THE SZEGŐ CONDITION FOR COULOMB JACOBI MATRICES

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ABSTRACT. A Jacobi matrix with $a_n \to 1$, $b_n \to 0$ and spectral measure $\nu'(x)dx + d\nu_{sing}(x)$ satisfies the Szegő condition if

$$\int_0^\pi \ln\big[\nu'(2\cos\theta)\big]\,d\theta$$

is finite. We prove that if

$$a_n \equiv 1 + \frac{\alpha}{n} + O(n^{-1-\varepsilon}) \qquad\qquad b_n \equiv \frac{\beta}{n} + O(n^{-1-\varepsilon})$$

with $2\alpha \geq |\beta|$ and $\varepsilon > 0$, then the corresponding matrix is Szegő.

1. INTRODUCTION

In this paper we discuss the Szegő condition for Jacobi matrices and orthogonal polynomials. A Jacobi matrix is the matrix

$$J = \begin{pmatrix} b_1 & a_1 & 0 & \dots \\ a_1 & b_2 & a_2 & \dots \\ 0 & a_2 & b_3 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix} \tag{1.1}$$

with $a_n > 0$ and $b_n \in \mathbb{R}$. We let ν be the spectral measure of J as an operator on $\ell^2(\{0, 1, \dots\})$, with respect to the vector δ_0. That is,

$$\langle \delta_0, (J-z)^{-1}\delta_0 \rangle = \int \frac{d\nu(x)}{x-z}$$

for $z \in \mathbb{C} \setminus \mathbb{R}$. We denote $\{P_n(x)\}_{n=0}^\infty$ the orthonormal polynomials for this measure, obtained from $\{x^n\}_{n=0}^\infty$ by the Gram-Schmidt procedure. Since $\nu(\mathbb{R}) = \|\delta_0\|^2 = 1$, we have $P_0(x) \equiv 1$. If we define $P_{-1}(x) \equiv 0$, then the P_n's obey the three term recurrence relation for $n \geq 0$

$$xP_n(x) = a_{n+1}P_{n+1}(x) + b_{n+1}P_n(x) + a_nP_{n-1}(x) \tag{1.2}$$

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Hence, $\{P_n(x)\}_{n=0}^{\infty}$ is the Dirichlet eigenfunction of J for energy x. This relationship establishes a one-to-one correspondence between bounded Jacobi matrices and polynomials orthonormal w.r.t. measures with bounded infinite support and total mass 1.

We will usually consider J such that $J - J_0$ is compact. Here J_0 is the free Jacobi matrix with $a_n \equiv 1$, $b_n \equiv 0$ and $d\nu_0(x) = (2\pi)^{-1}\chi_{[-2,2]}\sqrt{4-x^2}dx$. For such J we have $a_n \to 1$ and $b_n \to 0$ and $\sigma_{\rm ess}(J) = [-2,2]$. Outside of this interval J can only have simple isolated eigenvalues, with ± 2 the only possible accumulation points. We denote them $E_1^+ > E_2^+ > \cdots > 2$ and $E_1^- < E_2^- < \cdots < -2$.

The main object of our interest is the *Szegő integral*

$$Z(J) \equiv \frac{1}{2\pi}\int_{-2}^{2} \ln\left(\frac{\sqrt{4-x^2}}{2\pi\nu'(x)}\right)\frac{dx}{\sqrt{4-x^2}} \tag{1.3}$$

where $\nu'(x) \equiv d\nu_{ac}(x)/dx$. We say that J satisfies the *Szegő condition* if $Z(J)$ is finite. It can be proved that the negative part of the integral in (1.3) is always integrable and $Z(J) \geq -\frac{1}{2}\ln(2)$ (see Killip-Simon [9]). Hence, we are left with the question whether $Z(J) < \infty$. There is extensive literature on when this is the case (e.g. [1, 2, 7, 8, 12, 13, 14, 15, 16, 18, 19]), and so one is interested in answering this question.

Notice that

$$-2\pi Z(J) = \int_0^{\pi} \ln\left(\frac{\pi\nu'(2\cos\theta)}{\sin\theta}\right)d\theta = \int_0^{\pi} \ln\bigl(\nu'(2\cos\theta)\bigr)d\theta + const.$$

with $x = 2\cos\theta$. Many authors consider the last integral instead of $Z(J)$ and the question is whether $\int_0^{\pi}\ln\bigl(\nu'(2\cos\theta)\bigr)d\theta > -\infty$. For our purposes, $Z(J)$ is more suitable. Also notice that $Z(J) < \infty$ implies that the essential support of ν_{ac} is $[-2,2]$.

In this paper, we want to address a conjecture of Askey about Coulomb-type Jacobi matrices, reported by Nevai in [12]. Askey conjectured that if

$$a_n \equiv 1 + \frac{\alpha}{n} + O\left(\frac{1}{n^2}\right) \qquad\qquad b_n \equiv \frac{\beta}{n} + O\left(\frac{1}{n^2}\right) \tag{1.4}$$

with $(\alpha,\beta) \neq (0,0)$, then the Szegő condition fails (it has been known that it holds if $\alpha = \beta = 0$). Later, however, Askey-Ismail [1] found some explicit examples with $b_n \equiv 0$ and $\alpha > 0$ for which the Szegő condition holds! And in [7], Dombrowski-Nevai proved that the condition holds whenever $b_n \equiv 0$ and $a_n \equiv 1 + \alpha/n + o(n^{-2})$ with $\alpha > 0$. In conclusion, the conjecture had to be modified.

The "right" form of the conjecture can be guessed from Charris-Ismail [3], who computed the weights for certain Pollaczek-type polynomials (with a_n, b_n of the form (1.4)). Although they did not note it, their examples are Szegő if and only if $2\alpha \geq |\beta|$. We will see that this is true in general.

The first result which allows errors of the type (1.4) was proved by Simon and the author in [18], and is in-line with this picture. Indeed, the following appears in [18].

Proposition 1.1. *If*

$$a_n \equiv 1 + \frac{\alpha}{n} + E_a(n) \qquad\qquad b_n \equiv \frac{\beta}{n} + E_b(n) \tag{1.5}$$

with

$$\sum_{n=1}^{N} \big(|E_a(n)| + |E_b(n)|\big) = o\big(\ln(N)\big)$$

and $2\alpha < |\beta|$, then the Szegő condition fails.

So Askey was right in the case $2\alpha < |\beta|$. The present paper concentrates on the complementary region $2\alpha \geq |\beta|$ and shows that the Szegő condition holds there (see figure below). Here is our main result. We denote $a_+ \equiv \max\{a, 0\}$ and $a_- \equiv -\min\{a, 0\}$.

Theorem 1.2 (=Theorem 4.3)**.** *Let*

$$a_n \equiv c_n + O(n^{-1-\varepsilon}) \qquad\qquad b_n \equiv d_n + O(n^{-1-\varepsilon}) \tag{1.6}$$

for some $\varepsilon > 0$, where $c_n \geq 1 + \frac{|d_n|}{2}$ for $n > N$, $\lim_{n\to\infty} c_n = 1$ and

$$\sum_{n=1}^{\infty} n \left[c_{n+1}^2 - c_n^2 + \frac{c_{n+1}}{2}|d_{n+2} - d_{n+1}| + \frac{c_n}{2}|d_{n+1} - d_n| \right]_+ < \infty \tag{1.7}$$

Then the matrix J, given by (1.1)*, satisfies the Szegő condition.*

Remarks. 1. Notice that the sum in (1.7) cannot be simplified. We cannot replace the last two terms by $c_n|d_{n+1} - d_n|$ because we take positive parts of the summands in (1.7).

2. In particular one can take $c_n \equiv 1 + \alpha/n$ and $d_n \equiv \beta/n$ with $2\alpha \geq |\beta|$.

We will prove this theorem in two steps. The first one is an extension of the result in [7] and shows that J is Szegő whenever a_n, b_n satisfy the conditions for c_n, d_n in Theorem 1.2.

The second step lets us add $O(n^{-1-\varepsilon})$ errors to such c_n, d_n. Our tool here are the Case sum rules for Jacobi matrices, in particular the *step-by-step Z sum rule* (1.8) below (called C_0 in [9]). These were introduced by Case [2], recently extended in [9], and finally proved in the form we use here in [18] (see [4, 5] for related Schrödinger operators results). We let $\beta_j^\pm$ be such that $E_j^\pm = \beta_j^\pm + (\beta_j^\pm)^{-1}$ and $\pm\beta_j^\pm > 1$. If J has fewer than j eigenvalues above 2/below -2, we define $\beta_j^{+/-} \equiv +1/-1$. Let $J^{(n)}$ be the matrix obtained from J by removing n top rows and n leftmost columns. It was proved in [18] that if $J - J_0$ is compact, then we have

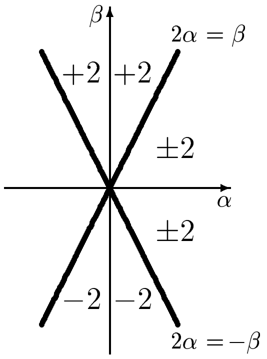
$$Z(J) = -\sum_{j=1}^{n} \ln(a_j) + \sum_{\pm} \sum_{j} \Big(\ln|\beta_j^{\pm}(J)| - \ln|\beta_j^{\pm}(J^{(n)})| \Big) + Z(J^{(n)}) \tag{1.8}$$

and that the double sum is always convergent with non-negative terms.

(1.8) holds even if $Z(J) = \infty$, and so J is Szegő if and only if $J^{(n)}$ is. In particular, the Szegő condition is stable under finite-rank perturbations. We will be able to pass to certain infinite-rank perturbations of J by representing them as limits of finite-rank perturbations and using lower semicontinuity of Z in J proved in [9]. To do this, we will need to control the change of the $E_j^{\pm}$'s under these perturbations, in order to estimate the double sum in (1.8) (or, more precisely, in (4.2) below).

The rest of the paper is organized as follows. In Section 2 we extend the above mentioned result from [7]. In Section 3 we prove results on the control of change of eigenvalues under certain finite-rank perturbations. In Section 4 we use these to prove Theorem 1.2, along with some related results.

Finally, Section 5 discusses some situations when the Szegő integral is allowed to diverge at one end (*one-sided Szegő conditions*). We study the case (1.4) with $O(n^{-1-\varepsilon})$ errors and establish the following picture.



± 2 – Szegő condition holds
$+2$ – Szegő condition at 2 holds
-2 – Szegő condition at -2 holds

The (α, β) plane is divided into 4 regions by the lines $2\alpha = \pm\beta$. Inside the right-hand region $Z(J)$ converges at both ends, inside the top and bottom regions $Z(J)$ converges only at, respectively, 2 and -2, and inside the left-hand region $Z(J)$ diverges at both ends. As for the borderlines $2\alpha = \pm\beta$, if $\alpha \geq 0$, then $Z(J)$ converges at both ends and if $\alpha < 0$, then $Z(J)$ diverges at ± 2 (convergence at ∓ 2 is left open). The divergence results follow from [18] and hold for more general errors, trace class in particular.

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2. On an argument of Dombrowski-Nevai

In this section we will improve a result of Dombrowski-Nevai [7]. We will closely follow their presentation and introduce an additional twist which will yield this improvement. The notation here is slightly different from [7] because their b_n's start with $n = 0$ and their "free" a_n's are $\frac{1}{2}$. We define

$$S_n(x) \equiv \sum_{j=0}^{n} \Big[(a_{j+1}^2 - a_j^2)P_j^2(x) + a_j(b_{j+1} - b_j)P_j(x)P_{j-1}(x)\Big] \tag{2.1}$$

where we take $a_0 = b_0 = 0$. Notice that the S_n obey the obvious recurrence relation

$$S_n(x) = S_{n-1}(x) + (a_{n+1}^2 - a_n^2)P_n^2(x) + a_n(b_{n+1} - b_n)P_n(x)P_{n-1}(x) \tag{2.2}$$

Using this and (1.2) one proves by induction the following formula from [6]

$$S_n(x) = a_{n+1}^2 \left[P_{n+1}^2(x) - \frac{x - b_{n+1}}{a_{n+1}} P_{n+1}(x)P_n(x) + P_n^2(x)\right] \tag{2.3}$$

The results in [7] are based on (2.2) and (2.3). Our simple but essential improvement is the introduction of a function closely related to S_n, but satisfying a recurrence relation which is more suitable for the purposes of this argument. We define

$$T_n(x) \equiv S_n(x) + \frac{a_{n+1}}{2}|b_{n+2} - b_{n+1}|P_n^2(x) \tag{2.4}$$

and then we have

$$\begin{aligned} T_n(x) = T_{n-1}(x) &+ (a_{n+1}^2 - a_n^2)P_n^2(x) + a_n(b_{n+1} - b_n)P_n(x)P_{n-1}(x) \\ &+ \frac{a_{n+1}}{2}|b_{n+2} - b_{n+1}|P_n^2(x) - \frac{a_n}{2}|b_{n+1} - b_n|P_{n-1}^2(x) \end{aligned}$$

The importance of this relation lies in the fact that it implies the crucial inequality

$$T_n(x) \le T_{n-1}(x) + \left[a_{n+1}^2 - a_n^2 + \frac{a_{n+1}}{2}|b_{n+2} - b_{n+1}| + \frac{a_n}{2}|b_{n+1} - b_n|\right] P_n^2(x) \tag{2.5}$$

by writing $|P_n(x)P_{n-1}(x)| \le \frac{1}{2}\big(P_n^2(x) + P_{n-1}^2(x)\big)$. Hence, our choice of T_n eliminated the unpleasant cross term in (2.2).

Now we are ready to apply the argument from [7], but to T_n in place of S_n. We define

$$\delta_n \equiv \left[a_{n+1}^2 - a_n^2 + \frac{a_{n+1}}{2}|b_{n+2} - b_{n+1}| + \frac{a_n}{2}|b_{n+1} - b_n|\right]_+ \tag{2.6}$$

Lemma 2.1. *If $a_n \ge 1 + \frac{|b_n|}{2}$ for $n > N$, then for $n > N$*

$$(4 - x^2)P_n^2(x) \le 4T_{n-1}(x) \qquad |x| \le 2 \tag{2.7}$$

$$\max_{|x|\le 2} P_n^2(x) \le (n+1)^2 \max_{|x|\le 2} T_{n-1}(x) \tag{2.8}$$

$$0 \le T_n(x) \le \exp\left(\frac{4\delta_n}{4 - x^2}\right) T_{n-1}(x) \qquad |x| < 2 \tag{2.9}$$

$$\max_{|x|\le 2} T_n(x) \le e^{(n+1)^2\delta_n} \max_{|x|\le 2} T_{n-1}(x) \tag{2.10}$$

Proof. From (2.3)

$$S_{n-1}(x) = a_n^2\left[P_{n-1}(x) - \frac{x - b_n}{2a_n}P_n(x)\right]^2 + \frac{1}{4}\left[4a_n^2 - (x - b_n)^2\right]P_n^2(x)$$

The assumption $2a_n \ge 2 + |b_n|$ implies $4a_n^2 - (x - b_n)^2 \ge 4 - x^2$ for $|x| \le 2$, and (2.4) implies $T_{n-1}(x) \ge S_{n-1}(x)$. This proves (2.7). (2.8) follows from (2.7) and a theorem of Bernstein [11, p.139], and (2.9)/(2.10) from (2.5), (2.6) and (2.7)/(2.8). ☐

In [7], similar statements are proved for S_n. The important difference is that the proofs use (2.2) rather than (2.5), and therefore involve $\delta_n' = [a_{n+1}^2 - a_n^2]_+ + a_n|b_{n+1} - b_n|$. This is a serious drawback because the condition $\sum n\delta_n < \infty$ will play a central role in our considerations. If, for example, $a_n = 1 + \alpha/n$ and $b_n = \beta/n$, then $\sum n\delta_n' < \infty$ only if $\alpha \ge 0$ and $\beta = 0$ (cf. the result from [7] mentioned in Section 1), but $\sum n\delta_n < \infty$ whenever $2\alpha \ge |\beta|$. This is because in δ_n (and not in δ_n') the contribution of the positive $|b_{n+1} - b_n|$ terms can be canceled by a decrease in a_n. Therefore T_n can sometimes be a better object to look at than S_n, for example in the case of Coulomb Jacobi matrices. The next result relates T_n and $Z(J)$.

Lemma 2.2. *Suppose $\lim_{n\to\infty} a_n = 1$, $\lim_{n\to\infty} b_n = 0$ and*

$$\sum_{n=1}^{\infty} \big(|a_{n+1} - a_n| + |b_{n+1} - b_n|\big) < \infty \tag{2.11}$$

Then for $|x| < 2$

$$\lim_{n\to\infty} T_n(x) = \frac{\sqrt{4-x^2}}{2\pi\nu'(x)} \tag{2.12}$$

Remarks. 1. The right-hand side appears in (1.3) and so one can use (2.12) and Fatou's lemma to obtain upper bounds on $Z(J)$ (see proof of Theorem 2.5).

2. Results relating density of the absolutely continuous part of the spectral measure and asymptotics of the solutions of difference (or differential) equations, under the assumption of finite variation of the potential, go back to Weidmann [22, 23].

Proof. If (2.11) holds, then it is proved in [10] that for $x \in (-2, 2)$

$$\lim_{n\to\infty}\left[P_{n+1}^2(x) - \frac{x-b_{n+2}}{a_{n+2}}P_{n+1}(x)P_n(x) + \frac{a_{n+1}}{a_{n+2}}P_n^2(x)\right] = \frac{\sqrt{4-x^2}}{2\pi\nu'(x)}$$

(in [10] $a_n \to \frac{1}{2}$ and the limit is $2\sqrt{1-x^2}/\pi\nu'(x)$). By [17] $\{P_n(x)\}_n$ is bounded for any fixed $x \in (-2, 2)$ when (2.11) holds. Hence $a_n \to 1$ and $b_n \to 0$ imply

$$\lim_{n\to\infty}\left[P_{n+1}^2(x) - xP_{n+1}(x)P_n(x) + P_n^2(x)\right] = \frac{\sqrt{4-x^2}}{2\pi\nu'(x)}$$

But by (2.3) and (2.4) this limit is the same as $\lim_n T_n(x)$. $\square$

In the light of the discussion preceeding the lemma, the following will be useful.

Lemma 2.3. *If* $\inf\{a_n\} > 0$ *and* $\sum_n \delta_n < \infty$, *then* (2.11) *holds.*

Proof. We have $0 \le [a_{n+1}^2 - a_n^2]_+ \le \delta_n$, hence $\sum[a_{n+1}^2 - a_n^2]_+ < \infty$. By telescoping $\sum[a_{n+1}^2 - a_n^2]_- \le a_1^2 + \sum[a_{n+1}^2 - a_n^2]_+ < \infty$ and so $\sum|a_{n+1}^2 - a_n^2| < \infty$. Since $\inf\{a_n\} > 0$, it follows that $\sum|a_{n+1} - a_n| < \infty$. Also, since

$$0 \le \frac{a_{n+1}}{2}|b_{n+2} - b_{n+1}| + \frac{a_n}{2}|b_{n+1} - b_n| \le \delta_n + |a_{n+1}^2 - a_n^2|$$

and a_n are bounded away from zero, $\sum|b_{n+1} - b_n| < \infty$. $\square$

These lemmas have the same consequences as in [7], but with δ_n in place of δ_n'. Thus we can prove the following two results.

Theorem 2.4. *Suppose* $a_n \ge 1 + \frac{|b_n|}{2}$ *for* $n > N$, $\lim_{n\to\infty} a_n = 1$ *and*

$$\sum_{n=1}^{\infty} n^2\left[a_{n+1}^2 - a_n^2 + \frac{a_{n+1}}{2}|b_{n+2} - b_{n+1}| + \frac{a_n}{2}|b_{n+1} - b_n|\right]_+ < \infty$$

Then there is $c > 0$ such that

$$\frac{d\nu_{ac}}{dx}(x) \geq c\sqrt{4-x^2} \qquad |x| < 2$$

Remarks. 1. In particular, the corresponding matrix J is Szegő.

2. Notice that the above conditions are satisfied for $a_n \downarrow 1$, $b_n \equiv 0$, as pointed out in [7].

Proof. By (2.6) and (2.10) we have for all $|x| \leq 2$ and $n > N$

$$T_n(x) \leq \exp\left(\sum_{j=N}^{\infty}(j+1)^2\delta_j\right) \max_{|x|\leq 2} T_N(x) \equiv \frac{1}{2\pi c} < \infty$$

Lemmas 2.3 and 2.2 finish the proof. □

The main result of this section is

Theorem 2.5. *Suppose $a_n \geq 1 + \frac{|b_n|}{2}$ for $n > N$, $\lim_{n\to\infty} a_n = 1$ and*

$$\sum_{n=1}^{\infty} n\left[a_{n+1}^2 - a_n^2 + \frac{a_{n+1}}{2}|b_{n+2}-b_{n+1}| + \frac{a_n}{2}|b_{n+1}-b_n|\right]_+ < \infty$$

Then J given by (1.1) is Szegő.

Proof. Once again, we closely follow [7]. By Lemmas 2.3 and 2.2 and Fatou's lemma

$$Z(J) \leq \lim_{\varepsilon\downarrow 0}\left(\liminf_{n\to\infty} \frac{1}{2\pi}\int_{-2+\varepsilon}^{2-\varepsilon} \ln_+(T_n(x))\,\frac{dx}{\sqrt{4-x^2}}\right)$$

and so it is sufficient to prove

$$\int_0^{2-n^{-2}} \ln_+(T_n(x))\,\frac{dx}{\sqrt{2-x}} + \int_{-2+n^{-2}}^{0} \ln_+(T_n(x))\,\frac{dx}{\sqrt{2+x}} \leq C$$

for some $C < \infty$. Let us consider the first integral, which we denote I_n (both can be treated similarly).

By (2.9) and (2.10), for $n > N$

$$\begin{aligned}
I_n \leq& I_{n-1} + 2\delta_n \int_0^{2-\frac{1}{(n-1)^2}} \frac{dx}{(2-x)^{\frac{3}{2}}} + \ln_+\left[\max_{|x|\leq 2} T_n(x)\right] \int_{2-\frac{1}{(n-1)^2}}^{2-\frac{1}{n^2}} \frac{dx}{\sqrt{2-x}} \\
=& I_{n-1} + 2\delta_n(2n-2-\sqrt{2}) + \left(\frac{2}{n-1} - \frac{2}{n}\right)\ln_+\left[\max_{|x|\leq 2} T_n(x)\right] \\
\leq& I_{n-1} + 4n\delta_n + \frac{2}{n-1}\ln_+\left[\max_{|x|\leq 2} T_{n-1}(x)\right] + \frac{2(n+1)^2\delta_n}{n-1} - \frac{2}{n}\ln_+\left[\max_{|x|\leq 2} T_n(x)\right] \\
\leq& I_{n-1} + 13n\delta_n + \frac{2}{n-1}\ln_+\left[\max_{|x|\leq 2} T_{n-1}(x)\right] - \frac{2}{n}\ln_+\left[\max_{|x|\leq 2} T_n(x)\right]
\end{aligned}$$

because $\ln_+(x) + \ln_+(y) \geq \ln_+(xy)$. By iterating this, we obtain

$$\begin{aligned} I_n \leq & I_N + 13 \sum_{j=N+1}^{n} j\delta_j + \frac{2}{N} \ln_+ \left[\max_{|x|\leq 2} T_N(x) \right] \\ \leq & 13 \sum_{n=1}^{\infty} n\delta_n + 5 \ln_+ \left[\max_{|x|\leq 2} T_N(x) \right] \equiv \frac{C}{2} \end{aligned}$$

as desired. □

In particular, if $a_n \equiv 1 + \alpha/n$ and $b_n \equiv \beta/n$ with $2\alpha \geq |\beta|$, then J is Szegő. Later we will add $O(n^{-1-\varepsilon})$ errors to these a_n, b_n.

For further reference we make

Definition 2.6. We call a pair of sequences $\{a_n, b_n\}_{n=1}^{\infty}$ *admissible*, if $a_n \geq 1 + \frac{|b_n|}{2}$ for $n > N$, $\lim_{n\to\infty} a_n = 1$ and

$$\sum_{n=1}^{\infty} n \left[a_{n+1}^2 - a_n^2 + \frac{a_{n+1}}{2} |b_{n+2} - b_{n+1}| + \frac{a_n}{2} |b_{n+1} - b_n| \right]_+ < \infty$$

Hence, if $\{a_n, b_n\}$ is admissible, then J is Szegő. We make some useful observations.

Lemma 2.7. *Suppose $\{a_n, b_n\}$ is admissible and $\{e_n, f_n\}$ is such that $2e_n \geq |f_n|$ for $n > N$, $e_n \to 0$ and $\sum n\big(|e_{n+1} - e_n| + |f_{n+1} - f_n|\big) < \infty$. Then $\{a_n + e_n, b_n + f_n\}$ is also admissible.*

Proof. We only need to show the last condition for admissibility. If

$$\begin{aligned} \varepsilon_n \equiv (a_{n+1} + e_{n+1})^2 - (a_n + e_n)^2 + \frac{a_{n+1} + e_{n+1}}{2} |b_{n+2} + f_{n+2} - b_{n+1} - f_{n+1}| \\ + \frac{a_n + e_n}{2} |b_{n+1} + f_{n+1} - b_n - f_n| \end{aligned}$$

then we want $\sum n[\varepsilon_n]_+ < \infty$. Notice that

$$\begin{aligned} \varepsilon_n \leq \delta_n &+ 2a_{n+1}|e_{n+1} - e_n| + 2|a_{n+1} - a_n||e_n| + |e_{n+1} + e_n||e_{n+1} - e_n| \\ &+ \frac{a_{n+1} + e_{n+1}}{2} |f_{n+2} - f_{n+1}| + \frac{a_n + e_n}{2} |f_{n+1} - f_n| \\ &+ \frac{e_{n+1}}{2} |b_{n+2} - b_{n+1}| + \frac{e_n}{2} |b_{n+1} - b_n| \end{aligned}$$

and so we only need to prove $\sum nX_n < \infty$ for X_n being any of the above terms. If X_n is δ_n or one of the terms containing $|e_{n+1} - e_n|$ or $|f_{n+1} - f_n|$, then this is obvious. For the remaining three terms the same is true by the fact that $\sum n|e_{n+1} - e_n| < \infty$ and $e_n \to 0$ imply $ne_n \to 0$, and by Lemma 2.3. □

Lemma 2.8. *Suppose $\{a_n, b_n\}$ is admissible and $e_n \downarrow 0$ is such that $\{ne_n|a_{n+1} - a_n|\}$ or $\{ne_n|b_{n+2} - b_{n+1}|\}$ is bounded. Then $\{a_n + e_n, b_n\}$ is also admissible.*

Proof. If

$$\varepsilon_n \equiv (a_{n+1} + e_{n+1})^2 - (a_n + e_n)^2 + \frac{a_{n+1} + e_{n+1}}{2}|b_{n+2} - b_{n+1}| + \frac{a_n + e_n}{2}|b_{n+1} - b_n|$$

then by $e_{n+1} \le e_n$

$$\begin{aligned} \varepsilon_n \le& \delta_n + 2a_{n+1}e_{n+1} - 2a_n e_n + e_{n+1}^2 - e_n^2 + \frac{e_{n+1}}{2}|b_{n+2} - b_{n+1}| + \frac{e_n}{2}|b_{n+1} - b_n| \\ \le& \delta_n + e_n\big(2(a_{n+1} - a_n) + \tfrac{1}{2}|b_{n+2} - b_{n+1}| + \tfrac{1}{2}|b_{n+1} - b_n|\big) \\ \le& \delta_n + \frac{2e_n}{a_{n+1} + a_n}\left(\delta_n + \frac{|a_n - a_{n+1}|}{4}|b_{n+2} - b_{n+1}| + \frac{|a_{n+1} - a_n|}{4}|b_{n+1} - b_n|\right) \end{aligned}$$

so $\sum n[\varepsilon_n]_+ < \infty$ by the hypotheses and Lemma 2.3. □

We conclude this section with an interesting corollary. Notice that in (1.8) one would like to take $n \to \infty$ to pass from the step-by-step sum rule to a "full size" sum rule not involving $J^{(n)}$. For this, one would need to separate the terms in (1.8) when taking $n \to \infty$. The following shows that there are many Jacobi matrices which are Szegő, but one cannot do this (see [18] for results on when it is possible).

Corollary 2.9. *Let $\{a_n, b_n\}$ be admissible and let $\tilde{J}$ be a matrix with $\tilde{a}_n \equiv a_n + c/n$ and $\tilde{b}_n \equiv b_n$ for some $c > 0$. Then $Z(\tilde{J}) < \infty$ but*

$$\bar{A}_0(\tilde{J}) \equiv \limsup_n \left(-\sum_{j=1}^n \ln(\tilde{a}_j) \right) = -\infty$$

and

$$\mathcal{E}_0(\tilde{J}) \equiv \sum_{j,\pm} \ln|\beta_j^\pm(\tilde{J})| = \infty$$

Proof. $\tilde{J}$ is Szegő by Lemma 2.8. Since $Z(J) < \infty$, (1.8) yields $\bar{A}_0(J) < \infty$ (because the other two terms in (1.8) are bounded from below). Since $a_n \to 1$ and $\sum \frac{c}{n} = \infty$, we obtain $\bar{A}_0(\tilde{J}) = -\infty$. By Theorem 4.1(d) in [18], this implies $\mathcal{E}_0(\tilde{J}) = \infty$. □

3. Control of change of eigenvalues under perturbations

In this section we will prove results on the behavior of eigenvalues under certain finite-rank perturbations of the a_n's and b_n's. Namely, we will show that these perturbations decrease E_j^+ and increase E_j^- for all but finitely many j. This, of course, means that we will not consider arbitrary perturbations. Indeed, in all the perturbations we can treat, the a_n's cannot increase. Immediately a question arises, how is this compatible with the possibility of $a_n > c_n$ in Theorem 1.2. The answer is in Lemma 2.7. Before doing a general $O(n^{-1-\varepsilon})$ perturbation of c_n, d_n, we will increase the c_n's by $Cn^{-1-\varepsilon}$ for some large C, so that the assumptions of Theorem 1.2 will stay valid and the new c_n will be larger than a_n. Then we will use results from this section. For details see the proof of Theorem 4.3.

For $j \geq 1$ and $n \geq -1$ we define

$$p_n(\pm j) \equiv \frac{P_n(E_j^\pm)}{\left(\sum_{m=0}^\infty P_m^2(E_j^\pm)\right)^{\frac{1}{2}}}$$

Hence $p(\pm j) \equiv \{p_n(\pm j)\}_{n=0}^\infty$ is the normalized eigenfunction for energy $E_j^\pm$. Therefore $p(\pm j)$ satisfies the same recurrence relation as $P(E_j^\pm)$, and so

$$p_{n+1}(\pm j) = \frac{E_j^\pm - b_{n+1}}{a_{n+1}} p_n(\pm j) - \frac{a_n}{a_{n+1}} p_{n-1}(\pm j) \tag{3.1}$$

In what follows, we will use the following result from first order perturbation theory for eigenvalues (see, e.g., [21, p.151]).

Proposition 3.1. *Let $J(t) \equiv J + tA$ for $t \in (-\varepsilon, \varepsilon)$ where J and A are bounded self-adjoint operators on a Hilbert space. Assume that $J(0)$ has a simple isolated eigenvalue $E(0) \notin \sigma_{ess}(J(0))$ and let $\varphi(0)$ be the corresponding normalized eigenfunction. Then there are analytic functions $E(t)$, $\varphi(t)$ defined on some interval $(-\varepsilon', \varepsilon')$ such that $E(t)$ is a simple isolated eigenvalue of $J(t)$ with normalized eigenfunction $\varphi(t)$, and we have $\frac{\partial}{\partial t} E(t) = \langle \varphi(t), A\varphi(t)\rangle$.*

In the case of Jacobi matrices, all eigenvalues outside $[-2, 2]$ are simple. Hence if $J(t) \equiv J + tA$ with A bounded self-adjoint matrix, then

$$\tfrac{\partial}{\partial t} E_j^\pm(t) = \langle p(\pm j; t), Ap(\pm j; t)\rangle \tag{3.2}$$

as long as $E_j^\pm(t)$ stays outside $[-2, 2]$.

We define $E_j^\pm \equiv \pm 2$ whenever J has less than j positive/negative eigenvalues. Then, of course, (3.2) does not apply when $E_j^\pm(t) = \pm 2$, but we at least have continuity of $E_j^\pm(t)$ in t by norm-continuity of $J(t)$.

Here is the main idea of this section. Fix n and take A to be the matrix with $A_{n-1,n} = A_{n,n-1} = -1$ and all other entries zero (the upper left-hand corner of A being $A_{0,0}$). Then increasing t corresponds to decreasing a_n. We have

$$\tfrac{\partial}{\partial t} E_j^\pm(t) = -2p_n(\pm j;t)p_{n-1}(\pm j;t)$$

Let us take $j=1$. Then by the Sturm oscillation theory [20] we know that $\operatorname{sgn}(p_n(1;t)) = \operatorname{sgn}(p_{n-1}(1;t))$ and $\operatorname{sgn}(p_n(-1;t)) = -\operatorname{sgn}(p_{n-1}(-1;t))$ for $n \geq 1$. Hence E_1^+ will decrease and E_1^- will increase when we decrease a_n. This is exactly what we want.

Unfortunately, this is not always the case for other eigenvalues. Indeed, let us consider a positive eigenvalue E_j^+. By oscillation theory, $p(j)$ changes sign $j-1$ times and so E_j^+ will grow at certain n. However, if $E_j^+ \approx 2$, $a_n \approx 1$ and $b_n \approx 0$, then by (3.1) $p_{n+1}(j) \approx 2p_n(j) - p_{n-1}(j)$, that is, $p(j)$ is (locally) close to a linear function of n. Therefore, if $\operatorname{sgn}(p_n(j)) = -\operatorname{sgn}(p_{n-1}(j))$, then $\operatorname{sgn}(p_m(j)) = \operatorname{sgn}(p_{m-1}(j))$ for $m \neq n$ but close to n. Hence, a suitable decrease of a_n along with some neighboring a_m's should always result into a decrease of E_j^+. This is the content of the present section.

Definition 3.2. Let $\delta > 0$. We say that $\tilde{J}$ *δ-minorates* J, if $|E_j^\pm(\tilde{J})| \leq |E_j^\pm(J)|$ whenever $|E_j^\pm(J)| < 2+\delta$.

Remarks. 1. This is well defined because $E_j^\pm \equiv \pm 2$ whenever J has less than j positive/negative eigenvalues.

2. Notice that for fixed δ this relation is transitive.

Lemma 3.3. *There exists $\delta > 0$ such that the following is true. If for some J we have $|a_m - 1| < \delta$ and $|b_m| < \delta$ for $m \in \{n, n+1, n+2\}$, and $\tilde{J}$ is obtained from J by decreasing a_n by $c > 0$ and a_{n+2} by $d > 0$ so that $|a_n - c - 1| < \delta$, $|a_{n+2} - d - 1| < \delta$ and $c/d \in [\frac{1}{13}, 13]$, then $\tilde{J}$ δ-minorates J.*

Remark. That is, decreasing both a_n and a_{n+2} results into decrease of all but finitely many $|E_j^\pm|$. The same trick applied to a_n and a_{n+1} fails.

Proof. Let $q \equiv c/d$. Let $E \equiv E_j^+$ and $p_n \equiv p_n(+j)$ for some $2 < E_j^+ < 2+\delta$. Then by (3.1)

$$p_{n+1} = 2p_n - p_{n-1} + \frac{E - 2a_{n+1} - b_{n+1}}{1 + (a_{n+1} - 1)} p_n + \frac{a_{n+1} - a_n}{1 + (a_n - 1)} p_{n-1}$$

$$=2p_n - p_{n-1} + O(\delta)(|p_n| + |p_{n-1}|) \tag{3.3}$$

with $|O(\delta)| \leq C\delta$ for some universal $C < \infty$ and all small δ. Similarly we obtain by iterating (3.1)

$$p_{n+2} = 3p_n - 2p_{n-1} + O(\delta)(|p_n| + |p_{n-1}|) \tag{3.4}$$

Let now $J(t) \equiv J + tA$ where A is such that $A_{n-1,n} = A_{n,n-1} = -q$, $A_{n+1,n+2} = A_{n+2,n+1} = -1$ and all other entries are 0. Then obviously $E_j^{\pm}(0) = E_j^{\pm}$ and $\tilde{J} = J(d)$. By (3.2)

$$\frac{\partial}{\partial t} E_j^+(0) = \langle p, Ap \rangle = -2(qp_n p_{n-1} + p_{n+2}p_{n+1})$$

By (3.3) and (3.4)

$$qp_n p_{n-1} + p_{n+2}p_{n+1} = 6p_n^2 - (7-q)p_n p_{n-1} + 2p_{n-1}^2 + O(\delta)(p_n^2 + p_{n-1}^2) \tag{3.5}$$

Since $6 \cdot 2 - \left(\frac{7-q}{2}\right)^2 > 0$ for $q \in (7 - 4\sqrt{3}, 7 + 4\sqrt{3}) \supset [\frac{1}{13}, 13]$, it follows that

$$6p_n^2 - (7-q)p_n p_{n-1} + 2p_{n-1}^2 > |O(\delta)|(p_n^2 + p_{n-1}^2)$$

for small enough δ (uniformly for all $q \in [\frac{1}{13}, 13]$). That is, $\frac{\partial}{\partial t} E_j^+(0) < 0$.

This argument obviously applies to all $t \in [0, d]$, not only to $t = 0$, as long as $E_j^+(t) > 2$. This is because for each such t, $J(t)$ satisfies the conditions of this lemma. Hence $E_j^+(t)$ can only decrease with t (and so stays smaller than $2 + \delta$). Also, no new eigenvalues can appear. Indeed – if $E_j^+(t_1) = 2$ and $E_j^+(t_2) > 2$ for some $t_2 > t_1$, then $E_j^+(t)$ would have to have a discontinuity in $[t_1, t_2]$, because by the above argument it has to decrease whenever it is larger than 2.

A similar argument applies to $E_j^-(0) > -2 - \delta$, with $p_{n+1} \approx -2p_n - p_{n-1}$ and $p_{n+2} \approx 3p_n + 2p_{n-1}$ in place of (3.3) and (3.4), and shows that such E_j^- increases with t. The result follows. $\square$

As mentioned earlier, same trick with a_{n+1} in place of a_{n+2} does not work. Indeed – in (3.5) we would have $2p_n^2 - (1-q)p_n p_{n-1} + O(\delta)(p_n^2 + p_{n-1}^2)$ which cannot be guaranteed to be positive for any $\delta > 0$. However, we can replace a_{n+2} by a_{n+k} for $k \geq 2$, and the lemma stays valid for some smaller $\delta = \delta(k) > 0$ and $c/d \in [(4k^2-3)^{-1}, 4k^2-3]$ (we use that $p_{n+k} \approx (k+1)p_n - kp_{n-1}$). Of course, the bounds on $|a_m - 1|$ and $|b_m|$ have to hold for $m \in \{n, \ldots, n+k\}$.

Before we start perturbing the b_n's, let us state one more result with the same flavor.

Lemma 3.4. *There exists $\delta > 0$ such that the following is true. If for some J we have $|a_m - 1| < \delta$ and $|b_m| < \delta$ for $m \in \{n, n+1, n+2\}$,* *and $\tilde{J}$ is obtained from J by decreasing a_n, a_{n+1} and a_{n+2} by $c>0$ so that $|a_m - c - 1| < \delta$ for $m \in \{n, n+1, n+2\}$, then $\tilde{J}$ δ-minorates J.*

Remark. Again, the result can be extended to decreasing $a_n, \dots, a_{n+k}$ (for $k \ge 2$) by $c > 0$, with a smaller $\delta = \delta(k) > 0$.

Proof. An argument as above yields for $A_{n-1,n} = A_{n,n-1} = A_{n,n+1} = A_{n+1,n} = A_{n+1,n+2} = A_{n+2,n+1} = -1$

$$\begin{aligned}\tfrac{\partial}{\partial t} E_j^+(0) &= -2(p_{n-1}p_n + p_n p_{n+1} + p_{n+1}p_{n+2}) \\ &= -2(8p_n^2 - 7p_n p_{n-1} + 2p_{n-1}^2 + O(\delta)(p_n^2 + p_{n-1}^2))\end{aligned}$$

which is negative for small enough δ, since $8 \cdot 2 - (\frac{7}{2})^2 > 0$. The rest of the previous proof applies. $\square$

Our next aim is to allow perturbations of the b_n's as well. If one decreases b_n, it is obvious that all E_j^+ decrease, but all E_j^- decrease as well. Hence, perturbing the b_n's alone will not move "in" all eigenvalues. To ensure that, we have to counter the undesired movement of E_j^- by decreasing a_n's.

Lemma 3.5. *There exists $\delta > 0$ such that the following is true. If for some J we have $|a_m - 1| < \delta$ and $|b_m| < \delta$ for $m \in \{n, n+1, n+2\}$, and $\tilde{J}$ is obtained from J by decreasing a_n and a_{n+2} by $c > 0$ and changing b_n by $d \in [-\frac{c}{2}, \frac{c}{2}]$ so that $|a_n - c - 1| < \delta$, $|a_{n+2} - c - 1| < \delta$ and $|b_n + d| < \delta$, then $\tilde{J}$ δ-minorates J.*

Proof. This time we have $A_{n-1,n} = A_{n,n-1} = A_{n+1,n+2} = A_{n+2,n+1} = -1$ and $A_{n-1,n-1} = q \equiv d/c$. We obtain

$$\begin{aligned}\tfrac{\partial}{\partial t} E_j^+(0) &= -2(p_{n-1}p_n + p_{n+1}p_{n+2}) + qp_{n-1}^2 \\ &= -2(6p_n^2 - 6p_n p_{n-1} + (2 - \tfrac{q}{2})p_{n-1}^2 + O(\delta)(p_n^2 + p_{n-1}^2))\end{aligned}$$

which is negative for small enough δ if $q < 1$ (i.e. if $6 \cdot (2 - \frac{q}{2}) - (\frac{6}{2})^2 > 0$). A similar argument for E_j^- requires $q > -1$, so there is a $\delta > 0$ which works for all $q \in [-\frac{1}{2}, \frac{1}{2}]$. $\square$

4. The main result

We will now outline an argument suggested in [18]. This shows how to use (1.8) to prove stability of the Szegő condition under certain trace class perturbations.

Let $\tilde{J}$ be a trace class perturbation of a matrix J which we know to be Szegő. That is

$$\sum_n \big(|\tilde{a}_n - a_n| + |\tilde{b}_n - b_n| \big) < \infty \tag{4.1}$$

Let $\tilde{J}_n$ be the matrix which we obtain from J by replacing a_j, b_j by $\tilde{a}_j, \tilde{b}_j$ for $j = 1, \dots, n$. Then $\tilde{J}_n \to \tilde{J}$ pointwise (and also in norm). Now by applying (1.8) to both $\tilde{J}_n$ and J and subtracting, we obtain

$$Z(\tilde{J}_n) = Z(J) - \sum_{j=1}^{n} (\ln(\tilde{a}_j) - \ln(a_j)) + \sum_{j,\pm} \Big(\ln |\beta_j^{\pm}(\tilde{J}_n)| - \ln |\beta_j^{\pm}(J)| \Big) \tag{4.2}$$

By lower semicontinuity of Z in J (in the topology of pointwise convergence of matrix elements; see [9]), we know that $Z(\tilde{J}) \leq \liminf Z(\tilde{J}_n)$. So taking $n \to \infty$ in (4.2) we obtain

$$Z(\tilde{J}) \leq Z(J) + \sum_{j=1}^{\infty} |\ln(\tilde{a}_j) - \ln(a_j)| + \liminf_n \sum_{j,\pm} \Big(\ln |\beta_j^{\pm}(\tilde{J}_n)| - \ln |\beta_j^{\pm}(J)| \Big) \tag{4.3}$$

If $\inf_j \{\tilde{a}_j, a_j\} > 0$, then the first sum is finite by (4.1). Hence, if we could show that the lim inf is smaller than $+\infty$, we would prove $\tilde{J}$ to be Szegő. Notice that this is true if for some $\delta > 0$ each $\tilde{J}_n$ δ-minorates J, because then $|\beta_j^{\pm}(\tilde{J}_n)| \leq |\beta_j^{\pm}(J)|$ whenever $|E_j^{\pm}(J)| < 2 + \delta$ and the other $|\beta_j^{\pm}(\tilde{J}_n)|$ are bounded. This is where results from the previous section enter the picture.

Unfortunately, we cannot treat general trace class perturbations at this moment. The reason is the necessity to use Lemma 2.7, as described in Section 3. It also needs to be said that in what follows, the "partial perturbations" $\tilde{J}_n$ will be slightly different from those above. They will differ in up to 4 matrix elements, but they will still converge to $\tilde{J}$ and so (4.3) will stay valid.

Let us now apply the above argument. We start with

Lemma 4.1. *Let J be Szegő with $a_n \to 1$, $b_n \to 0$, and let $e_n \downarrow 0$, $e_n < a_n$, $\sum_n e_n < \infty$. Then the matrix $\tilde{J}$ with $\tilde{a}_n \equiv a_n - e_n$ and $\tilde{b}_n \equiv b_n$ is also Szegő.*

Proof. Let $\delta \equiv \min\{\delta(2), \delta(3), \delta(4)\} > 0$ where $\delta(k)$ are as in the remark after Lemma 3.4 (that is, good for decreasing 3, 4 and 5 consecutive a_n's). Let N be such that for $j \geq N$ we have $|a_j - 1| < \delta$, $|\tilde{a}_j - 1| < \delta$ and $|b_j| < \delta$. For $n \geq N+1$ let $\tilde{J}_n$ be such that $b_j(\tilde{J}_n) \equiv b_j$ and

$$a_j(\tilde{J}_n) \equiv \begin{cases} \tilde{a}_j & j \leq N-1 \\ \tilde{a}_j + e_{n+1} & N \leq j \leq n \\ a_j & j \geq n+1 \end{cases}$$

Then $\tilde{J}_{N+1}$ is Szegő because it is a finite-rank perturbation of J.

Let $n \geq N+2$. Notice that $\tilde{J}_n$ is obtained from $\tilde{J}_{n-1}$ by decreasing $a_j(\tilde{J}_{n-1})$ by $c \equiv e_n - e_{n+1}$ for $j = N, \ldots, n$. This can be accomplished by successive decreases of 3, 4 or 5 neighboring a_j's by c, as in Lemma 3.4 (and the remark after it). It follows that $\tilde{J}_n$ δ-minorates $\tilde{J}_{n-1}$, and so by induction $\tilde{J}_n$ δ-minorates $\tilde{J}_{N+1}$. Then by (4.2) (with $\delta < \frac{1}{2}$)

$$Z(\tilde{J}_n) \leq Z(\tilde{J}_{N+1}) + 2\sum_{j=N}^{\infty} e_j + K \ln(M) < \infty$$

where K is the number of eigenvalues of $\tilde{J}_{N+1}$ outside $(-2-\delta, 2+\delta)$ and $M \equiv 3\sup_j\{a_j, |b_j|\} \geq \|\tilde{J}_n\|$. So $Z(\tilde{J}_n)$ are uniformly bounded and since $\tilde{J}_n \to \tilde{J}$ pointwise, lower semicontinuity of Z implies $Z(\tilde{J}) < \infty$. □

Corollary 4.2. *Suppose $\{a_n, b_n\}$ is admissible and $\{e_n, f_n\}$ is such that $e_n \to 0$, $f_n \to 0$, $e_n > -a_n$ and $\sum n\big(|e_{n+1}-e_n|+|f_{n+1}-f_n|\big) < \infty$. Then the matrix $\tilde{J}$ with $\tilde{a}_n \equiv a_n + e_n$, $\tilde{b}_n \equiv b_n + f_n$ is Szegő.*

Remark. This is almost like Lemma 2.7 with the condition $2e_n \geq |f_n|$ removed.

Proof. Let us define $\bar{e}_n \equiv \sum_{j=n}^{\infty} |e_{j+1} - e_j|$ and similarly for f_n. Notice that $\bar{e}_n \geq |e_n|$, $\bar{e}_n \downarrow 0$ and

$$\sum_{n=1}^{\infty} \bar{e}_n \leq \sum_{n=1}^{\infty} n|e_{n+1} - e_n| < \infty$$

Then if $\tilde{e}_n \equiv e_n + \bar{e}_n + \bar{f}_n$, we have $2\tilde{e}_n \geq |f_n|$, and so $\{a_n + \tilde{e}_n, b_n + f_n\}$ is admissible by Lemma 2.7. Then by Lemma 4.1 the result follows. □

We are now ready to prove Theorem 1.2.

Theorem 4.3. *Suppose that $\{a_n, b_n\}$ is admissible and $\varepsilon > 0$. Then the matrix $\tilde{J}$ with*

$$\tilde{a}_n \equiv a_n + O(n^{-1-\varepsilon}) \qquad\qquad \tilde{b}_n \equiv b_n + O(n^{-1-\varepsilon})$$

is Szegő.

Proof. Our strategy is as outlined in Section 3. We let

$$C \equiv \sup_n\{|\tilde{a}_n - a_n|n^{1+\varepsilon}, |\tilde{b}_n - b_n|n^{1+\varepsilon}\} < \infty \tag{4.4}$$

and increase a_n by $6Cn^{-1-\varepsilon}$ (we call these again a_n). Then by Lemma 2.7 (or Lemma 2.8), $\{a_n, b_n\}$ (with the new a_n) is also admissible. Thus, the new J is Szegő and we now have

$$a_n - \tilde{a}_n \in [5Cn^{-1-\varepsilon}, 7Cn^{-1-\varepsilon}] \tag{4.5}$$
$$b_n - \tilde{b}_n \in [-Cn^{-1-\varepsilon}, Cn^{-1-\varepsilon}]$$

Let δ be such that both Lemmas 3.3 and 3.5 hold. Let N be such that for $j \geq N$ we have $|a_j - 1| < \delta$, $|\tilde{a}_j - 1| < \delta$, $|b_j| < \delta$ and $|\tilde{b}_j| < \delta$. We let $\tilde{J}_{N-1}$ be such that

$$a_j(\tilde{J}_{N-1}) \equiv \begin{cases} \tilde{a}_j & j \leq N-1 \\ a_j & j \geq N \end{cases}$$

and similarly for $b_j(\tilde{J}_{N-1})$. Then $\tilde{J}_{N-1}$ is Szegő because it is a finite-rank perturbation of J.

We construct $\tilde{J}_N$ from $\tilde{J}_{N-1}$ by first decreasing a_N, a_{N+2} by $2|b_N - \tilde{b}_N|$ and changing b_N to $\tilde{b}_N$, and then decreasing a_N by $a_N - \tilde{a}_N$ and a_{N+2} by $(a_N - \tilde{a}_N)/13$ (in terms of the new a_N). Both these perturbations are δ-minorating by Lemmas 3.3 and 3.5, and the obtained matrix $\tilde{J}_N$ agrees with $\tilde{J}$ in first N couples $a_j(\tilde{J}_N)$, $b_j(\tilde{J}_N)$. The others are same as in J, only exception being $a_{N+2}(\tilde{J}_N)$, for which we know

$$a_{N+2}(\tilde{J}_N) - \tilde{a}_{N+2} \in [2C(N+2)^{-1-\varepsilon}, 7C(N+2)^{-1-\varepsilon}] \tag{4.6}$$

(if N is chosen so that $33N^{-1-\varepsilon} \leq 39(N+2)^{-1-\varepsilon}$).

Now we apply the same procedure to inductively construct $\tilde{J}_n$ from $\tilde{J}_{n-1}$ for $n \geq N+1$. Each $\tilde{J}_n$ will agree with $\tilde{J}$ up to index n, and other elements will be the same as in J, with the exception of $a_{n+1}(\tilde{J}_n)$ and $a_{n+2}(\tilde{J}_n)$. For these we will have (4.6) (with $n+1$ and $n+2$ in place of $N+2$), which is just enough so that we can change b_{n+1} to $\tilde{b}_{n+1}$ when passing to $\tilde{J}_{n+1}$ by the same method. Since $\tilde{J}_n$ δ-minorates $\tilde{J}_{n-1}$, we obtain by induction that each $\tilde{J}_n$ δ-minorates $\tilde{J}_{N-1}$.

Again, we have by (4.2) (with $\delta < \frac{1}{2}$)

$$Z(\tilde{J}_n) \leq Z(\tilde{J}_{N-1}) + 14C \sum_{j=N}^{\infty} j^{-1-\varepsilon} + K \ln(M) < \infty$$

with K and M as in the proof of Lemma 4.1. Since $\tilde{J}_n \to \tilde{J}$, the result follows. □

Corollary 4.4. *Let $2\alpha \geq |\beta|$, $e_n \downarrow 0$, $\varepsilon > 0$ and*

$$a_n \equiv 1 + \frac{\alpha}{n} + e_n + O(n^{-1-\varepsilon}) \qquad b_n \equiv \frac{\beta}{n} + O(n^{-1-\varepsilon})$$

Then J is Szegő.

Remarks. 1. This settles the $2\alpha \geq |\beta|$ case of Askey's conjecture.

2. The same is true when αn^{-1}, βn^{-1} are replaced by $\alpha n^{-\gamma}$, $\beta n^{-\gamma}$ for any $\gamma > 0$.

Proof. By Lemma 2.8, $\{1 + \alpha/n + e_n, \beta/n\}$ is admissible. Then use Theorem 4.3. $\square$

Let us now return to considering perturbations of a single a_n. As noted in Section 1, decreasing it can only guarantee decrease of $|E_1^{\pm}|$. However, if we know that J has *no* bound states (eigenvalues outside $[-2, 2]$), then this is sufficient to conclude that no new bound states can appear when decreasing a_n.

Theorem 4.5. *Assume that J with $a_n \to 1$, $b_n \to 0$ has only finitely many bound states and let $\tilde{J}$ have $\tilde{a}_n \leq a_n$ and $\tilde{b}_n = b_n$ with $\tilde{a}_n \to 1$. Then $\tilde{J}$ is Szegő if and only if J is Szegő and $\sum_n (a_n - \tilde{a}_n) < \infty$. In any case, $\tilde{J}$ also has only finitely many bound states.*

Proof. We only need to prove this theorem for J with no bound states. For by Sturm oscillation theory, J has finitely many of them iff $J^{(n)}$ has none for large enough n. And J is Szegő iff $J^{(n)}$ is. So let us assume that J has no bound states. Then by the above discussion, $\tilde{J}$ has none as well. Indeed – if we let $\tilde{J}_n$ have $a_j(\tilde{J}_n) \equiv \tilde{a}_j$ for $j = 1, \dots, n$ and all other entries same as J, then $\tilde{J}_n$ is created from $\tilde{J}_{n-1}$ by decreasing a_n. Since $\tilde{J}_{n-1}$ has no bound states, the same must be true for $\tilde{J}_n$. Since $\tilde{J}_n \to \tilde{J}$ in norm, $\tilde{J}$ also has no bound states.

If $Z(J) < \infty$ and $\sum (a_n - \tilde{a}_n) < \infty$, then $Z(\tilde{J}) < \infty$ by (4.3). No bound states and Theorem 4.1(d) in [18] imply $\bar{A}_0(J) > -\infty$. So if $\sum (a_n - \tilde{a}_n) = \infty$, we obtain $\bar{A}_0(\tilde{J}) = \infty$, and then $Z(\tilde{J}) = \infty$ by (1.8) (since $Z(\tilde{J}^{(n)}) \geq -\frac{1}{2}\ln(2)$). Finally, if $Z(J) = \infty$, then no bound states and Theorem 4.1(a) in [18] give $\bar{A}_0(J) = \infty$. This implies $\bar{A}_0(\tilde{J}) = \infty$ and so again $Z(\tilde{J}) = \infty$. $\square$

Since Theorem 4.1 in [18] does not distinguish between no bound states and $\mathcal{E}_0(J) < \infty$, we can extend the above result to that case, but we need to restrict it to δ-minorating perturbations of the a_n's only (e.g. decreasing a_n by $e_n \downarrow 0$). If $\mathcal{E}_0(J) = \infty$, then such a result cannot be generally true. For example, if $2\alpha > |\beta|$ in the Coulomb case, then decreasing α by $\alpha - |\beta|/2$ results into a non-summable change of the a_n's, but the matrix stays Szegő.

5. One-sided Szegő conditions

In this section we will discuss Jacobi matrices which are Szegő at 2 or -2. That is such, for which the Szegő integral (1.3) converges at ± 2, but is allowed to diverge at ∓ 2. This is particularly interesting for J which are Hilbert-Schmidt (i.e. L^2) perturbations of J_0. For such J we know from [9] that

$$Z_2^-(J) \equiv \frac{1}{4\pi} \int_{-2}^{2} \ln\left(\frac{\sqrt{4-x^2}}{2\pi \nu'(x)} \right) \sqrt{4-x^2}\, dx < \infty \tag{5.1}$$

(and $Z_2^-(J) \geq 0$ holds always; see [9]). That of course means that $Z(J)$ can only diverge at ± 2.

We define

$$Z_1^\pm(J) \equiv \frac{1}{4\pi} \int_{-2}^{2} \ln\left(\frac{\sqrt{4-x^2}}{2\pi \nu'(x)} \right) \frac{2 \pm x}{\sqrt{4-x^2}}\, dx \tag{5.2}$$

(the notation in (5.1) and (5.2) is from [18]). Again, $Z_1^\pm(J)$ is bounded below by some $c_0 > -\infty$ and it is lower semicontinuous in J [18]. If $J - J_0 \in L^2$, then by (5.1), the integral (1.3) converges at ± 2 if and only if $Z_1^\pm(J) < \infty$. Since we will mainly be interested in $J - J_0$ Hilbert-Schmidt, we use the following definition of one-sided Szegő conditions from [18].

Definition 5.1. We say that J is *Szegő at* ± 2 iff $Z_1^\pm(J) < \infty$.

We consider Z_2^- and $Z_1^\pm$ as above because they appear in sum rules similar to (1.8) [18]. Here we will only use the $Z_1^\pm$ sum rules. If we let $\xi^\pm(\beta) \equiv \ln|\beta| \pm \frac{1}{2}(\beta - \beta^{-1})$, then [18] proves for $J - J_0$ compact

$$Z_1^+(J) = -\sum_{j=1}^{n} \left(\ln(a_j) + \tfrac{1}{2} b_j \right) + \sum_{j,\pm} \Big[\xi^+\big(\beta_j^\pm(J)\big) - \xi^+\big(\beta_j^\pm(J^{(n)})\big) \Big] + Z_1^+(J^{(n)}) \tag{5.3}$$

$$Z_1^-(J) = -\sum_{j=1}^{n} \left(\ln(a_j) - \tfrac{1}{2} b_j \right) + \sum_{j,\pm} \Big[\xi^-\big(\beta_j^\pm(J)\big) - \xi^-\big(\beta_j^\pm(J^{(n)})\big) \Big] + Z_1^-(J^{(n)})$$

Just as with $Z(J)$, the infinite sums are always absolutely convergent and (5.3) holds even if $Z_1^\pm(J) = \infty$. This shows that the one-sided Szegő conditions are also stable under finite-rank perturbations.

We will only consider the Szegő condition at 2 and use the first of these identities. The reason for this is an obvious symmetry – a matrix J is Szegő at -2 iff $\tilde{J}$ with $\tilde{a}_n \equiv a_n$ and $\tilde{b}_n \equiv -b_n$ is Szegő at 2 (because $J \cong -\tilde{J}$). Therefore, our results for $+2$ will immediately translate into similar results for -2.

The main tool for handling trace class perturbations will be the following inequality, which we obtain from the first equation in (5.3) just as we obtained (4.3) from (1.8) (with the same $\tilde{J}_n$).

$$Z_1^+(\tilde{J}) \le Z_1^+(J) + \sum_{j=1}^{\infty} |\ln(\tilde{a}_j) - \ln(a_j)| + \tfrac{1}{2}\sum_{j=1}^{\infty} |\tilde{b}_j - b_j| \qquad (5.4)$$
$$+ \liminf_n \sum_{j,\pm} \Big(\xi^+(\beta_j^\pm(\tilde{J}_n)) - \xi^+(\beta_j^\pm(J))\Big)$$

Notice that $\xi^+(\beta)$ is increasing and positive on $[1,\infty)$, and increasing and negative on $(-\infty,-1]$. That of course means that the last sum in (5.4) will be negative whenever $\beta_j^\pm(\tilde{J}_n) \le \beta_j^\pm(J)$ for all $j,\pm$. In particular, if $\tilde{a}_j = a_j$ and $\tilde{b}_j \le b_j$ for all j.

Theorem 5.2. *Suppose $J - J_0$ is compact.*
(i) *If J is Szegő at 2, and $\tilde{J}$ has $\tilde{a}_n = a_n$, $\tilde{b}_n \le b_n$ with $\sum_n (b_n - \tilde{b}_n) < \infty$, then $\tilde{J}$ is also Szegő at 2.*
(ii) *If J is Szegő at -2, and $\tilde{J}$ has $\tilde{a}_n = a_n$, $\tilde{b}_n \ge b_n$ with $\sum_n (\tilde{b}_n - b_n) < \infty$, then $\tilde{J}$ is also Szegő at -2.*
(iii) *Let $\hat{J}$ have $\hat{a}_n = a_n$, $\hat{b}_n \ge b_n$ with $\sum_n (\hat{b}_n - b_n) < \infty$, and let both $J, \hat{J}$ be Szegő. If $\tilde{J}$ has $\tilde{a}_n = a_n$ and $b_n \le \tilde{b}_n \le \hat{b}_n$, then $\tilde{J}$ is also Szegő.*

Proof. (i) follows from the discussion above, (ii) from (i) by symmetry, and (iii) from (i) and (ii) and the fact that J is Szegő iff it is Szegő at both ± 2. ☐

When perturbing the a_n's as in Section 3, we have to be careful with negative eigenvalues. Indeed – decreasing all $|E_j^\pm|$ does not necessarily make the last sum in (5.4) negative, because $\xi^+(\beta)$ increases on $(-\infty,-1]$. This problem can be overcome if the contribution of the $\beta_j^-(J)$'s to that sum is finite. Since for $\beta \uparrow -1$

$$\xi^+(\beta) = O\big(|\beta+1|^3\big) = O\big(|E+2|^{\frac{3}{2}}\big)$$

this means that we need

$$\sum_j |E_j^- + 2|^{\frac{3}{2}} < \infty \qquad (5.5)$$

Then the $\liminf$ in (5.4) will be bounded from above if every change $\tilde{J}_{n-1} \to \tilde{J}_n$ decreases all $E_j^+ \in (2, 2+\delta)$, irrespective of what happens to E_j^- ($\xi^+(\beta)$ is negative on $(-\infty,-1]$). By [9], (5.5) holds whenever $J - J_0 \in L^2$.

But before we can use this idea to handle certain trace class perturbations as in Section 4, we first need to find some a_n, b_n to be perturbed. Our aim is to treat Coulomb Jacobi matrices with $2\alpha > \pm\beta$ and show they are Szegő at ∓ 2. To prove the next result, we will return to the methods of Section 2.

Lemma 5.3. *Suppose $a_n \to 1$, $b_n \to 0$.*

(i) *Let $\{a_n\}$ be eventually strictly monotone and*

$$\frac{a_n - a_{n-1}}{a_{n+1} - a_n} \to 1 \qquad\qquad \frac{b_{n+1} - b_n}{a_{n+1} - a_n} \to \omega \tag{5.6}$$

with ω finite. If eventually

$$\omega \operatorname{sgn}(a_{n+1} - a_n) < -2 \operatorname{sgn}(a_{n+1} - a_n)$$

then there are $\delta > 0$, $c > 0$ such that $\nu'(x) \geq c\sqrt{4 - x^2}$ in $(2-\delta, 2)$.

(ii) *Let $\{b_n\}$ be eventually strictly monotone and*

$$\frac{b_n - b_{n-1}}{b_{n+1} - b_n} \to 1 \qquad\qquad \frac{a_{n+1} - a_n}{b_{n+1} - b_n} \to \omega_1 \tag{5.7}$$

with ω_1 finite. If eventually

$$\omega_1 \operatorname{sgn}(b_{n+1} - b_n) < -\tfrac{1}{2} \operatorname{sgn}(b_{n+1} - b_n)$$

then there are $\delta > 0$, $c > 0$ such that $\nu'(x) \geq c\sqrt{4 - x^2}$ in $(2-\delta, 2)$.

Remarks. 1. (ii) is (i) with $\omega_1 = \omega^{-1}$. It handles the case $\omega = \pm\infty$.

2. In particular, such J are Szegő at 2 whenever $J - J_0 \in L^2$.

3. By symmetry, same result holds for Szegő condition at -2, with "< -2" and "$< -\frac{1}{2}$" replaced by "> 2" and "$< \frac{1}{2}$"

Proof. (i) First notice that (2.11) holds because a_n is (eventually) monotone, and either b_n is monotone (if $\omega \neq 0$) or $|b_{n+1} - b_n| \leq |a_{n+1} - a_n|$ (if $|\omega| < 1$). Hence, we can use Lemma 2.2. This time we will work with S_n instead of T_n, because it has a simpler recurrence relation (2.2). Notice that by the proof of Lemma 2.2, for every $|x| < 2$ we have $S_n(x) \to \sqrt{4 - x^2}/2\pi\nu'(x)$. The result will follow if we prove that $S_n(x) \leq C$ for some $C < \infty$, all $x \in (2 - \delta, 2)$ and all large n.

We will show this by proving that for some K and all large enough n we have $S_{n+K-1}(x) \leq S_{n-1}(x)$ for all $x \in (2 - \delta, 2)$. That is, we will iterate (2.2) K times at once. Here $K \geq 3$ and δ will be fixed, but they will not be specified until later.

We let n be large and such that for all $j \geq n$ we have $|a_j - 1| < \delta$ and $|b_j| < \delta$, and we take $x \in (2 - \delta, 2)$. Then by (1.2) in the form (3.1) we obtain for $P_n \equiv P_n(x)$ and $k \in \{0, \dots, K-1\}$

$$P_{n+k} = (k+1)P_n - kP_{n-1} + O(\delta)(|P_n| + |P_{n-1}|)$$

We also have

$$\begin{aligned} a_{n+k+1}^2 - a_{n+k}^2 =& (a_{n+k+1} - a_{n+k})(2 + o(1)) \\ a_{n+k}(b_{n+k+1} - b_{n+k}) =& (a_{n+k+1} - a_{n+k})(\omega + o(1)) \end{aligned}$$

with $o(1) = o(n^0)$ taken w.r.t. n. From these estimates we obtain

$$\begin{aligned} S_{n+k} - S_{n+k-1} =& (a_{n+k+1}^2 - a_{n+k}^2)P_{n+k}^2 + a_{n+k}(b_{n+k+1} - b_{n+k})P_{n+k}P_{n+k-1} \\ =& (a_{n+k+1} - a_{n+k})\Big\{ \big[(2 + o(1))(k+1)^2 + (\omega + o(1))k(k+1)\big] P_n^2 \\ & - \big[(2 + o(1))2k(k+1) + (\omega + o(1))(2k^2 - 1)\big] P_n P_{n-1} \\ & + \big[(2 + o(1))k^2 + (\omega + o(1))k(k-1)\big] P_{n-1}^2 + O(\delta)(P_n^2 + P_{n-1}^2)\Big\} \end{aligned}$$

where the $O(\delta)$ also depends on K and ω (but not on x or n). Using the identities $\sum_{k=0}^{K-1} k^2 = K(2K^2 - 3K + 1)/6$, $\sum_{k=0}^{K-1} k = K(K-1)/2$ and $a_{n+k+1} - a_{n+k} = (a_{n+1} - a_n)(1 + o(1))$, we obtain for $K \ge 3$

$$\begin{aligned} \frac{3}{K} \frac{S_{n+K-1} - S_{n-1}}{a_{n+1} - a_n} =& O(\delta)(P_n^2 + P_{n-1}^2) \\ & + \big[2K^2 + 3K + 1 + \omega(K^2 - 1) + o(1)\big] P_n^2 \\ & - \big[4K^2 - 4 + \omega(2K^2 - 3K - 2) + o(1)\big] P_n P_{n-1} \\ & + \big[2K^2 - 3K + 1 + \omega(K^2 - 3K + 2) + o(1)\big] P_{n-1}^2 \end{aligned}$$

where both $O(\delta)$ and $o(1)$ depend on K and ω. Let us denote by I, II, III the three square brackets in the above expression, without the $o(1)$ terms. If $I \cdot III - (II/2)^2 > 0$, then for small enough δ and large n (so that $O(\delta)$ and $o(1)$ are negligible) the above expression will have the same sign as I. We have $I \cdot III - (II/2)^2 > 0$ whenever

$$\omega \notin [c_1(K), c_2(K)] \equiv \left[-2 - \frac{6 + 2\sqrt{3}\sqrt{K^2 - 1}}{K^2 - 4}, -2 - \frac{6 - 2\sqrt{3}\sqrt{K^2 - 1}}{K^2 - 4} \right]$$

Also, $I > 0$ when $\omega > d(K) \equiv -(2K^2 + 3K + 1)/(K^2 - 1)$ and $I < 0$ when $\omega < d(K)$. Since $c_1(K), c_2(K), d(K) \to -2$ and by the above

$$\operatorname{sgn}(S_{n+K-1} - S_{n-1}) = \operatorname{sgn}(a_{n+1} - a_n)\operatorname{sgn}(I)$$

one only needs to take K large enough so that $\omega > \max\{c_2(K), d(K)\}$ (if $\operatorname{sgn}(a_{n+1} - a_n) < 0$) or $\omega < \min\{c_1(K), d(K)\}$ (if $\operatorname{sgn}(a_{n+1} - a_n) > 0$). Then for small enough δ and all large n one obtains $\operatorname{sgn}(S_{n+K-1}(x) - S_{n-1}(x)) = -1$ whenever $x \in (2 - \delta, 2)$. The result follows.

(ii) The proof is as in (i), but with the role of $a_{n+1} - a_n$ played by $b_{n+1} - b_n$. We obtain $I = \omega_1(2K^2 + 3K + 1) + K^2 - 1$ and $\operatorname{sgn}(S_{n+K-1} -$

$S_{n-1}) = \operatorname{sgn}(b_{n+1} - b_n) \operatorname{sgn}(I)$ whenever

$$\omega_1 \notin \left[-\frac{1}{2} - \frac{\sqrt{3}}{2\sqrt{K^2-1}}, -\frac{1}{2} + \frac{\sqrt{3}}{2\sqrt{K^2-1}} \right]$$

□

Now we are ready to introduce errors and state the main result of this section.

Theorem 5.4. *Suppose $\tilde{J}$ has*

$$\tilde{a}_n \equiv a_n + O(n^{-1-\varepsilon}) \qquad \tilde{b}_n \equiv b_n + O(n^{-1-\varepsilon})$$

where $\sum_{n=1}^\infty (a_n - 1)^2 + \sum_{n=1}^\infty b_n^2 < \infty$ and $\varepsilon > 0$.

(i) *Assume a_n, b_n satisfy (5.6) and $n^{2+\varepsilon}|a_{n+1} - a_n| \to \infty$. If eventually*

$$\omega \operatorname{sgn}(a_{n+1} - a_n) < -2 \operatorname{sgn}(a_{n+1} - a_n)$$

then $\tilde{J}$ is Szegő at 2. If eventually

$$\omega \operatorname{sgn}(a_{n+1} - a_n) > 2 \operatorname{sgn}(a_{n+1} - a_n)$$

then $\tilde{J}$ is Szegő at -2.

(ii) *Assume a_n, b_n satisfy (5.7) and $n^{2+\varepsilon}|b_{n+1} - b_n| \to \infty$. If eventually*

$$\omega_1 \operatorname{sgn}(b_{n+1} - b_n) < -\tfrac{1}{2} \operatorname{sgn}(b_{n+1} - b_n)$$

then $\tilde{J}$ is Szegő at 2. If eventually

$$\omega_1 \operatorname{sgn}(b_{n+1} - b_n) < \tfrac{1}{2} \operatorname{sgn}(b_{n+1} - b_n)$$

then $\tilde{J}$ is Szegő at -2.

Remark. Notice that if $\sup\{n^{2+\varepsilon}|a_{n+1} - a_n|\} < \infty$, then $|a_n - 1| \lesssim n^{-1-\varepsilon}$ and since (in (i)) ω is finite, we also have $|b_n| \lesssim n^{-1-\varepsilon}$. Hence, $J - J_0$ is trace class and hence Szegő by [9].

Proof. (i) We follow the proof of Theorem 4.3. First we increase a_n by $6Cn^{-1-\varepsilon}$ with C from (4.4). We have

$$\frac{a_n + \frac{6C}{n^{1+\varepsilon}} - a_{n-1} - \frac{6C}{(n-1)^{1+\varepsilon}}}{a_{n+1} + \frac{6C}{(n+1)^{1+\varepsilon}} - a_n - \frac{6C}{n^{1+\varepsilon}}} - \frac{a_n - a_{n-1}}{a_{n+1} - a_n} = \frac{O(1)}{n^{2+\varepsilon}(a_{n+1} - a_n) + O(1)} \to 0$$

So if we call $a_n + 6Cn^{-1-\varepsilon}$ again a_n, we still have $(a_n - a_{n-1})/(a_{n+1} - a_n) \to 1$. Similarly, $(b_{n+1} - b_n)/(a_{n+1} - a_n) \to \omega$. And, of course, $\{a_n\}$ has the same type of monotonicity as before, by the assumption $n^{2+\varepsilon}|a_{n+1} - a_n| \to \infty$. We call J the matrix with these new a_n, b_n. By hypothesis $J - J_0 \in L^2$, so J is Szegő at 2 by Lemma 5.3(i) and (5.1).

Now we consider the same $\tilde{J}_n$ as in the proof of Theorem 4.3. The first of them is $\tilde{J}_{N-1}$ and it is Szegő at 2 because it is a finite-rank perturbation of J. Each next $\tilde{J}_n$ will δ-minorate $\tilde{J}_{n-1}$. That proves that in (5.4) (with $\tilde{J}_{N-1}$ in place of J) the sum involving β_j^+ will be bounded above by $K\xi^+(M)$ with K and M as in Lemma 4.1. The sum with β_j^- will be bounded above by $\sum_j(-\xi^+(\beta_j^-(\tilde{J}_{N-1})))$ and this is finite by (5.5) (which holds because $\tilde{J}_{N-1} - J_0 \in L^2$). So the lim inf in (5.4) cannot be $+\infty$ and the result follows.

(ii) The proof is identical. $\square$

Corollary 5.5. *Let $\varepsilon > 0$ and*

$$a_n \equiv 1 + \frac{\alpha}{n} + O(n^{-1-\varepsilon}) \qquad\qquad b_n \equiv \frac{\beta}{n} + O(n^{-1-\varepsilon}) \tag{5.8}$$

If $2\alpha > \pm\beta$, then J given by (1.1) *is Szegő at ∓ 2.*

Proof. Use Theorem 5.4(i) (if $\alpha \neq 0$) or (ii) (if $\alpha = 0$) with $a_n \equiv 1+\alpha/n$, $b_n \equiv \beta/n$. $\square$

As for other pairs (α, β) in (5.8), Theorem 4.4(ii) in [18] shows that if $2\alpha < \pm\beta$, then J cannot be Szegő at ∓ 2. Hence, the (α, β) plane is divided into 4 regions by the lines $2\alpha = \pm\beta$. Inside the right-hand region J is Szegő, inside the top and bottom regions J is Szegő only at, respectively, 2 and -2, and inside the left-hand region J is Szegő neither at 2 nor at -2. On the borderlines the situation is as follows. If $2\alpha = \pm\beta$ and $\alpha \geq 0$, then Corollary 4.4 shows that J is Szegő, and so Szegő at both 2 and -2. If $2\alpha = \pm\beta$ and $\alpha < 0$, then J cannot be Szegő at ± 2 by Theorem 4.4(ii) in [18]. I think that such J *is* Szegő at ∓ 2.

Finally, it should be mentioned that although we have mainly considered Coulomb behavior of a_n, b_n, the above picture is valid in more general setting as well. For example in the case $a_n \equiv 1 + \alpha n^{-\gamma} + O(n^{-1-\varepsilon})$ and $b_n \equiv \beta n^{-\gamma} + O(n^{-1-\varepsilon})$ with $\frac{1}{2} < \gamma \leq 1$, $\varepsilon > 0$, as implied by results of [18] and this paper.

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