Feynman-Kac Representation and Markovianity of Semigroups Generated by Noncommutative Elliptic Operators

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Abstract

We construct Markovian semigroups generated by noncommutative elliptic operators on a von Neumann algebra. We first introduce a general framework of quantum Feynman-Kac formula in terms of unitary evolutions and multipliers, and then apply the general result to our problem. We construct multipliers which are determined by operator valued stochastic differential equations and satisfy cocycle relations with respect to a randomized flow on a von Neumann algebra. Our works are greatly influenced by that of Lindsay and Sinha [LS]. In order to ensure Markovianity of the semigroups, we modify the method employed by Lindsay and Sinha [LS].

1 Introduction

Recently Lindsay and Sinha [LS] constructed semigroups generated by noncommutative elliptic operators on a von Neumann algebra through a quantum version

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of Feynman-Kac formula. However, in general the semigroups they constructed do not preserve the completely positive property. In this paper, we introduce an another quantum version of Feynman-Kac formula such that the completely positive preserving property of semigroups becomes manifest. We then use the Feynman-Kac formula and a construction method similar to that employed in [LS] to construct Markovian semigroups generated by noncommutative elliptic operators. The form of the quantum Feynman-Kac formula we introduce is closely related to that given in [AFL].

In order to describe the content of the paper and the problems involved, let us recall the Feynman-Kac formula for diffusion type semigroups. For given $b \in L^\infty(\mathbb{R})$, let $H = -\frac{1}{2}\frac{d^2}{dx^2} - b\frac{d}{dx}$. Then the Feynman-Kac formula [RS, PS] yields that for $f \in L^2(\mathbb{R})$,

$$\begin{aligned} (e^{-tH}f)(x) &= \int_\Omega m_t^b(x,\omega) f(x+B_t(\omega))\, d\mathbb{P}(\omega), \\ m_t^b(x,\omega) &= \exp\left\{ \int_0^t b(x+B_\tau)\, dB_\tau - \frac{1}{2}\int_0^t b^2(x+B_\tau)\, d\tau \right\}, \end{aligned} \tag{1.1}$$

in which $(\Omega, \mathcal{B}, \mathbb{P})$ is the standard Wiener space, and $B_\tau(\omega) = \omega(\tau)$, $\omega \in \Omega$. Now, let us consider the semigroup $P_t^b = e^{-tH}$ as an one-parameter semigroup on the abelian von Neumann algebra $L^\infty(\mathbb{R})$. Then, the above representation can be written as

$$P_t^b(f) = \mathbb{E}[\, j_t(f) m_t^b\,], \tag{1.2}$$

where $j_t(f)(x) = f(x+B_t)$, and

$$m_t^b = \exp\left\{ \int_0^t j_\tau(b)\, dB_\tau - \frac{1}{2}\int_0^t j_\tau(b^2)\, d\tau \right\}, \tag{1.3}$$

and $\mathbb{E}$ is the Wiener expectation. Thus the translation group on $L^\infty(\mathbb{R})$ is randomized and then the shifted element is acted by a randomized multiplication operator, before being averaged to realize the semigroup (P_t^b). The expression (1.2) can be written in more symmetric way as follow:

$$P_t^b(f) = \mathbb{E}\left[\, \overline{\left((m_t^b)^{1/2}\right)}\, j_t(f)(m_t^b)^{1/2} \,\right], \tag{1.4}$$

where the bar denotes the complex conjugation.

These structures together with various noncommutative extensions have been studied by many authors ([AFL], [AMV], [Fag], [HIP], [LS], [LW] and references there in). In particular, Lindsay and Sinha [LS] replaced the function space on which the semigroup acts by an operator algebra. More precisely, let $\mathcal{A}$ be a von Neumann algebra acting on a separable Hilbert space $\mathfrak{h}$ and let $(\alpha_t : t \in \mathbb{R})$ be a weak*- continuous group of *-automorphisms of $\mathcal{A}$. For each $t \geq 0$ and $x \in \mathcal{A}$, define $j_t(x) : \Omega \to \mathcal{A}$ by

$$(j_t(x))(\omega) = \alpha_{\omega(t)}(x).$$

Then each j_t defines a weak*-continuous *-homomorphism of $\mathcal{A}$ into $\mathcal{M} = \mathcal{A} \otimes L^\infty(\Omega)$. The maps (j_t) are extended to $\mathcal{M}$ by

$$J_t : x \otimes v \mapsto j_t(x)v, \quad x \in \mathcal{A},\, v \in L^\infty(\Omega).$$

Notice that the multiplier $(m_t^b : t \geq 0)$ given (1.3) satisfies the following stochastic differential equations

$$d_t m_t^b(f) = j_t(b) m_t^b(f) dB_t, \quad m_0^b f = f,$$

for $f \in L^2(\mathbb{R})$ (cf., [PS]). For each $a = a^* \in \mathcal{A}$, Lindsay and Sinha [LS] constructed the multiplier $(m_t^a : t \geq 0)$ as the unique solution of the stochastic differential equations

$$d_t m_t^a \varphi = J_t(x) m_t^a \varphi \, dB_t, \quad m_0^a \varphi = \varphi, \tag{1.5}$$

for $\varphi \in \mathfrak{h}$, and defined

$$P_t^a(x)\varphi := \mathbb{E}[\, J_t(x) m_t^a \varphi\,], \quad \varphi \in \mathfrak{h}. \tag{1.6}$$

It turned out that $\{P_t^a\}$ is a weak*-continuous semigroup on $\mathcal{A}$ whose generator is an extension of $\frac{1}{2}\delta^2 + R_a\delta$, where δ is the generator of (α_t) and R_a is right multiplication operator : $R_a x = xa$, $x \in \mathcal{A}$. For the details, see [LS].

It is clear that in general the semigroup (P_t^a) defined in (1.6) does not preserve reality and positivity. Thus we consider the Feynman-Kac formula given in (1.4). One may check that the multiplier $\big((m_t^b)^{1/2} : t \geq 0\big)$ satisfies the stochastic differential equations

$$\begin{aligned} d_t (m_t^b)^{1/2} f &= \frac{1}{2} j_t(b) (m_t^b)^{1/2} f \, dB_t - \frac{1}{8} j_t(b^2) (m_t^b)^{1/2} f \, dt, \\ (m_0^b)^{1/2} f &= f, \end{aligned} \tag{1.7}$$

for $f \in L^2(\mathbb{R})$. For given $a = a^* \in \mathcal{A}$, let $(\tilde{m}_t^a : t \geq 0)$ be the multiplier corresponding to the classical counter part $\big((m_t^b)^{1/2} : t \geq 0\big)$. For each t, one may extend $\tilde{m}_t^a$ to $\mathcal{H} = \mathfrak{h} \otimes L^2(\Omega)$ by $\tilde{M}_t^a(\varphi \otimes v) := (\tilde{m}_t^a \varphi)v$, $\varphi \in \mathfrak{h}$, $v \in L^2(\Omega)$. It can be checked that for each t, $\tilde{M}_t^a$ is unbounded operator on $\mathcal{H}$, and so one has domain problems.

Thus we are looking for a bounded extension of $(\tilde{m}_t^a)$. Let $(\mathrm{E}_t : t \geq 0)$ be the family of projection operators on $\mathcal{H}$ defined by

$$\mathrm{E}_t : \varphi \otimes v \mapsto \varphi \otimes \mathbb{E}[\, v | \mathcal{B}_t \,], \quad \varphi \in \mathfrak{h},\, v \in L^2(\Omega),$$

where $\mathcal{B}_t$ is the σ-algebra generated by $B_s - B_0$, $0 \leq s \leq t$. Consider the multiplier $(M_t^a : t \geq 0)$ satisfying the stochastic differential equations

$$\begin{aligned} d_t M_t^a \xi &= J_t(a)(\mathrm{E}_t M_t^a \xi)\, dB_t - \frac{1}{2} J_t(a^2) \mathrm{E}_t M_t^a \xi \, dt, \\ M_0^a \xi &= \xi, \end{aligned} \tag{1.8}$$

for $\xi \in \mathcal{H}$. One may compare (1.8) and (1.7) with $a = \frac{b}{2}$. It turns out that for each $a \in \mathcal{A}$ and $t \geq 0$, there exists a unique solution $M_t^a \in B(\mathcal{H})$ satisfying (1.8). For given $a = a^* \in \mathcal{A}$, define

$$P_t^a(x)\varphi := \mathrm{E}_0[\, (M_t^a)^* J_t(x) M_t^a \varphi \,], \quad x \in \mathcal{A}, \varphi \in \mathfrak{h}.$$

Then $(P_t^a : t \geq 0)$ becomes a weak*-continuous Markovian semigroup on $\mathcal{A}$ whose generator is an extension of the following operator:

$$G^a = \frac{1}{2}\delta^2 + L_a \delta + R_a \delta + L_a R_a - \frac{1}{2} L_a^2 - \frac{1}{2} R_a^2, \tag{1.9}$$

where L_a and R_a are the left and right multiplication operators respectively: $L_a x = ax$, $R_a x = xa$, $x \in \mathcal{A}$. For the details, see Theorem 3.1 in Section 3.

Let us describe briefly the content of the paper. In Section 2, we introduce a general formalism of quantum Feynman-Kac formula in the framework of a symmetric Fock space. Let Φ be the symmetric Fock space over $L^2(\mathbb{R}_+ : dx)$ and $\mathcal{H} = \mathfrak{h} \otimes \Phi$. We first introduce the notion of a unitary evolution [HIP] and a multiplier (cocycle) of a given unitary evolution, and then give conditions (Assumption 2.1 and 2.2) under which the Feynman-Kac formula given in (2.5) represents a weak*-continuous Markovian semigroup on $\mathcal{A}$. We should mention that, if one reformulates the Feynman-Kac formula we introduce in the algebraic language, it is essentially same as that given in [AFL]. However we formulate the Feynman-Kac formula so that it becomes very convenient in application.

In Section 3, we construct a multiplier $(M_t^a : t \geq 0)$ for each $a \in \mathcal{A}$ and show that it satisfies all the necessary conditions listed in Section 2. We use the construction method similar to that employed in [LS] with necessary modifications. In Section 4, we investigate some properties of noncommutative elliptic operators of the type given in (1.9). Let δ be a *-derivation on a von Neumann algebra $\mathcal{A}$, and $a = a^* \in \mathcal{A}$. We show that G^a is a complete dissipation(Theorem 4.1) [Bra].

Before closing the introduction, we remark that all the results in Section 2 and Section 3 can be generalized to the case in which Φ is the symmetric Fock space over $L^2(\mathbb{R}_+ : \mathbf{k})$, where $\mathbf{k}$ is a finite dimensional Hilbert space called the *noise dimensional space.*

2 General Formalism

Let $\mathfrak{h}$ be a separable Hilbert space, called the *initial space.* We denote by Φ the boson Fock space over $L^2(\mathbb{R}_+)$. To each $t \in \mathbb{R}_+$ corresponds a tensor product decomposition $\Phi = \Phi_{t]} \otimes \Phi_{[t}$. More generally, if $s < t$, we have a decomposition

$$\Phi = \Phi_{s]} \otimes \Phi_{[s,t]} \otimes \Phi_{[t}.$$

The space $\Phi_{s]}, \Phi_{[s,t]}$ and $\Phi_{[t}$ are the boson Fock spaces over $L^2([0, s])$, $L^2([s, t])$ and $L^2([t, \infty))$ respectively. Denote by $\mathbf{1}_\Phi$ the vacuum vector of Φ. For each $0 \le s \le t$, we identify $\Phi_{[s,t]}$ as a subspace of Φ by $u \in \Phi_{[s,t]}$ being identified with $\mathbf{1}_{s]} \otimes u \otimes \mathbf{1}_{[t}$ where $\mathbf{1}_{s]}$ and $\mathbf{1}_{[t}$ are the vacuum vectors of $\Phi_{s]}$ and $\Phi_{[t}$ respectively. We use the following notations : for any $0 \le s \le t$

$$\mathcal{H} := \mathfrak{h} \otimes \Phi, \quad \mathcal{H}_{t]} := \mathfrak{h} \otimes \Phi_{t]}, \quad \mathcal{H}_{[s,t]} := \mathfrak{h} \otimes \Phi_{[s,t]}, \quad \mathcal{H}_{[t} := \mathfrak{h} \otimes \Phi_{[t}.$$

The space $\mathcal{H}$ has also other tensor product decompositions. First we have $\mathcal{H} = \mathcal{H}_{t]} \otimes \Phi_{[t}$. On the other hand, we also have $\mathcal{H} = \Phi_{t]} \otimes \mathcal{H}_{[t}$. We generally omit the tensor product sign in these decompositions, writing simply $\varphi u \,(\varphi \in \mathfrak{h}, u \in \Phi)$ or even $u_{t]}\varphi u_{[t}\,(u_{t]} \in \Phi_{t]}, \varphi \in \mathfrak{h}, u_{[t} \in \Phi_{[t})$ as if initial vectors really did commute with vectors of Fock space. We do not distinguish $\varphi \mathbf{1}_\Phi$ from φ, so that the initial space $\mathfrak{h}$ is identified with $\mathcal{H}_{0]}$. For details, we refer the reader to [Mey] and [Par].

Denote by E_t the projection on the closed subspace $\mathcal{H}_{t]}$ of $\mathcal{H}$. In particular, E_0 may be considered as the projection onto the initial space $\mathfrak{h}$.

Let τ_t, $t \in \mathbb{R}_+$ be the shift operators on $L^2(\mathbb{R}_+)$ acting as follows:

$$\tau_t f(s) = \begin{cases} f(s-t) & \text{if } t \le s \\ 0 & \text{otherwise.} \end{cases}$$

Denote by θ_t the second quantization of τ_t. The shift operators on Φ are isometric. We extend the shift operators θ_t to $\mathcal{H}$ by

$$\Theta_t(\varphi \otimes u) = \varphi \otimes \theta_t u, \quad \varphi \in \mathfrak{h},\, u \in \Phi.$$

Between the operators E_t and Θ_t, we have the relation [Mey]

$$\Theta_s \mathrm{E}_t = \mathrm{E}_{s+t} \Theta_s. \tag{2.1}$$

In particular, $\mathrm{E}_s \Theta_s = \Theta_s \mathrm{E}_0 = \mathrm{E}_0$.

We consider a family $\{U_{s,t} : 0 \le s \le t\}$ of unitary operators acting on $\mathcal{H}$ and satisfying the following conditions:

$$\begin{aligned} &(a) \quad U_{s,t} \text{ and } U_{s,t}^* \text{ are strongly continuous in } t \text{ for fixed } s, \\ &(b) \quad U_{s,s} = \mathbf{1}_{\mathrm{B}(\mathcal{H})} \text{ for all } s, \\ &(c) \quad U_{r,s} U_{s,t} = U_{r,t} \text{ for } r \le s \le t. \end{aligned} \tag{2.2}$$

Such family $\{U_{s,t} : s \le t\}$ is called a *unitary evolution* [HIP].

Let $\mathcal{A}$ be a von Neumann algebra acting on the initial space $\mathfrak{h}$. Denote by $\mathcal{A}'$ the commutant of $\mathcal{A}$. Put

$$J_{s,t}(x) := U_{s,t}(x \otimes \mathbf{1}_{\mathrm{B}(\Phi)}) U_{s,t}^*, \quad x \in \mathcal{A}. \tag{2.3}$$

In the following, we use simplified notations $\Phi_t := \Phi_{t]}$, $\mathcal{H}_t = \mathcal{H}_{t]}$, $U_t := U_{0,t}$ and $J_t(x) = J_{0,t}(x)$. Any operator defined on a tensorial factor of $\mathcal{H}$ will be often identified with its canonical amplification to $\mathcal{H}$ and denoted by the same symbol, i.e., $x = x \otimes \mathbf{1}_{\mathrm{B}(\Phi)}$, $x \in \mathcal{A}$.

Assumption 2.1 *Let $\{U_{s,t} : s \le t\}$ be a unitary evolution and $\mathcal{A}$ a von Neumann algebra acting on $\mathfrak{h}$. We assume that the following properties hold: for any $0 \le s \le t$,*

(a) $U_{s,t}$ maps $\mathcal{H}_{[s,t]}$ into itself, and acts as follow on decomposed vectors $u_{s]}\phi_{[s,t]}v_{[t}$ ($u_{s]} \in \Phi_s$, $\phi_{[s,t]} \in \mathcal{H}_{[s,t]}$, $v_{[t} \in \Phi_{[t}$, $\otimes$ omitted)

$$U_{s,t}(u_{s]}\phi_{[s,t]}v_{[t}) = u_{s]}(U_{s,t}\phi_{[s,t]})v_{[t},$$

(b) $\Theta_s U_{t-s} = U_{s,t} \Theta_s$,

(c) for any $x \in \mathcal{A}$ and $y \in \mathcal{A}'$,

$$[J_t(x), y] = 0,$$

where $[A, B] := AB - BA$, $A, B \in \mathrm{B}(\mathcal{H})$.

One notice that Assumption 2.1 (a) implies that $U_{s,t}$ is t-adapted (to the past) and s-adapted (to the future) respectively [Mey]. The semigroup property in (2.2) and Assumption 2.1 (a)-(b) imply that U_t satisfies the left cocycle [Mey].

A family $\{M_{s,t} : s \leq t\}$ of bounded operators on $\mathcal{H}$ is called a *multiplier* (or *cocycle*) for the unitary evolution $\{U_{s,t} : s \leq t\}$ if the following conditions are satisfied:

$$\begin{array}{ll} (a) & M_{s,t} \text{ is strongly continuous in } t \text{ for fixed } s, \\ (b) & M_{s,s} = \mathbf{1}_{\mathrm{B}(\mathcal{H})} \text{ for all } s \in \mathbb{R}_+, \\ (c) & U_s M_{s,t} U_s^* M_s = M_t \text{ for } s \leq t, \end{array} \tag{2.4}$$

where $M_t := M_{0,t}$.

Assumption 2.2 *Let $\{M_{s,t}\}$ be a multiplier (cocycle) for the unitary evolution $\{U_{s,t}; s \leq t\}$. We assume that the following properties hold: for $0 \leq s \leq t$,*

(a) $M_{s,t}$ maps $\mathcal{H}_{[s}$ into $\mathcal{H}_{[s}$ and acts as follow on decomposed vectors $u_{s]}\phi_{[s}$ ($u_{s]} \in \Phi_s$, $\phi_{[s} \in \mathcal{H}_{[s}$, $\otimes$ omitted)

$$M_{s,t}(u_{s]}\phi_{[s}) = u_{s]} M_{s,t}\phi_{[s},$$

(b) $\Theta_s M_{t-s} = M_{s,t}\Theta_s$,

(c) $\mathrm{E}_t M_{s,t} = M_{s,t}\mathrm{E}_t$,

(d) for any $y \in \mathcal{A}'$, $[M_{s,t}, y] = 0$,

(e) there is a continuous function $K : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\|M_t\| \leq K(t)$.

We remark that $M_{s,t}$ is s-adapted, but not necessary t-adapted. Instead of t-adaptedness of $M_{s,t}$, we have imposed the property (c).

The following is the main result in this section.

Theorem 2.1 *Let $\mathcal{A}$ be a von Neumann algebra acting on $\mathfrak{h}$. Let $\{M_{s,t} : s \leq t\}$ be a multiplier (cocycle) for given unitary evolution $\{U_{s,t} : s \leq t\}$. For any $x \in \mathcal{A}$ and $0 \leq t$, define*

$$P_t(x)\varphi := \mathrm{E}_0[M_t^* J_t(x) M_t \varphi], \quad \varphi \in \mathfrak{h} \subset \mathcal{H}. \tag{2.5}$$

Under Assumption 2.1 and Assumption 2.2, the following properties hold:

(a) $\{P_t\}_{t\geq 0}$ is a completely positive preserving semigroup on $\mathcal{A}$.

(b) *reality preserving :* $P_t(x^*) = P_t(x)^*$ *for any* $x \in \mathcal{A}$ *and* $t \geq 0$.

(c) *The semigroup* $\{P_t\}_{t\geq 0}$ *is weak*-continuous.*

Remark 2.1 *The expression (2.5) can be considered as a Feynman-Kac representation of the semigroup* $\{P_t\}_{t\geq 0}$*. See also Theorem 2.3 of [AFL].*

Proof of Theorem 2.1. (a) By Assumption 2.2 (e), $\|P_t(x)\| \leq K(t)^2\|x\|$, and so P_t is bounded for each $t \in \mathbb{R}_+$. By (2.2) (b) and (2.4) (b), $P_0(x) = x$ for any $x \in \mathcal{A}$. Let us prove the semigroup property. Notice that by Assumption 2.2 (a) and (b), $M^*_{t-s} = \Theta^*_s M^*_{s,t}\Theta_s$, and so $\Theta_s M^*_{t-s} = M^*_{s,t}\Theta_s$. It follows from (2.3), Assumption 2.1 (a) -(b) and Assumption 2.2 (a)-(b) that for any $x \in \mathcal{A}$, $w_{s]}\mathbf{1}_{[s} \in \Phi_s \subset \Phi$, $\varphi \in \mathfrak{h}$ and $s \leq t$,

$$\begin{aligned} & M^*_{s,t}J_{s,t}(x)M_{s,t}w_{s]}\mathbf{1}_{[s}\varphi \\ &\quad = \left[M^*_{s,t}J_{s,t}(x)M_{s,t}\Theta_s\varphi\mathbf{1}_\Phi\right] w_{s]} \\ &\quad = \left[\Theta_s M^*_{t-s}J_{t-s}(x)M_{t-s}\varphi\mathbf{1}_\Phi\right] w_{s]}, \end{aligned}$$

and so by (2.1)

$$\begin{aligned} & \mathrm{E}_s\left[M^*_{s,t}J_{s,t}(x)M_{s,t}w_{s]}\mathbf{1}_{[s}\varphi\right] \\ &\quad = \Theta_s\left[\mathrm{E}_0 M^*_{t-s}J_{t-s}(x)M_{t-s}\varphi\mathbf{1}_\Phi\right] w_{s]} \\ &\quad = (P_{t-s}(x)\varphi)w_{s]}\mathbf{1}_{[s}. \end{aligned}$$

Let us use the abbreviated notation, $\varphi := \varphi\mathbf{1}_{[s}$, etc., for $\varphi \in \mathfrak{h}$. Since $\{w_{s]}\varphi : w_{s]} \in \Phi_s, \varphi \in \mathfrak{h}\}$ is total in $\mathcal{H}_s$, it follows from the above relation that for $\varphi \in \mathfrak{h}$

$$\begin{aligned} & \mathrm{E}_s\left[M^*_{s,t}J_{s,t}(x)M_{s,t}U^*_s M_s\varphi\right] \\ &\quad = P_{t-s}(x)U^*_s M_s\varphi. \end{aligned}$$

Here we have used the fact that U^*_s and M_s map $\mathcal{H}_s$ into $\mathcal{H}_s$ by Assumption 2.1 (a) and Assumption 2.2 (c). We use the cocycle relation (c) in (2.4), Assumption 2.1 (a), Assumption 2.2 (c) and the fact that $U^*_s J_t(x)U_s = J_{s,t}(x)$ to obtain that

$$\begin{aligned} P_t(x)\varphi &= \mathrm{E}_0\left[M^*_s U_s M^*_{s,t}U^*_s J_t(x)U_s M_{s,t}U^*_s M_s\varphi\right] \\ &= \mathrm{E}_0\left\{M^*_s U_s \mathrm{E}_s\left[M^*_{s,t}J_{s,t}(x)M_{s,t}U^*_s M_s\varphi\right]\right\} \\ &= \mathrm{E}_0\left\{M^*_s U_s P_{t-s}(x)U^*_s M_s\varphi\right\} \\ &= \mathrm{E}_0\left[M^*_s J_s\left(P_{t-s}(x)\right)M_s\varphi\right] \\ &= P_s\left(P_{t-s}(x)\right)\varphi. \end{aligned}$$

Thus the semigroup property holds. Since $J_t(x)$, $x \in \mathcal{A}$, and M_t commute with any $y \in \mathcal{A}'$ by assumption 2.1 (c) and Assumption 2.2 (d), $P_t(x) \in \mathcal{A}$ for any $x \in \mathcal{A}$ by (2.5).

The completely positive preserving property follows from the definition P_t in (2.5). For any $x_1, x_2, \cdots, x_n \in \mathcal{A}$, $y_1, y_2, \cdots, y_n \in \mathcal{A}$, $n \in \mathbb{N}$, and $\varphi \in \mathfrak{h}$,

$$
\begin{aligned}
&\langle \varphi, \sum_{i,j} y_i^* P_t(x_i^* x_j) y_j \varphi \rangle_{\mathfrak{h}} \\
&= \langle \varphi \otimes \mathbf{1}_\Phi, (\sum_{i=1}^n y_i^* M_t^* U_t x_i^*)(\sum_{j=1}^n x_j U_t^* M_t y_j)(\varphi \otimes \mathbf{1}_\Phi) \rangle_{\mathcal{H}} \\
&\geq 0.
\end{aligned}
$$

(b) Since $\mathrm{E}_0 \varphi = \varphi$ for $\varphi \in \mathfrak{h}$, it follows from (2.5) that for any $x \in \mathcal{A}$ and $t \geq 0$

$$
P_t(x) = \mathrm{E}_0 M_t^* J_t(x) M_t \mathrm{E}_0.
$$

Thus (b) follows from the above.

(c) We next consider weak*-continuity of P_t. We will use the argument similar to that used in [LS]. We exploit the coincidence of σ-weak and weak operator topologies on the bounded subsets of $\mathcal{A}$ [BR]. First fix $t \geq 0$ and let (x_n) be be a sequence in the closed unit ball of $\mathcal{A}$, converging in the weak operator topology to x. Then, for $\varphi, \xi \in \mathfrak{h}$,

$$
\langle \varphi, P_t(x_n) \xi \rangle_{\mathfrak{h}} = \langle U_t^* M_t \varphi, (x_n \otimes \mathbf{1}_\Phi) U_t^* M_t \xi \rangle_{\mathcal{H}}.
$$

Since $(x_n \otimes \mathbf{1}_\Phi)$ converge in the weak operator topology to $x \otimes \mathbf{1}_\Phi$, $P_t(x_n)$ converges in the weak operator topology to $P_t(x)$. Since $\|P_t(x_n)\| \leq \|M_t\|^2$, $P_t(x_n)$ converges in the σ-weak topology. Thus $\phi \circ P_t$ is weak operator continuous on the closed unit ball of $\mathcal{A}$, and therefore σ-weakly continuous for each $\phi \in \mathcal{A}_*$. Hence each P_t is σ-weakly continuous.

Next fix $x \in \mathcal{A}$ and (t_n) be a real positive sequence converging to 0. Then for $\varphi, \xi \in \mathfrak{h}$, we write

$$
\begin{aligned}
&\langle \varphi, \{P_{t_n}(x) - x\} \xi \rangle_{\mathfrak{h}} \\
&\quad = \langle M_{t_n} \varphi, J_{t_n}(x) M_{t_n} \xi \rangle_{\mathcal{H}} - \langle \varphi, x \xi \rangle_{\mathcal{H}} \\
&\quad = \langle M_{t_n} \varphi - \varphi, J_{t_n}(x) M_{t_n} \xi \rangle_{\mathcal{H}} + \langle \varphi, J_{t_n}(x) M_{t_n} \xi \rangle_{\mathcal{H}} - \langle \varphi, x \xi \rangle_{\mathcal{H}} \\
&\quad = \langle M_{t_n} \varphi - \varphi, J_{t_n}(x)(M_{t_n} \xi - \xi) \rangle_{\mathcal{H}} + \langle M_{t_n} \varphi - \varphi, J_{t_n}(x) \xi \rangle_{\mathcal{H}} \\
&\qquad + \langle \varphi, J_{t_n}(x)(M_{t_n} \xi - \xi) \rangle_{\mathcal{H}} + \langle \varphi, (J_{t_n}(x) - x) \xi \rangle_{\mathcal{H}}
\end{aligned}
$$

Notice that $\|J_{t_n}(x)\| \leq \|x\|$. Thus the first three terms in the above converge to 0 as $t_n \to 0$ by the continuity of M_t in (2.4). The last term converges to 0 as $t_n \to 0$ by the continuity of U_t and U_t^* in (2.2). Hence $t \mapsto P_t(x)$ is weak operator continuous. Since the family $(P_t(x))$ is locally bounded by Assumption 2.2 (e), $t \mapsto P_t(x)$ is σ-weakly continuous. Thus the semigroup $\{P_t\}_{t\geq 0}$ is weak* continuous. $\square$

3 Semigroups Generated by Noncommutative Elliptic Operators

Let $(\Omega, \mathcal{B}, \mathbb{P})$ be the standard Wiener space. Thus Ω is the space of continuous functions on $[0, \infty)$ which vanish at 0, $\mathcal{B}$ is a σ-algebra and $\mathbb{P}$ is Wiener measure. The coordinate process $\omega \in \Omega \mapsto \omega(t)$ is a standard Brownian motion which will be denoted B_t. It generates the Wiener filtration $(\mathcal{B}_{[s,t]})_{t\geq s>0}$, where $B_{[s,t]}$ is the σ-algebra generated by $B_\tau - B_s$, $s \leq \tau \leq t$.

The Fock space Φ over $L^2(\mathbb{R}_+)$ can be identified with $L^2(\Omega, \mathcal{B}, \mathbb{P})$ [Mey]. The space $\Phi_{[s,t]}$, for instance, is $L^2(\mathcal{B}_{[s,t]})$. The shift map on $L^2(\Omega)$ satisfies the relation

$$\theta_t B_s = B_{s+t} - B_t. \tag{3.1}$$

With the above identification, we will use the notation in Section 2. If necessary, we identify elements of $\mathcal{H} = \mathfrak{h} \otimes L^2(\Omega, \mathcal{B}, \mathbb{P})$ as functions on Ω: $\mathcal{H} = L^2(\Omega, \mathfrak{h})$.

Let $(\alpha_t : t \in \mathbb{R})$ be a weak*-continuous group of *-automorphisms of von Neumann algebra $\mathcal{A}$ acting on the initial Hilbert space $\mathfrak{h}$. We assume that the automorphism is implemented, $\alpha_t(a) = \tilde{U}_t a \tilde{U}_t^*$ by a strongly continuous one parameter group of unitary operators on $\mathfrak{h}$. As in [LS], we randomize $\tilde{U}_t$ and extend to $\mathcal{H}$ by continuous linear extension of the prescription

$$U_t : v \otimes \varphi \mapsto v\tilde{U}_{B_t}\varphi, \quad v \in \Phi,\ \varphi \in \mathfrak{h}. \tag{3.2}$$

Let $\{U_{s,t} : s \leq t\}$ be the family of unitary operators on $\mathcal{H}$, where for decomposed vectors $u_s\varphi v_{[s}$ $(u_s \in \Phi_s, \varphi \in \mathfrak{h}, v_{[s} \in \Phi_{[s})$, $U_{s,t}$ acts as

$$U_{s,t}u_s\varphi v_{[s} = u_s(\tilde{U}_{B_t - B_s}\varphi)v_{[s}. \tag{3.3}$$

It follows from (2.3) that for any $x \in \mathcal{A}$

$$J_{s,t}(x) = \alpha_{B_t - B_s}(x). \tag{3.4}$$

We recall that for decomposed vectors $\varphi \otimes v$ $(\varphi \in \mathfrak{h}, v \in \Phi)$

$$\begin{aligned} \Theta_t &: \varphi \otimes v \mapsto \varphi \otimes \theta_t v, \\ \mathrm{E}_t &: \varphi \otimes v \mapsto \varphi \otimes \mathbb{E}[v|\mathcal{B}_{[0,t]}], \end{aligned}$$

where $\mathbb{E}$ is the Wiener expectation. One notice that $\theta_t v = v \circ \gamma_t$, where γ_t is the shift on paths defined by $(\gamma_t \omega)(s) = \omega(t+s) - \omega(t)$. See (3.1).

Lemma 3.1 *The family $\{U_{s,t} : s \leq t\}$ of unitary operators is a unitary evolution satisfying the properties in (2.2). Moreover the properties (a) -(c) in Assumption 2.1 also hold.*

Proof. The strong continuity of $U_{s,t}$ and $U_{s,t}^*$ can be proved by using the spectral theorem for the infinitesimal generator of $\tilde{U}_t$ and an explicit computation on Wiener integral. We leave the details to the reader. The properties (b) and (c) in (2.2) follows from (3.3).

Next, we consider the properties in Assumption 2.1. Notice that for any $\varphi \in \mathfrak{h}$, $\tilde{U}_{B_t - B_s}\varphi \in \mathcal{H}_{[s,t]}$. Thus the properties (a) and (b) in Assumption 2.1 follow from (3.3) and (3.1). Since $\alpha_t(\mathcal{A}) \subset \mathcal{A}$, it follows from (3.4) that for any $x \in \mathcal{A}$ and $y \in \mathcal{A}'$, $J_{s,t}(x)$ and y commute. This completes the proof. □

Next we are looking for the multipliers (cocycles) for the unitary evolution $\{U_{s,t} : s \leq t\}$ corresponding to noncommutative elliptic operators on $\mathcal{A}$. In the following, we identify $\mathcal{H}$ and $\mathcal{H}_s$ with $L^2(\Omega : \mathfrak{h})$ and $L^2(\mathcal{B}_{[0,s]} : \mathfrak{h})$ respectively : $\mathcal{H} = L^2(\Omega : \mathfrak{h})$ and $\mathcal{H}_s = L^2(\mathcal{B}_{[0,s]} : \mathfrak{h})$, $s \in \mathbb{R}_+$. We also denote $L^\infty(\mathcal{B}_{[0,s]} : \mathcal{A})$ by $\mathcal{M}_s$. An $\mathfrak{h}$-valued stochastic process (f_t) said to be *adapted* if $f_t \in L^2(\mathcal{B}_{[0,t]} : \mathfrak{h})$ for each $t \in \mathbb{R}_+$ and $t \mapsto \|f_t\|_{\mathcal{H}}$ is measurable.

Proposition 3.1 *For $0 \leq s \leq t$, $a \in \mathcal{M}_s$, $\xi \in \mathcal{H}$, the stochastic integral equation*

$$f(a,\xi;s,t) = \xi + \int_s^t J_{s,\tau}(a)\left[\mathrm{E}_\tau f(a,\xi;s,\tau)\right] dB_\tau - \frac{1}{2}\int_s^t J_{s,\tau}(a^2)\left[\mathrm{E}_\tau f(a,\xi;s,\tau)\right] d\tau \tag{3.5}$$

has a unique solution in $\mathcal{H}$, where the first integral is an Ito integral for a function taking values in $\mathfrak{h}$, and the second integral is a Bochner integral defined pointwise for each path.

Proof. We use the method similar to that employed in the proof of Proposition 2.1 of [LS]. For given $\xi \in \mathcal{H}$ and $a \in \mathcal{M}_s$, put

$$f_{s,t}^{(0)} := \xi. \tag{3.6}$$

As an inductive hypothesis, assume that for $n \geq 1$ the recursive procedure

$$f_{s,t}^{(n)} := \int_s^t J_{s,\tau}(a) \left[\mathrm{E}_\tau f_{s,\tau}^{(n-1)}\right] dB_\tau - \frac{1}{2} \int_s^t J_{s,\tau}(a^2) \left[\mathrm{E}_\tau f_{s,\tau}^{(n-1)}\right] d\tau, \tag{3.7}$$

defines t-adapted, $\mathfrak{h}$-valued processes $(f_{s,t}^{(n)})_{t \geq s}$ satisfying

$$\|f_{s,t}^{(n)}\|_{\mathcal{H}}^2 \leq \frac{1}{n!}\left[\, 4m(\|a\|^2)\, m(t-s)\,\right]^n \|\xi\|^2, \tag{3.8}$$

where

$$\begin{aligned} \|a\| &:= \sup_{\omega \in \Omega} \|a(\omega)\|_{\mathcal{A}}, \\ m(t) &:= max\{t, t^2\}, \quad t \in \mathbb{R}_+. \end{aligned}$$

We note that for fixed $s \in \mathbb{R}_+$, $J_{s,\tau}(a) \in \mathcal{M}_\tau$ for any $a \in \mathcal{M}_s$ and $\mathrm{E}_\tau f_{s,\tau}^{(n-1)} \in \mathcal{H}_\tau$, $n \geq 1$. The inductive hypothesis (3.8) is clearly satisfied when $n = 0$. It is also easy to show that $(f_{s,t}^{(1)})_{t \geq s}$ is t-adapted. See the proof below.

Assume that the bound (3.8) holds for $n = k$. It follows from the Ito isometry and the Hölder inequality that

$$\begin{aligned} &\|f_{s,t}^{(k+1)}\|_{\mathcal{H}}^2 = \mathbb{E}[\, \|f_{s,t}^{(k+1)}\|_{\mathfrak{h}}^2\,] \\ &\leq \mathbb{E}\left[\, 2\| \int_s^t J_{s,\tau}(a)(\mathrm{E}_\tau f_{s,\tau}^{(k)})\, dB_\tau\|_{\mathfrak{h}}^2 + 2\|\frac{1}{2}\int_s^t J_{s,\tau}(a^2)\mathrm{E}_\tau f_{s,\tau}^{(k)}\, d\tau\|_{\mathfrak{h}}^2 \right] \\ &\leq 2\int_s^t \mathbb{E}[\, \|J_{s,\tau}(a)\mathrm{E}_\tau f_{s,\tau}^{(k)}\|_{\mathfrak{h}}^2\,]\, d\tau + (t-s)\int_s^t \mathbb{E}[\, \|J_{s,\tau}(a^2)\mathrm{E}_\tau f_{s,\tau}^{(k)}\|_{\mathfrak{h}}^2\,]\, d\tau. \end{aligned} \tag{3.9}$$

Consider the case for $t - s < 1$. By (3.8), the above is bounded by

$$\begin{aligned} &2\|a\|^2\|\xi\|^2\big(4m(\|a\|^2)\big)^k \int_s^t \frac{(\tau-s)^k}{k!}\, d\tau + \|a\|^4\|\xi\|^2\big(4m(\|a\|^2)\big)^k \int_s^t \frac{(\tau-s)^k}{k!} d\tau \\ &\leq \frac{\big(4m(\|a\|^2)\big)^{k+1}(t-s)^{k+1}}{(k+1)!}\|\xi\|^2 \end{aligned}$$

If $t - s \geq 1$, by (3.8) the expression (3.9) is bounded by

$$\begin{aligned} &2\|a\|^2\|\xi\|^2\big(4m(\|a\|^2)\big)^k \int_s^t \frac{(\tau-s)^{2k}}{k!}\, d\tau \\ &\qquad + \|a\|^4\|\xi\|^2\big(4m(\|a\|^2)\big)^k (t-s) \int_s^t \frac{(\tau-s)^{2k}}{k!}\, d\tau \\ &\leq \frac{\big(4m(\|a\|^2)(t-s)^2\big)^{k+1}}{(k+1)!}\|\xi\|^2. \end{aligned}$$

Therefore $(f_{s,t}^{(k+1)} : t \geq s)$ is well defined, t-adapted process satisfying (3.8). Thus the recursive procedure is justified, and

$$\sum_{n\geq 0} \|f_{s,t}^{(n)}\| \leq \|\xi\| \sum_{n\geq 0} \frac{1}{\sqrt{n!}} \big[4m(\|a\|^2)m(t-s)\big]^{n/2} < \infty.$$

Hence $\sum_{n\geq 0} f_{s,t}^{(n)}$ defined an $\mathfrak{h}$-valued process $(f_{s,t} : t \geq s)$. Since for each $N \geq 1$,

$$\sum_{n=0}^{N} f_{s,t}^{(n)} = \xi + \int_s^t J_{s,\tau}(a)\mathrm{E}_\tau\Big(\sum_{n=0}^{N-1} f_{s,\tau}^{(n)}\Big)\, dB_\tau - \frac{1}{2}\int_s^t J_{s,\tau}(a^2)\mathrm{E}_\tau\Big(\sum_{n=0}^{N-1} f_{s,\tau}^{(n)}\Big)\, d\tau,$$

$(f_{s,t})$ satisfies the equation (3.5).

In order to show the uniqueness, let (g_t) be any solution of (3.5). Then

$$\begin{aligned}(g_t - \xi) &= \int_s^t J_{s,\tau}(a)\mathrm{E}_\tau g_\tau\, dB_\tau - \frac{1}{2}\int_s^t J_{s,\tau}(a^2)\mathrm{E}_\tau g_\tau\, d\tau \\ &= \int_s^t J_{s,\tau}(a)\mathrm{E}_\tau \xi\, dB_\tau + \int_s^t J_{s,\tau}(a)\mathrm{E}_\tau(g_\tau - \xi)\, dB_\tau \\ &\quad -\frac{1}{2}\int_s^t J_{s,\tau}(a^2)\mathrm{E}_\tau \xi\, d\tau - \frac{1}{2}\int_s^t J_{s,\tau}(a^2)\mathrm{E}_\tau(g_\tau - \xi)\, d\tau.\end{aligned}$$

We use the triangle inequality, the Ito isometry and the Hölder inequality to obtain

$$\begin{aligned}&\mathbb{E}[\, \|g_t - \xi\|_{\mathfrak{h}}^2\,] \\ &\leq 4\|a\|^2(t-s)\|\xi\|^2 + 4\|a\|^2 \int_s^t \mathbb{E}[\, \|g_\tau - \xi\|_{\mathfrak{h}}^2\,]\, d\tau \\ &\qquad +2(t-s)^2\|a\|^4\|\xi\|^2 + 2(t-s)\|a\|^4 \int_s^t \mathbb{E}[\, \|g_\tau - \xi\|_{\mathfrak{h}}^2\,]\, d\tau \\ &\leq 8m(\|a\|^2)m(t-s)\|\xi\|^2 + (4+2(t-s))m(\|a\|^2)\int_s^t \mathbb{E}[\, \|g_\tau - \xi\|_{\mathfrak{h}}^2\,]\, d\tau.\end{aligned}$$

Applying the Gronwall Lemma leads to the bound

$$\begin{aligned}\|g_t - \xi\|_{\mathcal{H}}^2 &= \mathbb{E}[\|g_t - \xi\|_{\mathfrak{h}}^2] \\ &\leq 8m(\|a\|^2)m(t-s)\|\xi\|^2\, exp\{8m(\|a\|^2)m(t-s)\},\end{aligned} \tag{3.10}$$

Now let $h_t = f_{s,t} - g_t$. By the Ito isometry and the Hölder inequality,

$$\begin{aligned}\mathbb{E}[\, \|h_t\|_{\mathfrak{h}}^2\,] &\leq 2\int_s^t \mathbb{E}[\, \|J_{s,\tau}(a)\mathrm{E}_\tau h_\tau\|_{\mathfrak{h}}^2\,]\, d\tau + (t-s)\int_s^t \mathbb{E}[\, \|J_{s,\tau}(a^2)\mathrm{E}_\tau h_\tau\|_{\mathfrak{h}}^2\,]\, d\tau \\ &\leq (2+(t-s))m(\|a\|^2)\int_s^t \mathbb{E}[\, \|h_\tau\|_{\mathfrak{h}}^2\,]\, d\tau.\end{aligned}$$

Iterating this relation and applying the bound (3.10) to the estimate $\|h_{t_n}\|_{\mathfrak{h}}^2 \leq 2\{\|f_{s,t_n} - \xi\|_{\mathfrak{h}}^2 + \|g_{t_n} - \xi\|_{\mathfrak{h}}^2\}$ gives

$$\begin{aligned}
\mathbb{E}[\, \|h_t\|_{\mathfrak{h}}^2 \,] &\leq [(2+(t-s))m(\|a\|^2)]^n \int_s^t \cdots \int_s^{t_{n-1}} 32m(\|a\|^2)m(t_n - s) \\
&\qquad \cdot \|\xi\|^2 exp\big\{8m(\|a\|^2)m(t_n - s))\big\}\, dt_n \cdots dt_1 \\
&\leq [(2+(t-s))m(\|a\|^2)]^n 32m(\|a\|^2)m(t-s)\|\xi\|^2 \\
&\qquad \cdot exp\big\{8m(\|a\|^2)m(t-s))\big\}\frac{(t-s)^n}{n!}
\end{aligned}$$

for any $n \in \mathbb{N}$. Hence $h_t = 0$ and so $g_t = f_{s,t}$ on $\mathcal{H}$. $\square$

We now cast Proposition 3.1 in terms of operators. Notice that by (3.5) -(3.7), $\xi \mapsto f(a, \xi; s, t)$ is linear. Thus for each $a \in \mathcal{M}_s$ and $0 \leq s \leq t$, let us define

$$\begin{aligned}
M_{s,t}^a \ &: \ \mathcal{H} \to \mathcal{H} \\
&\quad\ \xi \mapsto f(a, \xi; s, t).
\end{aligned} \tag{3.11}$$

By Proposition 3.1 and the bound (3.10), $M_{s,t}^a$ is a bounded operator on $\mathcal{H}$.

Proposition 3.2 *(a) For $a \in \mathcal{A}$, the family of bounded operators $M_{s,t}^a$, $s \leq t$, defined as in (3.11) satisfies the stochastic differential equation*

$$\begin{aligned}
d_t M_{s,t}^a &= J_{s,t}(a)\mathrm{E}_t M_{s,t}^a dB_t - \frac{1}{2} J_{s,t}(a^2)\mathrm{E}_t M_{s,t}^a dt, \\
M_{s,s}^a &= \mathbf{1}_{\mathrm{B}(\mathcal{H})}
\end{aligned} \tag{3.12}$$

on $\mathcal{H}$ in the strong operator sense.

(b) For $a \in \mathcal{A}$, the operators $M_{s,t}^a$, $s \leq t$, are strongly continuous in t for fixed s, and satisfy the cocycle relation

$$M_t^a = U_s M_{s,t}^a U_s^* M_s^a$$

for $0 \leq s \leq t$, where $M_t^a = M_{0,t}^a$, $t \geq 0$.

Remark 3.1 *The expression (3.12) is somewhat misreading. The correct expression would be the following : for $\xi \in \mathcal{H}$*

$$\begin{aligned}
d_t M_{s,t}^a \xi &= J_{s,t}(a)[\, \mathrm{E}_t M_{s,t}^a \xi \,]\, dB_t - \frac{1}{2} J_{s,t}(a^2)[\, \mathrm{E}_t M_{s,t}^a \xi]\, dt, \\
M_{s,s}^a \xi &= \xi.
\end{aligned}$$

For a notational brevity, we will use the expression (3.12) in the sequel.

Proof of Proposition 3.2 (a) This is a consequence of Proposition 3.1 and the definition of $M^a_{s,t}$ in (3.11).

(b) Define $\{g_{r,t} : t \geq r\}$ by

$$g_{r,t} = \begin{cases} f(a,\xi;r,t) & \text{for } r \leq t \leq s \\ M^{J_{r,s}(a)}_{s,t} f(a,\xi;r,s) & \text{for } t \geq s. \end{cases}$$

Since $J_{s,\tau}(J_{r,s}(a)) = J_{r,\tau}(a)$ for $r \leq s \leq \tau$, $(g_{r,t})$ satisfies the same stochastic integral equation as $(M^a_{r,t}\xi)$ and so

$$M^a_{r,t} = M^{J_{r,s}(a)}_{s,t} M^a_{r,s}. \tag{3.13}$$

Thus it follows from (3.13) and (3.10) that for $r \leq s \leq t$

$$\begin{aligned} \|M^a_{r,t} - M^a_{r,s}\| &\leq \|M^{J_{r,s}(a)}_{s,t} - \mathbf{1}\| \, \|M^a_{r,s}\| \\ &\leq C_1 m(t-s)^{1/2} exp\{C_2 m(t-s) + C_3 m(s-r)\} \end{aligned}$$

for some constants C_1, C_2 and C_3 depending only on $\|a\|$. Hence $M^a_{s,t}$, $s \leq t$, are norm-continuous and so strongly continuous.

For $a \in \mathcal{A}$ and $\xi \in \mathcal{H}$, define $(g_t : t \geq 0)$ by

$$g_t = \begin{cases} f(a,\xi;0,t) & \text{for } 0 \leq t \leq s \\ U_s M^a_{s,t} U^*_s f(a,\xi;0,s) & \text{for } t \geq s \end{cases}$$

Since $U_s \mathrm{E}_\tau = \mathrm{E}_\tau U_s$ for $\tau \geq s$, we have

$$\begin{aligned} g_t &= U_s \Big\{ U^*_s f(a,\xi;0,s) + \int_s^t J_{s,\tau}(a) \left(\mathrm{E}_\tau M^a_{s,\tau} U^*_s f(a,\xi;0,s) \right) dB_\tau \\ &\quad - \frac{1}{2} \int_s^t J_{s,\tau}(a^2) \mathrm{E}_\tau M^a_{s,\tau} U^*_s f(a,\xi;0,s)\, d\tau \Big\} \\ &= f(a,\xi;0,s) + \int_s^t J_\tau(a) \left(\mathrm{E}_\tau U_s M^a_{s,\tau} U^*_s f(a,\xi;0,s) \right) dB_\tau \\ &\quad - \frac{1}{2} \int_s^t J_\tau(a^2) \mathrm{E}_\tau U_s M^a_{s,\tau} U^*_s f(a,\xi;0,s)\, d\tau \\ &= \xi + \int_0^s J_\tau(a) \mathrm{E}_\tau f(a,\xi;0,\tau)\, dB_\tau - \frac{1}{2} \int_0^s J_\tau(a^2) \mathrm{E}_\tau f(a,\xi;0,\tau)\, d\tau \\ &\quad + \int_s^t J_\tau(a)(\mathrm{E}_\tau g_\tau)\, dB_\tau - \frac{1}{2} \int_s^t J_\tau(a^2) \mathrm{E}_\tau g_\tau\, d\tau \\ &= \xi + \int_0^t J_\tau(a)(\mathrm{E}_\tau g_\tau)\, dB_\tau - \frac{1}{2} \int_0^t J_\tau(a^2) \mathrm{E}_\tau g_\tau\, d\tau. \end{aligned}$$

Hence, by the uniqueness for this stochastic integral equation (Proposition 3.1), $g_t = f(a,\xi;0,t) = M_t^a\xi$. This proved the cocycle relation. $\square$

Proposition 3.2 implies that for any $a \in \mathcal{A}$, $\{M_{s,t}^a : s \leq t\}$ is a multiplier (cocycle) for the unitary evolution $\{U_{s,t} : s \leq t\}$ defined in (3.3).

Proposition 3.3 *For any $a \in \mathcal{A}$, the multiplier (cocycle) $\{M_{s,t}^a : s \leq t\}$ for the unitary evolution $\{U_{s,t} : s \leq t\}$ satisfies the properties (a)-(e) in Assumption 2.2.*

Proof. For $a \in \mathcal{A}$ and $0 \leq s \leq t$, let $\{M_{s,t}^{(n)}\}_{n\geq 0}$ be the family of bounded operators on $\mathcal{H}$ defined by $\xi \mapsto f_{s,t}^{(n)}$, where for $\xi \in \mathcal{H}$, $\{f_{s,t}^{(n)}\}$ is the family of vectors in $\mathcal{H}$ defined recursively in (3.7). The results in the proof of Proposition 3.2 imply that the family $\{M_{s,t}^{(n)}\}$ satisfies the recursive relation (strong operator sense)

$$\begin{aligned} M_{s,t}^{(0)} &= \mathbf{1}_{\mathcal{H}}, \\ M_{s,t}^{(n)} &= \int_s^t J_{s,t}(a)\mathrm{E}_\tau M_{s,\tau}^{(n-1)}\, dB_\tau - \frac{1}{2}\int_s^t J_{s,t}(a^2)\mathrm{E}_\tau M_{s,\tau}^{(n-1)}\, dt, \end{aligned} \tag{3.14}$$

and

$$M_{s,t}^a = \sum_{n=0}^{\infty} M_{s,t}^{(n)}. \tag{3.15}$$

It is clear that $\{M_{s,t}^{(0)} : s \leq t\}$ satisfies (a) - (d) in Assumption 2.2. As an inductive hypothesis, assume that for given $n \geq 1$, $\{M_{s,t}^{(n-1)} : s \leq t\}$ satisfies the properties (a) -(d) in Assumption 2.2.

For $s \leq \tau \leq t$, $\mathrm{E}_\tau M_{s,t}^{(n-1)}$ maps $\mathcal{H}_{[s}$ into $\mathcal{H}_{[s}$ and acts as follow on decomposed vectors $u_s\varphi_{[s}$ ($u_s \in \Phi_s$, $\varphi_{[s} \in \mathcal{H}_{[s}$)

$$\mathrm{E}_\tau M_{s,t}^{(n-1)}(u_s\varphi_{[s}) = u_s\mathrm{E}_\tau M_{s,t}^{(n-1)}\varphi_{[s},$$

and so it follows from (3.14) and (3.15) that $M_{s,t}^a$ satisfies the property (a) in Assumption 2.2. By the inductive hypothesis and (2.1), we have

$$\begin{aligned} J_{s,t}(a)\mathrm{E}_\tau M_{s,\tau}^{(n-1)}\Theta_s &= J_{s,\tau}(a)\Theta_s\mathrm{E}_{\tau-s}M_{\tau-s}^{(n-1)} \\ &= \Theta_s J_{\tau-s}(a)\mathrm{E}_{\tau-s}M_{\tau-s}^{(n-1)}. \end{aligned}$$

Since $(B_\tau - B_s)\Theta_s = \Theta_s B_{\tau-s}$ by taking the adjoint of the relation of (3.1), it follows from (3.14) and the above relation that

$$\begin{aligned} M_{s,t}^{(n)}\Theta_s &= \Theta_s \left\{ \int_0^{t-s} J_\tau(a) \mathrm{E}_\tau M_\tau^{(n-1)}\, dB_\tau - \frac{1}{2}\int_s^{t-s} J_\tau(a^2)\mathrm{E}_\tau M_\tau^{(n-1)}\, d\tau \right\} \\ &= \Theta_s M_{t-s}^{(n)}. \end{aligned}$$

Thus by (3.15), $\{M_{s,t}^a\}$ satisfies the property (b) in Assumption 2.2.

Since $\{M_{s,t}^{(n-1)}; s \le t\}$ satisfies the properties (c) and (d) by the induction hypothesis, $\{M_{s,t}^a : s \le t\}$ also satisfies the properties by (3.14) and (3.15). The property (e) in Assumption 2.2 holds by the bound (3.10). This completes the proof of the proposition. □

The following is the main result in this section:

Theorem 3.1 *(a) Let $a \in \mathcal{A}$. If the weak*-continuous automorphism group $\{\alpha_t\}$ is unitarily implemented, then (P_t^a) defined by*

$$(P_t^a(x))\varphi = \mathrm{E}_0\left[(M_t^a)^* J_t(x) M_t^a \varphi\right], \quad \varphi \in \mathfrak{h}, \tag{3.16}$$

is a weak-continuous, completely positive preserving semigroup on $\mathcal{A}$.*

(b) Let $a = a^* \in \mathcal{A}$. The semigroup (P_t^a) satisfies the following properties : for any $t \ge 0$

(i) identity preserving : $P_t^a(\mathbf{1}) = \mathbf{1}$.

(ii) reality preserving : $P_t^a(x^*) = (P_t^a(x))^*$, $x \in \mathcal{A}$,

(iii) Markovianity : $0 \le x \le \mathbf{1} \implies 0 \le P_t^a(x) \le \mathbf{1}$, $x \in \mathcal{A}$.

(iv) contractivity : $\|P_t\| \le 1$, $t \ge 0$.

Thus (P_t^a) is a Markovian semigroup in the sense of [DL] and [AMV].

(c) The weak-generator of (P_t^a) is an extension of the operator G^a defined by*

$$\begin{aligned} D(G^a) &= D(\delta^2), \\ G^a &= \frac{1}{2}\delta^2 + L_{a^*}\delta + R_a\delta + L_{a^*}R_a - \frac{1}{2}(L_{a^*})^2 - \frac{1}{2}(R_a)^2 \end{aligned} \tag{3.17}$$

where δ is the generator of the automorphism (α_t), and L_a and R_a are the left and right multiplication operators respectively : $L_a x = ax$, $R_a x = xa$, $x \in \mathcal{A}$.

Proof. (a) This follows as a corollary of Theorem 2.1, Lemma 3.1 and Propositions 3.2 and 3.3. (b) (i) follows from the part (c) stated below, i.e., $G^a(\mathbf{1}) = 0$ if $a = a^* \in \mathcal{A}$. (ii) holds by Theorem 2.1 (b) for any $a \in \mathcal{A}$. (iii) follows from (i) and the positivity of (P_t^a). The positivity and the identity preserving property of (P_t^a) imply the contractivity (iv). See Corollary 3.2.6 of [BR]. Thus it remains to prove the part (c).

(c) The following result is Proposition 1.3 of [LS]. Let $x \in D(\delta^2)$ and $\varphi \in \mathfrak{h}$. Then

$$J_{s,t}(x)\varphi = x\varphi + \int_s^t J_{s,\tau}(\delta x)\varphi \, dB_\tau + \frac{1}{2}\int_s^t J_{s,\tau}(\delta^2 x)\varphi \, d\tau. \tag{3.18}$$

By the Ito product formula for Hilbert space valued Ito integrals, (3.18) and Proposition 3.2 (a) combine to give

$$\begin{aligned} J_t(x)M_t^a\varphi &= x\varphi + \int_0^t J_\tau(\delta x + xa)M_\tau^a\varphi \, dB_\tau \\ &\quad + \int_0^t J_\tau(\frac{1}{2}\delta^2 x + (\delta x)a - \frac{1}{2}xa^2)M_\tau^a\varphi \, d\tau. \end{aligned} \tag{3.19}$$

Here we have used the fact that $\mathrm{E}_\tau M_\tau^a\varphi = M_\tau^a\varphi$ (Proposition 3.3 and Assumption 2.2 (c)). We use the Ito product formula, Proposition 3.2 (a) and (3.19) to obtain that for any $\varphi_1, \varphi_2 \in \mathfrak{h}$

$$\begin{aligned} &\langle M_t^a\varphi_1, J_t(x)M_t^a\varphi_2\rangle_{\mathfrak{h}} \\ &= \langle \varphi_1, x\varphi_2\rangle_{\mathfrak{h}} + \int_0^t \langle M_\tau^a\varphi_1, J_\tau(\delta x + a^*x + xa)M_\tau^a\varphi_2\rangle_{\mathfrak{h}} \, dB_\tau \\ &\quad + \int_0^t \langle M_\tau^a\varphi_1, J_\tau(\frac{1}{2}\delta^2 x + a^*\delta x + (\delta x)a + a^*xa - \frac{1}{2}(a^*)^2 x - \frac{1}{2}xa^2)M_\tau^a\varphi_2\rangle_{\mathfrak{h}} \, d\tau. \end{aligned}$$

By taking the Wiener expectation of the above expression, we have that

$$\begin{aligned} P_t^a(x)\varphi &= x\varphi + \int_0^t P_\tau^a(G^a(x))\varphi \, d\tau, \\ G^a(x) &= \frac{1}{2}\delta^2(x) + a^*\delta(x) + \delta(x)a + a^*xa - \frac{1}{2}(a^*)^2 x - \frac{1}{2}xa^2, \end{aligned}$$

and so the generator (P_t^a) is a weak*-closed extension of G^a. This completes the proof. □

4 Properties of Noncommutative Elliptic Operators

Let δ be a densely defined *-derivation on $\mathcal{A}$ such that $\mathbf{1} \in D(\delta)$ and $\delta(\mathbf{1}) = 0$. For given self-adjoint element a of $\mathcal{A}$ ($a = a^* \in \mathcal{A}$), define and operator G^a of $\mathcal{A}$ by

$$\begin{aligned} D(G^a) &= D(\delta^2), \\ G^a(x) &= \frac{1}{2}\delta^2(x) + a\delta(x) + \delta(x)a - \frac{1}{2}[\,a, [a, x]\,], \quad x \in D(G^a). \end{aligned} \tag{4.1}$$

Recall that for $a = a^* \in \mathcal{A}$, the pre-generator (3.17) of the semigroup (P_t^a) has the above expression. Since G^a can be considered as a noncommutative version of classical elliptic operator $\frac{1}{2}\Delta + \beta \circ \nabla$, where ∇ and Δ are the gradient and Laplacian operators on $L^2(\mathbb{R}^n)$ respectively, and β is a $\mathbb{R}^n$- valued function on $\mathbb{R}^n$, it would be worth to study detail properties of G^a.

Theorem 4.1 *Let δ be a densely defined *-derivation on $\mathcal{A}$ such that $\mathbf{1} \in D(\delta)$ and $\delta(\mathbf{1}) = 0$. For given $a = a^* \in \mathcal{A}$, let G^a be the operator on $\mathcal{A}$ defined as in (4.1). Then the following properties hold:*

(a) $\mathbf{1} \in D(G^a)$ *and* $G^a(\mathbf{1}) = 0$,

(b) $G^a(x^*) = (G^a(x))^*$, $x \in D(G^a)$,

(c) complete dissipationity : the matrix inequality

$$[G^a(x_i^* x_j)] \geq [G^a(x_i^*)x_j + x_i^* G^a(x_j)]$$

holds for any finite sequence $\{x_1, x_2, \cdots, x_n\}$ *in* $D(G^a)$.

Proof. It is clear that the properties (a) and (b) hold. Let us prove (c). Put

$$\delta_a(x) := i[a, x], \quad x \in \mathcal{A}.$$

Then δ_a is a *-derivation on $\mathcal{A}$ since $a = a^* \in \mathcal{A}$. $G^a(x)$ can be written as

$$G^a(x) = \frac{1}{2}\delta^2(x) + a\delta(x) + \delta(x)a + \frac{1}{2}\delta_a^2(x). \tag{4.2}$$

Since δ and δ_a are derivations on $\mathcal{A}$, we get that for any finite sequence $\{x_1, x_2, \cdots, x_n\}$ in $D(G^a)$

$$\begin{aligned}\delta(x_i^* x_j) &= \delta(x_i^*)x_j + x_i^*\delta(x_j),\\ \delta_a(x_i^* x_j) &= \delta_a(x_i^*)x_j + x_i^*\delta_a(x_j),\end{aligned}$$

and

$$\begin{aligned}\delta^2(x_i^* x_j) &= \delta^2(x_i^*)x_j + 2\delta(x_i^*)\delta(x_j) + x_i^*\delta^2(x_j),\\ \delta_a^2(x_i^* x_j) &= \delta_a^2(x_i^*)x_j + 2\delta_a(x_i^*)\delta_a(x_j) + x_i^*\delta_a^2(x_j),\end{aligned}$$

and so it follows from (4.2) and the above equalities that

$$\begin{aligned}&G^a(x_i^* x_j) - G^a(x_i^*)x_j - x_i^* G^a(x_j)\\ &\quad = \delta(x_i^*)[x_j, a] + [a, x_i^*]\delta(x_j) + \delta(x_i^*)\delta(x_j) + \delta_a(x_i^*)\delta_a(x_j)\\ &\quad = \{\delta(x_i^*) - i\delta_a(x_i^*)\}\{\delta(x_j) + i\delta_a(x_j)\}\\ &\quad = (\delta(x_i) + i\delta_a(x_i))^*(\delta(x_j) + i\delta_a(x_j)).\end{aligned}$$

The complete dissipationity of G^a follows from the above equality. $\square$

Theorem 4.1 implies that, if G^a is closable in an appropriate topology and if a closed extension of G^a generates a (positive preserving) semigroup (P_t^a), then (P_t^a) is completely positive, and preserves identity and reality [Bra, BR]. For instance. if δ is a bounded *-derivation on $\mathcal{A}$, then G^a is bounded and so G^a generates a strongly continuous semigroup on $\mathcal{A}$ and the properties mentioned above hold. See Corollary 3.2.1 of [Bra]. The main result (Theorem 3.1) in Section 3 conformed that if δ generates a weak*-continuous one-parameter group of *-automorphisms of $\mathcal{A}$, a closed extension of G^a generates a weak*-continuous semigroup. It would be interest to know conditions on δ and $a \in \mathcal{A}$ under which G^a is closable and a closed extension generates a semigroup [BR, Kat].

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