lambda,T;\varepsilon^{(1)},\varepsilon'^{(1)},\varepsilon^{(2)},\varepsilon'^{(2)}}=<O^{(\alpha)}(\mathbf{x})O^{(\alpha)}(\mathbf{y})>_{\Lambda;\varepsilon^{(1)},\varepsilon'^{(1)},\varepsilon^{(2)},\varepsilon'^{(2)}}$$

$$-<O^{(\alpha)}(\mathbf{x})>_{\Lambda;\varepsilon^{(1)},\varepsilon'^{(1)},\varepsilon^{(2)},\varepsilon'^{(2)}}<O^{(\alpha)}(\mathbf{x})>_{\Lambda;\varepsilon^{(1)},\varepsilon'^{(1)},\varepsilon^{(2)},\varepsilon'^{(2)}} \tag{6.23}$$

so that

$$<O^{(\alpha)}(\mathbf{x})O^{(\alpha)}(\mathbf{x})>_{\Lambda,T}=\sum_{\varepsilon^{(1)},\varepsilon'^{(1)}}(-1)^{\delta_{\varepsilon^{(1)},\varepsilon'^{(1)}}}\sum_{\varepsilon^{(2)},\varepsilon'^{(2)}}(-1)^{\delta_{\varepsilon^{(2)},\varepsilon'^{(2)}}}$$

$$\frac{Z_{2I}^{\varepsilon^{(1)},\varepsilon'^{(1)},\varepsilon^{(2)},\varepsilon'^{(2)}} < O^{(\alpha)}(\mathbf{x})O^{(\alpha)}(\mathbf{y}) >_{\Lambda,T;\varepsilon^{(1)},\varepsilon'^{(1)},\varepsilon^{(2)},\varepsilon'^{(2)}}}{\sum_{\varepsilon^{(1)},\varepsilon'^{(1)}}(-1)^{\delta_{\varepsilon^{(1)},\varepsilon'^{(1)}}}\sum_{\varepsilon^{(2)},\varepsilon'^{(2)}}(-1)^{\delta_{\varepsilon^{(2)},\varepsilon'^{(2)}}} Z_{2I}^{\varepsilon^{(1)},\varepsilon'^{(1)},\varepsilon^{(2)},\varepsilon'^{(2)}}} \tag{6.24}$$

Suppose that $\mathbf{x}$ and $\mathbf{y}$ are fixed to an M independent value; then

$$\lim_{M\to\infty} < O^{(\alpha)}(\mathbf{x})O^{(\alpha)}(\mathbf{y}) >_{\Lambda,T;\varepsilon^1,\varepsilon'^1,\varepsilon^2,\varepsilon'^2} - < O^{(\alpha)}(\mathbf{x})O^{(\alpha)}(\mathbf{y}) >_{\Lambda,T;-,-,-,-} = 0. \tag{6.25}$$

In fact, as explained in §4.10, the l.h.s. of (6.25) can be written as a sum of trees, and in each of them there is a factor $e^{i\mathbf{p}_{\varepsilon,\varepsilon'}\mathbf{x}} - 1$ or an end-point associated to $\tilde{Q}_{\varepsilon^1,\varepsilon'^1,\varepsilon^2,\varepsilon'^2}$. With respect to the previous bounds, simply dimensional analysis says there is now an extra factor $\frac{\gamma^{-h^*}}{M}$ in the bounds so it is vanishing in the limit $M \to \infty$, if $\mathbf{x}, \mathbf{y}$ and $t - t_c$ are fixed to an M-independent value. We can then simply study $< O^{(\alpha)}(\mathbf{x})O^{(\alpha)}(\mathbf{y}) >_{\Lambda,T;-,-,-,-}$ which is given by the Grassman integral (6.22) with $O^{(\alpha)}$ given by (6.21). By performing the change of variables (2.31) and (2.59), we get a sum of averages of monomials in the ψ and χ fields; we integrate the χ-fields, as discussed in §3, and we obtain a sum of Grassman integrals of monomials in the ψ fields; remembering that

$$\bar{H}^\alpha_{\mathbf{x}} H^{(\alpha)}_{\mathbf{x}} \bar{H}^\alpha_{\mathbf{y}} H^{(\alpha)}_{\mathbf{y}} = \frac{1}{16}[\psi^-_{1,\mathbf{x}}\psi^-_{-1,\mathbf{x}}\psi^+_{1,\mathbf{y}}\psi^+_{-1,\mathbf{y}} + \psi^-_{1,\mathbf{y}}\psi^-_{-1,\mathbf{y}}\psi^+_{1,\mathbf{x}}\psi^+_{-1,\mathbf{x}} + \tag{6.26}$$

$$\psi^-_{1,\mathbf{x}}\psi^+_{-1,\mathbf{x}}\psi^+_{1,\mathbf{y}}\psi^-_{-1,\mathbf{y}} + \psi^-_{1,\mathbf{x}}\psi^+_{-1,\mathbf{x}}\psi^-_{1,\mathbf{y}}\psi^+_{-1,\mathbf{y}} + \psi^+_{1,\mathbf{x}}\psi^-_{-1,\mathbf{x}}\psi^-_{1,\mathbf{y}}\psi^+_{-1,\mathbf{y}} + \psi^+_{1,\mathbf{x}}\psi^-_{-1,\mathbf{x}}\psi^+_{1,\mathbf{y}}\psi^-_{-1,\mathbf{y}}]$$

we find the first two of (1.13) and (1.15) by (6.1), (6.13), (6.15) and (6.16),(6.17). In $\Omega^{(\alpha),c}$ we include to contributions to the correlation function corresponding to monomials with six or more ψ fields, and from (6.18), (6.19) the last of (1.13) follows.

Finally the specific heat is obtained in a similar way, noting that we have to sum over $\mathbf{x}-\mathbf{y}$, and this produces an extra γ^{2h} in the r.h.s. of (6.16),(6.17),(6.18),(6.19).

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