SPECTRAL THEORY OF TIME-PERIODIC MANY-BODY SYSTEMS

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Abstract. We study the spectrum of the monodromy operator for an N-body quantum system in a time-periodic external field with time-mean equal to zero. This includes AC-Stark and circularly polarized fields, and pair-potentials with a local singularity up to (and including) the Coulomb singularity. In the framework of Floquet theory we prove a local commutator estimate and use it to prove a Limiting Absorption Principle for the Floquet Hamiltonian as well as exponential decay estimates on non-threshold eigenfunctions. These two results are then used to obtain a second order perturbation theory for embedded eigenvalues. The principal tool is a new extended Mourre theory.

Dedicated to Ira Herbst on the occasion of his 60th birthday.

Contents

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1. Introduction and results

In this paper we consider a quantum system consisting of N interacting ν-dimensional particles, placed in a time-periodic electric field $\tilde{\mathcal{E}}$ with zero mean

$$\tilde{\mathcal{E}} \in L^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^\nu), \qquad \tilde{\mathcal{E}}(t+1) = \tilde{\mathcal{E}}(t) \text{ a.e.} \quad \text{and} \quad \int_0^1 \tilde{\mathcal{E}}(s)ds = 0.$$

This includes in particular the cases of linearly polarized fields, also called AC-Stark fields, and circularly polarized fields. The choice of 1 as period is made for convenience. The family of time-dependent Hamiltonians for such a system is given by

$$\tilde{h}(t) = \sum_{i=1}^N \left(\frac{p_i^2}{2m_i} - q_i \tilde{\mathcal{E}}(t) \cdot x_i \right) + v, \qquad v(x_1, \dots, x_N) = \sum_{1 \le i < j \le N} v_{ij}(x_i - x_j). \tag{1.1}$$

Here x_i, m_i and q_i are the position, mass and charge of the i'th particle and $p_i = -i\nabla_{x_i}$ is its momentum. Note that $\tilde{h}(t)$ are operators on $L^2(\mathbb{R}^{\nu N})$. We consider three sets of conditions on the pair-potentials v_{ij}, see Conditions 1.1–1.3 given below.

The model arises as a dipole approximation of the Hamiltonian

$$\sum_{i=1}^N \frac{1}{2m_i} \left(p_i - q_i A_t^i\right)^2 + v.$$

Here A_t^i denotes multiplication by $A_t(x_i)$, where A_t solves the wave equation. Under the assumption that the field varies slowly on atomic length scales one replaces $A_t(x_i)$ by $A_t(0)$ (the dipole approximation). After the time-dependent transformation $f \to e^{-iA_t(0)\cdot\sum_{i=1}^N q_i x_i} f$ one arrives at (1.1) with electric field $\tilde{\mathcal{E}}(t) = -\dot{A}_t(0)$. See for example the monograph [SSL]. The requirement that the field is periodic corresponds to the requirement that the electromagnetic field only contain modes which are multiples of $2\pi\omega$ for some frequency $\omega > 0$ (here put equal to 1).

Our aim is to study the evolution generated by the family of Hamiltonians (1.1).

We write $\tilde{U}(t,s)$ for the two-parameter family of unitary operators (the evolution) which solves the time-dependent Schrödinger equation

$$i\frac{d}{dt}\tilde{U}(t,s) = \tilde{h}(t)\tilde{U}(t,s), \qquad \tilde{U}(s,s) = I. \tag{1.2}$$

See Remark 1.4 below. The solution satisfies the Chapman Kolmogorov equations

$$\tilde{U}(s,r)\tilde{U}(r,t) = \tilde{U}(s,t), \qquad r,s,t \in \mathbb{R} \tag{1.3}$$

and a periodicity equation

$$\tilde{U}(t+1,s+1) = \tilde{U}(t,s), \qquad s,t \in \mathbb{R}. \tag{1.4}$$

We will as usual formulate the model in a more convenient form (see [Hu1] for more details) using the inner product

$$x \cdot y = \sum_{i=1}^N 2m_i (x_i, y_i),$$

where $(\cdot,\cdot)$ is the inner product on $\mathbb{R}^\nu$, $x=(x_1,\dots,x_N)$ and $y=(y_1,\dots y_N)$. We will consider the system in its center of mass frame

$$X=\left\{x\in\mathbb{R}^{3N}:\sum_{i=1}^N m_i x_i=0\right\}.$$

Then $\mathbb{R}^{\nu N}=X\oplus X_{\mathrm{CM}}$, where $X_{\mathrm{CM}}=\{x:x_1=\cdots=x_N\}$. We have the splitting

$$\tilde{h}(t)=h_{\mathrm{CM}}(t)\otimes I+I\otimes h(t),\qquad \text{on } L^2(X_{\mathrm{CM}})\otimes L^2(X),$$

where

$$h_{\mathrm{CM}}(t)=p_{\mathrm{CM}}^2-\mathcal{E}_{\mathrm{CM}}(t)\cdot x,\quad\text{and}\quad h(t)=p^2-\mathcal{E}(t)\cdot x+v.$$

Here

$$\mathcal{E}_{\mathrm{CM}}=\frac{Q}{2M}\left(\tilde{\mathcal{E}},\dots,\tilde{\mathcal{E}}\right)\quad\text{and}\quad \mathcal{E}=\left(\left(\frac{q_1}{2m_1}-\frac{Q}{2M}\right)\tilde{\mathcal{E}},\dots,\left(\frac{q_N}{2m_N}-\frac{Q}{2M}\right)\tilde{\mathcal{E}}\right),\tag{1.5}$$

where $Q=q_1+\cdots+q_N$ and $M=m_1+\cdots+m_N$ are the total charge and mass of the system. In the special case where all the particles have identical charge to mass ratio, we see that the center of mass Hamiltonian is just an ordinary time-independent N-body Hamiltonian. We can solve the time-dependent Schrödinger equation (1.2) for $h_{\mathrm{CM}}(t)$ and $h(t)$ as well and we write $U_{\mathrm{CM}}(t,s)$ and $U(t,s)$ for the respective two-parameter families, cf. Remark 1.4. The evolutions U_{CM} and U also satisfy (1.3) and (1.4), and furthermore

$$\tilde{U}(t,s)=U_{\mathrm{CM}}(t,s)\otimes U(t,s).\tag{1.6}$$

In this paper we are interested in the spectral study of the monodromy (or period) operator $U(1,0)$, for the system in its center of mass frame. For a time-periodic problem, this replaces the spectral study usually done directly on the Hamiltonian for a time-independent system. See for example [Ya1], [How1] and [KiY2]. In particular we will study the structure of thresholds, eigenvalues and the continuous spectrum.

We will work in the framework of generalized N-body systems, which we review briefly. Let $\mathcal{A}$ be a finite index set and X a finite dimensional real vector-space with inner product. There is an injective map from $\mathcal{A}$ into the subspaces of X, $\mathcal{A}\ni a\to X^a\subset X$, and we write $X_a=(X^a)^\perp$. We introduce a partial ordering on $\mathcal{A}$:

$$a\subset b\Leftrightarrow X^a\subset X^b\tag{1.7}$$

and assume the following

(1) There exist $a_{\min},a_{\max}\in\mathcal{A}$ with $X^{a_{\min}}=\{0\}$ and $X^{a_{\max}}=X$.
(2) For each $a,b\in\mathcal{A}$ there exists $c=a\cup b\in\mathcal{A}$ with $X_a\cap X_b=X_c$.

We will write x^a and x_a for the orthogonal projection of a vector x onto the subspaces X^a and X_a respectively.

The physical problem above fits into this framework as follows. Here $\mathcal{A}$ is the set of all cluster partitions $a=\{C_1,\dots,C_{\#a}\}$, $1\le\#a\le N$, each given by splitting the set of particles $\{1,\dots,N\}$ into non-empty disjoint clusters C_i. The spaces X_a, $a\in\mathcal{A}$, are the spaces of configurations of the $\#a$ centers of mass of the clusters C_i (in the center of mass frame). The complement

$$X^a=X^{C_1}\oplus\cdots\oplus X^{C_{\#a}}$$

is the space of relative configurations within each of the clusters C_i. More precisely

$$X^{C_i}=\{x\in X:x_j=0,j\notin C_i\}\quad\text{and}\quad X_a=\{x\in X:k,l\in C_i\Rightarrow x_k=x_l\}.$$

Notice that $a \subset b$ if and only if any cluster $C \in a$ is contained in some cluster $C' \in b$. We furthermore see that $a_{\min} = \{(1), \ldots, (N)\}$, $a_{\max} = \{(1, \ldots, N)\}$ and $a \cup b$ is the smallest c such that $a \subset c$ and $b \subset c$.

We will work with three classes of generalized potentials

$$V(t,x) = \sum_{a \in \mathcal{A}} V_a(t, x^a),$$

where V_a is a function on $\mathbb{R} \times X^a$. In our example above the V_a's are independent of t and $V_a = 0$ if $\#a \neq N-1$. Since $V_{a_{\min}}$ is a constant, we can fix it to be zero. The first class consists of regular potentials, with an explicit time-dependence.

Condition 1.1. *For each $a \neq a_{\min}$ the following holds. The pair-potential $\mathbb{R} \times X^a \ni (t,y) \to V_a(t,y)$ is a continuous real-valued function satisfying*

i) *Periodicity: $V_a(t+1, \cdot) = V_a(t, \cdot)$, $t \in \mathbb{R}$.*
ii) *Regularity: $V_a \in C^1(\mathbb{R} \times X^a)$ and for each $t \in \mathbb{R}$, $(\partial_t^k \partial_y^\alpha V_a)(t, \cdot) \in C^1(X^a)$, $k + |\alpha| = 1$.*
iii) *Bounds at infinity: $|V_a(t,y)| + |y \cdot \nabla_y V_a(t,y)| = o(1)$, $||y|^{|\alpha|} \partial_t^k \partial_y^\alpha V_a(t,y)| = O(1)$ for $0 \leq k \leq 1$ and $k + |\alpha| \leq 2$.*

Here $o(1)$ means $o(1) \to 0$, for $|y| \to \infty$, uniformly in t. (Similarly for $O(1)$.)

The second class consists of time-independent potentials with L^p singularities. This class just fails to include the case of Coulomb interactions

Condition 1.2. *We assume $V_a = V_a^1 + V_a^2$, $a \neq a_{\min}$, is time-independent and V_a^1 satisfies Condition 1.1 ii) and iii) (or equivalently Condition 1.3 ii)). As for the V_a^2's we assume that they have compact support and for each a, there exist p, with $p \geq 2$ and $p > \dim(X^a)$, and $\kappa > 0$ such that $V_a^2, |\nabla V_a^2|^{\frac{1}{2}} \in L^p(X^a)$, and for $|\alpha| = 1$*

$$\left\| \partial^\alpha V_a^2(\cdot - y^a) - \partial^\alpha V_a^2(\cdot) \right\|_{L^{\frac{p}{2}}(X^a)} = O\left(|y^a|^\kappa\right), \text{ as } y^a \to 0.$$

Finally we consider the following class of time-independent potentials which includes atomic and molecular potentials, given by $v_{ij}(x) = c_{ij}|x|^{-1}$ in (1.1).

Condition 1.3. *We assume $V_a = V_a^1 + V_a^2$ is time-independent. For each $a \neq a_{\min}$:*

i) *If $\dim(X^a) < 3$ then $V_a^2 = 0$.*
ii) *$V_a^1 \in C^2(X^a)$, $|V_a^1(y)| + |y \cdot \nabla V_a^1(y)| = o(1)$ and $||y|^{|\alpha|} \partial^\alpha V_a^1(y)| = O(1)$, for $|\alpha| \leq 2$.*
iii) *$V_a^2 \in C^2(X^a \backslash \{0\})$ and has compact support. There exists $C > 0$ such that*

$$|\partial^\alpha V_a^2(y)| \leq \frac{C}{|y|^{|\alpha|+1}} \quad \text{for} \quad 0 \leq |\alpha| \leq 2.$$

We also consider a generalized field (instead of the particular field given by (1.5))

$$\mathcal{E} \in L^1_{\text{loc}}(\mathbb{R}; X), \qquad \mathcal{E}(t+1) = \mathcal{E}(t), \text{ a.e. and } \int_0^1 \mathcal{E}(t)dt = 0. \tag{1.8}$$

Some of our results for Coulomb interactions will require the stronger assumption

$$\mathcal{E} \in L^\infty(\mathbb{R}; X), \qquad \mathcal{E}(t+1) = \mathcal{E}(t), \text{ a.e. and } \int_0^1 \mathcal{E}(t)dt = 0. \tag{1.9}$$

Remark 1.4. We note that under Condition 1.1, 1.2 or 1.3, and (1.8), the time-dependent potential $(t,x) \to V_c(t,x) = V(t, x+2\int_0^t \int_0^s \mathcal{E}(r)drds)$ satisfies the assumptions of [Ya2]. This yields a solution to the time-dependent Schrödinger equation (1.2) for the operator family $p^2 + V_c(t,\cdot)$. Note that equation (1.2) should be considered in the strong sense, see [Mø, Appendix B]. The Avron-Herbst formula, which is a time-dependent change of coordinates (see [CFKS]) now yields the existence of a solution to (1.2) for the family $h(t)$. See [Mø, Proposition 4.3] for details. A similar argument gives solutions to (1.2) for the operator families $\tilde{h}(t)$ and $h_{\mathrm{CM}}(t)$. The solutions are unique within a class of absolutely continuous operator-valued functions, which implies (1.3), (1.4) and (1.6). (See also [CMR] for a simpler argument which works for more regular fields.)

For $a \in \mathcal{A}$, we have a natural splitting of the Hilbert space into a tensor product $L^2(X) = L^2(X_a) \otimes L^2(X^a)$. We introduce the Hamiltonians

$$h(t) = p^2 - \mathcal{E}(t)\cdot x + V \quad \text{and} \quad h_a(t) = p^2 - \mathcal{E}(t)\cdot x + V^a, \qquad V^a = \sum_{b\subset a} V_b.$$

We have a splitting of h_a with respect to the tensor structure of $L^2(X)$

$$h_a(t) = (p_a^2 - \mathcal{E}(t)_a \cdot x_a)\otimes I + I \otimes h^a(t), \qquad h^a(t) = (p^a)^2 - \mathcal{E}(t)^a \cdot x^a + V^a.$$

Note that $h_a(t)$ and $h^a(t)$ are also generalized (families of) Schrödinger operators.

The set of thresholds is

$$\mathcal{F}(U(1,0)) = \bigcup_{a\neq a_{\max}} e^{-i\alpha_a}\sigma_{\mathrm{pp}}(U^a(1,0)), \tag{1.10}$$

where $U^a(t,s)$ is the solution to (1.2) for the operator family $h^a(t)$ (see Remark 1.4) and $\sigma_{\mathrm{pp}}(\cdot)$ refers to pure point spectrum. We use the convention $\sigma_{\mathrm{pp}}(U^{a_{\min}}(1,0)) = \{1\}$. The numbers α_a will be introduced later, see (1.21). We just note here that $\alpha_a = 0$ if $\mathcal{E}_a = 0$.

Our main results for the monodromy operator are the following.

Theorem 1.5. *Suppose V satisfies either Condition 1.1, 1.2 or 1.3, and $\mathcal{E}$ is as in* (1.8). *The threshold set $\mathcal{F}(U(1,0))$ is closed and countable, and non-threshold eigenvalues $z \in \sigma_{\mathrm{pp}}(U(1,0))\backslash\mathcal{F}(U(1,0))$ have finite multiplicity and can only accumulate at the threshold set.*

It is not important for this result, in the case of Condition 1.3, that the singularity is located at the origin. In fact, the theorem remains true for potentials of the form $V(x) = V_1(x-z_1)+\cdots+V_k(x-z_k)$, $z_i \in X$, where the V_i's satisfy Condition 1.3. This remark extends the class of potentials to include the Born-Oppenheimer approximation for a molecular potential. Our next two results however, do require the singularity to be located at the origin.

Theorem 1.6. *Suppose one of the following two assumptions hold*

i) *V satisfies either Condition 1.1 or 1.2 and $\mathcal{E}$ is as in* (1.8).

ii) *V satisfies Condition 1.3 and $\mathcal{E}$ is as in* (1.9).

Then the singular continuous spectrum $\sigma_{\mathrm{sc}}(U(1,0)) = \emptyset$.

In the case of the two-body problem, the theorems above extend those of [Ya1] in two ways (see also [KuY] and [Yo]). The most significant is that we include the physical model of Hydrogen. The second extension is that we handle a more general class of electric fields. (Note that our Avron-Herbst type transformation, to be presented below, combines with [Ya1] to cover the larger class of fields.) The spectral theory for the two-body problem was also considered in [Ko2]. In the case of the many-body problem however, our results are new. Asymptotic completeness for the two-body problem was proved in [Ya1], by stationary methods, and for more general time-dependent problems in [KiY1], using the Enss method. In [N] Nakamura proved an asymptotic completeness result for the three-body problem using Faddeev's method. A similar result was proven in [Ko1]. Asymptotic completeness for N-body AC-Stark systems (and the more general models considered in this paper) remains an open problem.

As a consequence of our analysis we also get the following basic integral propagation estimate.

Theorem 1.7. *Assume V and $\mathcal{E}$ is as in Theorem 1.6. Let $s > \frac{1}{2} > r \geq 0$ and let g be a bounded Borel-measurable function on the unit-circle with support away from $\sigma_{\rm pp}(U(1,0)) \cup \mathcal{F}(U(1,0))$. Then there exists $C > 0$ such that*

$$\int_{-\infty}^{\infty} \|(1+p^2)^{r/2}(1+x^2)^{-s/2}U(t,0)g(U(1,0))\psi\|^2 dt \leq C\|\psi\|^2,$$

for all $\psi \in L^2(X)$.

As for decay of eigenfunctions we have the following theorem.

Theorem 1.8. *Assume V satisfies Condition 1.3 and $\mathcal{E}$ is as in (1.8). Let $e^{-iE} \in \sigma_{\rm pp}(U(1,0))$ and $\varphi \in L^2(X)$ satisfy $U(1,0)\varphi = e^{-iE}\varphi$. Then $\varphi \in \mathcal{D}(p)$ and if furthermore $e^{-iE} \notin \mathcal{F}(U(1,0))$ then for every $\sigma > 0$ with*

$$E + \sigma^2 < \inf\{\lambda > E : e^{-i\lambda} \in \mathcal{F}(U(1,0))\}$$

we have $e^{\sigma|x|}\varphi \in L^2(X)$.

Polynomial decay has previously been proven for the two-body problem in [Ko2] and [KuY]. (In [KuY] the estimate was for the associated Floquet Hamiltonian only, see below.) Our result on exponential decay seems to be the first of its kind for the AC-Stark model (or models with more general time-periodic potentials). An exponential decay estimate was proved in [Mø] and [A] for the case of time-periodic electric fields with nonzero mean (used to prove absence of eigenfunctions in that framework). Our proof of Theorem 1.8 also works for the molecular Born-Oppenheimer model and also with Condition 1.3 replaced by Condition 1.1 or 1.2.

Our last result is concerned with perturbation theory. We write

$$h_\kappa(t) = p^2 - (\mathcal{E}_0(t) + \kappa\mathcal{E}(t)) \cdot x + V, \tag{1.11}$$

where V is time-independent and κ is a small real perturbation parameter. Write $U_\kappa(1,0)$ for the corresponding monodromy operator.

We work here only under the physically most interesting assumption, Condition 1.3 and (1.9), although one could also consider the cases of Condition 1.1 and Condition 1.2. We study what happens to non-threshold eigenvalues of $U_0(1,0)$ when the perturbing field is turned on; i.e. for small nonzero κ. We refer the reader to Theorem 9.5 for the precise statement of the result. Although the field is classical, we use a standard interpretation of the model as that of an atom coupled to a reservoir of photons of energies $2\pi n$, $n \in \mathbb{N}$. The coupling strength of photons with energy $2\pi n$ is of the order $\frac{\kappa}{n}|\hat{\mathcal{E}}_n|$. Here $\hat{\mathcal{E}}_n$ is the n'th Fourier coefficient of $\mathcal{E}$ as a function on $[0,1]$. By reservoir we mean that the model does not keep track of the number of photons of a given energy, but rather assumes an unlimited supply. Our results can, in the case $\mathcal{E}_0 = 0$, be interpreted as an analysis of the effect of one-photon and some two-photon processes on the bound state energies of a molecular system. In Appendix A we develop an analytic perturbation theory in the case $\mathcal{E}_0 = 0$.

In the case of the two-body problem, with $\mathcal{E}(t) = \mathcal{E}_0 \cos(2\pi t)$ (the AC-Stark field) and a dilation analytic potential, the result stated in Theorem 9.5 is due to Yajima, see [Ya3]. Yajima furthermore defines resonances and compute them to any order in the perturbation parameter; see also [How2]. Note that for the AC-Stark field, only $\hat{\mathcal{E}}_{\pm 1}$ are nonzero. In the case of a circularly polarized field $\mathcal{E}(t) = (\cos(2\pi t), \sin(2\pi t), 0)$ and a dilation analytic two-body potential, the result is due to Tip, see [Ti]. (See also references in [Ti].)

For recent works on N-body systems in a quantized field we refer to [BFS] and [BFSS].

Floquet Theory:

We will now in two steps simplify the problem. The first step formulates the problem in terms of Floquet theory and the second step consists of a periodic Avron-Herbst type transformation which moves the time-dependence into the potential. (The two steps are interchangeable.)

The Floquet Hamiltonian associated with $h(t)$ is

$$H = \tau + h(t), \qquad \text{on } \mathcal{H} = L^2\left([0,1]; L^2(X)\right).$$

Here τ is the self-adjoint realization of $-i\frac{d}{dt}$, with periodic boundary conditions. It is well known that the spectral properties of the monodromy operator and the Floquet Hamiltonian are equivalent. We have the following relations

$$\sigma_{\mathrm{pp}}(U(1,0)) = e^{-i\sigma_{\mathrm{pp}}(H)}, \qquad \sigma_{\mathrm{ac}}(U(1,0)) = e^{-i\sigma_{\mathrm{ac}}(H)}, \qquad \sigma_{\mathrm{sc}}(U(1,0)) = e^{-i\sigma_{\mathrm{sc}}(H)} \tag{1.12}$$

and the multiplicity of an eigenvalue $z = e^{-i\lambda}$ of $U(1,0)$ is equal to the multiplicity of λ as an eigenvalue of H (regardless of the choice of λ). See [Ya1], [How1] and [Mø]. We note that the Floquet Hamiltonian is the self-adjoint generator of the strongly continuous unitary one-parameter group on $\mathcal{H}$ given by

$$(e^{-isH}\psi)(t) = U(t, t-s)\psi(t - s - [t-s]), \tag{1.13}$$

where $[r]$ is the integer part of r. This construction is based on (1.3) and (1.4). (In fact this is how one should interpret the 'sum' $\tau + h(t)$ if $\mathcal{E} \notin L^2_{\mathrm{loc}}(\mathbb{R}; X)$, see [Mø, Example 3.7].) The potentials considered in this paper are all $(\tau + p^2 - \mathcal{E} \cdot x)$-bounded with relative bound 0, see Theorem 6.2. This appears to be a new result in the case of Coulomb interactions. We note that the Coulomb potential $|x|^{-1}$ fails to be $(\tau + p^2)$-compact in $\nu \geq 3$ dimensions (cf. Section 6) and this singularity is therefore at the borderline for relative boundedness.

A time-dependent transformation:

The second simplification of the problem makes use of an idea of [Mø]. One can construct time-periodic coordinate changes $S^a(t)$, $a \neq a_{\min}$, such that the time-dependency in the new frame is in the potential only. This coordinate change should be compared with the Avron-Herbst formula, see [CFKS]. A closely related formula was introduced by H. A. Kramers in [Kr]. See also [A] for an application of the transformation of [Mø] to the N-body case with $\int_0^1 \mathcal{E}(t)dt \neq 0$.

For $a \in \mathcal{A}$ we introduce cluster Floquet Hamiltonians

$$H^a = \tau + h^a(t) \qquad \text{on } L^2([0,1]; L^2(X^a))$$

and the following functions

$$\begin{aligned} b(t) &= \int_0^t \mathcal{E}(s)ds - b_0, & b_0 &= \int_0^1 \int_0^t \mathcal{E}(s)dsdt \\ c(t) &= 2\int_0^t b(s)ds - c_0, & c_0 &= 2\int_0^1 \int_0^t b(s)dsdt \\ \alpha^a(t) &= \int_0^t |b(s)^a|^2 ds - t\alpha_0^a, & \alpha_0^a &= \int_0^1 |b(s)^a|^2 ds. \end{aligned} \tag{1.14}$$

The transformations are

$$S^a(t) = e^{-i\alpha^a(t)} \exp(ib(t)^a \cdot x^a) \exp(-ic(t)^a \cdot p^a) \tag{1.15}$$

and

$$S^{a*} H^a S^a = \tau + (p^a)^2 + V^a(t, x^a + c(t)^a) + \alpha_0^a. \tag{1.16}$$

Here $(S^a\psi)(t) = S^a(t)\psi(t)$. In terms of the physical problem this corresponds to the identities

$$S^a(t)^* U^a(t,s) S^a(s) = e^{i(s-t)\alpha_0^a} U_c^a(t,s), \tag{1.17}$$

where U_c^a solves (1.2) for the operator family $(p^a)^2 + V^a(t, x^a + c(t)^a)$.

The equation (1.16) and the periodicity of the transformations ($S^a(1) = S^a(0)$) show that, up to a translation/rotation, the spectral structure of the two systems are equivalent. We note that in the case of a circularly polarized field (in 3 dimensions), one can bring the Hamiltonian onto the time-independent form $(p-a)^2 + V - 2\pi L_3$, where $a = (a_1, 0, 0)$ is a constant vector and $L_3 = x_1p_2 - x_2p_1$, the third component of the angular momentum observable. This observation was used in [Ti]. In the following we will, in the light of (1.16), assume $\mathcal{E} = 0$ and instead work with operators of the form

$$H = \tau + h(t), \quad \text{where} \quad h(t) = p^2 + V, \tag{1.18}$$

and the pair potentials $(t, x^a) \to V_a(t, x^a)$ are time-periodic. Note that if $\mathcal{E}$ is as in (1.8) and V satisfies Condition 1.1, then so does the potential $(t, x) \to V(t, x + c(t))$.

We now proceed to discuss the structure of the transformed Floquet Hamiltonian H. For $a \in \mathcal{A}$, we have a natural splitting of the Hilbert space $\mathcal{H}$ into a tensor product

$$\mathcal{H} = L^2(X_a) \otimes \mathcal{H}^a, \quad \text{where} \quad \mathcal{H}^a = L^2\left([0,1]; L^2(X^a)\right).$$

In particular $\mathcal{H}^{a_{\min}} = L^2([0,1]; \mathbb{C})$. We introduce Hamiltonians respecting the tensor structure of $\mathcal{H}$

$$H_a = p_a^2 \otimes I + I \otimes H^a.$$

Note that H^a is the Floquet Hamiltonian for the generalized Schrödinger operator $h^a(t)$. In particular $H^{a_{\min}} = \tau$.

We note that our perturbation problem (1.11), in the framework of Floquet theory and in the new set of coordinates, takes the form

$$H_\kappa = H_0 + W_\kappa, \qquad H_0 = \tau + p^2 + V(\cdot + c_0) \quad \text{and} \quad W_\kappa(t,x) = V(x + c_\kappa(t)) - V(x + c_0(t)), \tag{1.19}$$

where $c_\kappa = c_0 + \kappa c$ and the functions c_0 and c are given by $\mathcal{E}_0$ and $\mathcal{E}$ respectively.

We introduce thresholds for the Floquet Hamiltonian (without electric field)

$$\mathcal{F}(H) = \bigcup_{a \neq a_{\max}} \sigma_{\rm pp}(H^a),$$

which is a 2π-periodic set containing $\sigma_{\rm pp}(H^{a_{\min}}) = 2\pi\mathbb{Z}$.

In the case of time-independent pair-potentials we have the well-known threshold set $\mathcal{F}(h) = \cup_{a \neq a_{\max}} \sigma_{\rm pp}(h^a)$, $\sigma_{\rm pp}(h^{a_{\min}}) := \{0\}$. In this particular case there is the following relation.

$$\mathcal{F}(H) = \mathcal{F}(h) + 2\pi\mathbb{Z}. \tag{1.20}$$

The constants appearing in (1.10) are

$$\alpha_a = \alpha_0^{a_{\max}} - \alpha_0^a = \int_0^1 |b(s)_a|^2 ds. \tag{1.21}$$

With this definition it follows from (1.12) and (1.16) that Theorems 1.5, 1.6 and 9.5 are consequences of the corresponding results phrased with $U(1,0)$ replaced by H.

We note that considering periods $T > 0$ other than 1, amounts to replacing τ by $T^{-1}\tau$. The effect of this is that the spectrum of H becomes $2\pi/T$ periodic. In view of perturbation theory, Theorem 9.5, this has two consequences. One can tune the period of the field such as to trigger oscillations between two or more bound states (with possible ionization coming in at lower order in κ, i.e. at longer time scales). Secondly, by choosing the period small enough, we find that Fermi's Golden Rule governs the ionization process to the order κ^2, i.e. at the leading ionization time-scale. See the discussion at the end of Section 9 and [Ya3].

Ideas and techniques:

Our approach to the study of the spectrum of H is a Mourre theoretical analysis of the Floquet Hamiltonian H. It seems reasonable to expect that one can use, for example, the generator of dilations $A = \frac{1}{2}(x \cdot p + p \cdot x)$, as a conjugate operator for H. This is due to the similarity of the geometric structure of the Floquet problem and the usual N-body problem. There is however a central technical obstruction, which appears already in the two-body case. Here we have

$$i[H, A] = 2p^2 - x \cdot \nabla V.$$

It is easy to see that this is indeed positive, modulo a compact error, when localized away from the threshold set $2\pi\mathbb{Z}$. (Note that $F(|x| < R)$ is H-compact, see [Ya1].) The right-hand side is however not H-bounded, nor is the second commutator bounded as a form on $\mathcal{D}(H)$. This problem was overcome for the two-body problem by Yokoyama in [Yo], who considered the following modification of A

$$\tilde{A} = \frac{1}{2}\left(x \cdot \frac{p}{1+p^2} + \frac{p}{1+p^2} \cdot x\right).$$

The problems now disappear and one can use standard Mourre theory. (Following for example [Mo], [PSS] or [ABG1].)

In the existing proofs of the Mourre estimate for a time-independent N-body operator, one uses a reduction to subsystems argument, where the property that

$$A = A^a + A_a, \qquad A^a = \frac{1}{2}(x^a \cdot p^a + p^a \cdot x^a) \quad \text{and} \quad A_a = \frac{1}{2}(x_a \cdot p_a + p_a \cdot x_a),$$

is important. See [PSS], [FH] and [Hu1]. A modification like the one considered by Yokoyama would here have to be introduced in a much more subtle way, in order to preserve some kind of reduction argument. We have not been able to follow this approach.

Instead we turn to an idea of [Sk], where the form boundedness, with respect to H, of the second commutator was replaced by boundedness with respect to the H-unbounded part, M, of the first commutator. We use a splitting

$$i[H, A] = M + G,$$

where G is H-bounded and $M > \delta > 0$ is not H-bounded. The crucial properties used in [Sk] was, roughly speaking, that

$$i[H, M] \text{ is } H\text{-bounded} \quad \text{and} \quad i[M, A] \text{ is } M\text{-bounded}. \tag{1.22}$$

The following Mourre type commutator estimate was considered in [Sk]

$$M + f(H)Gf(H) \geq \gamma f(H)^2 - K - C(1 - f(H))^2, \tag{1.23}$$

for some $\gamma > 0$, $C > 0$ and compact K, provided f is supported nearby some fixed energy. The motivation for using this (weaker looking) estimate instead of the standard Mourre estimate, $f(H)i[H, A]f(H) \geq \gamma f(H)^2 - K$, lies in the use of Mourre's differential inequality technique. One is lead to consider the resolvent family

$$R_z(\epsilon) = (H - i\epsilon(M + f(H)Gf(H)) - z)^{-1} \tag{1.24}$$

instead of the usual resolvent family $(H - i\epsilon f(H)i[H, A]f(H) - z)^{-1}$. One should notice that $R_z(\epsilon)$ has improved properties because M appears without energy localization.

Our case does not fit into the models covered by [Sk]. In the case of Condition 1.1 we want to use A, the generator of dilation, as conjugate operator and we take $M = 2p^2 + \delta$, $\delta > 0$. We note that $i[M, A]$ is M-bounded as required in [Sk] but $i[H, M]$ is not H-bounded. It is however $M^{\frac{1}{2}}$-bounded and this may be used to make sense of the resolvent (1.24) (notice at this point the symmetry between H and M). This observation would enable us to prove a Limiting Absorption Principle for H, using ideas similar to the ones employed in [Sk]. However, in order to encompass local singularities, we develop a more refined theory involving a weaker condition on the commutator $i[H, M]$ than the ones discussed above. In the case of Condition 1.3 we will have to replace the generator of dilation with the operator $A_1 = 1/2((x + c) \cdot p + p \cdot (x + c))$, where $t \to c(t)$ is the function appearing in the transformation (1.15). This is necessary and convenient in order to handle second commutators. Nevertheless, since the Coulomb singularity is the border-line for H_0-boundedness (in $\mathbb{R}^\nu$, $\nu \geq 3$), one still needs some weaker assumption. See Assumption 2.1 for the precise formulation.

Our Limiting Absorption Principle (LAP) is of the form

$$\sup_{\operatorname{Im} z \neq 0, \operatorname{Re} z \in \mathcal{V}} \|\langle A\rangle^{-\alpha} M^\beta (H - z)^{-1} M^\beta \langle A\rangle^{-\alpha}\| < \infty, \tag{1.25}$$

where $\mathcal{V} \subset \mathbb{R}$ is an open neighbourhood, not containing thresholds nor eigenvalues, $\alpha > \frac{1}{2}$ and $\beta < \frac{1}{2}$. The fact that we can handle the extra weight M^β comes from our choice of commutator estimate (1.23), see also (1.24). As a consequence we furthermore get a LAP with $\langle A\rangle^{-\alpha} M^\beta$ replaced by $\langle x\rangle^{-s}\langle p\rangle^r$, $s > \frac{1}{2}$ and $r < \frac{1}{2}$. (The presence of the M^β factors, with $\beta > \frac{1}{4}$, is crucial for obtaining a LAP with x-weights.) One should note that the well-definedness of the expression above is not immediately clear, even for fixed z, $\operatorname{Im} z \neq 0$, since $M^{\frac{1}{2}}$ is not H-bounded. This type of LAP was also considered in [Hos], for some 'unperturbed' problems and in [KuY] for two-body time-periodic systems. The paper by [KuY] uses a method of Hörmander and is in the framework of Besov spaces. In both of these papers the LAP, with x-weights, is proved with the critical exponent $r = \frac{1}{2}$. Our use of Mourre's differential inequality technique does not allow us to include the critical cases $\beta = r = \frac{1}{2}$.

We note that an extension in the spirit of [ABG1] of the abstract approach to the LAP as formulated in [Sk] is being developed in [GGM]. This extension does not cover the results obtained here.

It is well-known that a LAP implies absence of singular continuous spectrum, see [RS, Section XIII.7]. In particular Theorem 1.6 with $U(1, 0)$ replaced by H follows from (1.25).

As a consequence of the LAP with x-weights we get Theorem 1.7, cf. [KiY2]. We note that in order to replace the A-weights with x-weights we use that $\langle A\rangle^s\langle p\rangle^{-s}\langle x\rangle^{-s}$ is bounded.

We furthermore show that the limit of the resolvents $\langle A\rangle^{-\alpha} M^\beta (H - E \mp i0)^{-1} M^\beta \langle A\rangle^{-\alpha}$ exist and are Hölder continuous in E; cf. Proposition 9.1 (for the case of x-weights see the remark after Proposition 9.1). This will be used in the context of second order perturbation theory.

Our proof of the commutator estimate (1.23) follows a proof given by Hunziker in [Hu1], for the usual N-body problem. In order to do the threshold analysis of [Hu1], we show that eigenfunctions are a priori in the domain of the momentum operator p. In our context this is a non-trivial statement; notice that $\mathcal{D}(p) \not\subset \mathcal{D}(H)$. The fact that the Hamiltonian is bounded from below is used in [Hu1] at a critical point of the reduction argument. Our Hamiltonian is not bounded from below but instead we utilize that its spectrum is 2π-periodic. This observation also plays an important role for an argument given in [KuY] and [A]. We finally note that replacing energy localizations $f(H)$ with $f(H_a)$, up to a compact error, is not as easy here as in the time-independent problem (if at all possible). Instead we take advantage of the robustness of the commutator estimate (1.23); it allows for a non-compact error of the form $-\sigma p^2$, with $\sigma > 0$ small. This will also be important for our threshold analysis and for treating the Coulomb singularity. As in Hunziker's paper, Theorem 1.5 with $U(1,0)$ replaced by H is verified simultaneously with the commutator estimate (1.23).

We employ a Froese-Herbst type argument to show exponential decay of non-threshold eigenfunctions for the Floquet operator, cf. [FH2], [CFKS] and [DG]. We furthermore argue that first order derivatives of these eigenfunctions also decay exponentially, which is used to obtain exponential decay of the corresponding non-threshold eigenfunctions for $U(1,0)$.

As for our result on perturbation theory, we follow an approach used in the paper [AHS] to study embedded (non-threshold) eigenvalues of N-body Schrödinger operators. Note that $\sigma(H) = \mathbb{R}$, so in our problem all eigenvalues are embedded. In particular we verify Fermi's Golden Rule. In Appendix A we develop an analytic perturbation theory which is valid for weak fields.

The paper is organized as follows. In Section 2 we present the abstract positive commutator method and prove the Limiting Absorption Principle (1.25). In Section 3 we verify that our example, under Condition 1.1, satisfies the technical assumptions used in Section 2 and in Section 4 we prove the commutator estimate (1.23). In Section 5 we discuss how to extend the results of the previous sections to include potentials with L^p singularities, see Condition 1.2. In Section 6 we extend the results of the previous sections to potentials satisfying Condition 1.3. In Section 7 we derive the Limiting Absorption Principle with x-weights and prove an integral propagation estimate for the physical system, cf. Theorem 1.7. In Section 8 we prove exponential decay of eigenfunctions and in Section 9 we study the perturbation problem (1.19). Finally in Appendix A an analytic perturbation theory is considered.

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2. Abstract theory

For a self-adjoint operator T on a Hilbert space we denote by $\mathcal{D}(T)$ and $\rho(T)$ its domain and resolvent set, respectively. We use the notation $E_\Omega(T)$ for the spectral projection corresponding to any given Borel set Ω. For $\phi \in \mathcal{D}(T)$ let $\langle T\rangle_\phi = \langle \phi, T\phi\rangle$. Let $\mathcal{F}^0$ denote the set of smooth real-valued functions $\chi(t)$ on $\mathbb{R}$ with the property

$$\left|\frac{d^k}{dt^k}\chi(t)\right| \leq C_k(1+|t|)^{-k}, \qquad k \in \mathbb{N}\cup\{0\}.$$

Assumption 2.1. *Let $H, H_0, M, A, A_n, n \in \mathbb{N}$ be self-adjoint operators on a Hilbert space $\mathcal{H}$ with $M \geq \delta I$ for some positive number δ. Suppose $\mathcal{D}(H) = \mathcal{D}(H_0)$ and $V = H - H_0$ is ϵ-bounded relatively to H. Suppose that for some core $\mathcal{C}$ of A with $\mathcal{C} \subseteq \mathcal{D}(A_n)$ the identity $\lim_{n\to\infty} A_n\phi = A\phi$ holds for all $\phi \in \mathcal{C}$. Suppose $\mathcal{D}(M^{\frac{1}{2}}) \subseteq \mathcal{D}(A_n)$ and:*

1) *(Compatibility) The set $\mathcal{D} = \mathcal{D}(H_0) \cap \mathcal{D}(M)$ is dense in $\mathcal{D}(H_0)$ as well as in $\mathcal{D}(M)$. The form $i[M, H_0]$ defined on $\mathcal{D}$ extends to an M-bounded operator, and $\mathcal{D}(H_0)$ is preserved by M^{-1}. Moreover for all $\chi \in \mathcal{F}^0$ the form $i[M, \chi(H)]$ defined on $\mathcal{D}$ may be identified as a sum of operators $i[M, \chi(H)] = T_1 + T_2$, where T_1 is $M^{\frac{1}{2}}$-bounded and T_2 is H-bounded.*
2) *(First commutators) Let for all $n \in \mathbb{N}$ the form $i[H, A_n]$ defined on $\mathcal{D}^{\frac{1}{2}} = \mathcal{D}(H) \cap \mathcal{D}(M^{\frac{1}{2}})$ be denoted by H_n. There exists an H-bounded (symmetric) form G such that for all $\phi_1, \phi_2 \in \mathcal{D}^{\frac{1}{2}}$*
$$\lim_{n\to\infty}\langle \phi_1, H_n\phi_2\rangle = \langle \phi_1, (M+G)\phi_2\rangle.$$
3) *(Second commutators) For all $n \in \mathbb{N}$ the form $i[M, A_n]$ defined on $\mathcal{D}(M)$ extends to an M-bounded operator M_n. It holds that*
$$\sup_n \left\|M_n M^{-1}\right\| =: C_M < \infty.$$
For all real-valued $f \in C_0^\infty(\mathbb{R})$
$$\sup_n \left\|M^{-\frac{1}{2}} i[f(H)Gf(H), A_n] M^{-\frac{1}{2}}\right\| < \infty.$$
4) *(Positivity at E) For a given $E \in \mathbb{R}$ there exist $\gamma > 0$, an open neighbourhood $\mathcal{U}$ of E and a compact operator K such that for all real-valued $f \in C_0^\infty(\mathcal{U})$ the form inequality*
$$M + f(H)Gf(H) \geq \gamma I - f(H)Kf(H) - (I - f(H))L(I - f(H))$$
holds on $\mathcal{D}$ for some symmetric and H-bounded form $L = L(f)$.

Remarks 2.2.

1) The condition of Assumption 2.1 1) that $\mathcal{D}(H_0)$ is preserved by M^{-1} is an alternative to the condition of [Mo], that
$$\sup_{|\sigma|<1}\left\|Me^{i\sigma H_0}M^{-1}\right\| < \infty.$$
Given the condition on the form $i[M, H_0]$ stated in Assumption 2.1 1) those conditions are equivalent, cf. [GG, Lemma 2].
2) The condition $\mathcal{D}(M^{\frac{1}{2}}) \subseteq \mathcal{D}(A_n)$ is convenient for the examples discussed in this paper particularly in the context of inclusion of local singularities, however, there are different but somewhat similar extensions of [Mo] which do not have this requirement. Notice for

example that it does not appear in [Sk] (for good reasons). The requirement that A is self-adjoint was relaxed in [Sk], but this is not needed for the examples of this paper. On the other hand as mentioned in Section 1 the explicit H-boundedness assumption of $i[M,H]$ appearing in [Sk] does not comply with our examples. The stated somewhat weaker condition of Assumption 2.1 1) (involving the commutator with $\chi(H)$) will be a useful substitute in the present context.

3) By repeated use of the form inequality

$$2\,\mathrm{Re}\{D_1^* B D_2\} \le \epsilon D_1^* D_1 + \epsilon^{-1}\|B\|^2 D_2^* D_2, \qquad \epsilon > 0, \tag{2.1}$$

one verifies readily that the following two statements (with the inequality meant to hold on $\mathcal{D}$ or equivalently on $\mathcal{D}^{\frac{1}{2}}$) and Assumption 2.1 4) are equivalent (for fixed given $E \in \mathbb{R}$).

Assumption 2.1 4)'

$$\exists \gamma > 0 \exists \textit{ open nbh. } \mathcal{U}_E \exists \textit{ compact operator } K:$$

$$M + G \ge \gamma I - K - E^c L E^c;$$

$$E^c = E_{\mathbb{R}\setminus \mathcal{U}_E}(H), L \textit{ is a symmetric and } H\textit{-bnd. form.}$$

Assumption 2.1 4)"

$$\exists \gamma > 0 \exists f_E \in C_0^\infty(\mathbb{R}), f_E = 1 \textit{ on a nbh. of } E \textit{ and real-valued:}$$

$$M + f_E(H) G f_E(H) \ge \gamma f_E(H)^2 - K - (I - f_E(H)) L (I - f_E(H));$$

$$K \textit{ is a compact operator, } L \textit{ is a symmetric and } H\textit{-bnd. form.}$$

Similarly, if also M is an H-bounded form then Assumption 2.1 4) reduces to the standard Mourre estimate of [Mo] (by putting $H = H_0, M = I, A_n = n^2 A (n^2 + A^2)^{-1}$ and $\mathcal{C} = \mathcal{D}(A)$); consequently indeed the theory of this section is an extension of the one of [Mo].

Lemma 2.3 (Virial type theorem). *Suppose Assumption 2.1 2) and 4). Suppose* $\{\phi_m\} \subseteq \mathcal{D}(H)$ *is a sequence of eigenstates,* $(H - E_m)\phi_m = 0, \|\phi_m\| = 1,$ *such that* $\phi_m \to 0$ *weakly and* $E_m \to E$. *Then there exists* $m_0 \in \mathbb{N}$ *such that* $\phi_m \notin \mathcal{D}(M^{\frac{1}{2}})$ *for* $m \ge m_0$.

Proof. We pick $f \in C_0^\infty(\mathcal{U})$ equal to one on a neighbourhood of E and compute for $\phi_m \in \mathcal{D}(M^{\frac{1}{2}})$ the expectation

$$\langle M + G \rangle_{\phi_m} = \lim_{n\to\infty} \langle H_n \rangle_{\phi_m} = 0.$$

On the other hand if m is large enough

$$\langle M + G \rangle_{\phi_m} \ge \gamma - \langle K \rangle_{\phi_m} \ge \frac{\gamma}{2}$$

yielding a contradiction. □

We define and use throughout this section the notation

$$\langle T \rangle = (\lambda^2 + T^2)^{\frac{1}{2}}, \qquad \lambda \ge 2C_M + 1 \text{ fixed}, \tag{2.2}$$

where the constant C_M is given in Assumption 2.1 3).

Our main result is

Theorem 2.4 (LAP). *Suppose Assumption 2.1 with E not being an eigenvalue of H. Then there exists a neighbourhood $\mathcal{V}$ of E such that for $0 \le \beta < \frac{1}{2} < \alpha \le 1$*

$$\sup_{\operatorname{Im} z \neq 0, \operatorname{Re} z \in \mathcal{V}} \left\| \langle A\rangle^{-\alpha} M^{\beta} (H-z)^{-1} M^{\beta} \langle A\rangle^{-\alpha} \right\| < \infty. \tag{2.3}$$

We remark that the correct interpretation of (2.3) is in terms of extension by continuity as a form from $\mathcal{D}(M^{\beta})$. This extension makes sense by the following result.

Lemma 2.5. *Suppose Assumption 2.1 3) and $0 < \beta, \alpha \le 1$. Then*

$$M^{\beta} \langle A\rangle^{-\alpha} M^{-\beta} \textit{ is bounded.}$$

Proof. By interpolation we may assume that $\beta = 1$. We notice that

$$A_n \to A \text{ in the strong resolvent sense.} \tag{2.4}$$

To show this we let z with $\operatorname{Im} z \neq 0$ and $\phi \in \mathcal{C}_1 := (A-z)\mathcal{C}$ be given. Then

$$(A_n - z)^{-1}\phi - (A-z)^{-1}\phi = (A_n - z)^{-1}(A - A_n)\psi, \quad \text{where} \quad \psi = (A-z)^{-1}\phi,$$

and since $\psi \in \mathcal{C}$ the right hand side converges to zero in norm. Since $\mathcal{C}_1$ is dense in $\mathcal{H}$ we conclude (2.4).

As a consequence of (2.4) it suffices to show that

$$\sup_n \left\| \left[M, \langle A_n\rangle^{-\alpha}\right] M^{-1} \right\| < \infty.$$

For that we represent

$$\begin{aligned} \langle A_n\rangle^{-\alpha} &= c\int_0^{\infty} t^{-\frac{\alpha}{2}} (A_n^2 + \lambda^2 + t)^{-1} dt \\ &= c\int_0^{\infty} \frac{t^{-\frac{\alpha}{2}}}{(\lambda^2+t)^{\frac{1}{2}}} \operatorname{Im}\left(A_n - i(\lambda^2+t)^{\frac{1}{2}}\right)^{-1} dt, \qquad c = c\left(-\frac{\alpha}{2}\right), \end{aligned} \tag{2.5}$$

yielding by Assumption 2.1 3) and the proof of [Mo, Proposition II.3]

$$\begin{aligned} &\left[M, \langle A_n\rangle^{-\alpha}\right] M^{-1} = S_1 + S_2; \\ &S_1 = \frac{c}{2}\int_0^{\infty} \frac{t^{-\frac{\alpha}{2}}}{(\lambda^2+t)^{\frac{1}{2}}} \left(A_n - i(\lambda^2+t)^{\frac{1}{2}}\right)^{-1} M_n \left(A_n - i(\lambda^2+t)^{\frac{1}{2}}\right)^{-1} M^{-1} dt, \\ &S_2 = \frac{c}{2}\int_0^{\infty} \frac{-t^{-\frac{\alpha}{2}}}{(\lambda^2+t)^{\frac{1}{2}}} \left(A_n + i(\lambda^2+t)^{\frac{1}{2}}\right)^{-1} M_n \left(A_n + i(\lambda^2+t)^{\frac{1}{2}}\right)^{-1} M^{-1} dt. \end{aligned}$$

By the same ingredients

$$\left\| M\left(A_n - (+) i(\lambda^2+t)^{\frac{1}{2}}\right)^{-1} M^{-1} \right\| \le 2(\lambda^2+t)^{-\frac{1}{2}}. \tag{2.6}$$

We insert $I = M^{-1}M$ in front of the last resolvent of the integrands and estimate yielding in conjunction with (2.6) the bound

$$2C_M t^{-\frac{\alpha}{2}} (\lambda^2+t)^{-\frac{3}{2}} \tag{2.7}$$

for both integrands. Clearly the integral of this bound is of the form $C\lambda^{-1-\alpha}$, in particular finite and independent of n. □

Remarks.

(1) It is important for the uniform estimate (2.6) that the number $(\lambda^2 + t)^{\frac{1}{2}}$ is large (here accomplished by the requirement (2.2)). For example, there does not exist a uniform bound in the plus case for the example of Section 3 if $(\lambda^2 + t)^{\frac{1}{2}} < 2$. (Closely related to this point we notice that the identity [PSS, (6.5)] is wrong.)

(2) Our proof of (2.3) exhibits a Hölder continuity property (for the operator that is uniformly bounded) which will be useful in the context of perturbation theory in Section 9. We refer to Proposition 9.1.

In order to prove Theorem 2.4 we need various preliminary results.

Lemma 2.6. *Suppose Assumption 2.1 1). Consider, for $\epsilon \in \mathbb{R}\backslash\{0\}$, $H(\epsilon) = H - i\epsilon M$ on the domain $\mathcal{D} = \mathcal{D}(H) \cap \mathcal{D}(M)$. The operator is closed and the adjoint is given by*

$$H(\epsilon)^* = H(-\epsilon).$$

In particular $z \in \rho(H(\epsilon))$ for either $\operatorname{Im} z$ and ϵ both positive or both negative. Moreover in these cases the resolvent $R_z^0(\epsilon) = (H(\epsilon) - z)^{-1}$ obeys the bounds

$$\left\| R_z^0(\epsilon) \right\| \le |\operatorname{Im} z + \delta\epsilon|^{-1}, \tag{2.8}$$

$$\left\| M^{\frac{1}{2}} R_z^0(\epsilon) \right\| \le |\epsilon|^{-\frac{1}{2}} |\operatorname{Im} z + \delta\epsilon|^{-\frac{1}{2}}. \tag{2.9}$$

(Here δ refers to the delta of Assumption 2.1.)

Proof. By [RS, Theorem X.50] it suffices except for the statement (2.9) to know the lemma with H replaced by H_0. We refer to the proof of [Sk, Lemma 2.6] which readily may be modified under the present conditions. (Notice that the direct analogue of [Sk, Lemma 2.6] would require that $i[M, H_0]$ is H_0-bounded and not M-bounded.) As for (2.9), the estimate follows from (2.8) by squaring and using the identity

$$R_z^0(\epsilon)^*(2i\epsilon M + z - \bar{z}) R_z^0(\epsilon) = R_z^0(\epsilon) - R_z^0(\epsilon)^*. \tag{2.10}$$

□

The following result is a modified version of [Sk, Lemma 2.7]. The estimate is weaker than the one of [Sk] which reflects the weaker input of Assumption 2.1 1), cf. Remark 2.2 2).

Lemma 2.7. *Suppose the assumption of Lemma 2.6 and that $f \in C_0^\infty(\mathbb{R})$ is equal to one on a neighbourhood of a real number E. Then there exist constants $C, \epsilon_0 > 0$ and a neighbourhood $\mathcal{V}$ of E such that*

$$\left\| \langle H \rangle^{\frac{1}{2}} (I - f(H)) R_z^0(\epsilon) \langle H \rangle^{\frac{1}{2}} \right\| \le C$$

provided $|\epsilon| \le \epsilon_0, \epsilon \operatorname{Im} z > 0$ and $\operatorname{Re} z \in \mathcal{V}$.

Proof. Let $\mathcal{V}_r = (E - r, E + r)$ for $r > 0$. Pick $r > 0$ such that f is one on $\mathcal{V}_{2r}$ and define $\mathcal{V} = \mathcal{V}_r$. Decompose $I = \chi_- + \chi_0 + \chi_+$ in terms of non-negative functions in $\mathcal{F}^0$ with the properties: $\chi_0 \in C_0^\infty(\mathcal{V}_{2r})$, χ_- supported in $(-\infty, E - r)$, and χ_+ supported in $(E + r, \infty)$.

For any $\phi \in \mathcal{D}(\langle H \rangle^{\frac{1}{2}})$ let $\psi = R_z^0(\epsilon) \langle H \rangle^{\frac{1}{2}} \phi$. We define for $\epsilon \ne 0$, $\epsilon \operatorname{Im} z > 0$ and $\operatorname{Re} z \in \mathcal{V}$

$$g_z(\epsilon) = \left\| |H - \operatorname{Re} z|^{\frac{1}{2}} \chi_-(H) \psi \right\|^2 + \left\| |H - \operatorname{Re} z|^{\frac{1}{2}} \chi_+(H) \psi \right\|^2$$

and compute

$$g_z(\epsilon) = \mathrm{Re}\langle\psi, \chi_-^2\{(H - i\epsilon M - z) + i\epsilon M\}\psi\rangle - \mathrm{Re}\langle\psi, \chi_+^2\{(H - i\epsilon M - z) + i\epsilon M\}\psi\rangle.$$

In accordance with Assumption 2.1 1) there are bounded operators B_1^- and B_2^- corresponding to $\chi_- \in \mathcal{F}^0$ given by

$$B_1^- = T_1^- M^{-\frac{1}{2}} \quad \text{and} \quad B_2^- = \langle H\rangle^{-\frac{1}{2}} T_2^- \langle H\rangle^{-\frac{1}{2}}, \quad \text{where} \quad i[M, \chi_-] = T_1^- + T_2^-.$$

We introduce operators B_1^+ and B_2^+ in a similar fashion by replacing χ_- by χ_+.

Then estimating by the Cauchy Schwarz inequality yields for any $\kappa > 0$

$$\begin{aligned} g_z(\epsilon) \leq & \left\|\langle H\rangle^{\frac{1}{2}}\chi_-\psi\right\| \|\chi_-\phi\| + \left\|\langle H\rangle^{\frac{1}{2}}\chi_+\psi\right\| \|\chi_+\phi\| \\ & + \left|\left\langle i\epsilon[M, \chi_-^2]\right\rangle_\psi\right| + \left|\left\langle i\epsilon[M, \chi_+^2]\right\rangle_\psi\right| \\ \leq & \kappa\left(\left\|\langle H\rangle^{\frac{1}{2}}\chi_-\psi\right\|^2 + \left\|\langle H\rangle^{\frac{1}{2}}\chi_+\psi\right\|\right) + \kappa^{-1}\|\phi\|^2 \\ & + \|\epsilon B_1^-\| \|\chi_-\psi\| \left\|M^{\frac{1}{2}}\psi\right\| + \|\epsilon B_2^-\| \left\|\langle H\rangle^{\frac{1}{2}}\chi_-\psi\right\| \left\|\langle H\rangle^{\frac{1}{2}}\psi\right\| \\ & + \|\epsilon B_1^+\| \|\chi_+\psi\| \left\|M^{\frac{1}{2}}\psi\right\| + \|\epsilon B_2^+\| \left\|\langle H\rangle^{\frac{1}{2}}\chi_+\psi\right\| \left\|\langle H\rangle^{\frac{1}{2}}\psi\right\| \\ \leq & 2\kappa\left(\left\|\langle H\rangle^{\frac{1}{2}}\chi_-\psi\right\|^2 + \left\|\langle H\rangle^{\frac{1}{2}}\chi_+\psi\right\|^2\right) + \kappa^{-1}\|\phi\|^2 \\ & + C\kappa^{-1}\left\|\epsilon M^{\frac{1}{2}}\psi\right\|^2 + C'\kappa^{-1}\left\|\epsilon\langle H\rangle^{\frac{1}{2}}\psi\right\|^2. \end{aligned} \tag{2.11}$$

We use (2.10) to obtain

$$\left\|\epsilon M^{\frac{1}{2}}\psi\right\|^2 \leq 2\left|\epsilon\,\mathrm{Im}\left\langle\langle H\rangle^{\frac{1}{2}}\phi, R_z^0(\epsilon)\langle H\rangle^{\frac{1}{2}}\phi\right\rangle\right| \leq 2\|\phi\| \left\|\epsilon\langle H\rangle^{\frac{1}{2}}\psi\right\| \leq \|\phi\|^2 + \left\|\epsilon\langle H\rangle^{\frac{1}{2}}\psi\right\|^2.$$

This bound is inserted into the right hand side of (2.11). Next we use the estimate

$$\left\|\langle H\rangle^{\frac{1}{2}}\psi\right\| \leq \left\|\langle H\rangle^{\frac{1}{2}}\chi_-\psi\right\| + \left\|\langle H\rangle^{\frac{1}{2}}\chi_0\psi\right\| + \left\|\langle H\rangle^{\frac{1}{2}}\chi_-\psi\right\|$$

to get

$$\left\|\epsilon\langle H\rangle^{\frac{1}{2}}\psi\right\|^2 \leq 3\left(\left\|\epsilon\langle H\rangle^{\frac{1}{2}}\chi_-\psi\right\|^2 + \left\|\epsilon\langle H\rangle^{\frac{1}{2}}\chi_0\psi\right\|^2 + \left\|\epsilon\langle H\rangle^{\frac{1}{2}}\chi_-\psi\right\|^2\right). \tag{2.12}$$

Finally, we conclude from (2.12) and the previous estimates that

$$\begin{aligned} g_z(\epsilon) \leq & \left(2\kappa + 3\epsilon^2\kappa^{-1}(C + C')\right)\left(\left\|\langle H\rangle^{\frac{1}{2}}\chi_-\psi\right\|^2 + \left\|\langle H\rangle^{\frac{1}{2}}\chi_+\psi\right\|^2\right) \\ & + \kappa^{-1}(1 + C)\|\phi\|^2 + 3\kappa^{-1}(C + C')\left\|\epsilon\langle H\rangle^{\frac{1}{2}}\chi_0\psi\right\|^2. \end{aligned} \tag{2.13}$$

Obviously we can estimate

$$\left\|\langle H\rangle^{\frac{1}{2}}\chi_-\psi\right\|^2 + \left\|\langle H\rangle^{\frac{1}{2}}\chi_+\psi\right\|^2 \leq C'' g_z(\epsilon)$$

uniformly in $\operatorname{Re} z \in \mathcal{V}$.

Consequently we obtain from (2.13) by a subtraction and by choosing and henceforth fixing $\kappa > 0$ small that there exists $C > 0$ such that for all ϵ with $|\epsilon|$ sufficiently small and for all z with $\operatorname{Re} z \in \mathcal{V}$

$$g_z(\epsilon) \leq C\left(\|\phi\|^2 + \|\epsilon\psi\|^2\right). \tag{2.14}$$

Now suppose we can show that

$$\|\epsilon\psi\| \leq C'\|\phi\|. \tag{2.15}$$

Then by (2.14) the lemma follows since $\mathcal{D}(\langle H\rangle^{\frac{1}{2}})$ is dense in $\mathcal{H}$.

To show (2.15) we repeat the above arguments with ψ replaced by $\psi = R_z^0(\epsilon)\phi$. Notice that in this case (2.15) follows from (2.8). Now we prove (2.15) by the analogue of (2.14) (with the new ψ) and another application of (2.8):

$$\begin{aligned}
\|\epsilon\psi\| &\leq \left\|\epsilon R_z^0(\epsilon)\langle H\rangle^{\frac{1}{2}}\chi_-\phi\right\| + \left\|\epsilon R_z^0(\epsilon)\langle H\rangle^{\frac{1}{2}}\chi_0\phi\right\| + \left\|\epsilon R_z^0(\epsilon)\langle H\rangle^{\frac{1}{2}}\chi_+\phi\right\| \\
&\leq C\left(\left\|\epsilon\langle H\rangle^{\frac{1}{2}}\chi_- R_{\bar z}^0(-\epsilon)\right\| + 1 + \left\|\epsilon\langle H\rangle^{\frac{1}{2}}\chi_+ R_{\bar z}^0(-\epsilon)\right\|\right)\|\phi\| \\
&\leq C'\|\phi\|.
\end{aligned}$$

□

In the rest of this section we impose Assumption 2.1 with E not being an eigenvalue of H. We pick a real-valued $f \in C_0^\infty(\mathbb{R})$ equal to one on a neighbourhood of E such that the form inequality

$$M + f(H)Gf(H) \geq \frac{\gamma}{2}I - (I - f(H))L(I - f(H)) \tag{2.16}$$

holds on $\mathcal{D}$. We shall prove analogues of Lemmas 2.6 and 2.7 for the perturbed operator $H(\epsilon) - i\epsilon f(H)Gf(H)$. Introducing the notation

$$R_z(\epsilon) = (H - i\epsilon(M + f(H)Gf(H)) - z)^{-1}$$

for its resolvent we have

Lemma 2.8. *There exists constants $C, \epsilon_0 > 0$ and a neighbourhood $\mathcal{V}$ of E such that*

$$\|R_z(\epsilon)\| \leq C|\operatorname{Im} z + \delta\epsilon|^{-1}, \tag{2.17}$$

$$\left\|M^{\frac{1}{2}}R_z(\epsilon)\right\| \leq C|\epsilon|^{-\frac{1}{2}}|\operatorname{Im} z + \delta\epsilon|^{-\frac{1}{2}}, \tag{2.18}$$

$$\left\|\langle H\rangle^{\frac{1}{2}}(I - f(H))R_z(\epsilon)\langle H\rangle^{\frac{1}{2}}\right\| \leq C, \tag{2.19}$$

$$\left\|\langle H\rangle^{\frac{1}{2}}(I - f(H))R_z(\epsilon)M^{\beta}\right\| \leq C|\epsilon|^{-\beta}; \qquad \beta \in \left[0, \frac{1}{2}\right], \tag{2.20}$$

provided $|\epsilon| \leq \epsilon_0$, $\epsilon \operatorname{Im} z > 0$ and $\operatorname{Re} z \in \mathcal{V}$.

Proof. It is not obvious that the resolvent exists for all z in question. Clearly, a perturbation argument based on (2.8) gives the existence for large values of $|\operatorname{Im} z|$. Below we shall prove (2.17) in a domain of the desired form assuming that the resolvent exists. Then by a simple connectedness argument it follows that it exists in the whole domain.

So suppose z is given such that $R_z(\epsilon)$ exists. Then from

$$R_z(\epsilon) = R_z^0(\epsilon)(I + i\epsilon f(H)Gf(H)R_z(\epsilon)) \tag{2.21}$$

and Lemma 2.7 we obtain

$$\left\| \langle H\rangle^{\frac{1}{2}}(I - f(H))R_z(\epsilon)\right\| \leq C_1(1 + |\epsilon|\|R_z(\epsilon)\|). \tag{2.22}$$

Here and henceforth $\mathrm{Re}\, z \in \mathcal{V}$ with ϵ_0 and $\mathcal{V}$ chosen in agreement with Lemma 2.7.

By Lemma 2.6 the analogue of (2.10) for $R_z(\epsilon)$ is

$$R_z(\epsilon)^*(2i\epsilon(M + f(H)Gf(H)) + z - \bar{z})R_z(\epsilon) = R_z(\epsilon) - R_z(\epsilon)^*. \tag{2.23}$$

By (2.16) and (2.23)

$$R_z(\epsilon)^*R_z(\epsilon) \leq \frac{2}{\gamma}\left(\frac{\mathrm{Im}\, R_z(\epsilon)}{\epsilon} + R_z(\epsilon)^*(I - f(H))L(I - f(H))R_z(\epsilon)\right), \tag{2.24}$$

yielding

$$\|R_z(\epsilon)\|^2 \leq C_2\left(\frac{\|R_z(\epsilon)\|}{|\epsilon|} + \left\|\langle H\rangle^{\frac{1}{2}}(I - f(H))R_z(\epsilon)\right\|^2\right). \tag{2.25}$$

Combining (2.22) and (2.25) we obtain

$$\|R_z(\epsilon)\|^2 \leq C_2\left(\frac{\|R_z(\epsilon)\|}{|\epsilon|} + C_1^2(1 + |\epsilon|\|R_z(\epsilon)\|)^2\right). \tag{2.26}$$

We may assume that $C_1^2C_2\epsilon_0^2 < 1$. Then (by subtraction) (2.26) implies the bound $\|R_z(\epsilon)\| \leq C|\epsilon|^{-1}$, which in conjunction with (2.21) and (2.8) yields (2.17).

Upon combining with (2.9) we get (2.18).

As for (2.19) we use

$$\left\|R_z(\epsilon)\langle H\rangle^{\frac{1}{2}}\right\| \leq \left\|R_z(\epsilon)f(H)\langle H\rangle^{\frac{1}{2}}\right\| + \left\|R_z(\epsilon)(I - f(H))\langle H\rangle^{\frac{1}{2}}\right\|$$

and (2.22) to estimate

$$\begin{aligned}\left\|\langle H\rangle^{\frac{1}{2}}(I - f(H))R_z(\epsilon)\langle H\rangle^{\frac{1}{2}}\right\| &\leq C_1\left(1 + |\epsilon|\left\|R_z(\epsilon)\langle H\rangle^{\frac{1}{2}}\right\|\right)\\ &\leq C_2\left(1 + |\epsilon|\left\|\langle H\rangle^{\frac{1}{2}}(I - f(H))R_{\bar{z}}(-\epsilon)\right\|\right)\\ &\leq C_3.\end{aligned}$$

The estimate (2.20) for $\beta = \frac{1}{2}$

$$\left\|\langle H\rangle^{\frac{1}{2}}(I - f(H))R_z(\epsilon)M^{\frac{1}{2}}\right\| \leq C|\epsilon|^{-\frac{1}{2}} \tag{2.27}$$

follows by squaring and using (2.23) in combination with (2.19).

Finally, the general case (2.20) follows by interpolating (2.19) and (2.27). □

With $R_z(\epsilon)$ given as in Lemma 2.8 we introduce the operator

$$F_z(\epsilon) = \langle A\rangle^{-1}M^{\beta}R_z(\epsilon)M^{\beta}\langle A\rangle^{-1},$$

for $\beta < \frac{1}{2}$.

Lemma 2.9. *In addition to the bounds of Lemma 2.8 we have (with a possibly larger constant C)*

$$\left\| M^{\frac{1}{2}} R_z(\epsilon) M^\beta \langle A\rangle^{-1} \right\| \le C|\epsilon|^{-\frac{1}{2}} (1+\|F_z(\epsilon)\|)^{\frac{1}{2}}, \tag{2.28}$$

$$\left\| \langle H\rangle^{\frac{1}{2}} R_z(\epsilon) M^\beta \langle A\rangle^{-1} \right\| \le C|\epsilon|^{-\frac{1}{2}} (1+\|F_z(\epsilon)\|)^{-\frac{1}{2}}, \tag{2.29}$$

for all ϵ and z given as in the lemma. In particular

$$\|F_z(\epsilon)\| \le C|\epsilon|^{-1}. \tag{2.30}$$

Proof. We shall prove that

$$\left\| R_z(\epsilon) M^\beta \langle A\rangle^{-1} \right\| \le C|\epsilon|^{-\frac{1}{2}} (1+\|F_z(\epsilon)\|)^{\frac{1}{2}}. \tag{2.31}$$

Given (2.31), the estimate (2.28) follows by squaring and using (2.23).

To prove (2.31) we let for any $\phi \in \mathcal{D}(M^\beta), \psi = R_z(\epsilon) M^\beta \langle A\rangle^{-1}\phi$. The expectation of (2.24) in the state $M^\beta \langle A\rangle^{-1}\phi$ gives

$$\|\psi\|^2 \le \frac{2}{\gamma}\left(\frac{\|F_z(\epsilon)\|}{|\epsilon|}\|\phi\|^2 + C_1 \left\| \langle H\rangle^{\frac{1}{2}} (I-f(H))\psi \right\|^2 \right).$$

Upon combining with (2.27) we thus obtain

$$\|\psi\|^2 \le \left(\frac{2}{\gamma}\frac{\|F_z(\epsilon)\|}{|\epsilon|} + |\epsilon|^{-1} C_2 \right) \|\phi\|^2, \tag{2.32}$$

which clearly gives (2.31).

The statements (2.29) and (2.30) follow from (2.20) and (2.28). □

Proof of Theorem 2.4 for $\alpha = 1$.

We shall only consider the case $\alpha = 1$ in detail. The general case follows by modifying the proof below mimicking [PPS] and will be outlined at the end of this section.

Obviously we may assume that $\operatorname{Im} z > 0$.

We shall use Lemmas 2.8 and 2.9 to prove the differential inequality

$$\left\| \frac{d}{d\epsilon} F_z(\epsilon) \right\| \le C\epsilon^{-\frac{1}{2}-\beta} (1+\|F_z(\epsilon)\|), \quad \text{for} \quad \epsilon > 0. \tag{2.33}$$

In conjunction with (2.30) this will give Theorem 2.4 by repeated integrations with respect to ϵ using the fact that for any $\phi_1, \phi_2 \in \mathcal{D}$

$$\lim_{\epsilon\to 0^+} \langle \phi_1, R_z(\epsilon)\phi_2\rangle = \left\langle \phi_1, (H-z)^{-1}\phi_2 \right\rangle, \tag{2.34}$$

which in turn follows by the following computation using in the last step Assumption 2.1 1) with $\chi(t) = (t-\bar{z})^{-1}$:

$$\begin{aligned} \left\langle \phi_1, \left(R_z(\epsilon) - (H-z)^{-1}\right)\phi_2 \right\rangle &= i\epsilon \left\langle \phi_1, (H-z)^{-1}(M + f(H)Gf(H)) R_z(\epsilon)\phi_2 \right\rangle \\ &= i\epsilon \left\langle M(H-\bar{z})^{-1}\phi_1, R_z(\epsilon)\phi_2 \right\rangle + O(\epsilon) \\ &= O(\epsilon). \end{aligned}$$

To prove (2.33) we compute (for $\epsilon > 0$)

$$\frac{d}{d\epsilon}F_z(\epsilon) = \langle A\rangle^{-1}M^\beta R_z(\epsilon)i(M + f(H)Gf(H))R_z(\epsilon)M^\beta\langle A\rangle^{-1}. \tag{2.35}$$

The middle term is rewritten as

$$M + f(H)Gf(H) = (M + G) - (I - f(H))Gf(H) - G(I - f(H)). \tag{2.36}$$

Upon substituting (2.36) into the right hand side of (2.35) we obtain three terms. The second and third terms are bounded by

$$C\epsilon^{-\frac{1}{2}-\beta}(1 + \|F_z(\epsilon)\|)^{\frac{1}{2}} \left(\leq C\epsilon^{-\frac{1}{2}-\beta}(1 + \|F_z(\epsilon)\|)\right) \tag{2.37}$$

by (2.20) and (2.29).

It remains to bound the operator

$$\langle A\rangle^{-1}M^\beta R_z(\epsilon)i(M + G)R_z(\epsilon)M^\beta\langle A\rangle^{-1}$$

in accordance with (2.33). For that we introduce

$$B(\kappa) = \langle A\rangle^{-1}M_\kappa^\beta R_z(\epsilon)i(M + G)R_z(\epsilon)M_\kappa^\beta\langle A\rangle^{-1}, \tag{2.38}$$

where (symbolically)

$$M_\kappa^\beta = (1 + \kappa M)^{-1}M^\beta, \quad \text{for} \quad \kappa > 0.$$

By the density of $\mathcal{C}_1 := \langle A\rangle\mathcal{C}$ in $\mathcal{H}$ it suffices to show the bound

$$\lim_{\kappa\to 0}|\langle\phi_1, B(\kappa)\phi_2\rangle| \leq C\epsilon^{-\frac{1}{2}-\beta}(1 + \|F_z(\epsilon)\|)\|\phi_1\|\|\phi_2\|, \quad \text{for} \quad \phi_1, \phi_2 \in \mathcal{C}_1. \tag{2.39}$$

We substitute into (2.38) (cf. Assumption 2.1 2))

$$\begin{aligned} &M + G = \lim_{n\to\infty} H_n, \quad H_n = T_n^1 + T_n^2 + T_n^3; \\ &T_n^1 = i[H - i\epsilon(M + f(H)Gf(H)), A_n], \\ &T_n^2 = -\epsilon[M, A_n], \\ &T_n^3 = -\epsilon[f(H)Gf(H), A_n]. \end{aligned} \tag{2.40}$$

The contribution to the inner product $\langle\phi_1, B(\kappa)\phi_2\rangle$ from T_n^1 (before letting $n \to \infty$) is given by

$$\begin{aligned} &\langle\phi_1, \langle A\rangle^{-1}M_\kappa^\beta[R_z(\epsilon), A_n]M_\kappa^\beta\langle A\rangle^{-1}\phi_2\rangle \\ &= \left\langle M^{\frac{1}{2}}R_{\bar z}(-\epsilon)M_\kappa^\beta\langle A\rangle^{-1}\phi_1, B(\kappa, n)\phi_2\right\rangle - \left\langle B(\kappa, n)\phi_1, M^{\frac{1}{2}}R_z(\epsilon)M_\kappa^\beta\langle A\rangle^{-1}\phi_2\right\rangle, \end{aligned} \tag{2.41}$$

where $B(\kappa, n) = M^{-\frac{1}{2}}A_nM_\kappa^\beta\langle A\rangle^{-1}$, and hence it is bounded by

$$\left\|M^{\frac{1}{2}}R_{\bar z}(-\epsilon)M_\kappa^\beta\langle A\rangle^{-1}\phi_1\right\| \|B(\kappa, n)\phi_2\| + \|B(\kappa, n)\phi_1\| \left\|M^{\frac{1}{2}}R_z(\epsilon)M_\kappa^\beta\langle A\rangle^{-1}\phi_2\right\|. \tag{2.42}$$

We claim that

$$\limsup_{\kappa\to 0}\limsup_{n\to\infty}\|B(\kappa,n)\phi_j\|\le C\|\phi_j\|,\quad\text{for}\quad j=1,2. \tag{2.43}$$

Given (2.43) we conclude by taking the limit of (2.42) and using the estimate (2.28) that

$$\begin{aligned}&\limsup_{\kappa\to 0}\limsup_{n\to\infty}\left|\left\langle R_{\bar z}(-\epsilon)M_\kappa^\beta\langle A\rangle^{-1}\phi_1, T_n^1 R_z(\epsilon)M_\kappa^\beta\langle A\rangle^{-1}\phi_2\right\rangle\right|\\ &\qquad\le C\epsilon^{-\frac12}(1+\|F_z(\epsilon)\|)^{\frac12}\|\phi_1\|\|\phi_2\|.\end{aligned}$$

To show (2.43) we write

$$B(\kappa,n)=M_k^\beta M^{-\frac12}A_n\langle A\rangle^{-1}+M^{-\frac12}\left[A_n,M_\kappa^\beta\right]\langle A\rangle^{-1}.$$

We apply this identity to ϕ_2. For the first term on the right hand side we use the fact that $\phi_2\in\mathcal{C}_1$ and Assumption 2.1 to conclude that

$$\lim_{\kappa\to 0}\lim_{n\to\infty}M_\kappa^\beta M^{-\frac12}A_n\langle A\rangle^{-1}\phi_2=\lim_{\kappa\to 0}M_\kappa^\beta M^{-\frac12}A\langle A\rangle^{-1}\phi_2=M^{\beta-\frac12}A\langle A\rangle^{-1}\phi_2.$$

Since $\beta<\frac12$ we therefore only need to bound the contribution from the second term on the right hand side:

We compute

$$\begin{aligned}&M^{-\frac12}\left[A_n,M_\kappa^\beta\right]=S_1+S_2,\\ &S_1=-M^{-\frac12}(1+\kappa M)^{-1}\kappa[A_n,M](1+\kappa M)^{-1}M^\beta,\\ &S_2=-c_\beta M^{-\frac12}(1+\kappa M)^{-1}\int_0^\infty t^\beta(M+t)^{-1}[A_n,M](M+t)^{-1}dt,\end{aligned}$$

cf. (2.5).

Clearly

$$S_1=-i(1+\kappa M)^{-1}\left(M^{-\frac12}M_nM^{-\frac12}\right)(1+\kappa M)^{-1}\kappa M^{\frac12+\beta}$$

which in conjunction with the first bound of Assumption 2.1 3) (interpolated) yields an upper bound of the norm of S_1 that is independent of n and κ.

We notice that

$$S_2=-ic_\beta\int_0^\infty t^\beta(M+t)^{-1}(1+\kappa M)^{-1}M^{-\frac12}M_n(M+t)^{-1}dt.$$

By inserting $I=M^{-\frac12}M^{\frac12}$ in front of the last factor in the integrand and using Assumption 2.1 3) again we infer the following uniform bound:

$$\begin{aligned}t^\beta\left\|(M+t)^{-1}\right\|\left\|M^{-\frac12}M_nM^{-\frac12}\right\|\left\|M^{\frac12}(M+t)^{-1}\right\|&\le Ct^\beta(\delta+t)^{-\frac32}\left\|M^{-\frac12}M_nM^{-\frac12}\right\|\\ &\le CC_Mt^\beta(\delta+t)^{-\frac32}.\end{aligned}$$

Upon integrating we obtain a finite bound of the norm of S_2 that is independent of n and κ, and therefore we conclude the favourable bound (2.43) for $j=2$.

Obviously by symmetry (2.43) also holds for $j=1$. We are left with bounding the contributions to $\langle\phi_1,B(\kappa)\phi_2\rangle$ from T_n^2 and T_n^3.

As for the contribution from T_n^2 we use Assumption 2.1 3) and (2.28) to estimate

$$\begin{aligned}\limsup_{\kappa\to 0}\limsup_{n\to\infty} &\left|\langle R_{\bar z}(-\epsilon)M_\kappa^\beta\langle A\rangle^{-1}\phi_1, T_n^2R_z(\epsilon)M_\kappa^\beta\langle A\rangle^{-1}\phi_2\rangle\right|\\ &\le \epsilon C_M\left\|M^{\frac12}R_{\bar z}(-\epsilon)M^\beta\langle A\rangle^{-1}\phi_1\right\|\left\|M^{\frac12}R_z(\epsilon)M^\beta\langle A\rangle^{-1}\phi_2\right\|\\ &\le C(1+\|F_z(\epsilon)\|)\|\phi_1\|\|\phi_2\|.\end{aligned}$$

To get the same bound for the contribution from the term T_n^3 we write

$$\epsilon[f(H)Gf(H),A_n] = \epsilon M^{\frac12}\left(M^{-\frac12}[f(H)Gf(H),A_n]M^{-\frac12}\right)M^{\frac12}$$

and estimate the middle factor on the right hand side by using the second estimate of Assumption 2.1 3). Then we use (2.28) again to get the desired bound. We have shown (2.39). □

Sketch of proof of Theorem 2.4 for the case $\alpha\in\left(\frac12,1\right)$.

Following [PSS] we introduce the operator

$$F_z(\epsilon) = D(\epsilon)M^\beta R_z(\epsilon)M^\beta D(\epsilon),\quad \text{where}\quad D(\epsilon)=\langle A\rangle^{-\alpha}\langle\epsilon A\rangle^{\alpha-1}.$$

The results of Lemma 2.9 hold upon replacing $\langle A\rangle^{-1}$ by $D(\epsilon)$.

We shall show that

$$\left\|\frac{d}{d\epsilon}F_z(\epsilon)\right\|\le \epsilon^{-\frac12-\max(\beta,1-\alpha)}C(1+\|F_z(\epsilon)\|),\quad\text{for}\quad \epsilon>0. \tag{2.44}$$

Notice that we can then again integrate to obtain boundedness of $F_z(\epsilon)$. To apply (2.34) in a similar way as before we need the property

$$s-\lim_{\epsilon\to 0^+}M^\beta(D(\epsilon)-D(0))M^{-\beta}=0,$$

which follow from Lemma 2.5 and its proof. (Notice that the bounding constant (2.7) contains a factor ϵ when replacing A_n by ϵA_n.)

To show (2.44) we proceed as before:

The contribution from the term T_n^1 contains (after estimating) an extra factor

$$\|AD(\epsilon)\|\le C\epsilon^{\alpha-1},$$

which is in agreement with (2.44).

To deal with the "new terms"

$$\left(\frac{d}{d\epsilon}D(\epsilon)\right)M^\beta R_z(\epsilon)M^\beta D(\epsilon)\text{ and }D(\epsilon)M^\beta R_z(\epsilon)M^\beta\left(\frac{d}{d\epsilon}D(\epsilon)\right),$$

we use that

$$\left\|\frac{d}{d\epsilon}D(\epsilon)\right\|\le C\epsilon^{\alpha-1}$$

and the analogue of (2.28). We end up with bounds in agreement with (2.44). □

3. Our example (smooth case)

In this section we impose Condition 1.1. Singularities of L^p-type and Coulomb-type will be treated in Sections 5 and 6 respectively.

The notation $\langle T\rangle$ is used for $(1+T^2)^{\frac{1}{2}}$. (Notice that (2.2) is slightly different.)

We consider the following inputs in Assumption 2.1:

$$
\begin{aligned}
&\mathcal{H}=L^2([0,1])\otimes L^2(X)\\
&H_0=\tau+h_0,\\
&\tau=-i\frac{d}{dt}\text{ with periodic boundary condition, } h_0=p^2,\ p=-i\nabla,\\
&H=\tau+h=H_0+V,\\
&M=2p^2+\delta,\\
&G=-x\cdot\nabla V-\delta \qquad (\delta>0\text{ arbitrary}),\\
&A=\frac{1}{2}(x\cdot p+p\cdot x),\\
&A_n=\frac{1}{2}(F_n\cdot p+p\cdot F_n),\\
&F_n=F_n(x)=\langle x/n\rangle^{-1}x,\\
&\mathcal{C}=\mathcal{F}\otimes\mathcal{S}(X),
\end{aligned}
$$

where for the latter definition $\mathcal{F}$ is the set of 1-periodic trigonometric polynomials, $\mathcal{S}(X)$ is the set of Schwartz functions on X and the tensor product is the algebraic one.

We claim that Assumption 2.1 holds for E outside a set of thresholds. Leaving the verification of Assumption 2.1 4) to Section 4 we shall here give the arguments for Assumption 2.1 1)-3):

First of all by Nelson's commutator theorem [RS, Theorem X.37], $\mathcal{C}$ is a core of both A and A_n and the identity $\lim_{n\to\infty}A_n\phi=A\phi$ holds for all $\phi\in\mathcal{C}$. Obviously $\mathcal{D}(M^{\frac{1}{2}})\subseteq\mathcal{D}(A_n)$.

Also we notice that $\mathcal{C}\subseteq\mathcal{D}=\mathcal{D}(H_0)\cap\mathcal{D}(M)=\mathcal{D}(\tau)\cap\mathcal{D}(M)$ is a core of both H and M.

Computing in the representation where the operator τ is diagonalized it follows that

$$i[M,H_0]=0\text{ is }M\text{-bounded,}$$

and

$$(H_0-i)M^{-1}(H_0-i)^{-1}=M^{-1}\text{ is bounded.}$$

As a form on $\mathcal{D}$

$$i[M,\chi(H)]=2i[H-\tau-V,\chi(H)]=-2i[\tau+V,\chi(H)].$$

We pick an almost analytic extension $\tilde{\chi}\in C^\infty(\mathbb{C})$ of $\chi\in\mathcal{F}^0$ with a certain decay property so that we can represent (suitably interpreted), cf. [DG, Appendix C.3],

$$\chi(H)\phi=\frac{1}{\pi}\int_{\mathbb{C}}(\bar{\partial}\tilde{\chi})(\eta)(H-\eta)^{-1}\phi\,dudv,\quad\text{where}\quad\eta=u+iv. \tag{3.1}$$

This yields the representation

$$-2i[\tau,\chi(H)]=\frac{2}{\pi}\int_{\mathbb{C}}(\bar{\partial}\tilde{\chi})(\eta)(H-\eta)^{-1}\left(\frac{\partial}{\partial t}V\right)(H-\eta)^{-1}dudv.$$

The right hand side is identified as a bounded operator. Clearly $-2i[V,\chi(H)]$ is bounded. We have now verified Assumption 2.1 1).

As for Assumption 2.1 2) we compute using again the representation where the operator τ is diagonalized

$$i[H_0,A_n]=p\cdot 2F_n'p-\frac{1}{2}\Delta(\nabla\cdot F_n)\quad\text{and}\quad F_n'=\langle x/n\rangle^{-1}\left(I-\frac{|x/n\rangle\langle x/n|}{\langle x/n\rangle^2}\right). \tag{3.2}$$

Clearly $F_n(x)=nF(\frac{x}{n})$, where F and all its derivatives are bounded. Consequently the second term on the right hand side of (3.2) is $O(n^{-2})$. For the first term we notice that $F_n'\to I$ in the strong sense by the Lebesgue dominated convergence theorem. We conclude that for all $\phi_1,\phi_2\in\mathcal{D}^{\frac{1}{2}}$

$$\lim_{n\to\infty}\langle\phi_1,i[H_0,A_n]\phi_2\rangle=\lim_{n\to\infty}\langle p\phi_1,2F_n'p\phi_2\rangle=\langle\phi_1,(M-\delta)\phi_2\rangle. \tag{3.3}$$

For the potential part we compute

$$i[V,A_n]=-F_n\cdot\nabla V, \tag{3.4}$$

leading to the limit

$$\lim_{n\to\infty}\langle\phi_1,i[V,A_n]\phi_2\rangle=\langle\phi_1,-x\cdot\nabla V\phi_2\rangle. \tag{3.5}$$

We conclude from (3.3) and (3.5) that for all $\phi_1,\phi_2\in\mathcal{D}^{\frac{1}{2}}$

$$\lim_{n\to\infty}\langle\phi_1,H_n\phi_2\rangle=\langle\phi_1,(M+G)\phi_2\rangle. \tag{3.6}$$

As for Assumption 2.1 3) we use that the form $i[M,A_n]$ may be computed to be twice the expression on the right hand sides of (3.2). By a commutation we conclude that M_n exists as an M-bounded operator and that the first uniformity bound of Assumption 2.1 3) holds.

For the last bound of Assumption 2.1 3) we proceed in a fashion that to some extent anticipates inclusion of local singularities later on. We notice that for $\kappa>0$ and $\operatorname{Im}\eta\neq 0$

$$\begin{aligned}
&\left\|\frac{p_l}{I-i\kappa p_l}(H-\eta)^{-1}\langle p\rangle^{-1}\right\|\leq|\operatorname{Im}\eta|^{-1}\left\|p_l\langle p\rangle^{-1}\right\|+\left\|(H-\eta)^{-1}\left[H,\frac{p_l}{I-i\kappa p_l}\right](H-\eta)^{-1}\langle p\rangle^{-1}\right\|\\
&\quad\leq|\operatorname{Im}\eta|^{-1}\left\|p_l\langle p\rangle^{-1}\right\|+\left\|(H-\eta)^{-1}(I-i\kappa p_l)^{-1}(\partial_l V)(I-i\kappa p_l)^{-1}(H-\eta)^{-1}\langle p\rangle^{-1}\right\|\\
&\quad\leq|\operatorname{Im}\eta|^{-1}\left\|p_l\langle p\rangle^{-1}\right\|+\left\|(H-\eta)^{-1}(H_0+i)\right\|\left\|(H_0+i)^{-1}\partial_l V(H_0-i)^{-1}\right\|\left\|(H_0-i)(H-\eta)^{-1}\right\|\\
&\quad\leq|\operatorname{Im}\eta|^{-1}\left\|p_l\langle p\rangle^{-1}\right\|+C\left(\frac{1+|\eta|}{|\operatorname{Im}\eta|}\right)^2\left\|(H_0+i)^{-1}\partial_l V(H_0-i)^{-1}\right\|.
\end{aligned}$$

Letting $\kappa\to 0$ we get

$$\left\|\langle p\rangle(H-\eta)^{-1}\langle p\rangle^{-1}\right\|\leq C_1\left(\frac{1+|\eta|}{|\operatorname{Im}\eta|}\right)^2,\quad\text{where}\quad C_1=C_2\left(1+\left\||\nabla_x V|^{\frac{1}{2}}\langle H_0\rangle^{-1}\right\|^2\right). \tag{3.7}$$

At first we show that

$$\sup_n\left\|\langle p\rangle^{-1}i[f(H),A_n]Gf(H)\langle p\rangle^{-1}\right\|<\infty. \tag{3.8}$$

We represent as a form on $\mathcal{D}(M^{\frac{1}{2}})$

$$i[f(H),A_n]=-\frac{1}{\pi}\int_{\mathbb{C}}(\bar\partial\tilde f)(\eta)(H-\eta)^{-1}H_n(H-\eta)^{-1}dudv,$$

cf. (3.1).

We substitute the expressions (3.2) and (3.4) into this formula. For the contribution from (3.2) only the first term on the right hand side needs elaboration. Using (3.7), its derivation and the condition that G is an H-bounded operator it suffices to show that

$$\sup_{\kappa>0}\left\|(H+i)^{-1}\frac{p_l}{I-i\kappa p_l}Gf(H)\langle p\rangle^{-1}\right\|<\infty.$$

Obviously by commutation using (3.7) again this may be verified under the weak assumption

$$|\nabla_x G|^{\frac{1}{2}}\langle H_0\rangle^{-1} \text{ is bounded.} \tag{3.9}$$

As for the contribution to (3.8) from (3.4) we factorize

$$\begin{aligned}&\langle p\rangle^{-1}(H+i)^{-1}F_n\cdot\nabla V(H-i)^{-1}=B_n^*B;\\ &B_n=(H-i)f_n(H-i)^{-1}\langle p\rangle^{-1},\\ &B=(H+i)^{-1}x\cdot\nabla V(H-i)^{-1},\end{aligned} \tag{3.10}$$

and use the fact that B_n is uniformly bounded, cf. (3.7) (in fact $B_n\to\langle p\rangle^{-1}$ in the strong sense). Clearly for this step we also need B bounded, or equivalently

$$\langle H_0\rangle^{-1}x\cdot\nabla V\langle H_0\rangle^{-1} \text{ is bounded.} \tag{3.11}$$

We have verified (3.8) under the C^2-conditions in a fashion that allows inclusion of certain local singularities as exhibited by the bounds (3.7), (3.9) and (3.11). In addition we used the condition that G is an H-bounded operator. As for the latter we remark that in the context of inclusion of L^p-singularities in Section 5 it is relevant to consider G as an H-bounded form. Although we shall not elaborate, the arguments above can be modified under this form-boundedness assumption at the expense of strengthening (3.7) and (3.11) as follows

$$|\nabla_x V|^{\frac{1}{2}}\langle H_0\rangle^{-\frac{1}{2}} \text{ and } |x\cdot\nabla_x V|^{\frac{1}{2}}\langle H_0\rangle^{-\frac{1}{2}} \text{ are bounded.} \tag{3.12}$$

Next we show that

$$\sup_n\left\|\langle p\rangle^{-1}f(H)i[G,A_n]f(H)\langle p\rangle^{-1}\right\|<\infty. \tag{3.13}$$

We factorize using notation of (3.10)

$$\begin{aligned}&\langle p\rangle^{-1}(H+i)^{-1}F_n\cdot\nabla G(H-i)^{-1}=B_n^*B';\\ &B'=(H+i)^{-1}x\cdot\nabla G(H-i)^{-1},\end{aligned}$$

yielding for a constant C independent of n (but depending on the bound on the right hand side of (3.7))

$$\left\|\langle p\rangle^{-1}f(H)i[G,A_n]f(H)\langle p\rangle^{-1}\right\|\leq C\|B'\|. \tag{3.14}$$

Finiteness of the right hand side of (3.14) requires

$$\langle H_0\rangle^{-1}x\cdot\nabla G\langle H_0\rangle^{-1} \text{ is bounded.} \tag{3.15}$$

We have completed the verification of (3.13) and hence Assumption 2.1 3) under C^2-conditions. We obtained bounds that will be useful for inclusion of local singularities.

4. The commutator estimate

In this section we prove the commutator estimate in Assumption 2.1 4) for the example presented in Section 3. We will again work with regular N-body systems, that is, we suppose V satisfies Condition 1.1.

We introduce the reguralization operators

$$T_R = \left(1 - i\frac{\tau}{R}\right)^{-1} \quad \text{and} \quad \widetilde{T}_R = T_R^* T_R = \left(1 + \left(\frac{\tau}{R}\right)^2\right)^{-1}. \tag{4.1}$$

We note the properties

$$\mathrm{s} - \lim_{R\to\infty} T_R = I \quad \text{and} \quad (H_0 + i)^{-1} T_R = T_R (H_0 + i)^{-1}. \tag{4.2}$$

This reguralization will be used on several occasions to compute commutator forms.

We start with the following result (see also [Sk, Theorem 3.1])

Proposition 4.1. *Let P be an eigenprojection corresponding to an eigenvalue $E \in \sigma_{\mathrm{pp}}(H)$. Then pP is bounded (in fact with norm uniformly bounded in E).*

Proof. For $n > 0$ and $R > 1$ we consider 'vector-fields'

$$F_{n,R} = F_{n,R}(x, \tau) = \widetilde{T}_R \langle x/n\rangle^{-1} x.$$

We compute the 'x-derivative' (cf. (3.2))

$$F'_{n,R} = \widetilde{T}_R \langle x/n\rangle^{-1}\left(I - \frac{|x/n\rangle\langle x/n|}{\langle x/n\rangle^2}\right)$$

and write

$$A_{n,R} = \frac{1}{2}(F_{n,R}\cdot p + p \cdot F_{n,R}).$$

The cut-offs ensure that $A_{n,R}$ is H-bounded.

We compute as a form on $\mathcal{D} = \mathcal{D}(\tau) \cap \mathcal{D}(p^2)$

$$i[\tau + p^2, A_{n,R}] = 2p^t \,\mathrm{Re}\{F'_{n,R}\} p - \frac{1}{2}\mathrm{Re}\{\Delta(\nabla \cdot F_{n,R})\}. \tag{4.3}$$

Here

$$\sup_{R>1} \|\Delta(\nabla\cdot F_{n,R})\| \le Cn^{-2}. \tag{4.4}$$

As for the potential we have

$$i[V, A_{n,R}] = -\mathrm{Re}\{F_{n,R}\cdot \nabla V\} + 2R^{-1}\,\mathrm{Re}\left\{\widetilde{T}_R\,\mathrm{Re}\left\{\frac{\tau}{R}\,\partial_t V\right\} A_{n,R}\right\}. \tag{4.5}$$

When sandwiched between resolvents the second term on the right-hand side can be estimated as

$$2R^{-1}\left\|\widetilde{T}_R\,\mathrm{Re}\left\{\frac{\tau}{R}\,\partial_t V\right\} A_{n,R}(H_0+i)^{-1}\right\| \le CR^{-\frac{1}{2}} n. \tag{4.6}$$

For $\psi \in \mathcal{D}$ we combine (4.3)-(4.6) to get the lower bound

$$\begin{aligned}&\langle \psi, i[H, A_{n,R}]\psi\rangle \\ &\geq 2\left\langle p\psi, F'_{n,R}p\psi\right\rangle - \langle \psi, \mathrm{Re}\left\{F_{n,R}\cdot \nabla V\right\}\psi\rangle - C\left(n^{-2} + R^{-\frac{1}{2}}n\right)\left(\|H\psi\|^2 + \|\psi\|^2\right).\end{aligned}$$

Replacing ψ by $T_S\varphi$, $\varphi \in \mathcal{D}(H)$, and taking the limit $S \to \infty$ implies, see (4.2), that the above estimate holds in the sense of forms on $\mathcal{D}(H)$ as well. We now use, cf. (4.1), that $F'_{n,R} \geq F'_{n,R'}$, for a fixed $R' < R$. We thus obtain for $\varphi \in \mathcal{D}(H)$

$$\liminf_{n\to\infty}\liminf_{R\to\infty}\langle \varphi, i[H, A_{n,R}]\varphi\rangle \geq 2\left\|\widetilde{T}_{R'}^{\frac{1}{2}}p\varphi\right\|^2 - \langle \varphi, x\cdot\nabla V\varphi\rangle, \qquad R' > 1. \tag{4.7}$$

Now let φ be an eigenfunction for H. Then the left-hand side is identically zero and hence, by the Lebesgue monotone convergence theorem, we have

$$\|p\varphi\|^2 \leq \frac{1}{2}\sup_{t\in[0,1],x\in X}|x\cdot\nabla V(t,x)|\|\varphi\|^2.$$

By Condition 1.1 iii), this completes the proof. $\square$

We introduce the distance, $d(E)$, to the nearest threshold below E

$$d(E) = \inf_{E'\in\mathcal{F}(H), E'\leq E}(E - E').$$

For $a \in \mathcal{A}\backslash\{a_{\max}\}$ we write d^a for the distance function associated naturally with the Hamiltonian H^a and its threshold set $\mathcal{F}(H^a)$. Let d' denote the analogous distance function for the usual N-body problem. From (1.20) we get in this case the relation

$$d(E) = \inf_{n\in\mathbb{Z}} d'(E + 2\pi n).$$

We will abbreviate

$$M = 2p^2, \qquad G = -x\cdot\nabla V, \qquad M^a = 2(p^a)^2 \quad \text{and} \quad G^a = -x^a\cdot\nabla V^a, \quad a \neq a_{\min}, a_{\max}.$$

We will furthermore write

$$\mathcal{D} = \mathcal{D}(H)\cap\mathcal{D}(M) \quad \text{and} \quad \mathcal{D}^a = \mathcal{D}(H^a)\cap\mathcal{D}(M^a), \quad a \neq a_{\min}, a_{\max}.$$

Let $f_{0,1} \in C_0^\infty(\mathbb{R})$ satisfy

$$0 \leq f_{0,1} \leq 1 \qquad f_{0,1}(s) = 0, \quad |s| > 1 \quad \text{and} \quad f_{0,1} = 1, \quad |s| \leq \frac{1}{2}.$$

For $E \in \mathbb{R}$ and $\kappa > 0$ we write $f_{E,\kappa}(s) = f_{0,1}((s - E)/\kappa)$.

The following theorem combined with the results of Section 3 and Remarks 2.2 3) will imply Assumption 2.1 for any $E \notin \mathcal{F}(H)$. Notice that given $E \notin \mathcal{F}(H)$ it suffices to verify Assumption 2.1 4)" for M and G given as above, cf. (2.1).

Theorem 4.2. *Suppose V satisfies Condition 1.1. Let $E \in \mathbb{R}$ and $\epsilon > 0$. There exist $\kappa > 0$, $C \geq 0$ and a compact self-adjoint operator K such that*

i) *We have the commutator estimate, as forms on $\mathcal{D}$,*

$$M + f_{E,\kappa}(H)Gf_{E,\kappa}(H) \geq 2(d(E) - \epsilon)f_{E,\kappa}(H)^2 - K - C(I - f_{E,\kappa}(H))^2.$$

ii) *The threshold set $\mathcal{F}(H)$ is closed and countable, and non-threshold eigenvalues of H have finite multiplicity and can at most accumulate at $\mathcal{F}(H)$.*

The strategy of the proof will be the same as the one employed by Hunziker in [Hu1] for the usual N-body problem. In the following K will denote compact self-adjoint operators.

Lemma 4.3. *Assume i) is given. Suppose furthermore that ii) holds with H replaced by H^a for every $a \neq a_{\min}, a_{\max}$. Then ii) holds as well.*

Proof. First we note that $\mathcal{F}(H)$ is closed and countable, by the assumption on the subsystems. Let $\{E_n\}_{n\in\mathbb{N}} \subset \sigma_{\mathrm{pp}}(H)\backslash\mathcal{F}(H)$ with corresponding eigenfunctions $\{\varphi_n\}_{n\in\mathbb{N}}$. Suppose $E_n \to E$ and $\varphi_n \to 0$ weakly. By i), Lemma 2.3 and Proposition 4.1 we find that $E \in \mathcal{F}(H)$. This proves the lemma. □

Proof of Theorem 4.2 in the case 'N=2'.

By 'N=2' we mean $\mathcal{A} = \{a_{\min}, a_{\max}\}$. We start by proving i). Let $\epsilon > 0$ and $E \in \mathbb{R}$. In this case G is H-compact, see [Ya1] and [Mø], and we estimate for κ small

$$\begin{aligned} M + f_{E,\kappa}(H)Gf_{E,\kappa}(H) &\geq f_{E,\kappa}(H_0)Mf_{E,\kappa}(H_0) - K \\ &\geq 2(d(E) - \epsilon)f_{E,\kappa}(H_0)^2 - K \\ &= 2(d(E) - \epsilon)f_{E,\kappa}(H)^2 - K. \end{aligned}$$

In the last step we used an almost analytic extension of $f_{0,1}$, see (3.1), to verify that $f_{E,\kappa}(H_0) - f_{E,\kappa}(H)$ is compact. This argument is similar to the one used in [Yo]. Note that $C = 0$ in this case.

As for ii), we remark that $\mathcal{F}(H) = 2\pi\mathbb{Z}$ is closed and countable and the result follows from Lemma 4.3. □

We will proceed by induction, with respect to the ordering (1.7) on $\mathcal{A}$.

Induction Hypothesis. *For any $a \neq a_{\min}, a_{\max}$ the following holds: Let $E \in \mathbb{R}$ and $\epsilon > 0$. There exist $\kappa > 0$, $C \geq 0$ and a compact self-adjoint operator K^a such that*

i) *We have the commutator estimate, as forms on $\mathcal{D}^a$,*

$$M^a + f_{E,\kappa}(H^a)G^a f_{E,\kappa}(H^a) \geq 2(d^a(E) - \epsilon)f_{E,\kappa}(H^a)^2 - K^a - C(I - f_{E,\kappa}(H^a))^2. \tag{4.8}$$

ii) *The threshold set $\mathcal{F}(H^a)$ is closed and countable, and non-threshold eigenvalues of H^a have finite multiplicity and can at most accumulate at $\mathcal{F}(H^a)$.*

By Lemma 4.3 it is enough to prove Theorem 4.2 i).

We will break the proof into several steps, following the structure of [Hu1]. The first step contains the main difficulty, compared to Hunziker's proof.

We will frequently use the estimate (often with H replaced by H^a)

$$2\,\mathrm{Re}\{(I - f_{E,\kappa}(H))Bf_{E,\kappa}(H)\} \geq -\tilde{\epsilon}^{-1}\|B\|^2(I - f_{E,\kappa}(H))^2 - \tilde{\epsilon}f_{E,\kappa}(H)^2, \tag{4.9}$$

which holds for every bounded operator B and every $\tilde{\epsilon} > 0$. This is a special case of the estimate (2.1).

Lemma 4.4. *Let $a \neq a_{\min}, a_{\max}$, $E \in \mathbb{R}$ and $\epsilon > 0$. There exist $\kappa > 0$ and $C > 0$, such that*

$$M^a + f_{E,\kappa}(H^a) G^a f_{E,\kappa}(H^a) \geq 2(d(E) - \epsilon) f_{E,\kappa}(H^a)^2 - C(I - f_{E,\kappa}(H^a))^2,$$

as forms on $\mathcal{D}^a$.

Proof. In the case $E \notin \sigma_{\mathrm{pp}}(H^a)$, the estimate follows easily from the induction hypothesis (4.8), since $f_{E,\kappa'}(H^a) \to 0$ strongly, as $\kappa' \to 0$: First we estimate the second term on the right-hand side of (4.8) (with ϵ replaced by $\epsilon/3$)

$$\begin{aligned} -K^a &\geq -f_{E,\kappa'}(H^a) K^a f_{E,\kappa'}(H^a) - \frac{\epsilon}{6} f_{E,\kappa'}(H^a)^2 - C(\epsilon)(I - f_{E,\kappa'}(H^a))^2 \\ &\geq -\frac{\epsilon}{3} f_{E,\kappa'}(H^a)^2 - C(\epsilon)(I - f_{E,\kappa'}(H^a))^2. \end{aligned}$$

Next we use (4.9) repeatedly to replace $f_{E,\kappa}(H^a)$ by $f_{E,\kappa'}(H^a)$ in (4.8).

Let $E \in \sigma_{\mathrm{pp}}(H^a)$ and abbreviate

$$B^a_{E,\kappa} = M^a + f_{E,\kappa}(H^a) G^a f_{E,\kappa}(H^a).$$

Since $2(p^a)^2 = B^a_{E,\kappa} - f_{E,\kappa}(H^a) G^a f_{E,\kappa}(H^a)$ it follows that it suffice to prove the following statement: Let $\epsilon > 0$ and $\sigma > 0$. There exist $\kappa > 0$ and $C > 0$ such that

$$B^a_{E,\kappa} \geq -\epsilon f_{E,\kappa}(H^a)^2 - \sigma (p^a)^2 - C(I - f_{E,\kappa}(H^a))^2. \tag{4.10}$$

(We substitute the expression for $(p^a)^2$ and isolate $B^a_{E,\kappa}$ assuming that $\sigma > 0$ is small.)

Let $\epsilon' = \epsilon/50$. Write P for the eigenprojection associated with E and pick a sequence $\{P_n\}$ of finite rank projections with $P_n \leq P$, such that $P_n \to P$ strongly. We note that

$$P_n B^a_{E,\kappa} P_n = P_n B^a_{E,\kappa}(P - P_n) = 0.$$

This follows from the Virial Theorem (cf. Lemma 2.3 and Proposition 4.1). We can now compute

$$B^a_{E,\kappa} = 2\,\mathrm{Re}\{P_n B^a_{E,\kappa}(I - P)\} + (I - P_n) B^a_{E,\kappa}(I - P_n). \tag{4.11}$$

We now use Proposition 4.1 and the induction hypothesis (4.8) (extended from $\mathcal{D}$ to $\mathcal{D}^{\frac{1}{2}}$), with ϵ replaced by ϵ', on the last term and obtain

$$\begin{aligned} (I - P_n) B^a_{E,\kappa}(I - P_n) &\geq -2\epsilon' f_{E,\kappa}(H^a)^2 - (I - P_n) K (I - P_n) - C_1 (I - f_{E,\kappa}(H^a))^2 \\ &\geq -3\epsilon' f_{E,\kappa'}(H^a)^2 - f_{E,\kappa'}(H^a)(I - P) K (I - P) f_{E,\kappa'}(H^a) \\ &\quad - 2\,\mathrm{Re}\{(P - P_n) K (I - P)\} - (P - P_n) K (P - P_n) \\ &\quad - C_2 (I - f_{E,\kappa'}(H^a))^2. \end{aligned} \tag{4.12}$$

In the second step we used again (4.9) repeatedly. We are now in a position to choose the parameters n and κ'.

Pick n large such that $\|(P - P_n)K\| \leq \epsilon'$. By (4.9) this implies

$$2\,\mathrm{Re}\{(P - P_n) K (I - P)\} + (P - P_n) K (P - P_n) \leq 4\epsilon' f_{E,\kappa'}(H^a)^2 + C(I - f_{E,\kappa'}(H^a))^2. \tag{4.13}$$

As for the first term in (4.11) we introduce the indicator function χ_θ for the interval $[-\theta, \infty)$. We rewrite

$$\begin{aligned} P_n B^a_{E,\kappa}(I-P) &= P_n \chi_\theta B^a_{E,\kappa}(I-P) + P_n(I-\chi_\theta)B^a_{E,\kappa}(I-P) \\ &= P_n\chi_\theta B^a_{E,\kappa}(I - f_{E,\kappa'}(H^a)) + P_n\chi_\theta B^a_{E,\kappa}(I-P) f_{E,\kappa'}(H^a) \\ &\quad + P_n(I-\chi_\theta)B^a_{E,\kappa}(I-P). \end{aligned} \tag{4.14}$$

First we consider the third term on the right-hand side. Using Proposition 4.1, (2.1) and the compactness of $P_n p$ we get

$$\begin{aligned} 2\,\mathrm{Re}\{P_n(I-\chi_\theta)B^a_{E,\kappa}(I-P)\} &= o_\theta(1) - 4\,\mathrm{Re}\{P_n(I-\chi_\theta)p^a\cdot p^a\} \\ &\geq o_\theta(1) - \sigma(p^a)^2, \end{aligned}$$

where $o_\theta(1)\to 0$ for $\theta\to\infty$. Pick θ large enough such that $|o_\theta(1)|\leq \epsilon'$. To treat the two first terms on the right-hand side of (4.14) we use that $(p^a)^2\chi_\theta(H^a-i)^{-1}$ is bounded: Choose κ' so small that the norm of the second term is smaller than ϵ'. Then apply (4.9) to the first term. In conclusion

$$2\,\mathrm{Re}\{P_n B^a_{E,\kappa}(I-P)\} \geq -4\epsilon' f_{E,\kappa'}(H^a)^2 - \sigma(p^a)^2 - C(I - f_{E,\kappa'}(H^a))^2. \tag{4.15}$$

By possibly choosing κ' smaller we obtain in addition to (4.15) the estimate

$$-f_{E,\kappa'}(H^a)(I-P)K(I-P)f_{E,\kappa'}(H^a) \geq -\epsilon' f_{E,\kappa'}(H^a)^2.$$

Combining this with (4.11), (4.12), (4.13) and (4.15) proves (4.10), and hence the lemma. □

Notice that the distance function d satisfies

$$d(E+E') \leq d(E) + E', \quad E'\geq 0, \tag{4.16}$$

just as for the usual N-body case. This will be used in the following

Lemma 4.5. *Let $a \neq a_{\min}, a_{\max}$ and $\epsilon > 0$. There exist $\kappa > 0$ and $C > 0$, such that for all $E\in\mathbb{R}$ we have, as forms on $\mathcal{D}^a$,*

$$B^a_{E,\kappa} \geq 2(d(E+\epsilon) - 2\epsilon) f_{E,\kappa}(H^a)^2 - C(I - f_{E,\kappa}(H^a))^2.$$

Remark. Uniformity in E follows from local uniformity due to the periodic structure of the problem. More precisely because

$$e^{-i2\pi nt} H e^{i2\pi nt} = H + 2\pi n, \quad \text{for} \quad n\in\mathbb{Z}.$$

This observation was also used in [KuY] and [A].

Proof. First we note that Lemma 4.4 and (4.16) imply

$$\begin{aligned} B^a_{E,\kappa} &\geq 2(d(E) - \epsilon) f_{E,\kappa}(H^a)^2 - C(I - f_{E,\kappa}(H^a))^2 \\ &\geq 2(d(E+\epsilon) - 2\epsilon) f_{E,\kappa}(H^a)^2 - C(I - f_{E,\kappa}(H^a))^2. \end{aligned}$$

Here $\kappa = \kappa(E)$ and $C = C(E)$. What is left to prove is that we can choose κ and C independently of E. By the remark above it is enough to choose them independently of $E\in[0,2\pi]$.

Assume the lemma to be false. For $\kappa_n = n^{-1}$ and $C_n = n$ there exists $E_n \in [0, 2\pi]$ such that the estimate in the lemma does not hold. We extract a subsequence such that $E_n \to E \in [0, 2\pi]$, $\kappa_n \to 0$ and $C_n \to \infty$ as $n \to \infty$.

By Lemma 4.4 we have a $\kappa_0 > 0$ and a $C_0 \geq 0$ such that

$$B^a_{E,\kappa_0} \geq 2(d(E) - \epsilon/3) f_{E,\kappa_0}(H^a)^2 - C_0(I - f_{E,\kappa_0}(H^a))^2. \tag{4.17}$$

Choose n so large that $|E - E_n| < \epsilon/3$,

$$[E_n - \kappa_n, E_n + \kappa_n] \subset [E - \kappa_0/2, E + \kappa_0/2] \tag{4.18}$$

and

$$C_n \geq C_0 + (1 + 3\epsilon^{-1} C^a) C^a, \qquad C^a = \sup_{t \in [0,1], y \in X^a} |G^a(t, y)| + \sup_n 2|d(E_n + \epsilon) - 5\epsilon/3|. \tag{4.19}$$

(Note that $d(E) \leq 2\pi$.) By (4.16) we have

$$d(E) \geq d(E_n + \epsilon) - \epsilon + E - E_n \geq d(E_n + \epsilon) - 4\epsilon/3.$$

Combining this with (4.17) we get

$$B^a_{E,\kappa_0} \geq 2(d(E_n + \epsilon) - 5\epsilon/3) f_{E,\kappa_0}(H^a)^2 - C_0(I - f_{E,\kappa_0}(H^a))^2.$$

By (4.18) we have

$$f_{E,\kappa_0} = f_{E_n,\kappa_n} + f_{E,\kappa_0}(1 - f_{E_n,\kappa_n}) \quad \text{and} \quad 1 - f_{E,\kappa_0} \leq 1 - f_{E_n,\kappa_n}.$$

This, together with (4.9), applied with $\tilde{\epsilon} = \epsilon/3$ and $B = -f_{E,\kappa_0}(H^a)(G^a - 2(d(E_n + \epsilon) - 5\epsilon/3))$, implies

$$B^a_{E_n,\kappa_n} \geq 2(d(E_n + \epsilon) - 2\epsilon) f_{E_n,\kappa_n}(H^a)^2 - (C_0 + (1 + 3\epsilon^{-1} C^a) C^a)(I - f_{E_n,\kappa_n}(H^a))^2,$$

which contradicts our assumption by (4.19). □

In the next lemma we use the uniformity of the estimates above

Lemma 4.6. *Let $E \in \mathbb{R}$ and $\epsilon > 0$. There exist $\kappa > 0$ and $C > 0$ such that for all $a \neq a_{\min}, a_{\max}$, as forms on $\mathcal{D}$,*

$$M + f_{E,\kappa}(H_a) G^a f_{E,\kappa}(H_a) \geq 2(d(E + \epsilon) - 2\epsilon) f_{E,\kappa}(H_a)^2 - C(I - f_{E,\kappa}(H_a))^2.$$

Proof. By (4.9) it suffices to show the bound in the lemma for a fixed $a \neq a_{\min}, a_{\max}$, cf. the proof of Lemma 4.5.

Let $\kappa > 0$ and $C \geq 0$ be given by Lemma 4.5. Let $\mathcal{F}_a$ denote the partial Fourier transform with respect to x_a. For $\psi \in \mathcal{D} = \mathcal{D}(\tau) \cap \mathcal{D}(M) \subset \mathcal{H} \simeq L^2(X_a; \mathcal{H}^a)$, we estimate, noting that $\mathcal{F}_a \psi(k) \in \mathcal{D}^a$ almost everywhere,

$$\begin{aligned}
\langle M + f_{E,\kappa}(H_a) G^a f_{E,\kappa}(H_a) \rangle_\psi &= \int_{X_a} \langle B^a_{E-k^2,\kappa} + 2k^2 \rangle_{(\mathcal{F}_a\psi)(k)}\, dk \\
&\geq \int_{X_a} \left\{ 2(d(E - k^2 + \epsilon) - 2\epsilon) + 2k^2 \right\} \|f_{E-k^2,\kappa}(H^a)(\mathcal{F}_a\psi)(k)\|^2 dk \\
&\quad - C \int_{X_a} \|(I - f_{E-k^2,\kappa}(H^a))(\mathcal{F}_a\psi)(k)\|^2 dk.
\end{aligned}$$

Here we used that $k \to k(\mathcal{F}_a\psi)(k) = (\mathcal{F}_a p_a \psi)(k)$ and $k \to (M^a)^{\frac{1}{2}}(\mathcal{F}_a\psi)(k)$ are in $L^2(X_a; \mathcal{H}^a)$ to make sense of the first equality. Applying (4.16), with E replaced by $E - k^2 + \epsilon$ and $E' = k^2$, concludes the proof. □

Proof of Theorem 4.2.

As previously mentioned it suffices to show Theorem 4.2 i). As in the proof of Lemma 4.4, cf. (4.10), we find that it is enough to prove the following statement: For $E \in \mathbb{R}$, $\epsilon > 0$ and $\sigma > 0$, there exist $\kappa > 0$, $C > 0$ and K compact such that

$$M + f_{E,\kappa}(H) G f_{E,\kappa}(H) \geq 2(d(E) - \epsilon) f_{E,\kappa}(H)^2 - K - \sigma p^2 - C(I - f_{E,\kappa}(H))^2. \tag{4.20}$$

Let $\sigma' = \frac{\sigma}{6|\mathcal{A}|}$.

Let $\{j_a\}_{a \neq a_{max}} \subset C^\infty(X)$ be a partition of unity satisfying that

$$\sum_{a \neq a_{\max}} j_a^2 = 1 \quad \text{and} \quad \sup_{x \in X} \frac{\langle x \rangle}{\langle x^b \rangle} j_a < \infty, \quad b \not\subset a. \tag{4.21}$$

The functions j_a should furthermore be homogeneous of degree zero outside a compact set.

By the IMS localization formula (cf. [CFKS]) we get for $f \in C_0^\infty(\mathbb{R})$

$$M + f_{E,\kappa}(H) G f_{E,\kappa}(H) = \sum_{a \neq a_{max}} j_a M j_a + f_{E,\kappa}(H) j_a G j_a f_{E,\kappa}(H) - |\nabla j_a|^2.$$

As for the localization error we write

$$|\nabla j_a|^2 = K + (I - f_{E,\kappa}(H)) |\nabla j_a|^2 |(I - f_{E,\kappa}(H)),$$

with K compact. Using this and the observation that $j_a x \cdot \nabla V_b j_a$, $b \not\subset a$, is H-compact (see (4.21)) we get

$$M + f_{E,\kappa}(H) G f_{E,\kappa}(H) \geq \sum_{a \neq a_{\max}} (j_a M j_a + f_{E,\kappa}(H) j_a G^a j_a f_{E,\kappa}(H)) - K - C(I - f_{E,\kappa}(H))^2.$$

In the proof of the Mourre estimate for the usual N-body problem one uses here compactness of $f_{E,\kappa}(H) j_a - j_a f_{E,\kappa}(H_a)$, which is not an obvious statement for our problem (if true at all). Here we take a simpler path. Let $T(\eta) = (H - \eta)^{-1} j_a - j_a (H_a - \eta)^{-1}$, $\operatorname{Im} \eta \neq 0$. Write $I_a = V - V^a$ for the intercluster potential. We compute

$$\begin{aligned} T(\eta) &= (H - \eta)^{-1}(-j_a I_a + 2i \operatorname{Re}\{\nabla j_a \cdot p\})(H_a - \eta)^{-1} \\ &= K^1(\eta) + \sum_{j=1}^{\dim X} K_j^1(\eta) p_j \\ &= K^2(\eta) + \sum_{j=1}^{\dim X} p_j K_j^2(\eta), \end{aligned} \tag{4.22}$$

where the $K^i(\eta)$'s and the $K^i_j(\eta)$'s are compact operators with

$$\|K^i(\eta)\| + \|K^i_j(\eta)\| \leq C \left(\frac{1 + |\eta|}{|\operatorname{Im} \eta|} \right)^3. \tag{4.23}$$

This bound follows from (3.7). Let $\tilde{f}$ be an almost analytic extension of $f_{0,1}$, see (3.1). Then for $B \in \mathcal{B}(H)$

$$\begin{aligned}\mathrm{Re}\{(f_{E,\kappa}(H)j_a - j_a f_{E,\kappa}(H_a))B\} &= \kappa\,\mathrm{Re}\left\{\frac{1}{\pi}\int_{\mathbb{C}}(\bar{\partial}\tilde{f})(\eta)T(E+\kappa\eta)\,B\,du\,dv\right\}\\ &\geq -\sigma' p^2 - K. \end{aligned}\tag{4.24}$$

Here $\eta = u + iv$. Using this argument with $B = G^a j_a f_{E,\kappa}(H)$ and $B = G^a f_{E,\kappa}(H_a) j_a$ yields

$$\begin{aligned}&M + f_{E,\kappa}(H)Gf_{E,\kappa}(H)\\ &\quad\geq \sum_{a\neq a_{max}} j_a(M + f_{E,\kappa}(H_a)G^a f_{E,\kappa}(H_a))j_a - 2|\mathcal{A}|\sigma' p^2 - K - C(I - f_{E,\kappa}(H))^2.\end{aligned}\tag{4.25}$$

We now fix $\kappa > 0$ and $C > 0$ in accordance with Lemma 4.6 so that

$$\begin{aligned}M + f_{E,\kappa}(H)Gf_{E,\kappa}(H) \geq 2(d(E+\epsilon) - 2\epsilon) &\sum_{a\neq a_{max}} j_a f_{E,\kappa}(H_a)^2 j_a\\ &- 2|\mathcal{A}|\sigma' p^2 - K - C\sum_a j_a(I - f_{E,\kappa}(H_a))^2 j_a,\end{aligned}$$

with the convention that $j_{a_{\max}} = 1$. Using (4.24) on the first and the last term on the right-hand side (for $a \neq a_{\max}$) yields finally

$$M + f_{E,\kappa}(H)Gf_{E,\kappa}(H) \geq 2(d(E+\epsilon) - 2\epsilon)f_{E,\kappa}(H)^2 - K - 6|\mathcal{A}|\sigma' p^2 - C(I - f_{E,\kappa}(H))^2.$$

The estimate (4.20), and hence Theorem 4.2 i), now follows since $d(E+\epsilon) - 2\epsilon = d(E) - \epsilon$ for ϵ small enough, provided $E \notin \mathcal{F}(H)$. Here we used that $\mathcal{F}(H)$ is a closed set, which follows from Theorem 4.2 ii), applied with $a \neq a_{\max}$. □

In the following section, we will discuss how to extend our methods to potentials with L^p-type singularities. See Condition 1.2. As for Proposition 4.1, we refer the reader to the corresponding result in the case of Coulomb singularities, cf. Theorem 6.3. The crucial property is the H_0-boundedness of $|\nabla V_a^2|^{\frac{1}{2}}$, which follows from Lemma 5.1. As for Theorem 4.2, the proof is the same as above apart from two remarks: 1) Note that G is form H_0-compact in the two-body case and that $j_a G^a j_a$ is form H_0-compact in the general case. This follows from (5.4). 2) Replace $\sup_{t\in[0,1],y\in X^a}|G^a(t,x)|$ by $\|f_{E,1}(H^a)G^a f_{E,1}(H^a)\|$ in (4.19).

5. Singularities of L^p-type

In this section we impose Condition 1.2.

We shall outline a modification of the theory of Section 2 under this condition leading again to Theorem 2.4.

The principal tool will be the following boundedness result which applies to the L^p-functions V_2^a and $|\nabla V_2^a|^{\frac{1}{2}}$. We use notation from Section 3.

Lemma 5.1. *There exists a constant C such that for all $W \in L^p(X^a)$ with p as in Condition 1.2*

$$\|W^a((\cdot + c(t))^a)\langle H_0\rangle^{-\frac{1}{2}}\| \leq C\|W^a\|_{L^p(X^a)}. \tag{5.1}$$

Proof. We mimic the proof of [HMS, Lemma 6.5 (1)]: We abbreviate $f = f(x,t) = W((x + c(t))^a)$ and estimate with $f_s = f(x, t-s)$

$$\left\| fe^{-is(H_0 - \lambda - i\gamma)}f \right\| \leq \left\| fe^{-isp^2}f_s \right\| e^{-s\gamma} \leq \|W^a\|^2_{L^p(X^a)}(4\pi s)^{-\frac{\dim(X^a)}{p}}e^{-s\gamma}. \tag{5.2}$$

Using (5.2) as input we can from this point proceed exactly as in [HMS]. □

Remark. The finiteness of the left hand side of (5.1) is presumably wrong for the Coulomb singularity (in dimension ≥ 3). The above proof just misses to include that singularity. A similar problem was encountered in [HMS].

We can use Lemma 5.1 to mimic most of Section 3. For example Assumption 2.1 1) and 2) may easily be verified along the line of Section 3. However, for the last condition of Assumption 2.1 3) we meet problems at (3.9) and (3.15). (On the other hand clearly (3.12) is satisfied.) Those conditions are second order conditions and do not comply with Condition 1.2. In order to circumvent this problem we follow the idea of [ABG1, 2] and [Ta] that amounts to smearing out the singularities of G (due to those of V_2^a) in terms of the parameter ϵ that enters into the theory of Section 2. To be specific we define for a fixed $\phi^a \in C_0^\infty(X^a)$ with $\int \phi^a dx^a = 1$

$$\begin{aligned} V_\epsilon^a &= V_1^a + \phi_\epsilon^a * V_2^a; \qquad \phi_\epsilon^a(x^a) = |\epsilon|^{-\dim(X^a)}\phi^a(|\epsilon|^{-1}x^a), \\ G_\epsilon &= -\sum_{a\in\mathcal{A}} x \cdot \nabla V_\epsilon^a\left((\cdot + c(t))^a\right) - \delta. \end{aligned} \tag{5.3}$$

By Condition 1.2 and (5.1)

$$\left\|\langle H_0\rangle^{-\frac{1}{2}}(G_\epsilon - G)\langle H_0\rangle^{-\frac{1}{2}}\right\| = O\left(|\epsilon|^\kappa\right) \quad \text{and} \quad \left\|\langle H_0\rangle^{-\frac{1}{2}}\frac{d}{d\epsilon}G_\epsilon\langle H_0\rangle^{-\frac{1}{2}}\right\| = O\left(|\epsilon|^{\kappa-1}\right). \tag{5.4}$$

In particular for all $f \in C_0^\infty(\mathbb{R})$

$$\|f(H)(G_\epsilon - G)f(H)\| \to 0 \text{ for } \epsilon \to 0.$$

Hence since Assumption 2.1 4) is known for a fixed energy E for potentials of the form of this section (see the discussion at the end of Section 4) we may assume (2.16) with the factor G on the left hand side replaced by G_ϵ and with the right hand side independent of ϵ (assuming $|\epsilon|$ small).

Motivated by this observation we define

$$R_z(\epsilon) = (H - i\epsilon(M + f(H)G_\epsilon F(H)) - z)^{-1}$$

and mimic from this point the remaining part of Section 2.

Clearly the right hand side of (2.35) will now contain two error terms. One term comes from replacing G by $G_\epsilon - G$ and the other comes from differentiating G_ϵ with respect to ϵ. These new terms are treated by (5.4) and Lemma 2.9.

Continuing as before we get the conditions (3.9) and (3.15) now with G_ϵ (and not G). Using Condition 1.2 and (5.1) again, cf. (5.4), yields an additional factor $C\epsilon^{\kappa-1}$ to the bound $C(1+\|F_z(\epsilon)\|)$ obtained at the end of the proof of Theorem 2.4 (for $\alpha = 1$) in Section 2 for the term T_n^3. In the process we need the bound

$$\sup_{\epsilon>0} \left\| \langle H_0 \rangle^{-\frac{1}{2}} G_\epsilon \langle H_0 \rangle^{-\frac{1}{2}} \right\| < \infty,$$

cf. (5.4). Again we obtain a favourable differential inequality.

6. The Coulomb singularity

In this section we impose Condition 1.3. Here the potential has the form $V(t,x) = V(x + c(t))$, after the transformation $S^{a_{\max}}$ (see (1.15)).

We start by proving that the Coulomb singularity is relatively bounded with respect to the free Floquet Hamiltonian (with relative bound zero). Secondly we show how to extend our commutator estimate, and hence our result on the structure of the point spectrum, to systems with Coulomb singularities. Finally we prove the Limiting Absorption Principle. As a consequence we have absence of singular continuous spectrum. For the last part we need to impose the stronger condition (1.9) on the electric field.

We will use the integral kernel for the resolvent of the free Laplacian on $\mathbb{R}^3$

$$\left(p^2 + \kappa^2\right)^{-1}(x,y) = (4\pi)^{-1}\frac{e^{-\kappa|x-y|}}{|x-y|}, \tag{6.1}$$

and the following Hardy inequality

$$\left\|\frac{1}{|x|}\frac{1}{|p|}\right\|_{\mathcal{B}(L^2(\mathbb{R}^3))} < \infty, \tag{6.2}$$

see [RSII]. Let $\chi \in C_0^\infty(\mathbb{R}^3)$. We recall the following well-known estimate, see for example [Sa],

$$\left\|\chi\left(p^2 - \lambda - i\mu\right)^{-1}\chi\right\|_{\mathcal{B}(L^2(\mathbb{R}^3))} \le C\langle\lambda\rangle^{-\frac{1}{2}}, \tag{6.3}$$

which holds uniformly in $\mu > 1$. Here χ is considered as a multiplication operator and we use again the notation $\langle\lambda\rangle = (1+\lambda^2)^{1/2}$.

Lemma 6.1. *Let $\lambda \in \mathbb{R}$, $\mu > 1$ and $\chi \in C_0^\infty(\mathbb{R}^3)$. Then there exists $C > 0$ independent of λ and μ such that*

$$\left\|\chi\left(p^2 - \lambda - i\mu\right)^{-1}\right\|_{\mathcal{B}(L^2(\mathbb{R}^3))} \le C\langle\lambda\rangle^{-\frac{1}{4}}\mu^{-\frac{1}{2}} \tag{6.4}$$

and

$$\left\|\frac{1}{|x|}\left(p^2 - \lambda - i\langle\lambda\rangle^{\frac{1}{2}}\mu^{\frac{1}{2}}\right)^{-1}\right\|_{\mathcal{B}(L^2(\mathbb{R}^3))} \le C\langle\lambda\rangle^{-\frac{1}{4}}\mu^{-\frac{1}{4}}. \tag{6.5}$$

Proof. The estimate (6.4) follows from (6.3) after an application of the C^*-identity and the first resolvent formula.

As for (6.5) we employ a trick from [HMS], which is originally due to Agmon. By the C^*-identity and the first resolvent formula, we get,

$$\left\|\frac{1}{|x|}\left(p^2 - \lambda - i\langle\lambda\rangle^{\frac{1}{2}}\mu^{\frac{1}{2}}\right)^{-1}\right\|_{\mathcal{B}(L^2(\mathbb{R}^3))} \le \langle\lambda\rangle^{-\frac{1}{4}}\mu^{-\frac{1}{4}}\left\|\frac{1}{|x|}\left(p^2 - \lambda - i\langle\lambda\rangle^{\frac{1}{2}}\mu^{\frac{1}{2}}\right)^{-1}\frac{1}{|x|}\right\|^{\frac{1}{2}}_{\mathcal{B}(L^2(\mathbb{R}^3))}.$$

Using the explicit formula (6.1) for the integral kernel of the resolvent we can estimate

$$\left\|\frac{1}{|x|}\left(p^2 - \lambda - i\langle\lambda\rangle^{\frac{1}{2}}\mu^{\frac{1}{2}}\right)^{-1}\frac{1}{|x|}\right\|_{\mathcal{B}(L^2(\mathbb{R}^3))} \le \left\|\frac{1}{|x|}\left(p^2\right)^{-1}\frac{1}{|x|}\right\|_{\mathcal{B}(L^2(\mathbb{R}^3))},$$

which by (6.2) is finite. □

Let $Y \subset X$ and suppose $\dim Y \ge 3$. Write P_Y for the orthogonal projection on Y. Let V be a measurable function on Y which satisfies

$$|V(y)| \le C\max\left\{1, \frac{1}{|y|}\right\},$$

for some $C > 0$. Recall the functions b and c appearing in the transformation (1.15). We have

Theorem 6.2. *Let Y and V be as above and let $\mathcal{E}$ be as in* (1.8)*. Then V', given by $V'(t,x) = V(P_Y(x+c(t))$, satisfies the estimate*

$$\left\| V'\left(\tau + p^2 + i\mu\right)^{-1} \right\|_{\mathcal{B}(\mathcal{H})} \leq C|\mu|^{-\frac{1}{4}}, \quad \textit{for} \quad |\mu| > 1.$$

In particular V' is H_0-bounded with relative bound 0, where $H_0 = \tau + p^2$.

Remark. We note that (under Condition 1.3) this result together with the second resolvent equation implies the estimate

$$\left\| V'(H-\eta)^{-1} \right\| \leq C \max\left\{ |\operatorname{Im}\eta|^{-1}, |\operatorname{Im}\eta|^{-\frac{1}{4}} \right\}, \tag{6.6}$$

uniformly in $\eta \in \mathbb{C}$. This follows from Theorem 6.2 and the following application (6.7) of the remark after Lemma 4.5. Let L be an H-bounded operator which satisfy

$$e^{i2\pi t} L e^{-i2\pi t} = L.$$

Then (with $\operatorname{Im}\eta$ fixed)

$$\sup_{\operatorname{Re}\eta\in\mathbb{R}} \left\| L(H-\eta)^{-1} \right\| = \sup_{\operatorname{Re}\eta\in[0,2\pi]} \left\| L(H-\eta)^{-1} \right\|, \tag{6.7}$$

and consequently

$$\sup_{E\in\mathbb{R}} \|L(H-E-i)^{-1}\| < \infty. \tag{6.8}$$

Proof. We start by reducing the problem. Write $Y = Y_1 \oplus Y_2$ where $\dim Y_1 = 3$ and $y = (y_1, y_2)$. Let P_{Y_1} denote the orthogonal projection on Y_1. We use a cutoff of the form $\chi(y) = \chi_1(y_1)\chi_2(y_2)$, with $\chi_i \in C_0^\infty(Y_i)$ and $\chi_i(0) = 1$. We write $\chi'(t,y) = \chi(y + P_Y c(t))$ (and similarly for χ_1). By boundedness at infinity, it is sufficient to consider $\chi' V'$ instead of V' (c is bounded). We estimate, for $\mu > 1$, using the assumption and the estimate $\frac{1}{|y|} \leq \frac{1}{|y_1|}$, for $y = (y_1, y_2) \in Y$:

$$\left\| \chi' V'(\tau + p^2 - i\mu)^{-1} \right\|_{\mathcal{B}(\mathcal{H})} \leq C \left\| \frac{\chi_1'}{|y_1 + P_{Y_1}c(t)|} \left(\tau + p^2 - i\mu\right)^{-1} \right\|_{\mathcal{B}(\mathcal{H})}.$$

We diagonalize the part of p acting on $L^2(X \ominus Y_1)$ and use the coordinate change $\exp(iP_{Y_1}c(t) \cdot P_{Y_1}p)$, to move the time-dependence out of the potential and into the free energy.

$$\begin{aligned} &\left\| \chi' V'(\tau + p^2 - i\mu)^{-1} \right\|_{\mathcal{B}(\mathcal{H})} \\ &\leq C \sup_{\sigma\geq 0} \left\| \frac{\chi_1}{|y_1|} \left(\tau + p^2 + 2P_{Y_1}b(t)\cdot p + \sigma - i\mu\right)^{-1} \right\|_{\mathcal{B}(L^2([0,1];L^2(Y_1)))}. \end{aligned}$$

Recall that $2b = \dot{c}$. By (6.7), it is enough to consider $\sigma \in [0, 2\pi]$ and hence it suffices to show that

$$\left\| \frac{\chi}{|x|} \left(\tau + p^2 + 2b\cdot p - i\mu\right)^{-1} \right\|_{\mathcal{B}(L^2([0,1];L^2(\mathbb{R}^3))}} \leq C\mu^{-\frac{1}{4}}, \tag{6.9}$$

where $b : [0,1] \to \mathbb{R}^3$ is a periodic and absolutely continuous function and $\chi \in C_0^\infty(\mathbb{R}^3)$.

Abbreviate

$$R_0(\mu) := \left(\tau + p^2 - i\langle\tau\rangle^{\frac{1}{2}}\mu^{\frac{1}{2}}\right)^{-1} \quad \text{and} \quad R_b(\mu) := \left(\tau + (p-b)^2 - i\mu\right)^{-1}.$$

We write, using (6.5) in the last step,

$$\begin{aligned}\frac{\chi}{|x|}R_b(\mu) &= \frac{1}{|x|}R_0(\mu)\left(\tau + p^2 - i\langle\tau\rangle^{\frac12}\mu^{\frac12}\right)\chi R_b(\mu)\\ &= \frac{1}{|x|}R_0(\mu)\left\{\left(2b\cdot p - |b|^2 - i\langle\tau\rangle^{\frac12}\mu^{\frac12} - i\mu\right)\chi - 2i(p-b)\cdot\nabla\chi + \Delta\chi\right\}R_b(\mu)\\ &\quad + \frac{1}{|x|}R_0(\mu)\chi\\ &= \frac{1}{|x|}R_0(\mu)\left\{2b\cdot p\chi + i\langle\tau\rangle^{\frac12}\mu^{\frac12}\chi - 2ip\cdot\nabla\chi\right\}R_b(\mu) + O\left(\mu^{-\frac14}\right).\end{aligned} \tag{6.10}$$

By (6.5) we also get (abbreviating $\|\cdot\| := \|\cdot\|_{\mathcal{B}(L^2([0,1];L^2(\mathbb{R}^3)))}$)

$$\begin{aligned}\left\|\frac{1}{|x|}R_0(\mu)\langle p\rangle^{\frac12}\right\| &\le \left\|\frac{1}{|x|}R_0(\mu)\langle p\rangle^{\frac12}F(p^2 > 2|\tau|)\right\| + \left\|\frac{1}{|x|}R_0(\mu)\left\langle\sqrt{2|\tau|}\right\rangle^{\frac12}\right\|\\ &= \left\|\frac{1}{|x|}R_0(\mu)\langle p\rangle^{\frac12}F(p^2 > 2|\tau|)\right\| + O\left(\mu^{-\frac14}\right).\end{aligned}$$

As for the first term we use Hardy's inequality (6.2) and estimate

$$\begin{aligned}\left\|\frac{1}{|x|}R_0(\mu)\langle p\rangle^{\frac12}F(p^2 > 2|\tau|)\right\| &\le C\sup_{\sigma\ge0}\left(\frac{\langle\sigma\rangle^3}{\frac14\sigma^4+\mu}\right)^{\frac12}\\ &= O\left(\mu^{-\frac18}\right).\end{aligned}$$

In particular

$$\left\|\frac{1}{|x|}R_0(\mu)\langle p\rangle^{\frac12}\right\| = O(1). \tag{6.11}$$

By (6.5), (6.10) and (6.11), it suffices to show the following two estimates

$$\begin{aligned}\|\langle p\rangle^{\frac12}\chi R_b(\mu)\| &\le C\mu^{-\frac14}\\ \|\langle\tau\rangle^{\frac14}\chi R_b(\mu)\| &\le C\mu^{-\frac12}.\end{aligned} \tag{6.12}$$

As for the first estimate we conjugate with the unitary operator $e^{ic(t)\cdot p}$ and use that $\langle p\rangle^{\frac12}\chi(x + c(t))\langle p\rangle^{-\frac12}$ is bounded, to estimate

$$\left\|\langle p\rangle^{\frac12}\chi R_b(\mu)\right\| \le C\left\|\langle p\rangle^{\frac12}\tilde\chi\left(\tau + p^2 - i\mu\right)^{-1}\right\|,$$

where $\tilde\chi \in C_0^\infty$ is chosen such that $\tilde\chi(x)\chi(x + c(t)) = \chi(x + c(t))$.

Using the IMS localization formula, the C^*-identity and (6.4), we get the following estimate (viewing τ as a real parameter)

$$\begin{aligned}&\left\|\langle p\rangle^{\frac12}\tilde\chi\left(\tau + p^2 - i\mu\right)^{-1}\right\|^4_{\mathcal{B}(L^2(\mathbb{R}^3))}\\ &\le \left\|\tilde\chi\left(\tau + p^2 - i\mu\right)^{-1}\right\|^2_{\mathcal{B}(L^2(\mathbb{R}^3))} \times \left\|\left(\tau + p^2 + i\mu\right)^{-1}\tilde\chi\left(p^2+1\right)\tilde\chi\left(\tau + p^2 - i\mu\right)^{-1}\right\|_{\mathcal{B}(L^2(\mathbb{R}^3))}\\ &\le C_1\langle\tau\rangle^{-\frac12}\mu^{-1}\left\|\left(\tau + p^2 + i\mu\right)^{-1}\left\{\frac12\left(p^2\tilde\chi^2 + \tilde\chi^2p^2\right) + \tilde\chi^2 + |\nabla\tilde\chi|^2\right\}\left(\tau + p^2 - i\mu\right)^{-1}\right\|_{\mathcal{B}(L^2(\mathbb{R}^3))}\\ &\le C_2\mu^{-1}.\end{aligned} \tag{6.13}$$

This proves the first estimate of (6.12).

As for the second estimate we again conjugate with $e^{ic(t)\cdot p}$ and notice that we get the term $\langle \tau + 2b(t)\cdot p\rangle^{\frac{1}{4}}$. By interpolation and the estimate $(\tau + 2b(t)\cdot p)^2 \leq C(\tau^2 + p^2)$ we find that

$$\langle \tau + 2b(t)\cdot p\rangle^{\frac{1}{4}}(1+\tau^2+p^2)^{-\frac{1}{8}} \in \mathcal{B}(L^2([0,1];L^2(\mathbb{R}^3))).$$

Since $(1+\tau^2+p^2)^{\frac{1}{8}} \leq \langle\tau\rangle^{\frac{1}{4}} + \langle p\rangle^{\frac{1}{4}}$ we are reduced to the two estimates

$$\begin{aligned} \left\| \langle p\rangle^{\frac{1}{4}}\tilde{\chi}(\tau+p^2-i\mu)^{-1}\right\| &\leq C\mu^{-\frac{1}{2}} \\ \left\| \langle \tau\rangle^{\frac{1}{4}}\tilde{\chi}(\tau+p^2-i\mu)^{-1}\right\| &\leq C\mu^{-\frac{1}{2}}. \end{aligned}$$

The first estimate obviously follow from the first estimate in (6.12) (applied with $b=0$). The second follows from (6.4). ☐

Remark. Before we continue with verifying the positive commutator estimate we pause to show that the Coulomb potential is not H_0-compact. This is in contrast with singularities of L^p-type (cf. Lemma 5.1). Let $\psi \in C_0^\infty(\mathbb{R}^\nu)$, $\nu \geq 3$, and write

$$\psi_n = D_n\psi \in L^2(\mathbb{R}^\nu) \quad \text{and} \quad \varphi_n = e^{-i2\pi nt}\psi_n \in L^2\left([0,1];L^2(\mathbb{R}^\nu)\right),$$

where $(D_n\psi)(x) = n^{-\nu/2}\psi(x/n)$. Then

$$\|\varphi_n\|_{L^2([0,1];L^2(\mathbb{R}^\nu))} = 1 \quad \text{and} \quad \mathrm{w}-\lim_{n\to\infty}\varphi_n = 0.$$

On the other hand we compute, for $n>0$,

$$\begin{aligned} \left\| \left(\tau+p^2-i\right)^{-1}\frac{1}{|x|}\varphi_n\right\|^2_{L^2([0,1];L^2(\mathbb{R}^\nu))} &= \left\| \left(p^2-2\pi n-i\right)^{-1}\frac{1}{|x|}\psi_n\right\|^2_{L^2(\mathbb{R}^\nu)} \\ &= \frac{1}{n}\left\| \left(p^2-2\pi-in^{-1}\right)^{-1}\frac{1}{|x|}\psi\right\|^2_{L^2(\mathbb{R}^\nu)} \\ &= \left\langle \mathrm{Im}\left\{\left(p^2-2\pi-in^{-1}\right)^{-1}\right\}\frac{1}{|x|}\psi, \frac{1}{|x|}\psi\right\rangle_{L^2(\mathbb{R}^\nu)}. \end{aligned}$$

By the Limiting Absorption Principle for p^2 we thus have

$$\lim_{n\to\infty}\left\| \left(\tau+p^2-i\right)^{-1}\frac{1}{|x|}\varphi_n\right\|^2_{L^2([0,1];L^2(\mathbb{R}^\nu))} = \pi\left\langle \delta\left(p^2-2\pi\right)\frac{1}{|x|}\psi, \frac{1}{|x|}\psi\right\rangle_{L^2(\mathbb{R}^\nu)}$$

and hence, $\frac{1}{|x|}$ is not H_0-compact. (Note that the Coulomb singularity is in $L^2_{\mathrm{loc}}(\mathbb{R}^\nu)$ for $\nu\geq 3$.)

As a consequence of this remark potentials V satisfying $|x|V(x)\to\infty$ as $|x|\to 0$ are not H_0-bounded. (If V was H_0-bounded, then by a simple argument $|x|^{-1}$ would be relatively compact.)

We have the following result, which implies Theorem 1.5 in the case of Condition 1.3.

Theorem 6.3. *Suppose V satisfies Condition 1.3 and $\mathcal{E}$ is as in* (1.8). *Then the consequences of Proposition 4.1 and Theorem 4.2 hold for $H = \tau + p^2 + V(x + c(t))$.*

Proof. As for Proposition 4.1 we proceed in the same manner. The only place were we need to pay special attention is when treating the term $i[V, A_{n,R}]$, viewed as a form on $\mathcal{D} = \mathcal{D}(\tau) \cap \mathcal{D}(p^2)$. See (4.5).

The first part of the problem is computational. Here we note that $A_{n,R}$ maps $\mathcal{D}$ into $\mathcal{D}(p)$. This together with (6.2) shows that (4.5) holds for V's satisfying Condition 1.3 as well. Using the reguralization T_S, as in the proof of Proposition 4.1, we get as a form on $\mathcal{D}(H)$ (see (4.2))

$$\begin{aligned} i[V, A_{n,R}] &= \lim_{S\to\infty} T_S^* i[V, A_{n,R}] T_S \\ &= -\operatorname{Re}\{F_{n,R} \cdot \nabla V\} + 2R^{-1} \operatorname{Re}\left\{\widetilde{T}_R \operatorname{Re}\left\{\frac{\tau}{R} b \cdot \nabla V\right\} A_{n,R}\right\}. \end{aligned} \tag{6.14}$$

Here we used Theorem 6.2. The next step will be to verify that the second term on the right-hand side vanishes when the cutoffs in x and τ are removed. Note that the estimate (4.6) cannot be expected to hold. Consider the operators on $\mathcal{D}(H)$

$$B_k^1(R) = \sqrt{|b \cdot \nabla V|} \left(\frac{\tau}{R}\right)^k \widetilde{T}_R, \quad \text{for} \quad k \in \{0, 1\}.$$

By Condition 1.3 iii) and Theorem 6.2 we find that

$$\sup_{R>1} \|B_k^1(R)(H_0 + i)^{-1}\| < \infty, \quad \text{for} \quad k \in \{0, 1\}. \tag{6.15}$$

As for the remaining part of the second term of (6.14) we write

$$B_k^2(n, R) = \left\{\sqrt{|b \cdot \nabla V|} \langle p \rangle^{-1}\right\} \left\{\frac{\langle \tau \rangle}{R} \left(\frac{\tau}{R}\right)^k \widetilde{T}_R\right\} \left\{\langle \tau \rangle^{-1} \langle p \rangle A_n\right\}, \quad \text{for} \quad k \in \{0, 1\},$$

as operators on $\mathcal{D}(H)$. Here $A_n = \operatorname{Re}\{\langle x/n \rangle^{-1} x \cdot p\}$. The term in the first bracket is bounded by Condition 1.3 iii) and Hardy's inequality (6.2). As for the term in the second bracket, we note that it goes strongly to 0 as $R \to \infty$. The term in the last bracket, which is independent of R, is H-bounded. Hence for $k \in \{0, 1\}$

$$B_k^2(n, R)\psi \to 0, \quad \text{for} \quad R \to \infty,$$

for any $\psi \in \mathcal{D}(H)$ and n. This together with (6.15) shows that the second term of (6.14) vanishes in the limit $R \to \infty$.

As for the first term of (6.14) we note that $F_{n,R} \cdot \nabla V \to \langle x/n \rangle^{-1} x \cdot \nabla V$ as $R \to \infty$ in the sense of forms on $D(H)$. Finally $\langle x/n \rangle^{-1} x \cdot \nabla V \to x \cdot \nabla V$ as $n \to \infty$ again in the sense of forms on $\mathcal{D}(H)$. (We used here the Lebesgue dominated convergence theorem twice.)

We have thus verified that the estimate (4.7) holds under Condition 1.3 as well, and hence also Proposition 4.1.

As for Theorem 4.2, there are two problems we need to address. The first comes from the fact that we cannot expect G to have good compactness properties, as a form on $\mathcal{D}(H)$. See the discussion after the proof of Theorem 6.2. It enters at three places. We treat the first two occurrences of the problem simultaneously. Let $\tilde{G}$ denote G in the '$N = 2$' case and $j_a G_a j_a$ in the general case. In the proof of Theorem 4.2 for regular potentials, we used that $f(H)\tilde{G}f(H)$ was compact. Here we proceed differently. Let $R_0 > 0$ be such that $\tilde{G}$ is C^2 outside B_{R_0}, the ball of radius R_0. Let

$\chi \in C_0^\infty(B_{2R_0})$, $\chi = 1$ on B_{R_0}. Write for $R > 1$: $\chi_R(x) = \chi(x/R)$. Then, for $f \in C_0^\infty(\mathbb{R})$, (see (3.1)):

$$\begin{aligned} f(H)\tilde{G}f(H) &= f(H)\tilde{G}\chi_R f(H) - K \\ &= f(H)\tilde{G}[\chi_R, f(H)] - K \\ &= \frac{1}{\pi}\int_{\mathbb{C}} (\bar{\partial}\tilde{f})(\eta) f(H)\tilde{G}(H-\eta)^{-1} 2(\nabla\chi_R)\cdot p(H-\eta)^{-1} du dv - K \\ &= -R^{-1}B - \sum_{i=1}^{\dim X} K_i p_i - K. \end{aligned} \tag{6.17}$$

Here we used Theorem 6.2 and Condition 1.3 iii) in the last step. The operators K_i and K are compact and $B \in \mathcal{B}(\mathcal{H})$, with norms bounded uniformly in $R > 1$. We can now estimate (see also (4.24)),

$$f(H)\tilde{G}f(H) \geq -\frac{C(f)}{R} - \sigma p^2 - K,$$

for some $C(f) > 0$ and any $\sigma > 0$ (note that $K = K(R, \sigma)$). Choosing R large enough (depending on E) solves the problem.

The third occurrence of the compactness problem appears in the verification of (4.24), with singular potentials. Here one should replace j_a by $j_{a,R}(x) = j_a(x/R)$. Instead of (4.22) (with j_a replaced by $j_{a,R}$) we write

$$\begin{aligned} T_R(\eta) &= (H-\eta)^{-1}\left\{K^1(\eta) + R^{-1}B^1(\eta, R) + \sum_{j=1}^{\dim X} K_j^1(\eta, R)p_j\right\} \\ &= \left\{K^2(\eta) + R^{-1}B^2(\eta, R) + \sum_{j=1}^{\dim X} p_j K_j^2(\eta, R)\right\}(H_a - \eta)^{-1} \end{aligned}$$

where the $K^i(\eta)$'s and the $K^i_j(\eta)$'s are compact, $B^i(\eta, R) \in \mathcal{B}(H)$ and they all satisfy the bound (4.23), uniformly in $R > 1$. Let $S = S(R)$ be an H-bounded operator with $\|S(H_0 + i)^{-1}\|$ bounded uniformly in $R > 1$. We can now proceed as above, using the representation of $T_R(\eta)$ above, instead of (4.22). We replace the estimate (4.24) by

$$\mathrm{Re}\,\{(f_{E,\kappa}(H)j_a - j_a f_{E,\kappa}(H_a))\, S\} \geq -\frac{C(\kappa)}{R} - \sigma' p^2 - K. \tag{6.18}$$

Applied with $S = G^a j_{a,R} f_{E,\kappa}(H)$ and $S = G^a f_{E,\kappa}(H_a) j_{a,R}$, this estimate yields (4.25) with an extra $-\frac{C(\kappa)}{R} f_{E,\kappa}(H)^2$ term. By choosing R large (after having fixed κ) completes the handling of the compactness problem in the proof of Theorem 4.2.

The remaining point is to replace $\sup_{t,y}|G^a(t,y)|$ by $C^a = \|f_{E,1}(H^a)G^a f_{E,1}(H^a)\|$ in (4.19). By (6.8), C^a is bounded uniformly in E. □

As for the Limiting Absorption Principle Theorem 2.4, we observe that the Coulomb singularity is too strong for H, A to satisfy Assumption 2.1 3). Furthermore, the procedure described in Section 5 to circumvent this problem for L^p singularities fails as well. We elect instead to replace the generator of dilation by the operator

$$A_1 = \frac{1}{2}((x+c)\cdot p + p\cdot(x+c)). \tag{6.19}$$

(The reader should not confuse A_1 with the first element of the approximating family A_n.)

We consider, for $\delta' \geq 0$, decompositions of the commutator

$$i[H, A_1] = M + \hat{G}_1 = M_1 + G_1,$$

where

$$\begin{aligned} M &= 2p^2, & \hat{G}_1 &= -(x+c)\cdot\nabla V(\cdot + c) + 2b\cdot p,\\ M_1 &= 2p^2 + 2b\cdot p + \delta' \quad \text{and} & G_1 &= -(x+c)\cdot\nabla V(\cdot + c) - \delta'. \end{aligned}$$

We fix δ' large enough, such that there exists $\delta > 0$ (cf. Assumption 2.1) with

$$M_1 \geq \delta(M + I) \geq \delta I. \tag{6.20}$$

We have

Proposition 6.4. *Let $E \in \mathbb{R}$ and $\epsilon > 0$. There exist $\kappa > 0$, a compact operator K and a constant $C > 0$ such that as forms on $\mathcal{D}(H) \cap \mathcal{D}(M)$*

$$M + f_{E,\kappa}(H)\hat{G}_1 f_{E,\kappa}(H) \geq 2(d(E) - \epsilon) f_{E,\kappa}(H)^2 - K - C(1 - f_{E,\kappa}(H))^2 \tag{6.21}$$

and

$$M_1 + f_{E,\kappa}(H) G_1 f_{E,\kappa}(H) \geq 2(d(E) - \epsilon) f_{E,\kappa}(H)^2 - K - C(1 - f_{E,\kappa}(H))^2. \tag{6.22}$$

Proof. We start with (6.21) and follow the proof of the commutator estimate in Theorem 6.3.

Note first that $\hat{G}_1$ is not H-bounded. Instead we compute (as a form on $\mathcal{D}(p)$)

$$f(H) b\cdot p f(H) = f(H) b f(H)\cdot p + f(H) b\cdot [p, f(H)]. \tag{6.23}$$

Let $\{b_n\} \subset C^1([0,1]; X)$ with $b_n'(0) = b_n'(1)$ be chosen such that $b_n \to b$ in $L^\infty([0,1]; X)$. Then $f(H) b_n f(H) \to f(H) b f(H)$ in operator norm. On the other hand the Fourier series for each b_n converge to b_n in $L^\infty([0,1]; X)$ and hence by the first remark after Lemma 4.5 we find $f(H) b f(H) = 0$ if the support of f is sufficiently narrow. Since $[p, f(H)]$ is bounded we find that $f(H)\hat{G}_1 f(H)$ is bounded if the support of f is sufficiently narrow.

Next we consider the two-body problem. Let χ_R be as in (6.17). Write, for $R > 0$ large enough,

$$\begin{aligned} i[p, f(H)] &= i[p, f(H) - f(H_0)]\\ &= \frac{1}{\pi}\int_{\mathbb{C}} (\bar{\partial}\tilde{f})(\eta)\{-(H-\eta)^{-1}\nabla V (H-\eta)^{-1} V \langle p\rangle^{-1} + (H-\eta)^{-1}\nabla V\langle p\rangle^{-1}\}\\ &\qquad \times \{\langle p\rangle \chi_R (H_0 - \eta)^{-1}\langle p\rangle^{-1}\} du\, dv \langle p\rangle + K_2\\ &= K_1\langle p\rangle + K_2, \end{aligned}$$

where K_1 and K_2 are compact operators. Here we used (6.2) and Theorem 6.2 to bound the term in the first pair of brackets and the fact that the term in the second pair of brackets is compact. Combining this with (6.23) we get, for any $\sigma > 0$, a compact $K = K(\sigma)$ such that

$$f(H)\, b\cdot p\, f(H) \geq -\sigma p^2 - K.$$

This implies the commutator estimate for the two-body problem.

As for the commutator estimate in the general case, we focus on the reduction to subsystems argument, which is the only place where a problem occurs. More precisely, we need to estimate the following term in addition to (6.18)

$$\begin{aligned} T :&= f(H)j_{a,R}\, b\cdot p\, j_{a,R}f(H) - j_{a,R}f(H_a)\, b^a\cdot p^a\, f(H_a)j_{a,R} \\ &= 2\,\mathrm{Re}\,\{j_{a,R}f(H_a)\, b\cdot p\,(j_{a,R}f(H)-f(H_a)j_{a,R})\} \\ &\quad + (j_{a,R}f(H)-f(H_a)j_{a,R})\, b\cdot p\,(j_{a,R}f(H)-f(H_a)j_{a,R}). \end{aligned}$$

We used here that $f(H_a)b_a\cdot p_a f(H_a)=0$ if the support of f is sufficiently narrow. We recall that $j_{a,R}(x)=j_a(x/R)$. By the first equality in (4.22) one can verify that for any $\sigma>0$, there exist $C=C(f)>0$ and compact $K=K(\sigma,R,f)$ such that

$$T\geq -\left(\frac{C}{R}+\sigma\right)p^2-K.$$

This estimate is of the same type as (6.18) and thus implies (6.21).

As for (6.22) we estimate for any $\sigma>0$, using (2.1) and (6.20),

$$\begin{aligned} M_1 + f(H)G_1f(H) &- M - f(H)\hat{G}_1f(H) \\ &= \delta'(I-f(H)^2) - (b\cdot p - f(H)b\cdot pf(H)) \\ &\geq -\sigma f(H)^2 - \frac{\sigma}{\delta}M_1 - C(\sigma)(I-f(H))^2, \end{aligned}$$

for some $C(\sigma)>0$. From this estimate we find that (6.22) is a consequence of (6.21). See the argument leading up to (4.10). □

We have the following result, which implies Theorem 1.6 in the case of Condition 1.3.

Theorem 6.5. *Suppose V satisfies Condition 1.3 and $\mathcal{E}$ is as in* (1.9)*. For any $E\notin\sigma_{\mathrm{pp}}(H)\cup\mathcal{F}(H)$ the Limiting Absorption Principle (2.3) with A replaced by A_1 holds.*

Proof. We verify the technical conditions 1)-3) of Assumption 2.1 for the pair M_1, G_1. The result is then a consequence of (6.22) and Theorem 2.4.

As approximating family we take

$$A_{1,n}=\mathrm{Re}\{f_n(x)(x+c(t))\cdot p\},$$

where $f_n(x)=\langle x/n\rangle^{-1}$ as in Section 3.

As for 1) we compute, as a form on $\mathcal{D}(H_0)\cap\mathcal{D}(M_1)=\mathcal{D}(\tau)\cap\mathcal{D}(p^2)$,

$$i[M_1,H_0]=2i[b\cdot p,\tau]=-2\mathcal{E}\cdot p.$$

Hence, $i[M_1,H_0]$ extends from a form on $\mathcal{D}$ to an M_1-bounded operator. By viewing p as a multiplication operator, we find that $M_1^{-1}:\mathcal{D}(H_0)\to\mathcal{D}(H_0)$.

Before we continue we recall a property of the almost analytic extension $\tilde{\chi}$, where $\chi\in\mathcal{F}^0$. See Assumption 2.1 for the function class $\mathcal{F}_0$. We have, writing $\eta=u+iv$,

$$|\bar{\partial}\tilde{\chi}(\eta)|\leq C_N\langle\eta\rangle^{-N-1}|v|^N, \tag{6.24}$$

for any $N\in\mathbb{N}\cup\{0\}$.

We compute for $\chi \in \mathcal{F}^0$, see (3.1), as a form on $\mathcal{D}$:

$$i[M_1, \chi(H)] = i[-2\tau - 2V + 2b \cdot p, \chi(H)]. \tag{6.25}$$

In order to handle the right-hand side, we need to be more careful than in Section 3. Since $\mathcal{D}(H) = \mathcal{D}(H_0)$ and $[\tau, H_0] = 0$ we find that $(\tau + i)^{-1}$ leaves $\mathcal{D}(H)$ invariant. We will again need the regularization operator T_R, see (4.1). We compute, using (3.1) and the identity $\tau T_R = iR(I - T_R)$:

$$\begin{aligned} i[\tau T_R, \chi(H)] &= -\frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}\tilde{\chi}(\eta)(H-\eta)^{-1} i[\tau T_R, V](H-\eta)^{-1} du dv \\ &= iR\frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}\tilde{\chi}(\eta)(H-\eta)^{-1} i[T_R, V](H-\eta)^{-1} du dv \\ &= \frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}\tilde{\chi}(\eta)(H-\eta)^{-1} T_R b \cdot \nabla V T_R (H-\eta)^{-1} du dv. \end{aligned}$$

Here we used (6.6) with $V' = V$ and (6.24) to make sense of the first equality. Note that the term under the integral is bounded but not obviously integrable, cf. (6.24). Instead we write

$$\begin{aligned} &(H-\eta)^{-1} T_R b \cdot \nabla V T_R (H-\eta)^{-1} \\ &\qquad = \big\{(H-\eta)^{-1}(H_0 - \eta)\big\}\, T_R\, \big\{(H_0-\eta)^{-1} b \cdot \nabla V\big(H_0 + i\big)^{-1}\big\} T_R \\ &\qquad\quad \times \big\{(H_0 + i)(H+i)^{-1}\big\}\, (H-\eta)^{-1}(H+i). \end{aligned}$$

We thus find, by (4.2), (6.24), (6.6) (with $V' = |b \cdot \nabla V|^{\frac{1}{2}}$) and the Lebesgue dominated convergence theorem, that the form $i[\tau, \chi(H)]$ extends to an H-bounded operator. (Note that it is important that we use the decay in $\operatorname{Im}\eta$ coming from (6.6).) We thus take, see (6.25),

$$T_2' = -2i[\tau, \chi(H)] - i[V, \chi(H)]. \tag{6.26}$$

Note that the last term is bounded.

As for $i[b \cdot p, \chi(H)] = b \cdot i[p, \chi(H)] + [b, \chi(H)] \cdot p$. The last term is clearly $M_1^{\frac{1}{2}}$-bounded and we take

$$T_1 = 2i[b, \chi(H)] \cdot p.$$

We are reduced to considering $[p_i, \chi(H)]$ as a form on $\mathcal{D}$, since (multiplication with) b_i leaves $\mathcal{D}$ invariant. We proceed exactly as for $i[\tau, \chi(H)]$ and conclude that the form extends to an H-bounded operator. See also the argument which led up to (3.7). We finally take

$$T_2 = T_2' + 2b(t) \cdot [p, \chi(H)].$$

This concludes the verification of 1).

The condition 2) follows from Theorem 6.2.

The first part of 3) is clear. As for the last part of 3) we note that it is enough to verify (3.8). This follows since $i[G_1, A_n]$ is no more singular than V itself and hence H-bounded (uniformly in n). The estimate (3.8) can be verified in exactly the same manner as in Section 3, since (3.7) and (3.9) (with G replaced by G_1) both hold for potentials satisfying Condition 1.3. $\square$

7. Smoothness results

Let x, p and A be given as in Section 3, under Conditions 1.1 and 1.2; and $A = A_1$ be given by (6.19), under Condition 1.3. We introduce $\langle x\rangle = (1+x^2)^{\frac{1}{2}}, \langle p\rangle = (1+p^2)^{\frac{1}{2}}$ while $\langle A\rangle$ is given as in Theorem 2.4. We shall prove the following technical result and then use it to change the weights in the Limiting Absorption Principle statement which in turn implies two smoothness statements.

Lemma 7.1. *For all $s \in [0,\infty)$*

$$\langle A\rangle^s \langle p\rangle^{-s} \langle x\rangle^{-s} \text{ is bounded on } \mathcal{H}. \tag{7.1}$$

Proof. We introduce the following (standard) class of smooth symbols on X:

$$a \in S_l^m \leftrightarrow |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha,\beta} \langle x\rangle^{l-|\alpha|} \langle \xi\rangle^{m-|\beta|}; \quad x \in X, \xi \in X'.$$

For a $a \in S_l^m$ the notation $a^w(x,p)$ signifies the corresponding Weyl-quantization. See [Hö, Chapter 18] for definition and properties of this class of operators (treated in a more general context).

Using that A is the Weyl-quantization of the symbol $x\cdot\xi \in S_1^1$ (or $(x+c)\cdot\xi$) (7.1) for s even follows by the calculus [Hö, Theorems 18.5.4 and 18.6.3]. (Notice that in this case $\langle A\rangle^s$ is the Weyl-quantization of a symbol in S_s^s.) By the same arguments it suffice to prove (7.1) for $s \in (0,2)$ which in turn goes as follows:

Consider the analytic family

$$B(z) = e^{z^2} \langle A\rangle^z \langle p\rangle^{-z} \langle x\rangle^{-z}$$

on the strip $0 \le \operatorname{Re} z \le 2$. We notice that by the calculus the operator

$$\langle A\rangle^2 \langle p\rangle^{-(2+iy)} \langle x\rangle^{-2}$$

is polynomially bounded in $y \in \mathbb{R}$. On the other hand the first factor e^{z^2} is rapidly decaying as $z \to \infty$ in the strip. Consequently we infer from the three line theorem, cf. [RS, Appendix to IX.4], that indeed $B(s)$ is bounded. □

Remark. The above proof is due to Jensen [J]. It replaces our own and somewhat longer proof (based on a parametric construction) which appeared in a preliminary version of this paper. Although (7.1) appears in the literature we have not there been able to find a correct proof. The result is stated in [JMP] without proof and with an improper reference. Moreover another interpolation proof is given in [Hos], but it is wrong.

Corollary 7.2 (LAP). *For the examples of Sections 3, 5 and 6 suppose E is not an eigenvalue nor a threshold of H. Then there exists a neighbourhood $\mathcal{V}$ of E such that for $0 \le r < \frac{1}{2} < s \le 1$*

$$\sup_{\operatorname{Im} z\neq 0, \operatorname{Re} z \in \mathcal{V}} \left\| \langle p\rangle^r \langle x\rangle^{-s} (H-z)^{-1} \langle x\rangle^{-s} \langle p\rangle^r \right\| < \infty. \tag{7.2}$$

Proof. We may assume that $r < 1-s$ (cf. [Hö, Theorems 18.5.4 and 18.6.3]). Let $\alpha = s$ and $\beta = 2^{-1}(r+s)$. For any function $\phi \in \mathcal{C} = \mathcal{F} \otimes \mathcal{S}(X)$ (as defined in Section 3) we compute

$$\langle A\rangle^\alpha M^{-\beta} \langle x\rangle^{-s} \langle p\rangle^r \phi = B_2 B_1 \phi,$$

where $B_1 = b_1^w(x,p)$ for $b_1 \in S_0^0$ and $B_2 = \langle A\rangle^s \langle p\rangle^{-s}\langle x\rangle^{-s}$. In particular B_1 and B_2 are bounded. We compute for $\psi \in \mathcal{C}$ using (3.7) an Lemma 2.5

$$\begin{aligned}
&\langle \psi, (H-z)^{-1}\langle x\rangle^{-s}\langle p\rangle^r \phi\rangle \\
&= \langle \langle A\rangle^{-\alpha} M^\beta (H-\bar z)^{-1}\psi, B_2 B_1 \phi\rangle \\
&= \lim_{k\to 0} \langle \psi, (H-z)^{-1} M^\beta \langle A\rangle^{-\alpha}(1+\kappa M)^{-1} B_2 B_1 \phi\rangle .
\end{aligned}$$

Putting now $\psi = \langle x\rangle^{-s}\langle p\rangle^r \tilde\phi$ and repeating this argument we obtain

$$\begin{gathered}
\Big\langle \langle x\rangle^{-s}\langle p\rangle^r \tilde\phi, (H-z)^{-1}\langle x\rangle^{-s}\langle p\rangle^r \phi\Big\rangle = \lim_{\tilde\kappa\to 0}\lim_{\kappa\to 0} \Big\langle (1+\tilde\kappa M)^{-1} B_2 B_1 \tilde\phi, F_z(0)(1+\kappa M)^{-1} B_2 B_1 \phi\Big\rangle ;\\
F_z(0) = \langle A\rangle^{-\alpha} M^\beta (H-z)^{-1} M^\beta \langle A\rangle^{-\alpha}
\end{gathered}$$

Notice that we used (3.7) and Lemma 2.5 again (the latter with β replaced by $\beta + \frac12$).

Obviously we can apply this representation in combination with Theorem 2.4 to conclude that indeed

$$\Big|\Big\langle \langle x\rangle^{-s}\langle p\rangle^r \tilde\phi, (H-z)^{-1}\langle x\rangle^{-s}\langle p\rangle^r \phi\Big\rangle\Big| \le C\|\tilde\phi\|\|\phi\|.$$

□

For an account of Kato's theory of smooth operators we refer to [RS, Section XIII.7]. The following result is a consequence of Corollary 7.2 and [RS, Theorems XIII.25 and XIII.30].

Corollary 7.3. *Under the condition of Corollary 7.2 on r and s the operator $\langle p\rangle^r\langle x\rangle^{-s}$ is H-smooth on any closed 2π-periodic Borel set Ω not containing eigenvalues nor thresholds. In particular for any bounded Borel-measurable 2π-periodic function f supported in Ω*

$$\int_{-\infty}^{\infty} \big\|\langle p\rangle^r\langle x\rangle^{-s} e^{-i\sigma H} f(H)\phi\big\|^2\, d\sigma \le C\|\phi\|^2. \tag{7.3}$$

A small computation similar to the one in the proof of [KiY2, Theorem 2.4] converts the bound (7.3) to a similar one for the propagator $U(t,0)$ generated by the family $h(t)$ given by (1.18).

Corollary 7.4. *Under the conditions of Corollary 7.2 on r and s, for any $\psi \in L^2(X)$ and function g on the unit-circle with the property that $f(E) = g(e^{-iE})$ fulfills the requirements of Corollary 7.3*

$$\int_{-\infty}^{\infty} \|\langle p\rangle^r\langle x\rangle^{-s} U(t,0) g(U(1,0))\psi\|^2 dt \le C\|\psi\|^2.$$

Remark. By a conjugation, cf. (1.17), we obtain Theorem 1.7 from Corollary 7.4.

8. Decay properties of eigenfunctions

In this section we prove Theorem 1.8 and derive some regularity properties of eigenfunctions. We will work under Condition 1.3 and (1.8) and first study eigenfunctions of the Floquet Hamiltonian (1.18).

We introduce regularized weights

$$\theta_m(t) = \frac{t}{1+\frac{t}{m}}, \qquad m \geq 1,$$

and compute the t-derivative

$$\theta_m'(t) = \theta_m^{(1)}(t) = \left(1+\frac{t}{m}\right)^{-2}. \tag{8.1}$$

The notation θ_m and $\theta_m^{(k)}$ will denote (multiplication by) the functions $\theta_m(\langle x\rangle)$ and $\theta_m^{(k)}(\langle x\rangle)$ respectively. Note that we have the following properties uniformly in $m \geq 1$

$$\theta_m'\theta_m^{-1} \leq \langle x\rangle^{-1} \quad \text{and} \quad |\partial_x^\alpha \theta_m| \leq C_\alpha \langle x\rangle^{1-|\alpha|}. \tag{8.2}$$

We introduce furthermore

$$\Theta_m(t) = \Theta_m^{\sigma,\delta}(t) = \sigma t + \delta\theta_m(t), \qquad \sigma, \delta \geq 0,$$

and write as above Θ_m and $\Theta_m^{(k)}$ for $\Theta_m(\langle x\rangle)$ and $\Theta_m^{(k)}(\langle x\rangle)$.

We write $E_s(\cdot,\cdot) = E_s(\cdot,\cdot;m,n,\sigma,\delta) : [0,1]\times X \to \mathbb{R}$, $s \in \mathbb{R}$, for smooth bounded function families indexed by $n \geq m \geq 1$ and $\sigma, \delta \geq 0$ satisfying

$$|(\partial_t^k \partial_x^\alpha E_s)(t,x)| \leq C_{k,\alpha}\langle x\rangle^{s-|\alpha|}, \tag{8.3}$$

uniformly in $n \geq m \geq 1$ and locally uniformly in $\sigma, \delta \geq 0$. Note that $E_s E_t = E_{s+t}$.

We abbreviate $\hat{x} = x/\langle x\rangle$ and introduce some observables

$$\begin{aligned}
A &= \frac{1}{2}(x\cdot p + p\cdot x), & B &= \frac{1}{2}(\hat{x}\cdot p + p\cdot\hat{x}),\\
A_n &= \frac{1}{2}\left(F_n\cdot p + p\cdot F_n\right), & F_n &= \langle x/n\rangle^{-1}x,\\
\widetilde{A}_n &= \frac{1}{2}\left(\Theta_n\hat{x}\cdot p + p\cdot\hat{x}\Theta_n\right), & \widetilde{B}_m &= \frac{1}{2}\left(\Theta_m'\hat{x}\cdot p + p\cdot\hat{x}\Theta_m'\right).
\end{aligned}$$

We have the properties, see (8.2) and (8.3),

$$A_n = B\frac{\langle x\rangle}{\langle x/n\rangle} + iE_0 = \frac{\langle x\rangle}{\langle x/n\rangle}B - iE_0, \tag{8.4}$$

$$\widetilde{A}_n = B\Theta_n + iE_0 = \Theta_n B - iE_0, \tag{8.5}$$

$$\widetilde{B}_m = B\Theta_m' + iE_{-1} = \Theta_m' B - iE_{-1}, \tag{8.6}$$

$$(\widetilde{B}_m)^2 = B(\Theta_m')^2 B + E_{-2}. \tag{8.7}$$

Using the identity $i[H,\Theta_m] = 2\widetilde{B}_m$ and (8.2) we get, for $\lambda \in \mathbb{R}$ and $\psi \in \mathcal{C}$

$$\begin{aligned}
i(H-\lambda-\sigma^2)e^{\Theta_m}\psi &= ie^{\Theta_m}(H-\lambda)\psi + \left\{2\widetilde{B}_m + i(\delta^2(\theta_m')^2 + 2\sigma\delta\theta_m')\hat{x}^2 - i\sigma^2\langle x\rangle^{-2}\right\}e^{\Theta_m}\psi\\
&= ie^{\Theta_m}(H-\lambda)\psi + \left\{2\widetilde{B}_m + i\delta E_0 + iE_{-2}\right\}e^{\Theta_m}\psi.
\end{aligned} \tag{8.8}$$

Here $\mathcal{C}$ is as in Section 3.

In the following we write $\psi_m = e^{\Theta_m}\psi$, for $\psi \in \mathcal{D}(e^{\sigma\langle x\rangle})$. We will furthermore use the notation χ_S for $\chi_S(x) = \chi(x/S)$ where $\chi \in C_0^\infty$, $0 \leq \chi \leq 1$ and $\chi(x) = 1$ for $|x| < 1$. We have

Lemma 8.1. *Let $\sigma = 0$ and $\delta \geq 0$.*

i) *We have the inequality between forms on $\mathcal{C}$*

$$\mathrm{Re}\{A_n\widetilde{B}_m\} \geq \delta B\frac{\langle x\rangle\theta'_m}{\langle x/n\rangle}B - E_{-1}, \quad \textit{for} \quad n \geq m \geq 1.$$

ii) *Let $\epsilon > 0$. There exists $C > 0$ independent of $m \geq 1$ such that for $\psi \in \mathcal{C}$*

$$\|(H-\lambda)\psi_m\|^2 \leq C(m)\|(H-\lambda)\psi\|^2 + \epsilon\delta\left\langle B\frac{\langle x\rangle\theta'_m}{\langle x/n\rangle}B\right\rangle_{\psi_m} + C\left(\|\psi_m\|^2 + \|p\psi\|^2\right).$$

(The constants C and $C(m)$ can be chosen locally uniformly in $\delta \geq 0$.)

Proof. We estimate using (8.3), (8.4) and (8.6)

$$\begin{aligned}\mathrm{Re}\{A_n\widetilde{B}_m\} &= \delta\,\mathrm{Re}\,\{A_n\theta'_m B\} - \frac{1}{2}i[A_n, E_{-1}]\\ &= \delta B\frac{\langle x\rangle\theta'_m}{\langle x/n\rangle}B - \frac{\delta}{2}i[B, E_0] - E_{-1}\\ &\geq \delta B\frac{\langle x\rangle\theta'_m}{\langle x/m\rangle}B - E_{-1}\end{aligned}$$

Here we used monotonicity of $n \to \langle x/n\rangle^{-1}$ in the last step. This proves i).

As for ii) we estimate first using (8.1)

$$(\theta'_m)^2 \leq \langle x\rangle^{-1}\frac{\langle x\rangle\theta'_m}{\langle x/m\rangle} \leq S^{-1}\frac{\langle x\rangle\theta'_m}{\langle x/m\rangle} + \chi_S. \tag{8.9}$$

Now use (8.3), (8.7) and (8.8) to estimate

$$\begin{aligned}\|(H-\lambda)\psi_m\|^2 &\leq 3\|e^{\delta\theta_m}(H-\lambda)\psi\|^2 + 12\|\widetilde{B}_m\psi_m\|^2 + 3\|(\delta E_0 + E_{-2})\psi_m\|^2\\ &\leq C(m)\|(H-\lambda)\psi\|^2 + 12\langle\delta^2 B(\theta'_m)^2 B\rangle_{\psi_m} + C\|\psi_m\|^2.\end{aligned}$$

The result now follows by inserting (8.9) and choosing S large enough. □

Proposition 8.2. *Let $E \in \sigma_{\mathrm{pp}}(H)$ and $\varphi \in \mathcal{D}(H)$ with $H\varphi = E\varphi$. Suppose there exists $\delta > 0$ such that $e^{\delta|x|}\varphi \in \mathcal{H}$. Then $e^{\delta|x|}p\varphi \in \mathcal{H}$.*

Proof. We first derive two a priori estimates.

The first is an estimate of the regularized commutator. By (8.8) and Lemma 8.1 i) we get for $\psi \in \mathcal{C}$, writing E_0 for $\delta E_0 + E_{-2}$

$$\begin{aligned}&\langle i[H, A_n]\rangle_{\psi_m} = -2\,\mathrm{Re}\langle A_n\psi_m, e^{\delta\theta_m}i(H-E)\psi\rangle - 4\,\mathrm{Re}\langle A_n\widetilde{B}_m\rangle_{\psi_m} - \langle i[A_n, E_0]\rangle_{\psi_m}\\ &\leq -4\delta\left\langle B\frac{\langle x\rangle\theta'_m}{\langle x/n\rangle}B\right\rangle_{\psi_m} + 2\|A_n\psi_m\|\|e^{\delta\theta_m}(H-E)\psi\| + \langle E_0\rangle_{\psi_m}\\ &\leq \|p\psi_m\|^2 - 4\delta\left\langle B\frac{\langle x\rangle\theta'_m}{\langle x/n\rangle}B\right\rangle_{\psi_m} + C\left(\|\psi_m\|^2 + \|p\psi\|^2\right) + C(m,n)\|(H-E)\psi\|^2.\end{aligned} \tag{8.10}$$

As for the second a priori estimate we use Theorem 6.2, (3.2) and (3.4) and obtain

$$2\left\langle p^t F_n' p\right\rangle_{\psi_m} \le \langle i[H, A_n]\rangle_{\psi_m} + C\|\psi_m\|^2 + C_0\|(H-E)\psi_m\|^2. \tag{8.11}$$

We note that a similar estimate was used in the proof of Proposition 4.1 in the case of Coulomb singularities (cf. the proof of Theorem 6.3). We now apply Lemma 8.1 ii) with $\epsilon = 4/C_0$ and obtain

$$2\left\langle p^t F_n' p\right\rangle_{\psi_m} \le \langle i[H, A_n]\rangle_{\psi_m} + 4\delta\left\langle B\frac{\langle x\rangle\theta_m'}{\langle x/n\rangle}B\right\rangle_{\psi_m} + C(\|\psi_m\|^2 + \|p\psi\|^2) + C(m)\|(H-E)\psi\|^2. \tag{8.12}$$

Combining (8.10) and (8.13) we get

$$2\left\langle p^t F_n' p\right\rangle_{\psi_m} \le \|p\psi_m\|^2 + C\left(\|\psi_m\|^2 + \|p\psi\|^2\right) + C(m,n)\|(H-E)\psi\|^2. \tag{8.13}$$

Let $\{\phi_\ell\}_{\ell\in\mathbb{N}} \subset \mathcal{C}$, such that ϕ_ℓ converge to φ (the eigenfunction) in $\mathcal{D}(H)$. In order to deal with the $\|p\phi_\ell\|$ term, we introduce the two-parameter sequence $T_R\phi_\ell \in \mathcal{C}$, cf. (4.1) and (4.2). We have (using that $T_R = (H_0+i)^{-1}T_R(H_0+i)$ and that pT_R and $pe^{\delta\theta_m}T_R$ are H-bounded) the following limits

$$\begin{aligned}
&\lim_{R\to\infty}\lim_{\ell\to\infty}(H-E)T_R\phi_\ell = \lim_{R\to\infty}(H-E)T_R\varphi = (H-E)\varphi = 0,\\
&\lim_{R\to\infty}\lim_{\ell\to\infty}pT_R\phi_\ell = \lim_{R\to\infty}pT_R\varphi = \lim_{R\to\infty}T_R p\varphi = p\varphi,\\
&\lim_{R\to\infty}\lim_{\ell\to\infty}pe^{\delta\theta_m}T_R\phi_\ell = \lim_{R\to\infty}T_R p\varphi_m = p\varphi_m,
\end{aligned} \tag{8.14}$$

where as above $\varphi_m = e^{\delta\theta_m}\varphi$.

Substituting $\psi = T_R\phi_\ell$ and applying these limits to (8.13) and subsequently taking the limit $n\to\infty$ on the left-hand side yields (note that $\mathrm{s}-\lim_{n\to\infty}F_n' = I$)

$$\|p\varphi_m\|^2 \le C\left(\|\varphi_m\|^2 + \|p\varphi\|^2\right).$$

The observation $\|[p, e^{\delta\theta_m}]\varphi\| \le C\|\varphi_m\|$ now implies the result by letting m go to infinity. $\square$

Lemma 8.3. *Let $\sigma \ge 0$ and $0 \le \delta \le 1$.*

i) *We have the inequality between forms on $\mathcal{C}$*

$$\mathrm{Re}\{\widetilde{A}_n\widetilde{B}_m\} \ge B\Theta_m\Theta_m'B - (\sigma+\delta)E_{-1}, \quad \text{for} \quad n \ge m \ge 1.$$

ii) *Let $\epsilon, \epsilon' > 0$ and $0 < \sigma_l < \sigma_u$. There exist $C_1, C_2 > 0$ such that for $\psi \in \mathcal{C}$*

$$\begin{aligned}
&\|(H-\lambda-\sigma^2)\psi_m\|^2\\
&\le C(m)\|e^{\sigma\langle x\rangle}(H-\lambda)\psi\|^2 + (\epsilon+\delta C_1)\|\psi_m\|^2 + \epsilon'\langle B\Theta_m\Theta_m'B\rangle_{\psi_m} + C_2\left(\|\psi\|^2 + \|p\psi\|^2\right).
\end{aligned}$$

The constants C_1 and C_2 can be chosen independently of $n \ge m \ge 1$, $0 \le \delta \le 1$ and $\sigma \in \{0\} \cup [\sigma_l, \sigma_u]$.

Proof. As for i) observe first using (8.3) that

$$i[\widetilde{A}_n, E_s] = (\sigma+\delta)E_s \quad \text{and} \quad i[B, \Theta'_m E_0] = (\sigma+\delta)E_{-1}. \tag{8.15}$$

Now use this observation (with $s=-1$), (8.5) and (8.6) together with monotonicity of $n \to \Theta_n$ as in the proof of Lemma 8.1 i).

As for ii) we estimate first for $\sigma \in [\sigma_l, \sigma_u]$, $0 \le \delta \le 1$ and $S > 1$

$$(\Theta'_m)^2 \le \frac{\sigma+\delta}{\sigma}\langle x\rangle^{-1}\Theta_m\Theta'_m \le \frac{\sigma_u+1}{\sigma_l S}\Theta_m\Theta'_m + C(\sigma_l,\sigma_u)\chi_S. \tag{8.16}$$

For $\sigma = 0$ we estimate using (8.2)

$$(\Theta'_m)^2 \le \langle x\rangle^{-1}\Theta_m\Theta'_m \le S^{-1}\Theta_m\Theta'_m + C\chi_S. \tag{8.17}$$

Secondly we use (8.7) and (8.8) to estimate

$$\begin{aligned} \|(H-\lambda-\sigma^2)\psi_m\|^2 &\le 4\|e^{\Theta_m}(H-\lambda)\psi\|^2 + 16\|\widetilde{B}_m\psi_m\|^2 + 4\delta^2\|E_0\psi_m\|^2 + 4\|E_{-2}\psi_m\|^2 \\ &\le C(m)\|e^{\sigma\langle x\rangle}(H-\lambda)\psi\|^2 + 16\langle B(\Theta'_m)^2B\rangle_{\psi_m} + \langle E_{-2}\rangle_{\psi_m} + \delta C\|\psi_m\|^2. \end{aligned}$$

Write $E_{-2} = (1-\chi_S)E_{-2} + \chi_S E_{-2}$. The result now follows by inserting (8.16) if $\sigma \in [\sigma_l, \sigma_u]$ and (8.17) if $\sigma = 0$, and subsequently choosing S large enough. □

Proposition 8.4. *Let $E \in \sigma_{\mathrm{pp}}(H)\backslash\mathcal{F}(H)$ and suppose $\varphi \in \mathcal{D}(H)$ satisfies $H\varphi = E\varphi$. Then for any $\sigma > 0$ satisfying $E + \sigma^2 < \inf\{\mathcal{F}(H)\cap(E,\infty)\}$ we have $e^{\sigma|x|}\varphi \in \mathcal{H}$.*

Proof. Let E and φ be as in the formulation of the proposition. Define

$$\sigma_0 = \sup\left\{\sigma \ge 0 : e^{\sigma|x|}\varphi \in \mathcal{H}\right\}. \tag{8.18}$$

Assume that $\sigma_0 < \sigma_1 = (\inf\{\mathcal{F}(H)\cap(E,\infty)\} - E)^{1/2}$. Below we will work with σ of the form

$$\sigma = 0 \text{ if } \sigma_0 = 0 \quad \text{and} \quad \frac{\sigma_0}{2} \le \sigma < \sigma_0 \text{ if } \sigma_0 > 0.$$

The lower bound $\sigma_0/2$ is chosen for convenience. Let $\sigma' = (\sigma+\sigma_0)/2$. (Note that $\sigma \le \sigma' \le \sigma_0$ with equalities if and only if $\sigma_0 = 0$.)

We will use the commutator estimate in Theorem 6.3. Since $E + \sigma^2 \notin \mathcal{F}(H)$ there exist $\gamma > 0$, $\kappa > 0$, $C \ge 0$ and a compact operator $K(\sigma)$ such that the commutator estimate

$$M + f_{E+\sigma^2,\kappa}(H)Gf_{E+\sigma^2,\kappa}(H) \ge \gamma f_{E+\sigma^2,\kappa}(H)^2 - K(\sigma) - C(I - f_{E+\sigma^2,\kappa}(H))^2 \tag{8.19}$$

holds. Note that we can choose $\gamma = d(E)$, κ and C independently of $\sigma_0/2 \le \sigma \le \sigma_0$, cf. (2.1) and a compactness argument.

We start with two a priori estimates which are uniform in $n \ge m \ge 1$, $\sigma_0/2 \le \sigma \le \sigma_0$ and $0 < \delta \le 1$.

The first is an estimate of the regularized commutator. By the first part of (8.15) (applied with $s \in \{0, -2\}$), (8.8) and Lemma 8.3 i) we get for $\psi \in \mathcal{C}$

$$\begin{aligned}\langle i[H, \widetilde{A}_n]\rangle_{\psi_m} &= -2\,\mathrm{Re}\langle \widetilde{A}_n\psi_m, e^{\Theta_m} i(H-E)\psi\rangle - 4\,\mathrm{Re}\langle \widetilde{A}_n\widetilde{B}_m\rangle_{\psi_m} - \langle i[\widetilde{A}_n, \delta E_0 + E_{-2}]\rangle_{\psi_m}\\ &\le -4\,\langle B\Theta_m\Theta_m' B\rangle_{\psi_m} + 2\|\widetilde{A}_n\psi_m\|\|e^{\Theta_m}(H-E)\psi\| + (\sigma+\delta)\langle \delta E_0 + E_{-1}\rangle_{\psi_m}\\ &\le -4\,\langle B\Theta_m\Theta_m' B\rangle_{\psi_m} + (\sigma+\delta)\left(\frac{\gamma}{10} + \delta C_1\right)\|\psi_m\|^2 + C_2(\|e^{\sigma'\langle x\rangle}\psi\|^2 + \|e^{\sigma'\langle x\rangle}p\psi\|^2)\\ &\quad + C(m,n,\sigma)\|e^{\sigma\langle x\rangle}(H-E)\psi\|^2. \qquad (8.20)\end{aligned}$$

In the last step we used that $|(1-\chi_S)E_{-1}| \le \gamma/10$ for S large enough and the estimate

$$\Theta_n e^{\sigma\langle x\rangle} \le \begin{cases} nC & \sigma_0 = 0\\ (\sigma_0-\sigma)^{-1}Ce^{\sigma'\langle x\rangle} & \sigma_0 > 0\end{cases}.$$

The second a priori estimate uses (8.19) and the support properties of $f = f_{E+\sigma^2,\kappa}$. We obtain for $\psi \in \mathcal{C}$

$$\begin{aligned}\|\psi_m\|^2 &\le 2\|f(H)\psi_m\|^2 + 2\|(I-f(H))\psi_m\|^2\\ &\le \frac{2}{\gamma}\langle M + f(H)Gf(H)\rangle_{\psi_m} + \langle K(\sigma)\rangle_{\psi_m} + C\|(H-E-\sigma^2)\psi_m\|^2.\end{aligned}$$

Here we have absorbed $2/\gamma$ into the compact operator $K(\sigma)$. Write

$$f(H)Gf(H) = G - (I-f(H))G(I-f(H)) - 2\,\mathrm{Re}\{f(H)G(I-f(H))\}.$$

This identity, (2.1) and form boundedness of G on $\mathcal{D}(H)$ imply

$$\|\psi_m\|^2 \le \frac{2}{\gamma}\langle M+G\rangle_{\psi_m} + \frac{1}{10}\|\psi_m\|^2 + \langle K(\sigma)\rangle_{\psi_m} + C_0\|(H-E-\sigma^2)\psi_m\|^2.$$

Using Lemma 8.3 ii) with $\epsilon = 1/(10C_0)$ and $\epsilon' = 8/(\gamma C_0(\sigma_0+1))$ now gives

$$\begin{aligned}\|\psi_m\|^2 &\le \frac{2}{\gamma}\langle M+G\rangle_{\psi_m} + \frac{8}{\gamma(\sigma+\delta)}\langle B\Theta_m\Theta_m' B\rangle_{\psi_m} + \left(\frac{1}{5} + \delta C_3\right)\|\psi_m\|^2 + \langle K(\sigma)\rangle_{\psi_m}\\ &\quad + C_4(\|\psi\|^2 + \|p\psi\|^2) + C(m)\|e^{\sigma\langle x\rangle}(H-E)\psi\|^2. \qquad (8.21)\end{aligned}$$

Here we used that $(\sigma_0+1)^{-1} \le (\sigma+\delta)^{-1}$.

Let $\{\phi_\ell\}_{\ell\in\mathbb{N}} \subset \mathcal{C}$, such that ϕ_ℓ converge to φ (the eigenfunction) in $\mathcal{D}(H)$. In order to deal with the error terms, we introduce the three-parameter sequence $\chi_S T_R \phi_\ell \in \mathcal{C}$, cf. (4.1) and (4.2). We have (using that $T_R = (H_0+i)^{-1}T_R(H_0+i)$ and that $p\chi_S T_R$ is H-bounded) the following limits

$$\begin{aligned}&\lim_{\ell\to\infty} e^{\sigma\langle x\rangle}(H-E)\chi_S T_R\phi_\ell = e^{\sigma\langle x\rangle}\chi_S(H-E)T_R\varphi - 2iT_R e^{\sigma\langle x\rangle}\,\mathrm{Re}\{\nabla\chi_S\cdot p\}\varphi\\ &\lim_{\ell\to\infty} e^{\sigma'\langle x\rangle}p\chi_S T_R\phi_\ell = e^{\sigma'\langle x\rangle}\chi_S T_R p\varphi - ie^{\sigma'\langle x\rangle}\nabla\chi_S T_R\varphi\\ &\lim_{\ell\to\infty}\langle M+G\rangle_{e^{\Theta_m}\chi_S T_R\phi_\ell} = \langle M+G\rangle_{\chi_S T_R\varphi_m}\\ &\lim_{\ell\to\infty}\langle B\Theta_m\Theta_m' B\rangle_{e^{\Theta_m}\chi_S T_R\phi_\ell} = \langle B\Theta_m\Theta_m' B\rangle_{\chi_S T_R\varphi_m}\\ &\lim_{\ell\to\infty}\langle i[H,\widetilde{A}_n]\rangle_{e^{\Theta_m}\chi_S T_R\phi_\ell} = \langle i[H,\widetilde{A}_n]\rangle_{\chi_S T_R\varphi_m},\end{aligned}$$

where $\varphi_m = e^{\Theta_m}\varphi$. Removing the cutoffs using that $\nabla\chi_S = O(S^{-1})$ and Proposition 8.2 we obtain

$$\begin{aligned}
&\lim_{S\to\infty}\lim_{R\to\infty}\lim_{\ell\to\infty} e^{\sigma\langle x\rangle}(H-E)\chi_S T_R\phi_\ell = -\lim_{S\to\infty} 2ie^{\sigma\langle x\rangle}\operatorname{Re}\{\nabla\chi_S\cdot p\}\varphi = 0\\
&\lim_{S\to\infty}\lim_{R\to\infty}\lim_{\ell\to\infty} e^{\sigma'\langle x\rangle}p\chi_S T_R\phi_\ell = \lim_{S\to\infty}\left(e^{\sigma'\langle x\rangle}\chi_S p\varphi - ie^{\sigma'\langle x\rangle}\nabla\chi_S\varphi\right) = e^{\sigma'\langle x\rangle}p\varphi\\
&\lim_{S\to\infty}\lim_{R\to\infty}\lim_{\ell\to\infty}\langle M+G\rangle_{e^{\Theta_m}\chi_S T_R\phi_\ell} = \langle M+G\rangle_{\varphi_m}\\
&\lim_{S\to\infty}\lim_{R\to\infty}\lim_{\ell\to\infty}\langle B\Theta_m\Theta_m' B\rangle_{e^{\Theta_m}\chi_S T_R\phi_\ell} = \langle B\Theta_m\Theta_m' B\rangle_{\varphi_m}\\
&\lim_{S\to\infty}\lim_{R\to\infty}\lim_{\ell\to\infty}\langle i[H,\widetilde{A}_n]\rangle_{e^{\Theta_m}\chi_S T_R\phi_\ell} = \langle i[H,\widetilde{A}_n]\rangle_{\varphi_m}.
\end{aligned}\tag{8.22}$$

Substituting $\psi = \chi_S T_R\phi_\ell$ and applying first these limits to the two a priori estimates (8.20) and (8.21) and subsequently the limit $n\to\infty$ yields

$$\langle M+G\rangle_{\varphi_m} \le -\frac{4}{\sigma+\delta}\langle B\Theta_m\Theta_m' B\rangle_{\varphi_m} + \left(\frac{\gamma}{10}+\delta C_1\right)\|\varphi_m\|^2 + \frac{C_2}{\sigma+\delta}(\|e^{\sigma'\langle x\rangle}\varphi\|^2 + \|e^{\sigma'\langle x\rangle}p\varphi\|^2) \tag{8.23}$$

and

$$\begin{aligned}
\|\varphi_m\|^2 \le{}& \frac{2}{\gamma}\langle M+G\rangle_{\varphi_m} + \frac{8}{\gamma(\sigma+\delta)}\langle B\Theta_m\Theta_m' B\rangle_{\varphi_m}\\
&+\left(\frac{1}{5}+\delta C_3\right)\|\varphi_m\|^2 + \langle K(\sigma)\rangle_{\varphi_m} + C_4(\|\varphi\|^2+\|p\varphi\|^2).
\end{aligned}\tag{8.24}$$

Here we used that $\lim_{n\to\infty}\langle i[H,\widetilde{A}_n]\rangle_{\varphi_m} = (\sigma+\delta)\langle M+G\rangle_{\varphi_m}$.

Now choose $0<\delta\le 1$ such that

$$\delta\left(\frac{2C_1}{\gamma}+C_3\right) \le \frac{1}{5}. \tag{8.25}$$

If $\sigma_0>0$ we furthermore choose σ such that

$$\frac{\sigma_0}{2} \le \sigma < \sigma_0 < \sigma+\delta. \tag{8.26}$$

There exists (since σ has now been fixed) $S>0$ large such that

$$\langle K(\sigma)\rangle_{\varphi_m} = \langle K(\sigma)(I-\chi_S)\rangle_{\varphi_m} + \langle K(\sigma)\chi_S\rangle_{\varphi_m} \le \frac{1}{5}\|\varphi_m\|^2 + C\|\varphi\|^2. \tag{8.27}$$

Combining (8.23)–(8.25) and (8.27) yields uniformly in $m\ge 1$

$$\frac{1}{5}\|\varphi_m\|^2 \le C(\|e^{\sigma'\langle x\rangle}\varphi\|^2 + \|e^{\sigma'\langle x\rangle}p\varphi\|^2).$$

This proves, cf. Proposition 8.2, that $e^{(\sigma+\delta)|x|}\varphi\in\mathcal{H}$.

By (8.18) and (8.26) we have thus arrived at a contradiction. □

We are now in a position to translate the exponential decay estimate obtained for non-threshold eigenfunctions of H into a statement for non-threshold eigenfunctions of the monodromy operator $U(1,0)$.

Proof of Theorem 1.8.

By (1.17) is suffices to prove the statement with $U(1,0)$ given by the monodromy operator associated with $h(t)$ given by (1.18).

Recall from [Ya1] that any eigenfunction ψ of H with eigenvalue E can be written as

$$\psi(t) = e^{itE}U(t,0)\varphi, \tag{8.28}$$

where $U(1,0)\varphi = e^{-iE}\varphi$. In particular, by strong continuity of the map $t \to U(t,0)$ we find that $t \to \psi(t)$ is continuous.

Since $\psi \in \mathcal{D}(p)$, there exists t such that $\psi(t) \in \mathcal{D}(p)$. Furthermore we note that by [Ya2] and an interpolation argument, the map

$$t \to U(t,0) \in \mathcal{B}(\mathcal{H}^s(X)), \qquad 0 \le s \le 2, \tag{8.29}$$

(and its adjoint) is strongly continuous. Here $\mathcal{H}^s(X) = \langle p\rangle^{-s}L^2(X)$, $s \ge 0$, are Sobolev spaces. By (8.28) we conclude $\varphi \in \mathcal{D}(p)$ and that $t \to (p\psi)(t)$ is continuous.

Let $0 \le \delta < (\inf\{\mathcal{F}(H) \cap (E,\infty)\} - E)^{1/2}$. There exists, by Proposition 8.4, $0 \le t_0 \le 1$ such that $e^{\delta\langle x\rangle}\psi(t_0) \in L^2(X)$ and hence

$$\sup_{m\ge 1} \|e^{\delta\theta_m}\psi(t_0)\|_{L^2(X)} < \infty. \tag{8.30}$$

We write $\beta_m(t) = \|e^{\delta\theta_m}\psi(t)\|^2$ and $\beta_{m,\epsilon}(t) = \|(1+\epsilon p^2)^{-\frac{1}{2}}e^{\delta\theta_m}\psi(t)\|^2$. Due to the presence of the resolvent of ϵp^2 we get from [Ya2] that $\beta_{m,\epsilon}$ is C^1 and

$$\frac{d}{dt}\beta_{m,\epsilon}(t) = \left\langle \psi(t), \left(T_1 + \epsilon^{\frac{1}{2}}T_2\right)\psi(t)\right\rangle,$$

where

$$T_1 = 4\,\mathrm{Re}\left\{\mathrm{Re}\left\{\nabla\left(e^{\delta\theta_m}\right)\cdot p\right\}(1+\epsilon p^2)^{-1}e^{\delta\theta_m}\right\}$$
$$T_2 = e^{\delta\theta_m}(1+\epsilon p^2)^{-1} i\left[\epsilon^{\frac{1}{2}}p^2, V\right](1+\epsilon p^2)^{-1}e^{\delta\theta_m}.$$

Note that both T_1 and T_2 are bounded uniformly in $0 < \epsilon < 1$ as forms on $\mathcal{H}^1(X)$. This follows from (6.2) and uniform boundedness of $\epsilon^{\frac{1}{2}}\langle p\rangle(1+\epsilon p^2)^{-1}$. By strong continuity of the map (8.29) (with $s = 1$) and the Lebesgue dominated convergence theorem we thus find that β_m is C^1 and

$$\beta_m(t) = \int_{t_0}^{t} \langle \psi(s), 2\,\mathrm{Re}\{p\cdot\nabla(e^{2\delta\theta_m})\}\psi(s)\rangle\, ds + \beta_m(t_0).$$

From this identity we get the estimate

$$\beta_m(t) \le C\left(\|e^{\delta\langle x\rangle}\psi\|^2 + \|e^{\delta\langle x\rangle}p\psi\|^2\right) + \sup_{m\ge 1}\beta_m(t_0).$$

By Propositions 8.2 and 8.4, (8.30) and the Lebesgue monotone convergence theorem we thus obtain the result. $\square$

We end this section with a regularity result for eigenfunctions which will be useful (in the case of non-threshold eigenfunctions) in the following section on perturbation theory.

Let $g \in C^\infty(\mathbb{R})$ obey

$$g' \ge 0, \qquad g(t) = \begin{cases} 2 & \text{for } t > 3 \\ t & \text{for } |t| < 1 \\ -2 & \text{for } t < -3 \end{cases} \quad \text{and} \quad \sqrt{tg'g} \in C_0^\infty(\mathbb{R}). \tag{8.31}$$

We are in the following going to use a procedure which was introduced in [FS] for a technically somewhat similar problem.

Let $h(t) = g(t)/t$. We pick an almost analytic extension of h, denoted by $\tilde{h}$, with the properties:

$$\forall N:\ |\bar\partial\tilde{h}(\eta)| \le C_N\langle\eta\rangle^{-N-2}|v|^N \quad\text{and}\quad \tilde{h}(\eta) = \begin{cases} 2/\eta & \text{for } u>6,\ |v|<\rho(u-6) \\ -2/\eta & \text{for } u<-6,\ |v|<\rho(6-u) \end{cases}, \tag{8.32}$$

for some $\rho > 0$ (recall from (3.1) the notation $\eta = u + iv$). We furthermore choose $\tilde{h}$ such that $\overline{\tilde{h}(\eta)} = \tilde{h}(\bar\eta)$. This gives the representation, cf. (3.1),

$$g(t) = \frac{1}{\pi}\int_{\mathbb{C}} (\bar\partial\tilde{h})(\eta)t(t-\eta)^{-1}du\,dv.$$

We will use the following observables in addition to A and B

$$A_1 = \frac{1}{2}((x+c)\cdot p + p\cdot(x+c)), \qquad B_1 = \frac{1}{2}(\widehat{x+c}\cdot p + p\cdot\widehat{x+c}),$$
$$A_{1;n} = \frac{1}{2}\left(F_{1;n}\cdot p + p\cdot F_{1;n}\right), \qquad F_{1;n} = \frac{x+c}{\langle(x+c)/n\rangle}.$$

Let $g_m(t) = mg(t/m)$, for $m \ge 1$. We note that by a commutator argument of [Mo], cf. the proof of Lemma 2.5, there exists $\sigma > 0$ such that for $|v| \ge \sigma/m$: $(B_1/m - \eta)^{-1}$ and $(A_1/m-\eta)^{-1}$ preserve $\mathcal{D}(p^2)$ and $\mathcal{D}(\langle x\rangle)$ and we have the estimates

$$\left\|\langle p\rangle^2\left(\frac{B_1}{m}-\eta\right)^{-1}\langle p\rangle^{-2}\right\| + \left\|\langle p\rangle^2\left(\frac{A_1}{m}-\eta\right)^{-1}\langle p\rangle^{-2}\right\| \le C|v|^{-1} \tag{8.33}$$

and

$$\left\|\langle x\rangle\left(\frac{B_1}{m}-\eta\right)^{-1}\langle x\rangle^{-1}\right\| + \left\|\langle x\rangle\left(\frac{A_1}{m}-\eta\right)^{-1}\langle x\rangle^{-1}\right\| \le C|v|^{-1} \tag{8.34}$$

uniformly in $m \ge 1$ and η with $|v| \ge \sigma/m$. This motivates the decomposition into smooth bounded real valued functions $g_m = g_{1m} + g_{2m}$, where

$$g_{1m}(t) = \frac{m}{\pi}\int_{|v|>\sigma/m}(\bar\partial\tilde{h})(\eta)\left(1+\eta\left(\frac{t}{m}-\eta\right)^{-1}\right)du\,dv$$
$$g_{2m}(t) = \frac{m}{\pi}\int_{|v|\le\sigma/m}(\bar\partial\tilde{h})(\eta)\left(1+\eta\left(\frac{t}{m}-\eta\right)^{-1}\right)du\,dv. \tag{8.35}$$

Note that due to (8.32), the integral in the expression for g_{2m} is over a compact (and shrinking) set. This implies the properties

$$\sup_{t\in\mathbb{R}}|g_{2m}(t)| \le C < \infty \quad\text{and}\quad \sup_{t\in\mathbb{R}} t^{k+1}|g_{2m}^{(k)}(t)| \le C_k < \infty \quad\text{for}\quad k\ge 1. \tag{8.36}$$

(In fact the two suprema are $O(m^{-N})$ for any N.)

We have

$$g_{1m}(B_1), g_{1m}(A_1) \text{ maps } \mathcal{C} \text{ into } \mathcal{D}(\tau)\cap\mathcal{D}(p^2), \tag{8.37}$$

which will be used without comment in the following to make sense of forms on $\mathcal{C}$. This observation is a consequence of (8.33), in fact

Lemma 8.5. *Let $0 \le s \le 2$. There exists $C > 0$ independent of m such that*

$$\left\| \langle p \rangle^{s-|\alpha|} [g_{1m}(B_1), p^\alpha] \langle p \rangle^{-s} \right\| + \left\| \langle p \rangle^{s-|\alpha|} [g_{1m}(A_1), p^\alpha] \langle p \rangle^{-s} \right\| \le C,$$

for any multiindex α with $1 \le |\alpha| \le 2$.

Remark. The weighted commutators above should be viewed as the extension by continuity of forms on $\mathcal{D}(p^2)$.

Proof. We have as a form on $\mathcal{D}(p^2)$, cf. (8.35),

$$\begin{aligned}&\langle p \rangle^{s-|\alpha|} [g_{1m}(B_1), p^\alpha] \langle p \rangle^{-s}\\ &\quad = -\frac{1}{\pi} \int_{|v|>\sigma/m} (\bar{\partial}\tilde{h})(\eta)\eta \langle p \rangle^{s-|\alpha|} \left(\frac{B_1}{m} - \eta \right)^{-1} [B_1, p^\alpha] \left(\frac{B_1}{m} - \eta \right)^{-1} \langle p \rangle^{-s} du\, dv.\end{aligned}$$

We bound the expression under the integral using (8.32) (with $N = 2$) and (8.33) by a constant times $\langle \eta \rangle^{-3}$. This implies the lemma in the case of B_1. The statement with A_1 follows similarly. □

In the following we will again use the notation E_s, see (8.3). Only the parameters n and m are needed here and the estimate in (8.3) should be uniform in $n, m \ge 1$.

Lemma 8.6. *Let $\epsilon > 0$. There exists $C > 0$ independent of $n, m \ge 1$ such that the following estimates hold in the sense of forms on $\mathcal{C}$*

$$\mathrm{Re}\left\{ g_{1m}(B_1) A_{1;n} i[p^2, g_{1m}(B_1)] \right\} \ge -C\langle p \rangle^2 - \epsilon g_{1m}(B_1) p^2 g_{1m}(B_1) \tag{8.38}$$

and

$$\mathrm{Re}\left\{ g_{1m}(A_1) A_{1;n} i[p^2, g_{1m}(A_1)] \right\} \ge -C\langle p \rangle^2 - \epsilon g_{1m}(A_1) p^2 g_{1m}(A_1). \tag{8.39}$$

Proof. We drop the subscript 1 from A_1, $A_{1;n}$ and B_1 for the purpose of this proof.

First we compute (see also (3.2) and (8.11))

$$i[p^2, g_{1m}(B)] = -\frac{1}{\pi} \int_{|v|>\sigma/m} (\bar{\partial}\tilde{h})(\eta)\eta \left(\frac{B}{m} - \eta \right)^{-1} \left\{ p^t T(1) p + E_{-3} \right\} \left(\frac{B}{m} - \eta \right)^{-1} du\, dv, \tag{8.40}$$

where

$$T(n) = \frac{1}{\langle (x+c)/n \rangle} \left(I - \frac{|x+c\rangle\langle x+c|}{\langle x+c \rangle^2} \right) \ge 0. \tag{8.41}$$

We proceed in several steps.

Step I: Estimate using (8.32)–(8.34)

$$-\frac{1}{\pi} \mathrm{Re} \left\{ \int_{|v|>\sigma/m} (\bar{\partial}\tilde{h})(\eta)\eta g_{1m}(B) A_n \left(\frac{B}{m} - \eta \right)^{-1} E_{-3} \left(\frac{B}{m} - \eta \right)^{-1} du\, dv \right\} \ge -C\langle p \rangle^2. \tag{8.42}$$

Step II: Write $A_n = B\langle x+c\rangle\langle (x+c)/n\rangle^{-1} + iE_0$ (see (8.4)). Since $\overline{\tilde{h}(\eta)} = \tilde{h}(\bar{\eta})$ we can write

$$\begin{aligned}
&\frac{1}{\pi}\operatorname{Im}\left\{\int_{|v|>\sigma/m} (\bar{\partial}\tilde{h})(\eta)\eta g_{1m}(B)E_0\left(\frac{B}{m}-\eta\right)^{-1} p^t T(1)p\left(\frac{B}{m}-\eta\right)^{-1} du\,dv\right\}\\
&= -\frac{i}{2\pi}\int_{|v|>\sigma/m} (\bar{\partial}\tilde{h})(\eta)\eta\left\{g_{1m}(B)E_0\left(\frac{B}{m}-\eta\right)^{-1} p^t T(1)p\left(\frac{B}{m}-\eta\right)^{-1}\right.\\
&\quad \left. - \left(\frac{B}{m}-\eta\right)^{-1} p^t T(1)p\left(\frac{B}{m}-\eta\right)^{-1} E_0 g_{1m}(B)\right\} du\,dv. \qquad (8.43)
\end{aligned}$$

When reversing the order of the operators constituting the first term in the brackets above, we get a number of commutators which all remove one power of B, cf. Lemma 8.5 and the identity $B = m\{(B/m-\eta)+\eta\}$. We are thus left with two powers of p. Applying (8.32), (8.33) and combining with (8.42) now yields

$$\begin{aligned}
&\operatorname{Re}\{g_{1m}(B)A_n i[p^2, g_{1m}(B)]\}\\
&\geq -\frac{1}{\pi}\operatorname{Re}\left\{\int_{|v|>\sigma/m} (\bar{\partial}\tilde{h})(\eta)\eta g_{1m}(B)B\frac{\langle x+c\rangle}{\langle (x+c)/n\rangle}\left(\frac{B}{m}-\eta\right)^{-1} p^t T(1)p\left(\frac{B}{m}-\eta\right)^{-1} du\,dv\right\}\\
&\quad - C\langle p\rangle^2. \qquad (8.44)
\end{aligned}$$

Step III: We proceed by moving p^t to the left, p to the right and collecting functions of B to the left of functions of x. Arguing as for (8.43) we find that all errors come in the form of double commutators as indicated by the identity $LST + TSL = SLT + TLS + [[L,S],T]$. Each double commutator will remove two powers of B from the expression under the integral in (8.44). (We leave the somewhat tedious details to the reader.) We thus get

$$\operatorname{Re}\{g_{1m}(B)A_n i[p^2, g_{1m}(B)]\} \geq p^t \operatorname{Re}\{Bg'_{1m}(B)g_{1m}(B)T(n)\}p - Cp^2. \qquad (8.45)$$

Here we used that $\langle x+c\rangle\langle (x+c)/n\rangle^{-1}T(1) = T(n)$, cf. (8.41), and the identity

$$g'_{1m}(t) = -\frac{1}{\pi}\int_{|v|>\sigma/m} (\bar{\partial}\tilde{h})(\eta)\eta\left(\frac{t}{m}-\eta\right)^{-2} du\,dv. \qquad (8.46)$$

Step IV: Abbreviate $S_m = \sqrt{tg'_m g_m} \in C_0^\infty(\mathbb{R})$ and write $g_{1m} = g_m - g_{2m}$ and $g'_{1m} = g'_m - g'_{2m}$. By (8.36) we get for any $\sigma > 0$

$$\begin{aligned}
p^t \operatorname{Re}\{Bg'_{1m}(B)g_{1m}(B)T(n)\}p &\geq p^t \operatorname{Re}\{S_m(B)^2 T(n)\}p\\
&\quad - p^t \operatorname{Re}\{Bg'_m(B)g_{2m}(B)T(n)\}p - Cp^2\\
&\geq p^t \operatorname{Re}\{S_m(B)^2 T(n)\}p - \sigma p^t\left(Bg'_m(B)\right)^2 p - C(\sigma)p^2.
\end{aligned} \qquad (8.47)$$

There exists, cf. (8.31), $C_g > 0$ such that

$$\left(tg'_m(t)\right)^2 \leq C_g g_m(t)^2. \qquad (8.48)$$

We furthermore estimate

$$p^t g_m(B)^2 p \leq 2p^t g_{1m}(B)^2 p + Cp^2 \leq 2g_{1m}(B)p^2 g_{1m}(B) + Cp^2.$$

In the last step we used an argument similar to the one used to obtain (8.45). Inserting this estimate and (8.48) into (8.47) (applied with $\sigma = \epsilon/(2C_g)$) gives

$$p^t \operatorname{Re}\{Bg'_{1m}(B)g_{1m}(B)T(n)\} p \geq p^t \operatorname{Re}\{S_m(B)^2 T(n)\} p - \epsilon g_{1m}(B)p^2 g_{1m}(B) - C\langle p\rangle^2. \tag{8.49}$$

To complete the proof we pick an almost analytic extension of S_m and symmetrize

$$\operatorname{Re}\{S_m(B)^2 T(n)\} \geq S_m(B)T(n)S_m(B) - C \geq -C.$$

Here we used (8.41) in the last step. This estimate in conjunction with (8.49) and (8.45) yields (8.38).

The second estimate (8.39) is proved similarly and we leave it to the reader. □

Proposition 8.7. *Let φ be an eigenfunction of H with eigenvalue E. We have*
 i) *$p\varphi \in \mathcal{D}(B_1)$.*
 ii) *If $E \notin \mathcal{F}(H)$, then $p\varphi \in \mathcal{D}(A_1)$.*

Proof. For $\psi \in \mathcal{H}$ we abbreviate $\psi_m = g_{1m}(B)\psi$.

We compute (as forms on $\mathcal{C}$)

$$g_{1m}(B_1)A_{1;n}i[H, g_{1m}(B_1)] = g_{1m}(B_1)A_{1;n}\left\{i[\tau, g_{1m}(B_1)] + i[p^2, g_{1m}(B_1)] + i[V, g_{1m}(B_1)]\right\}. \tag{8.50}$$

Using the identity

$$i[\tau, B_1] = 2\operatorname{Re}\left\{\langle x+c\rangle^{-1} b^t \left(I - \frac{|x+c\rangle\langle x+c|}{\langle x+c\rangle^2}\right) p\right\}$$

together with (8.32)–(8.35) gives

$$\operatorname{Re}\{g_{1m}(B_1)A_{1;n}i[\tau, g_{1m}(B_1)]\} \geq -\frac{1}{12} g_{1m}(B_1)p^2 g_{1m}(B_1) - C\langle p\rangle^2. \tag{8.51}$$

As for the commutator with the potential we use (6.2) to write

$$i[V, B_1] = \langle x\rangle^{-1}\left\{\frac{\langle x\rangle}{\langle x+c\rangle}(x+c)\cdot\nabla V\langle p\rangle^{-1}\right\}\langle p\rangle$$

and obtain as above

$$\operatorname{Re}\{g_{1m}(B_1)A_{1;n}i[V, g_{1m}(B_1)]\} \geq -\frac{1}{12} g_{1m}(B_1)p^2 g_{1m}(B_1) - C\langle p\rangle^2. \tag{8.52}$$

Let $\psi \in \mathcal{C}$. Using (8.50)–(8.52) in conjunction with (8.38) (applied with $\epsilon = 1/12$) yields the following estimate

$$\begin{aligned}\langle i[H, A_{1;n}]\rangle_{\psi_m} &= -2\operatorname{Re}\{\langle A_{1;n}\psi_m, g_{1m}(B)i(H-E)\psi\rangle\} - 2\left\langle \operatorname{Re}\{g_{1m}(B)A_{1;n}i[H, g_{1m}(B)]\}\right\rangle_\psi \\ &\leq \frac{1}{2}\|p\psi_m\|^2 + C(\|\psi_m\|^2 + \|\psi\|^2 + \|p\psi\|^2) + C(m,n)\|(H-E)\psi\|^2.\end{aligned} \tag{8.54}$$

Here we used that $\|A_{1;n}\psi_m\| \le C(n)\|B_1\psi_m\| + C\|\psi_m\| \le C(m,n)(\|\psi_m\| + \|\psi\| + \|p\psi\|)$.

Next we estimate as for (8.11) but using (6.2) instead of Theorem 6.2. This gives

$$2\left\langle p^t F'_{1;n} p\right\rangle_{\psi_m} \le \langle i[H, A_{1;n}]\rangle_{\psi_m} + \frac{1}{2}\|p\psi_m\|^2 + C\|\psi_m\|^2. \tag{8.55}$$

Note that $F'_{1;n}(x) = F'_n(x+c)$, cf. (3.2). Inserting (8.54) into (8.55) yields

$$2\left\langle p^t F'_{1;n} p\right\rangle_{\psi_m} \le \|p\psi_m\|^2 + C\left(\|\psi\|^2 + \|p\psi\|^2\right) + C(m,n)\|(H-E)\psi\|^2. \tag{8.56}$$

Here we used in addition that $\|\psi_m\|^2 \le C(\|\psi\|^2 + \|p\psi\|^2)$.

Using the two first limits in (8.14) to replace ψ with φ in (8.56) and subsequently taking the limit $n \to \infty$ gives

$$\|p\varphi_m\|^2 \le C(\|\varphi\|^2 + \|p\varphi\|^2).$$

This implies, cf. (8.36) and Lemma 8.5, the estimate

$$\|g_m(B_1)p\varphi\|^2 \le C(\|\varphi\|^2 + \|p\varphi\|^2)$$

(uniformly in $m \ge 1$) from which i) follows.

As for ii) we note that the estimate (8.54) (with an extra $\|\langle x+c\rangle\langle (x+c)/n\rangle^{-1}p\psi\|^2$ term on the right-hand side) can be proved in a similar fashion with $\psi_m = g_{1m}(B_1)\psi$ replaced by $g_{1m}(A_1)\psi$, using (8.39) instead of (8.38). As above this together with Propositions 8.2 and 8.4 implies $p\varphi \in \mathcal{D}(A_1)$. $\square$

9. Perturbation theory

In this section we study the perturbation problem, (1.19), for the Floquet Hamiltonian and in the end our results are translated into a statement about the perturbation problem, (1.11), for the physical system. We use perturbation theory following [AHS]; this approach appears more suited for the N-body problem than the well-known Feshback method applied in, for example, [DJ].

Throughout the section it is assumed that V satisfies Condition 1.3 (although the same methods would give similar results under Conditions 1.1 or 1.2) and fields (1.9). More specific let $\mathcal{E}_0$ and $\mathcal{E}$ be given fields obeying (1.9). We define $b_0 = b_0(t)$, $c_0 = c_0(t)$ and $b = b(t)$, $c = c(t)$ to be the corresponding integrated fields, cf. (1.14). We are going to study an eigenvalue perturbation problem defined in terms of the fields $\mathcal{E}_\kappa = \mathcal{E}_0 + \kappa\mathcal{E}$ with κ a real (perturbation) parameter. The corresponding integrated fields are $b_\kappa = b_0 + \kappa b$ and $c_\kappa = c_0 + \kappa c$. The corresponding Floquet Hamiltonians are given by $H_\kappa = \tau + h_\kappa(t)$, where $h_\kappa(t) = p^2 + V(\cdot + c_\kappa)$, cf. (1.11), (1.18) and (1.19). Now let

$$E_0 \in \sigma_{\mathrm{pp}}(H_0)\backslash\mathcal{F}(H_0) \tag{9.1}$$

be given. The corresponding finite-dimensional projection is denoted P_0. We will study the eigenvalue perturbation problem for H_κ at E_0 using three ingredients from previous sections: 1) The Mourre estimate from Section 6. 2) Decay properties of P_0 from Section 8. 3) A uniform LAP obtained from the method of Section 2.

It is convenient first to change frame, $x \to x - c_\kappa$, as in the proof of Theorem 6.2: Thus we consider the operators

$$H_\kappa^1 = e^{-ip\cdot c_\kappa} H_\kappa e^{ip\cdot c_\kappa} = \tau + p^2 + 2b_\kappa\cdot p + V,$$

where $V = V(x)$ is independent of t. (Here and in the following the superindex 1 is used without parentheses.) Similarly we introduce $P_0^1 = e^{-ip\cdot c_0} P_0 e^{ip\cdot c_0}$. By the results of Section 8

$$\langle p\rangle A P_0^1,\ \langle p\rangle\langle x\rangle^2 P_0^1 \in \mathcal{B}(\mathcal{H}). \tag{9.2}$$

Here $\mathcal{B}(\mathcal{H})$ signifies the bounded operators on the Floquet Hilbert space $\mathcal{H}$, and A is the generator of dilations as given in Section 3.

We introduce

$$\widetilde{H}_0^1 = H_0^1 + P_0^1, \quad \text{and} \quad \widetilde{H}_\kappa^1 = H_\kappa^1 + P_0^1.$$

Note that $\widetilde{H}_0^1$ has no eigenvalues in a small open neighbourhood $\mathcal{V}$ of E_0. We write

$$i[\widetilde{H}_\kappa^1, A] = M_\kappa + \widetilde{G},$$

where

$$M_\kappa = 2p^2 + 2b_\kappa\cdot p + \delta' \quad \text{and} \quad \widetilde{G} = -x\cdot\nabla V(x) + i[P_0^1, A] - \delta'.$$

Here $\delta' > 0$ is fixed such that (6.20) holds for M_1 replaced by M_κ uniformly in $\kappa \in [-1, 1]$. By (9.2) and the finiteness of $m_0 = \dim\mathrm{Range}(P_0^1)$, we conclude that $[P_0^1, A]$ and $\langle p\rangle^{-1}[[P_0^1, A], A]\langle p\rangle^{-1}$ extend to compact operators. In particular, seen by first conjugating (6.22) by $e^{-ip\cdot c_0}$, there exist $\epsilon > 0$, $\gamma > 0$ and $C > 0$ such that:

$$M_0 + f_{E_0,\epsilon}(\widetilde{H}_0^1)\widetilde{G} f_{E_0,\epsilon}(\widetilde{H}_0^1) \geq \gamma I - C(I - f_{E_0,\epsilon}(\widetilde{H}_0^1))^2. \tag{9.3}$$

Here we used that $f_{E_0,\epsilon}(\widetilde{H}_0^1) - f_{E_0,\epsilon}(H_0^1)$ is compact and that $\mathrm{s}-\lim_{\epsilon\to 0} f_{E_0,\epsilon}(\widetilde{H}_0^1) = 0$. Clearly the technical conditions Assumption 2.1 1)-3) are satisfied for $\widetilde{H}_\kappa^1, A, M_\kappa$ and $\widetilde{G}$ for $|\kappa| \leq 1$ (see Section 6).

As in [AHS] we may perturb (9.3) as to obtain the existence of $\kappa_0 > 0$, $\epsilon > 0$, $\gamma > 0$ and $C > 0$ (with a new $\gamma > 0$ and a new $C > 0$) such that for all κ with $|\kappa| \leq \kappa_0$ we have:

$$M_\kappa + f_{E_0,\epsilon}(\widetilde{H}^1_\kappa)\widetilde{G} f_{E_0,\epsilon}(\widetilde{H}^1_\kappa) \geq \gamma I - C(I - f_{E_0,\epsilon}(\widetilde{H}^1_\kappa))^2. \tag{9.4}$$

Notice for example that for the term

$$T = f_{E_0,\epsilon}(\widetilde{H}^1_\kappa)\widetilde{G} f_{E_0,\epsilon}(\widetilde{H}^1_\kappa) - f_{E_0,\epsilon}(\widetilde{H}^1_0)\widetilde{G} f_{E_0,\epsilon}(\widetilde{H}^1_0), \tag{9.5}$$

we may write $T = \langle p \rangle T' + T' \langle p \rangle$ where

$$\|T'\| \leq C|\kappa| \tag{9.6}$$

and then estimate by the Cauchy-Schwarz inequality, cf. (4.10).

Consequently Assumption 2.1 is verified, and the Limiting Absorption Principle of Theorem 2.4 holds for $\widetilde{H}^1_\kappa$ in a small neighbourhood $\mathcal{V}$ of E_0 , cf. the Virial Theorem and Proposition 4.1. Due to the uniformity of (9.4) with respect to $\kappa \in [-\kappa_0, \kappa_0]$ we may obtain resolvent bounds from the method of Section 2 that are uniform in numerically small κ, see Lemma 9.2 stated below.

We will also need the following abstract Hölder continuity statement. If Q is a densely defined form which extends from $\mathcal{D}(Q)$ to a bounded form we write Q^0 for the corresponding bounded operator.

Proposition 9.1. *Suppose Assumption 2.1 is satisfied for some $E = E_0$ that is not an eigenvalue of H. Let $\mathcal{V}_\pm = \{z \in \mathbb{C} : \operatorname{Re} z \in \mathcal{V}, \pm \operatorname{Im} z \geq 0\}$ where $\mathcal{V}$ is a small neighbourhood of E_0, and let $0 \leq \beta < \frac{1}{2} < \alpha \leq 1$. Then (in addition to the conclusion of Theorem 2.4) the maps*

$$\mathcal{V}_\pm \ni z \to \left[\langle A \rangle^{-\alpha} M^\beta (H - z)^{-1} M^\beta \langle A \rangle^{-\alpha}\right]^0 \in \mathcal{B}(H)$$

are uniformly Hölder continuous with exponent $\eta := \frac{1-2\max\{\beta,1-\alpha\}}{3-2\max\{\beta,1-\alpha\}}$. Consequently, the limits

$$\langle A \rangle^{-\alpha} M^\beta (H - E \mp i0)^{-1} M^\beta \langle A \rangle^{-\alpha} := \lim_{\epsilon \to 0} \left[\langle A \rangle^{-\alpha} M^\beta (H - E \mp i\epsilon)^{-1} M^\beta \langle A \rangle^{-\alpha}\right]^0$$

exist and are uniformly Hölder continuous in $E \in \mathcal{V}$ with exponent η.

Remark. Under the conditions of Corollary 7.2 (in particular $0 \leq r < 1/2 < s \leq 1$) we obtain from Lemma 7.1 and Proposition 9.1 that the maps

$$E \to \langle p \rangle^r \langle x \rangle^{-s} (H - E \mp i0)^{-1} \langle x \rangle^{-s} \langle p \rangle^r$$

are uniformly Hölder continuous on $\mathcal{V}$ with exponent η.

Proof. We only consider the case of $\mathcal{V}_+$. Let $z, z' \in \mathcal{V}_+$ with $z \neq z'$ and let $\epsilon = |z - z'|^\gamma$ where $\gamma = \frac{2}{3-2\mu}$ and $\mu = \max\{\beta, 1 - \alpha\}$. By (2.44) and the comments accompanying (2.44) we have, uniformly in $z \in \mathcal{V}_+$,

$$\|F_z - F_z(\epsilon)\| \leq C\epsilon^{\frac{1}{2}-\mu}, \tag{9.7}$$

where

$$F_z = \left[\langle A \rangle^{-\alpha} M^\beta (H - z)^{-1} M^\beta \langle A \rangle^{-\alpha}\right]^0 \quad \text{and} \quad F_z(\epsilon) = D(\epsilon) M^\beta R_z(\epsilon) M^\beta D(\epsilon).$$

Recall that $D(\epsilon) = \langle A \rangle^{-\alpha} \langle \epsilon A \rangle^{\alpha-1}$. From Lemma 2.9 we furthermore find that

$$\|F_z(\epsilon) - F_{z'}(\epsilon)\| \leq C\epsilon^{-1}|z - z'|. \tag{9.8}$$

Combining (9.7) and (9.8) we get

$$\|F_z - F_{z'}\| \leq C(\epsilon^{\frac{1}{2}-\mu} + \epsilon^{-1}|z - z'|) = C|z - z'|^\eta.$$

□

Taking the uniformity of (9.4) into account, the proof of Proposition 9.1 yields uniform Hölder continuity results for our problem. We are only going to need the result for the case $\beta = 0$ and $\alpha = 1$.

Lemma 9.2. *There exists a neighbourhood $\mathcal{V}$ of E_0 and $\kappa_0 > 0$ such for $|\kappa| \leq \kappa_0$ the maps*

$$\mathcal{V}_\pm \ni z \to \langle A\rangle^{-1}(\widetilde{H}^1_\kappa - z)^{-1}\langle A\rangle^{-1} \in \mathcal{B}(H)$$

are Hölder continuous with exponent $1/3$, uniformly in $z \in \mathcal{V}_\pm$ and κ with $|\kappa| \leq \kappa_0$. Consequently, the limits

$$\langle A\rangle^{-1}(\widetilde{H}^1_\kappa - E \mp i0)^{-1}\langle A\rangle^{-1} := \lim_{\epsilon\to 0}\langle A\rangle^{-1}(\widetilde{H}_\kappa - E \mp i\epsilon)^{-1}\langle A\rangle^{-1}$$

exist and are Hölder continuous with exponent $1/3$, uniformly for $E \in \mathcal{V}$ and $|\kappa| \leq \kappa_0$.

We are going to use Lemma 9.2 as follows. We define the following operator on the finite dimensional space Range(P^1_0):

$$Q_\kappa(z) = P^1_0(\widetilde{H}^1_\kappa - z)^{-1}P^1_0, \qquad z \in \mathcal{V}_+ \cup \mathcal{V}_-.$$

By (9.2) and Lemma 9.2 the limits $Q_\kappa(E \pm i0) = \lim_{\epsilon\to 0} Q_\kappa(E \pm i\epsilon)$ exist.

We have the formula

$$(H^1_\kappa - z)^{-1}P^1_0(I - Q_\kappa(z)) = (\widetilde{H}^1_\kappa - z)^{-1}P^1_0,$$

which relates the resolvents of H^1_κ and $\widetilde{H}^1_\kappa$. It implies that the non-existence of an eigenvalue at $E \in \mathcal{V}$ is equivalent to the invertibility of $I - Q_\kappa(E + i0)$ as an operator on Range(P^1_0) (see [AHS]). We are thus led to study the operator $Q_\kappa(z)$.

Expanding the resolvent we get

$$Q_\kappa(z) = \gamma_z P^1_0 - 2\kappa\gamma_z^2 P^1_0 p\cdot bP^1_0 + 4\kappa^2\gamma_z^2 P^1_0 p\cdot b(\widetilde{H}^1_\kappa - z)^{-1}p\cdot bP^1_0, \tag{9.9}$$

where $\gamma_z = (1 + E_0 - z)^{-1}$. Note that (9.2) and Lemma 9.2 imply the existence of the limits of all the terms on the right hand side of (9.9) as $\operatorname{Im} z \to \pm 0$.

We estimate the last term on the right hand side of (9.9).

Lemma 9.3. *There exist a small neighbourhood $\mathcal{V}$ of E_0 and $C, \kappa_0 > 0$ such that for $E \in \mathcal{V}$ and $|\kappa| \leq \kappa_0$*

$$\left\| P^1_0 p\cdot b(\widetilde{H}^1_\kappa - E \mp i0)^{-1}p\cdot bP^1_0 - P^1_0 p\cdot b(\widetilde{H}^1_0 - E \mp i0)^{-1}p\cdot bP^1_0 \right\| \leq C|\kappa|^{1/3}.$$

Proof. We only give the proof for the "minus case"; it goes along the same line as the one of Proposition 9.1. Let $\epsilon = |\kappa|^{\frac{2}{3}}$. We use (9.2) and (9.7) with $\beta = 0$ and $\alpha = 1$ to estimate (uniformly in $z \in \mathcal{V}_+$ and $\kappa \in [-\kappa_0, \kappa_0]$)

$$\left\| P^1_0 p\cdot b(\widetilde{H}^1_\kappa - z)^{-1}p\cdot bP^1_0 - P^1_0 p\cdot b\tilde{R}^1_{\kappa,z}(\epsilon)p\cdot bP^1_0 \right\| \leq C\epsilon^{\frac{1}{2}}. \tag{9.10}$$

Compute

$$\tilde{R}^1_{\kappa,z}(\epsilon) - \tilde{R}^1_{0,z}(\epsilon) = \tilde{R}^1_{\kappa,z}(\epsilon)\{-2(1 - i\epsilon)\kappa p\cdot b + i\epsilon T\}\tilde{R}^1_{0,z}(\epsilon),$$

where T is given by (9.5).

Combining this with (9.2) and (2.28) yields

$$\left\| P^1_0 p\cdot b\tilde{R}^1_{\kappa,z}(\epsilon)p\cdot bP^1_0 - P^1_0 p\cdot b\tilde{R}^1_{0,z}(\epsilon)p\cdot bP^1_0 \right\| \leq C|\kappa|\epsilon^{-1}, \tag{9.11}$$

uniformly in $z \in \mathcal{V}_+$. □

Using Lemma 9.2, Lemma 9.3 and (9.9) we may write

$$\begin{aligned} &I - Q_\kappa(E + i0) \\ &= (1-\gamma_E)I + \gamma_E^2 \kappa A^{(1)} + {\gamma_E}^2 \kappa^2 A^{(2)} - i{\gamma_E}^2\kappa^2 B + O(|\kappa|^{\frac{7}{3}}) + |E-E_0|^{\frac{1}{3}} O(\kappa^2), \end{aligned} \tag{9.12}$$

where (considered as operators on $\mathrm{Range}(P_0^1)$)

$$A^{(1)} = 2P_0^1 b\cdot pP_0^1,\ A^{(2)} = -4\,\mathrm{Re}\{P_0^1 b\cdot p(\widetilde{H}_0^1 - E_0 - i0)^{-1} b\cdot pP_0^1\} \tag{9.13}$$

and

$$B = 4\,\mathrm{Im}\{P_0^1 b\cdot p(\widetilde{H}_0^1 - E_0 - i0)^{-1} b\cdot pP_0^1\}. \tag{9.14}$$

We conclude from (9.12), using the fact that B is non-negative, that indeed the left hand side is invertible if B is invertible (for κ numerically small and $E \in \mathbb{R}$ close to E_0). More generally it follows that the nullspace of the operator on the left hand side is a subset of the nullspace of its (negative) imaginary part, and from this and the invariance of the pure point spectrum (i.e. $\sigma_{\mathrm{pp}}(H_\kappa) = \sigma_{\mathrm{pp}}(H_\kappa^1)$) one readily obtains the following result.

Proposition 9.4. *There exist $\kappa_0 > 0$, a neighbourhood $\mathcal{V}$ of E_0 and $C > 0$ such that for $0 < |\kappa| \le \kappa_0$ we have*

1. *If $0 \notin \sigma(B)$, then $\sigma_{\mathrm{pp}}(H_\kappa) \cap \mathcal{V} = \emptyset$.*
2. *Any $E \in \sigma_{\mathrm{pp}}(H_\kappa) \cap \mathcal{V}$ must satisfy*

$$\left|E - E_0 - \kappa E_\kappa^1 - (\kappa E_\kappa^1)^2\right| \le C|\kappa|^{\frac{7}{3}}$$

for some

$$E_\kappa^1 \in \sigma\left(A^{(1)} + \kappa A^{(2)}\right).$$

We can readily translate Proposition 9.4 into a statement for the monodromy operator $U_\kappa(1,0)$ for the physical problem; see Section 1 (in particular (1.10), (1.12) and (1.16)). Let

$$\beta_\kappa = 2\kappa \int_0^1 b_0 \cdot b(s)ds + \kappa^2 \int_0^1 |b(s)|^2 ds.$$

Theorem 9.5. *Suppose Condition 1.3 and* (1.9). *Let $e^{-i\lambda_0} \in \sigma_{\mathrm{pp}}(U_0(1,0)) \backslash \mathcal{F}(U_0(1,0))$ and set $E_0 = \lambda_0 - \int_0^1 |b_0(s)|^2 ds$. There exist $\kappa_0 > 0$, a neighbourhood $\mathcal{V}$ of λ_0 and $C > 0$ such that for $0 < |\kappa| \le \kappa_0$ we have, with $A^{(1)}, A^{(2)}$ and B specified in* (9.13) *and* (9.14),

1. *If $0 \notin \sigma(B)$, then $\sigma_{\mathrm{pp}}(U_\kappa(1,0)) \cap e^{-i\mathcal{V}} = \emptyset$.*
2. *Any $e^{-i\lambda} \in \sigma_{\mathrm{pp}}(U_\kappa(1,0)) \cap e^{-i\mathcal{V}}$ must satisfy*

$$\left|e^{-i\lambda} - e^{-i(\lambda_0 + \beta_\kappa + \kappa E_\kappa^1 + (\kappa E_\kappa^1)^2)}\right| \le C|\kappa|^{\frac{7}{3}}$$

for some

$$E_\kappa^1 \in \sigma\left(A^{(1)} + \kappa A^{(2)}\right).$$

Remarks 9.6. One may view Theorem 9.5 as a first step in an attempt to show that $U(1,0)$ for a "generic" field $\mathcal{E}$ has purely absolutely continuous spectrum, cf. [AHS]. Progress in this direction would depend on verification of the condition of (1), $0 \notin \sigma(B)$, for a perturbation κb. We remark that [Ya3] has a discussion on absence of eigenvalues for generic frequencies in the two-body problem. The techniques of this section could also be used to discuss perturbation in frequency, although we have fixed the frequency to be one throughout the paper. Finally, in Appendix A we indicated how the statements of Theorem 9.5 for the special case $b_0 = 0$ can be obtained in a different way under an additional dilation-analyticity assumption on the potential. One virtue of that method is that expansions to higher order comes out easily; in fact it follows (for example) that any perturbed eigenvalue (that persists as an eigenvalue for all numerically small κ) is analytic in κ.

We end this section with a brief discussion of a physical interpretation of Theorem 9.5 for the case $b_0 = 0$. See [Ya3] for a related discussion of what is there phrased as the "photon property" of the electro-magnetic field. Since $b_0 = 0$ the operator $h_0 = p^2 + V$ is just the usual (time-independent) Hamiltonian with its threshold set $\mathcal{F}(h_0)$ defined as usual, see (1.20). The condition (9.1) reads

$$E_0 \in (\sigma_{\mathrm{pp}}(h_0) + 2\pi\mathbb{Z}) \backslash (\mathcal{F}(h_0) + 2\pi\mathbb{Z}).$$

In the interpretation of an atom (or more general a system of molecules) coupled to a reservoir of photons of energies $2\pi n$, this choice of unperturbed boundstate energy implies that any number of emissions or absorptions of photons does not land the atom at a threshold energy. We write

$$b = \sum_{n\in\mathbb{Z}} \hat{b}_n e^{i2\pi nt}, \quad \text{where} \quad \hat{b}_n = \int_0^1 e^{-i2\pi nt} b(t)dt.$$

Note that $\hat{b}_0 = 0$ and $\hat{b}_n = \bar{\hat{b}}_{-n}$. Motivated by the following explicit formulas for the operators $A^{(1)}, A^{(2)}$ and B of (9.13) and (9.14) we interpret the absolute value of the $\hat{b}_n$'s, $n > 0$, as the strength with which "photons" of energy $k_n = 2\pi n$ couple with the atom. The sign of n distinguishes between emission and absorption of a photon of energy $k_{|n|}$. In terms of the n'th Fourier coefficient of $\mathcal{E}$, $\hat{b}_n = \frac{1}{2\pi in}\hat{\mathcal{E}}_n$. Let $\mathcal{P} = (E_0 + 2\pi\mathbb{Z}) \cap \sigma_{\mathrm{pp}}(h_0)$. The operators $A^{(1)}$, $A^{(2)}$ and B are considered as acting on the space

$$E_{\mathcal{P}}(h_0)L^2(X) = \oplus_{\lambda\in\mathcal{P}} E_{\{\lambda\}}(h_0)L^2(X) \ (\simeq \mathrm{Range}(P_0))$$

.

We have

$$A^{(1)} = \{A^{(1)}_{\lambda_1,\lambda_2}\}_{\lambda_1,\lambda_2\in\mathcal{P}}$$

where

$$A^{(1)}_{\lambda_1,\lambda_2} = 2E_{\{\lambda_1\}}(h_0)\hat{b}_{(2\pi)^{-1}(\lambda_2-\lambda_1)} \cdot p\, E_{\{\lambda_2\}}(h_0).$$

This term can be interpreted as describing 1-photon interaction between different eigenspaces:

$$\lambda_2 \xrightarrow{k_n} \lambda_1 = \lambda_2 - 2\pi n.$$

Note that $A^{(1)} = 0$ if $|\mathcal{P}| = 1$.

Secondly

$$A^{(2)} = \{A^{(2)}_{\lambda_1,\lambda_2}\}_{\lambda_1,\lambda_2\in\mathcal{P}}$$

where

$$A^{(2)}_{\lambda_1,\lambda_2} = -4\,\mathrm{Re} \sum_{\substack{n,m\in\mathbb{Z} \\ 2\pi(n+m)=\lambda_2-\lambda_1}} E_{\{\lambda_1\}}(h_0)\hat{b}_n \cdot p\, \tilde{r}_0(\lambda_2 - 2\pi m)\, \hat{b}_m \cdot pE_{\{\lambda_2\}}(h_0).$$

with

$$\tilde{r}_0(\lambda) = \big(h_0 + E_{\{\lambda\}}(h_0) - \lambda - i0\big)^{-1}.$$

This term accounts for 2-photon interactions between eigenstates:

$$\lambda_2 \stackrel{k_m,k_n}{\longrightarrow} \lambda_1 = \lambda_2 - 2\pi(m+n).$$

Finally we have

$$B = \{B_{\lambda_1,\lambda_2}\}_{\lambda_1,\lambda_2\in\mathcal{P}}$$

where

$$B_{\lambda_1,\lambda_2} = 4\,\mathrm{Im} \sum_{\substack{n,m\in\mathbb{Z} \\ 2\pi(n+m)=\lambda_2-\lambda_1}} E_{\{\lambda_1\}}(h_0)\hat{b}_n \cdot p\,\tilde{r}_0(\lambda_2 - 2\pi m)\,\hat{b}_m \cdot pE_{\{\lambda_2\}}(h_0).$$

This term describes interactions between two eigenstates via the absolutely continuous spectrum:

$$\lambda_2 \xrightarrow{k_m} \lambda_2 - 2\pi m\ (\in \sigma_{\mathrm{ac}}(h_0) \cap (E_0 + 2\pi\mathbb{Z})) \xrightarrow{k_n} \lambda_1 = \lambda_2 - 2\pi(m+n).$$

Appendix A. An analytic perturbation theory

In this appendix we shall outline an analytic perturbation theory valid in the small field regime and under an additional dilation-analyticity assumption on the potential. In particular the theory leads to an alternative proof of Proposition 9.4 (and therefore also of Theorem 9.5) in the case $b_0 = 0$, see (9.14). As in Section 9 we consider $\mathcal{E}$ obeying (1.9). We are going to study the spectrum of operators of the form

$$H^1 = \tau + p^2 + 2b \cdot p + V, \tag{A.1}$$

where $b = b(t)$ is the integrated field, cf. (1.14), and V is independent of t. We shall replace b by κb and study corresponding eigenvalue problems, cf. Section 9.

First we study properties of H^1 given without any parameter-dependence (and with b arbitrary). We do not need V to be local. Our assumption on V is given in terms of the Combes class introduced using the notation $A^a = \mathrm{Re}\{x^a \cdot p^a\}$ and $\mathcal{H}_2^a = ((p^a)^2) + 1)^{-1} L^2(X^a)$

Definition A.1. *Fix $\phi > 0$. A symmetric operator V_a on $L^2(X^a)$ is in C_ϕ if and only if:*

(1) $\mathcal{D}((p^a)^2) \subseteq D(V_a)$.
(2) $V_a((p^a)^2) + 1)^{-1}$ *is compact.*
(3) *The $\mathcal{B}(\mathcal{H}_2^a, L^2(X^a))$-valued function*

$$\mathbb{R} \ni \theta \to V_a(\theta) = e^{i\theta A^a} V_a e^{-i\theta A^a} \in \mathcal{B}(\mathcal{H}_2^a, L^2(X^a))$$

has an analytic extension to the strip $\{\theta \mid |\mathrm{Im}\,\theta| < \phi\}$. (This extension is henceforth denoted by $V_a(\theta)$.)

We impose the condition that for some $\phi \in (0, \frac{\pi}{2})$

$$V = \sum_{a \in \mathcal{A}} V_a, \quad \text{where} \quad V_a \in C_\phi. \tag{A.2}$$

Under the above conditions the family of operators $h(t) = p^2 + 2b \cdot p + V$ on $L^2(X)$ has a dynamics U that preserves $\mathcal{D}(p^2)$, cf. [RS, Theorem X.70] (or [CMR]). Consequently by (1.13) H^1 is essentially self-adjoint on the subspace $\mathcal{D}(\tau) \cap \mathcal{D}(p^2)$ of $\mathcal{H}$. Next we consider the family of dilated operators

$$H^1(\theta) = \tau + e^{-2\theta} p^2 + 2e^{-\theta} b \cdot p + V(\theta), \tag{A.3}$$

where $0 \le \mathrm{Im}\,\theta < \phi$ and $V(\theta) = \sum_{a \in \mathcal{A}} V_a(\theta)$. For $0 < \mathrm{Im}\,\theta < \phi$,

$$\mathcal{D}(H^1(\theta)) = \mathcal{D}(H^1(\theta)^*) = \mathcal{D}(\tau) \cap \mathcal{D}(p^2).$$

Also note that for $\eta \in \mathbb{R}$, $e^{i\eta A} H^1(\theta) e^{-i\eta A} = H^1(\eta + \theta)$.

We need the class of analytic vectors N_ϕ: A vector $\phi \in \mathcal{H}$ is in N_ϕ if and only if the $\mathcal{H}$-valued function

$$\mathbb{R} \ni \theta \to \psi(\theta) = e^{i\theta A} \psi \in \mathcal{H}$$

has an analytic extension (denoted by $\psi(\theta)$) to the strip $\{\theta \mid |\mathrm{Im}\,\theta| < \phi\}$.

Lemma A.2. *Let $C > 0$ be given. There exists $\theta_0 \in (0, \phi)$ such that if $\|b\|_{L^\infty} \le C$, $\mathrm{Im}\,\theta \in [0, \theta_0]$ and $\mathrm{Im}\,z \ge 1$, then $z \in \rho(H^1(\theta))$ with*

$$\left\|(H^1(\theta) - z)^{-1}\right\| \le 3, \tag{A.4}$$

and for all $\psi \in N_\phi$
$$\langle \psi, (H^1 - z)^{-1}\psi\rangle = \langle \psi(\bar\theta), (H^1(\theta) - z)^{-1}\psi(\theta)\rangle. \tag{A.5}$$

Proof. We mimic a numerical range argument of [HS]. It suffices to consider $\theta \to i\theta$ with $\operatorname{Im}\theta = 0$. Let $\psi \in \mathcal{D}(H^1(i\theta))$ be any given normalized vector. We abbreviate $\langle\psi, A\psi\rangle = \langle A\rangle_\psi$ for relevant operators A.

$$\operatorname{Im}\langle H^1(i\theta) - z\rangle_\psi = \sin(-2\theta)\langle p^2\rangle_\psi + 2\sin(-\theta)\langle b\cdot p\rangle_\psi + \operatorname{Im}\langle V(i\theta) - V(0)\rangle_\psi - \operatorname{Im} z. \tag{A.6}$$

We assume $\theta \le \frac{\phi}{2}$. Then $\sin(-2\theta) \le -\frac{4\theta}{\pi}$ and $|\sin(-\theta)| \le \theta$, and (by a Cauchy estimate and interpolation) for all $y' > 1$

$$\begin{aligned}|\operatorname{Im}\langle V(i\theta) - V(0)\rangle_\psi| &\le \theta \sup_{0\le\theta'\le\theta}\left\|(p^2+y')^{-\frac12}\frac{d}{d\theta'}V(i\theta')(p^2+y')^{-\frac12}\right\|\langle p^2+y'\rangle_\psi\\ &\le \theta C(y')\langle p^2+y'\rangle_\psi,\ C(y') = \frac{24}{\phi}\sup_{|\eta|\le\frac23\phi}\left\|V(\eta)(p^2+y')^{-1}\right\|.\end{aligned} \tag{A.7}$$

Clearly $C(y') \to 0$ for $y' \to \infty$.

From (A.6), (A.7) and the Cauchy-Schwarz inequality we get the estimate

$$\operatorname{Im}\langle H^1(i\theta) - z\rangle_\psi \le -\theta\left(\frac{4}{\pi} - 1 - C(y')\right)\langle p^2\rangle_\psi + \theta\,\|b\|^2_{L^\infty} + \theta C(y')y' - \operatorname{Im} z.$$

Now we fix a large y' such that $C(y') \le \frac{4}{\pi} - 1$ making the first term on the right hand side non-positive. Then $-1/3$ is a total upper bound for all $\theta \le \theta_0$ where

$$\theta_0 = \min\left(\frac{\phi}{2}, \frac{1}{3C(y')y'}, \frac{1}{3C^2}\right),$$

yielding (A.4). As for (A.5) we consider for given z with $\operatorname{Im} z \ge 1$ and b with $\|b\|_{L^\infty} \le C$ the function
$$\theta \to \left\langle \psi(\bar\theta), (H^1(\theta) - z)^{-1}\psi(\theta)\right\rangle.$$
It is analytic and constant on the open strip $\{\theta \in \mathbb{C}|\ \operatorname{Im}\theta \in (0,\theta_0)\}$. Using the fact that $\mathcal{D}(\tau)\cap\mathcal{D}(p^2)$ is a core for H^1 one readily obtains in combination with (A.4) that

$$s - \lim_{\theta\downarrow 0}(H^1(i\theta) - z)^{-1} = (H^1 - z)^{-1},$$

which (together with a similar argument for $\theta \uparrow \theta_0$) shows (A.5) in the case $\operatorname{Im} z \ge 1$. □

Now we replace b by $b_\kappa = \kappa b$ in (A.1) and let E_0 be given as in (9.1) (using the same notation as in Section 9). To obtain an alternative proof of Proposition 9.4 in this case we need to extend the formula (A.5) for fixed θ, say $\theta = i\theta_0$, and all real, numerically small κ to $z \in S_\epsilon = \{z|\operatorname{Im} z > 0,\ \operatorname{Re} z \in [-\epsilon+E_0, E_0+\epsilon]\}$. Here $\epsilon > 0$ is chosen such that $[-\epsilon+E_0, E_0+\epsilon]\cap(\mathcal{F}(h_0)+2\pi\mathbb{Z}) = \emptyset$. Our argument is perturbative. It follows from [BC] that E_0 is a discrete eigenvalue of $H^1_0(i\theta_0)$ and that $\sigma(H^1_0(i\theta_0))\cap\{z|\operatorname{Im} z > 0\} = \emptyset$. Since $2e^{-i\theta_0}b_\kappa\cdot p$ is a relatively bounded (and analytic) perturbation we may pick a small circle C_δ centered at E_0 and a small $\kappa_0 > 0$ such that $C_\delta \cap \sigma(H^1_\kappa(i\theta_0)) = \emptyset$ for $|\kappa| \le \kappa_0$. The dimension of the range of the corresponding Riesz projection

$$\frac{i}{2\pi}\int_{C_\delta}(H^1_\kappa(i\theta_0) - z)^{-1}dz$$

is exactly equal to the multiplicity m_0 of the eigenvalue E_0 of H_0^1. In particular if $E \in S_\epsilon \cap \sigma(H_\kappa^1(i\theta_0))$ then E is a discrete eigenvalue of $H_\kappa^1(i\theta_0)$. By extending (A.5) to a small neighbourhood of E (excluding E) and invoking the density of N_ϕ we then conclude that indeed such E cannot exist. Thus the formula

$$\langle \psi, (H_\kappa^1 - z)^{-1}\psi \rangle = \langle \psi(-i\theta_0), (H_\kappa^1(i\theta_0) - z)^{-1}\psi(i\theta_0) \rangle \tag{A.8}$$

is valid for $z \in S_\epsilon$ and for all real κ with $|\kappa| \le \kappa_0$. Mimicking [Si] under use of (A.8) leads to the following conclusion. Note that the operator B of (9.14) is well-defined in the present context too.

Proposition A.3. *There exist $\kappa_0 > 0$ and a neighbourhood $\mathcal{V}$ of E_0 such that for all real κ with $0 < |\kappa| \le \kappa_0$ we have*

(1) *If $0 \notin \sigma(B)$, then $\sigma_{\mathrm{pp}}(H_\kappa^1) \cap \mathcal{V} = \emptyset$.*
(2) *Any $E \in \sigma_{\mathrm{pp}}(H_\kappa^1) \cap \mathcal{V}$ belongs to a family of $m \le m_0$ analytic functions (in κ) of eigenvalues of H_κ^1. Expansion to second order of those functions agrees with Proposition 9.4 (2).*
(3) *There are $m_0 - m$ branches of "resonances" (Puiseux series in κ) each of which has a polynomial expansion up to some even order $2p$; the coefficients of the powers κ^q with $q < 2p$ are real while the coefficient of the power κ^{2p} has negative imaginary part.*

Remark. We mention two open problems: 1) Extend the perturbation theory of this section to the case $b_0 \neq 0$ (as in Section 9). The problem here is to control the essential spectrum of $H_\kappa^1(\theta)$. Such "control" is provided for the two-body problem in [How2]. The general N-body problem with Coulomb singularities would probably (if feasible at all) require some kind of "exterior scaling", cf. [Hu2]. 2) Find an analytic perturbation theory for Born-Oppenheimer molecules in the small field regime (like in Proposition A.3). One may suspect that the distortion technique of [Hu2] would provide an analogue of Proposition A.3, although there is a technical difficulty in using the same scheme of proof.

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