PERIODIC MANIFOLDS WITH SPECTRAL GAPS

OLAF POST

Abstract. We investigate spectral properties of the Laplace operator on a class of non-compact Riemannian manifolds. For a given number N we construct periodic manifolds such that the essential spectrum of the corresponding Laplacian has at least N open gaps. We use two different methods. First, we construct a periodic manifold starting from an infinite number of copies of a compact manifold, connected by small cylinders. In the second construction we begin with a periodic manifold which will be conformally deformed. In both constructions, a decoupling of the different period cells is responsible for the gaps.

1. Introduction

The spectra of periodic Schrödinger or divergence type operators have been extensively studied. In particular, it is well-known that the spectrum of a periodic elliptic operator on $L_2(\mathbb{R}^d)$ with smooth coefficients is the locally finite union of compact intervals, called *(spectral) bands*. We are mainly interested in the question whether these bands are separated by *spectral gaps* or not. By a *gap* in the (essential) spectrum of a positive operator H we mean an interval (a,b) such that

$$(a,b) \cap \operatorname{spec} H = \emptyset.$$

To exclude trivial cases we assume that a is greater than the infimum of the essential spectrum of H. The number of gaps is given by the number of components of the intersection of the resolvent set $\mathbb{C} \setminus \operatorname{spec} H$ with $\mathbb{R}$. Results on spectral gaps where H is a Schrödinger or divergence type operator can be found for example in [15, 16, 17, 18] (see also the references therein).

In this paper, we want to give examples of (non-compact) periodic manifolds $\mathcal{M}$ such that the corresponding Laplacian $\Delta_{\mathcal{M}}$ *without* potential has spectral gaps. Here, *periodicity* means that a finitely generated abelian group Γ acts isometrically and properly discontinuously on $\mathcal{M}$ (cf. for example [2, 4, 5, 10, 20], periodic manifold are also called *covering manifolds*). Therefore we obtain the same qualitative results *only* by the periodic geometry.

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As in the Schrödinger operator case a decoupling of the different period cells is responsible for the gaps. In the Schrödinger operator case, the decoupling is achieved by a high potential barrier separating each period cell from the others. In the geometric case, decoupling means that the junction between two period cells is small. Here, a period cell is the closure of a fundamental domain (see the next section).

From a physical point of view the Laplacian on a manifold is the Hamiltonian of an electron confined to this manifold (at least in a semi-classical limit, cf. for example [12, 23]). Periodic structures like a periodically curved cylinder or a quantum wire could have applications in solid state physics. A quantum wire (or quantum wave guide) is a planar strip, cf. for example [11]. The knowledge of the band-gap structure of $\operatorname{spec} H$ is important for the conductive properties of the periodic material described by the Hamiltonian H. In particular, the existence and size of the first gap decide whether the material is a conductor or an insulator.

Basic ideas and results. The construction of Γ-periodic manifolds with spectral gaps will be given later on in detail (cf. Section 3). Here, we sketch the ideas

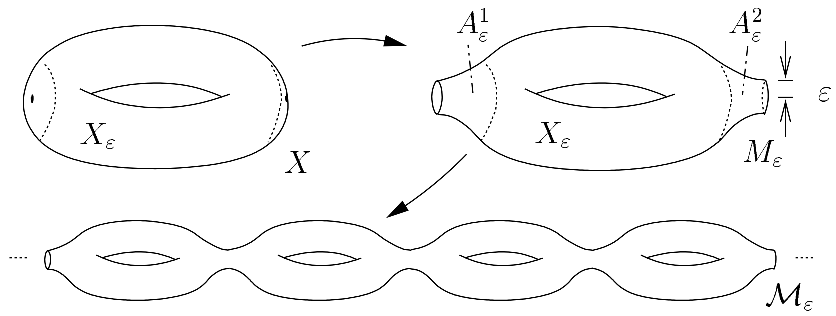

FIGURE 1. Construction of the period cell M_ε and the periodic manifold $\mathcal{M}_\varepsilon$ with $\Gamma = \mathbb{Z}$.

and fix the notation. We start our construction from a compact Riemannian manifold X of dimension $d \geq 2$ (for simplicity without boundary). Here, Γ is an abelian group with r generators. We choose $2r$ different points $x_1, \dots, x_{2r} \in X$ and endow each point x_i with a cylindrical end A^i_ε (the boundary being isometric to a sphere of radius $\varepsilon > 0$). We call the resulting manifold M_ε. By glueing together Γ copies of the period cell M_ε we obtain a Γ-periodic manifold $\mathcal{M}_\varepsilon$ (see Figure 1 for $r = 1$ generators). Our first result is the following:

Theorem 1.1. *Each spectral band of the periodic Laplacian $\Delta_{\mathcal{M}_\varepsilon}$ on $\mathcal{M}_\varepsilon$ tends to an eigenvalue of the Laplacian Δ_X on X as $\varepsilon \to 0$. In particular, for each $N \in \mathbb{N}$ there exist at least N gaps in the spectrum of $\Delta_{\mathcal{M}_\varepsilon}$ provided ε is small enough.*

We also prove a similar result in the case where the cells M_ε are joined by long thin cylinders of fixed length. Let N_ε be the manifold obtained from M_ε by identifying $\partial A_\varepsilon^{2i-1}$ with a component of the boundary of a cylinder C_ε^i of length $L_i \geq 0$ and radius $\varepsilon > 0$. The periodic manifold $\mathcal{N}_\varepsilon$ is obtained in the same way by glueing together Γ copies of N_ε.

Theorem 1.2. *Each spectral band of the periodic Laplacian $\Delta_{\mathcal{N}_\varepsilon}$ on $\mathcal{N}_\varepsilon$ tends to an eigenvalue of the Laplacian Δ_X on X or to an eigenvalue of the Laplacian with Dirichlet boundary conditions on $[0, L_i]$ for an $i = 1, \dots, r$ with $L_i > 0$ as $\varepsilon \to 0$. In particular, for each $N \in \mathbb{N}$ there exist at least N gaps in the spectrum of $\Delta_{\mathcal{N}_\varepsilon}$ provided ε is small enough.*

The proofs of these two results will be given in Section 3. Both results are related to articles of Chavel and Feldman [7] and Anné [1] where compact manifolds with small handles resp. compact manifolds joined by small cylinders are analysed. Note that the joined manifolds in [1] are not smooth in contrast to our construction.

The second construction is in some sense the reverse of the first construction. Starting with a given periodic manifold $\mathcal{M}$ of dimension $d \geq 2$ we deform the (periodic) metric g by a (periodic) conformal factor ρ_ε to obtain spectral gaps in the spectrum of the Laplacian. The idea is to let the conformal factor converge to the indicator function of a closed periodic set $\mathcal{X} = \bigcup_{\gamma \in \Gamma} \gamma X$ where $X \subset \mathcal{M}$ is a closed subset disjoint from all translates γX, $\gamma \neq 0$. This convergence is of course not uniform because of the discontinuity of the indicator function. We denote the manifold $\mathcal{M}$ with metric $\rho_\varepsilon^2 g$ by $\mathcal{M}_\varepsilon$. We have the following result:

Theorem 1.3. *Suppose that the dimension of $\mathcal{M}$ is greater or equal to 3. Then each band of the periodic Laplacian $\Delta_{\mathcal{M}_\varepsilon}$ on the conformally deformed periodic manifold $\mathcal{M}_\varepsilon$ tends to an eigenvalue of the Neumann Laplacian Δ_X^{N} on X as $\varepsilon \to 0$. In particular, for each $N \in \mathbb{N}$ there exist at least N gaps in the spectrum of $\Delta_{\mathcal{M}_\varepsilon}$ provided ε is small enough.*

The assumption $d \geq 3$ is essential here, since in dimension 2 it is no longer true that the first band of $\Delta_{\mathcal{M}_\varepsilon}$ tends to a single point as $\varepsilon \to 0$. Even in the limit case $\varepsilon = 0$, a nontrivial interval remains due to the special structure of the conformal Laplacian, cf. Equation (4.6). Nevertheless, we can prove the existence of gaps in a simple example by direct calculations (see Example 4.6). The proof of Theorem 1.3 and the example in dimension 2 can be found in Section 4.

The proofs of our theorems basically use the variational max-min characterisation of the eigenvalues of an operator with purely discrete spectrum (called Min-max Principle) to compare the eigenvalues of operators defined on parameter-depending Hilbert spaces (see the Main Lemma 2.2). The basic idea is taken from [1] and [13] even if the Main Lemma is more general. Furthermore, Floquet Theory allows to analyse the spectrum of a periodic operator. Details are given in the next section.

Davies and Harrell II [9] proved the existence of at least one gap in the periodic conformally flat case (transforming the conformal Laplacian in a corresponding

Schrödinger operator). This result is a special case of Theorem 1.3. Using similar methods, Green showed in [14] the existence of a finite number of gaps in the 2-dimensional conformally flat case. Furthermore, Yoshitomi [28] proved the existence of spectral gaps for the (Dirichlet) Laplacian on periodically curved quantum wave guides. In all these three papers the existence of gaps is established basically by analysing a one-dimensional problem. Here, in contrast, we directly study the multi-dimensional problem.

Green conjectured that a necessary requirement for a large number of gaps is for the curvature to be large in absolute value at some points. In the examples given here the same phenomenon occurs (see Remarks 3.5 and 4.1). Therefore the results of Fukaya [13] cannot be applied here. Fukaya showed the continuity of the k-th eigenvalue of the Laplacian on M_ε where M_ε is a convergent family of manifolds (in a certain sense) with given bound on the curvature.

There are also results on periodic operators on manifolds with *non-commutative* groups Γ. For example, Brüning and Sunada proved in [4] that the Laplacian on a periodic manifold still has band structure even for certain non-commutative groups Γ. Furthermore, Sunada [20] showed that — in contrast to the Schrödinger operator case on $\mathbb{R}^d$ (cf. [25]) — there exist (non-compact!) periodic manifolds with commutative group Γ such that the corresponding Laplacian has an eigenvalue (possibly embedded in another band).

Finally note that Lott [22] has constructed a (non-periodic) 2-dimensional complete non-compact finite-volume manifold such that the corresponding Laplacian has an infinite number of gaps. In the periodic case in contrast we would expect that the Generalized Bethe-Sommerfeld conjecture is true, i.e. that there are only finitely many gaps in the spectrum of $\Delta_{\mathcal{M}}$ if $d \geq 2$. Skriganov [27] proved this conjecture for periodic Schrödinger operators in Euclidean space. Furthermore, an asymptotic upper bound on the number of gaps have been established in [4] (not implying the Generalized Bethe-Sommerfeld conjecture). Note that our results do not say anything whether the number of gaps is finite or not.

2. Preliminaries

Laplacian on a manifold. Throughout this article we study manifolds of dimension $d \geq 2$. For a Riemannian manifold M (compact or not) without boundary we denote by $L_2(M)$ the usual L_2-space of square integrable functions on M with respect to the volume measure on M. Locally in a chart, the volume measure has the density $(\det g)^{\frac{1}{2}}$ with respect to the Lebesgue measure, where $\det g$ is the determinant of the metric tensor (g_{ij}) in this chart. The norm of $L_2(M)$ will be denoted by $\|\cdot\|_M$. For $u \in C_c^\infty(M)$, the space of compactly supported smooth functions, we set

$$\check{q}_M(u) := \int_M |\mathrm{d}u|^2.$$

Here $\mathrm{d}u$ denotes the exterior derivate of u, which is a section in the cotangent bundle T^*M over M. Locally in a chart, $|\mathrm{d}u|^2 = \sum_{i,j} g^{ij} \partial_i u \, \partial_j \overline{u}$ where (g^{ij}) is the inverse of (g_{ij}).

We denote the closure of the non-negative quadratic form $\check{q}_M$ by q_M. Note that the domain $\operatorname{dom} q_M$ of the closed quadratic form q_M consists of those functions in $L_2(M)$ such that the weak derivative $\mathrm{d}u$ is also square integrable (i.e., $q_M(u) < \infty$).

We define the *Laplacian* Δ_M (for a manifold without boundary) as the unique self-adjoint and non-negative operator associated with the closed quadratic form q_M, i.e., operator and quadratic form are related by

$$q_M(u) = \langle \Delta_M u, u \rangle$$

for $u \in C_{\mathrm{c}}^\infty(M)$ (for details on quadratic forms see e.g. [19, Chapter VI], [26] or [8]).

If M is a compact manifold with (piecewise) smooth boundary $\partial M \neq \emptyset$ we can define the Laplacian with *Dirichlet* resp. *Neumann boundary conditions* in the same way. Here, we start from the (closure of the) quadratic form $\check{q}_M$ defined on $C_{\mathrm{c}}^\infty(M)$, the space of smooth functions with support *away* from the boundary, resp. on $C^\infty(M)$, the space of functions smooth up to the boundary. We denote the closure of the quadratic form by q_M^{D} resp. q_M^{N} and the corresponding operator by Δ_M^{D} resp. Δ_M^{N}.

If M is compact the spectrum of Δ_M (with any boundary condition if $\partial M \neq \emptyset$) is purely discrete. We denote the corresponding eigenvalues by $\lambda_k(M)$ (resp. $\lambda_k^{\mathrm{D}}(M)$ or $\lambda_k^{\mathrm{N}}(M)$ in the Dirichlet or Neumann case) written in increasing order and repeated according to multiplicity.

Periodic manifolds and Floquet Theory. Let Γ be an abelian group of infinite order with r generators and neutral element 1. Such groups are isomorphic to $\mathbb{Z}^{r_0} \times \mathbb{Z}_{p_1}^{r_1} \times \cdots \times \mathbb{Z}_{p_a}^{r_a}$ with $r_0 > 0$ and $r_0 + \cdots + r_a = r$. Here, $\mathbb{Z}_p$ denotes the abelian group of order p. A d-dimensional (non-compact) Riemannian manifold $\mathcal{M}$ will be called Γ-*periodic* or a *covering manifold* if Γ acts properly discontinuously, isometrically and cocompactly, i.e., the quotient $\mathcal{M}/\Gamma$ is a d-dimensional compact Riemannian manifold such that the quotient map is a local isometry (cf. e.g. [2, 4, 5, 10, 20]). For simplicity we assume that $\mathcal{M}$ has no boundary.

A compact subset M of $\mathcal{M}$ is called a *period cell* if M is the closure of a fundamental domain D, i.e., $M = \overline{D}$, D is open and connected, D is disjoint from any translate γD for all $\gamma \in \Gamma$, $\gamma \neq 1$, and the union of all translates γM is equal to $\mathcal{M}$.

Now we want to analyse the spectrum of periodic elliptic operators on $\mathcal{M}$. Here, *periodicity* means that the operator commutes with all translation operators on $L_2(\mathcal{M})$ induced by the group action (cf. e.g. [4, 20]). In particular the Laplacian on $\mathcal{M}$ is periodic. From Floquet theory it suffices to analyse the spectra of the periodic operator restricted to a period cell M with quasi-periodic boundary conditions (cf. e.g. [10, 25]). In order to do this, we define θ-periodic boundary conditions. Let θ be an element of the dual group $\hat{\Gamma} = \operatorname{Hom}(\Gamma, \mathbb{T}^1)$ of Γ, which is isomorphic to a subgroup of the r-dimensional torus $\mathbb{T}^r = \{\theta \in \mathbb{C}^r;\, |\theta_i| = 1 \text{ for all } i\}$. We denote by q_M^θ the closure of the quadratic form $\check{q}_M$ defined on the space of those functions $u \in C^\infty(M)$ that satisfy

$$u(\gamma x) = \overline{\theta(\gamma)}\, u(x)$$

for all $x \in \partial M$ and all $\gamma \in \Gamma$ such that $\gamma x \in \partial M$. The corresponding operator is denoted by Δ_M^θ. Again, Δ_M^θ has purely discrete spectrum denoted by $\lambda_k^\theta(M)$. The eigenvalues depend continuously on θ. Furthermore $\lambda_k^\theta(M)$ depends even analytically on θ if we exclude those $\theta \in \hat{\Gamma}$ for which $\lambda_k^\theta(M)$ is a multiple eigenvalue (cf. e.g. [3, 25]). From Floquet theory we obtain

$$\operatorname{spec} \Delta_{\mathcal{M}} = \bigcup_{\theta \in \hat{\Gamma}} \operatorname{spec} \Delta_M^\theta = \bigcup_{k \in \mathbb{N}} B_k(\mathcal{M})$$

where $B_k = B_k(\mathcal{M}) = \{\lambda_k^\theta(M);\, \theta \in \hat{\Gamma}\}$ is a compact interval, called *k-th band* (cf. e.g. [25]). In general, we do not know whether the intervals B_k overlap or not. But we can show the existence of gaps by proving that $\lambda_k^\theta(M)$ does not vary too much in θ.

Remark 2.1. Note that the first band cannot be trivial (i.e. $B_1 = \{0\}$) since the first eigenvalue $\lambda_1^\theta(M)$ is 0 if and only if we are in the case of periodic boundary conditions, i.e., $\theta = 1$. There are no constant *and* θ-periodic functions if M is connected. This means that the first band consists of absolutely continuous spectrum provided the first and second band do not overlap or more precisely, that $B_1 \cap B_2^c$ is absolutely continuous. In general it is not true that all the spectrum is absolutely continuous. In [20] one can find an example where a band reduces to a point (not necessarily being an isolated eigenvalue).

Main Lemma and Min-max principle. Here we state a formal result on how to deal with parameter-depending Hilbert spaces and operators with purely discrete spectrum on these spaces. In particular, we are interested in the dependence of the eigenvalues on the parameter. Such Hilbert spaces occur in the next section when we construct a period cell depending on a parameter ε. The basic idea of the Main Lemma 2.2 is taken from [1] and [13]. Nevertheless the Main Lemma is more general: here we allow even non-uniform convergence, i.e., the convergence assumed in Conditions (2.2) and (2.3) could depend on (u_ε). Furthermore, we can choose alternatively between the convergence or the inequality in the assumptions.

We first quote the *Min-max Principle.* Suppose that q is a closed, non-negative quadratic form on the separable Hilbert space $\mathcal{H}$ such that the corresponding operator Q has purely discrete spectrum denoted by $\operatorname{spec} Q = \{\lambda_k \,|\, k \in \mathbb{N}\}$. Note that $\lambda_k \geq 0$ for all k. Throughout this article we assume that the sequence of eigenvalues (λ_k) is written in increasing order and repeated according to multiplicity. We then have

$$\lambda_k = \inf_{L_k} \sup_{u \in L_k, u \neq 0} \frac{q(u)}{\|u\|^2} \tag{2.1}$$

where the infimum is taken over all k-dimensional subspaces L_k of $\operatorname{dom} q$ (for this version of the Min-max principle see e.g. [8]).

Suppose now that for each $\varepsilon > 0$ we have separable Hilbert spaces $\mathcal{H}_\varepsilon$ and $\mathcal{H}'_\varepsilon$. Furthermore suppose that q_ε and q'_ε are non-negative, closed quadratic forms on $\mathcal{H}_\varepsilon$ and $\mathcal{H}'_\varepsilon$. Finally suppose that the corresponding operators have purely discrete spectrum denoted by $\lambda_k(\varepsilon)$ and $\lambda'_k(\varepsilon)$, $k \in \mathbb{N}$ (written in increasing order and repeated according to multiplicity). A corresponding orthonormal basis of eigenfunctions of Q_ε is denoted by $(\varphi_k^\varepsilon)_k$ and the linear span of the first k eigenvalues by $L_k(\varepsilon)$.

Lemma 2.2 (Main Lemma)**.** *Suppose that for each $\varepsilon > 0$ a linear map $\Phi_\varepsilon \colon \operatorname{dom} q_\varepsilon \longrightarrow \operatorname{dom} q'_\varepsilon$ is given such that for all $u_\varepsilon \in L_k(\varepsilon)$ Conditions* (2.2) *and* (2.3) *are satisfied:*

$$\lim_{\varepsilon \to 0} (\|\Phi_\varepsilon u_\varepsilon\|^2_{\mathcal{H}'_\varepsilon} - \|u_\varepsilon\|^2_{\mathcal{H}_\varepsilon}) = 0 \qquad \textit{or} \qquad \|u_\varepsilon\|^2_{\mathcal{H}_\varepsilon} \leq \|\Phi_\varepsilon u_\varepsilon\|^2_{\mathcal{H}'_\varepsilon} \tag{2.2}$$

$$\lim_{\varepsilon \to 0} (q'_\varepsilon(\Phi_\varepsilon u_\varepsilon) - q_\varepsilon(u_\varepsilon)) = 0 \qquad \textit{or} \qquad q_\varepsilon(u_\varepsilon) \geq q'_\varepsilon(\Phi_\varepsilon u_\varepsilon). \tag{2.3}$$

Furthermore, we assume that for each $k \in \mathbb{N}$ there exist a constant $c_k > 0$ such that

$$\lambda_k(\varepsilon) \leq c_k \qquad \textit{for all } \varepsilon > 0. \tag{2.4}$$

Then for each $k \in \mathbb{N}$ there exists a function $\delta_k(\varepsilon) \geq 0$ converging to 0 as $\varepsilon \to 0$ such that

$$\lambda'_k(\varepsilon) \leq \lambda_k(\varepsilon) + \delta_k(\varepsilon) \tag{2.5}$$

for small enough $\varepsilon > 0$.

Proof. For $u = u_\varepsilon = \sum_{i=1}^k \alpha_i^\varepsilon \varphi_i^\varepsilon$ with complex numbers $\alpha_i = \alpha_i^\varepsilon$ we have

$$\frac{q'_\varepsilon(\Phi_\varepsilon u)}{\|\Phi_\varepsilon u\|^2} - \frac{q_\varepsilon(u)}{\|u\|^2} = \frac{1}{\|\Phi_\varepsilon u\|^2} \Big(\frac{q_\varepsilon(u)}{\|u\|^2} \big(\|u\|^2 - \|\Phi_\varepsilon u\|^2\big) + \big(q'_\varepsilon(\Phi_\varepsilon u) - q_\varepsilon(u)\big)\Big).$$

Furthermore, we estimate

$$\begin{aligned} \|u\|^2 - \|\Phi_\varepsilon u\|^2 &= \sum_{i,j=1}^k \alpha_i \overline{\alpha_j} \big(\delta_{ij} - \langle \Phi_\varepsilon \varphi_i^\varepsilon, \Phi_\varepsilon \varphi_j^\varepsilon \rangle\big) \\ &\leq \delta'_k(\varepsilon) \sum_{j=1}^k |\alpha_j|^2 = \delta'_k(\varepsilon)\, \|u\|^2 \end{aligned} \tag{2.6}$$

where

$$\delta'_k(\varepsilon) := k \max_{i,j=1,\dots,k} |\delta_{ij} - \langle \Phi_\varepsilon \varphi_i^\varepsilon, \Phi_\varepsilon \varphi_j^\varepsilon \rangle|$$

by the Cauchy-Schwarz Inequality. The Polarisation Identity together with Condition (2.2) yields $\delta'_k(\varepsilon) \to 0$ as $\varepsilon \to 0$. If we are in the second alternative of Condition (2.2) we simply set $\delta'_k(\varepsilon) = 0$. By a similar argument we can show the existence of a function $\delta''_k(\varepsilon) \geq 0$ converging to 0 as $\varepsilon \to 0$ such that

$$q'_\varepsilon(\Phi_\varepsilon u) - q_\varepsilon(u) \leq \delta''_k(\varepsilon)\|u\|^2. \tag{2.7}$$

From Estimate (2.6) we also conclude

$$\|u\|^2 \leq \frac{1}{1 - \delta'_k(\varepsilon)} \|\Phi_\varepsilon u\|^2 \tag{2.8}$$

provided ε is small enough. Applying Condition (2.4) we obtain the estimate

$$\frac{q'_\varepsilon(\Phi_\varepsilon u)}{\|\Phi_\varepsilon u\|^2} - \frac{q_\varepsilon(u)}{\|u\|^2} \leq \delta_k(\varepsilon) := \frac{1}{1 - \delta'(\varepsilon)} \big(c_k \delta'(\varepsilon) + \delta''(\varepsilon) \big). \tag{2.9}$$

Estimate (2.8) also yields the injectivity of $\Phi_\varepsilon\restriction_{L_k(\varepsilon)}$, i.e., $\Phi_\varepsilon(L_k(\varepsilon))$ is a k-dimensional subspace of $\operatorname{dom} q'_\varepsilon$. Finally, the Min-max Principle (2.1) implies the desired estimate on the eigenvalues. ☐

3. Construction of a Periodic Manifold

Suppose that X is a compact oriented and connected Riemannian manifold of dimension $d \geq 2$ (for simplicity without boundary). We want to construct a Γ-periodic manifold where Γ is an abelian group with r generators $e_1, \dots, e_r$. We choose $2r$ distinct points $x_1, \dots, x_{2r}$. For each point x_i, denote by B^i_ε the open geodesic ball around x_i of radius $\varepsilon > 0$. Suppose further that $B^i_{\varepsilon_0}$ are pairwise disjoint where $\varepsilon_0 > 0$ denotes the injectivity radius of X. Denote by B_ε the union of all balls B^i_ε, $i = 1, \dots, 2r$. Let $X_\varepsilon := X \setminus B_{2\varepsilon}$ for $0 < 2\varepsilon < \varepsilon_0$ with metric inherited from X.

On $B^i_{\varepsilon_0}$, the metric of X is given in polar coordinates $(s, \sigma) \in]0, \varepsilon_0[\times \mathbb{S}^{d-1}$ by

$$g = \mathrm{d}s^2 + h^i_s \tag{3.1}$$

where h^i_s denotes a metric on $\{s\} \times \mathbb{S}^{d-1}$ (see Figure 2). Here, $\mathbb{S}^{d-1}$ denotes the $(d-1)$-dimensional sphere with standard metric $\mathrm{d}\sigma^2$.

Let r_ε be a smooth monotone function with $r_\varepsilon(s) = \varepsilon$ for $0 \leq s \leq \varepsilon/2$ and $r_\varepsilon(s) = s$ for $2\varepsilon \leq s \leq \varepsilon_0$. Furthermore, let χ_ε be a smooth cut-off function having values between 0 and 1 and satifying $\chi_\varepsilon(s) = 0$ if $s \leq \varepsilon$ and $\chi_\varepsilon(s) = 1$ if $s \geq 2\varepsilon$. Now we let the modified metric $h^i_{\varepsilon,s}$ be a convex combination of the original metric and the spherical metric $(r_\varepsilon(s))^2 \mathrm{d}\sigma^2$, i.e.,

$$h^i_{\varepsilon,s} := \chi_\varepsilon(s) h^i_s + \big(1 - \chi_\varepsilon(s)\big)\big(r_\varepsilon(s)\big)^2 \mathrm{d}\sigma^2. \tag{3.2}$$

We denote the completion of $X \setminus \{x_1, \dots, x_{2r}\}$ together with the modified metric

$$g^i_\varepsilon := \mathrm{d}s^2 + h^i_{\varepsilon,s}$$

on $B^i_{\varepsilon_0}$ by M_ε (see Figure 2). Since $g^i_\varepsilon(s,\sigma) = g(s,\sigma)$ for $s \ge 2\varepsilon$, the punched manifold X_ε is embedded in M_ε. Furthermore, since $g^i_\varepsilon(s,\sigma) = \mathrm{d}s^2 + \varepsilon^2 \mathrm{d}\sigma^2$ for $s \le \varepsilon/2$, there exists a neighbourhood of $\partial M_\varepsilon \cap B^i_\varepsilon$ given in coordinates by $[0, \varepsilon/2] \times \mathbb{S}^{d-1}$ which is isometric to a cylinders of radius ε and length $\varepsilon/2$. Let

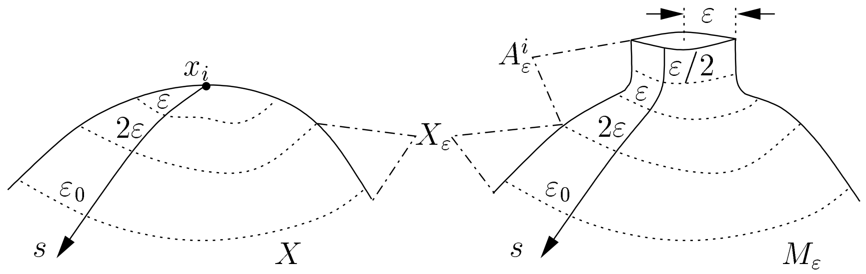

FIGURE 2. Modification of the metric g of the 2-dimensional manifold X near the point x_i.

A^i_ε be the *cylindrical end* of the manifold M_ε near x_i given in coordinates by $[0, 2\varepsilon] \times \mathbb{S}^{d-1}$. Next, let $A^i_{\varepsilon,x}$ be the sphere with distance s from the boundary given in coordinates by $\{s\} \times \mathbb{S}^{d-1}$. Finally, let A_ε be the union of all cylindrical ends A^i_ε, $i = 1, \dots, 2r$.

Finally, we construct the corresponding periodic manifold $\mathcal{M}_\varepsilon$: Let γM_ε be an isometric copy of M_ε with identification $x \mapsto \gamma x$ for each $\gamma \in \Gamma$. We construct a new (non-compact) manifold $\mathcal{M}_\varepsilon$ by identifying $\gamma \partial A^{2i-1}_\varepsilon$ with $e_i \gamma \partial A^{2i}_\varepsilon$ for each $\gamma \in \Gamma$ and $i = 1, \dots, r$. Remember that e_i denotes the i-th generator of Γ. Since in a neighbourhood of ∂A^i_ε the manifold is isometric to a cylinder of radius ε, we can choose a smooth atlas and a smooth metric on the glued manifold $\mathcal{M}_\varepsilon$. We therefore obtain a (non-compact) Γ-periodic manifold $\mathcal{M}_\varepsilon$ and M_ε is a period cell for $\mathcal{M}_\varepsilon$.

Now we are able to state the following theorem (Theorem 1.1 follows via Floquet Theory):

Theorem 3.1. *As $\varepsilon \to 0$ we have $\lambda^\theta_k(M_\varepsilon) \to \lambda_k(X)$ uniformly in $\theta \in \hat{\Gamma}$.*

Therefore, the k-th band $B_k(\mathcal{M}_\varepsilon)$ reduces to the point $\{\lambda_k(X)\}$ as $\varepsilon \to 0$. Note that the convergence is *not* uniform in k since there are topological obstructions (see the discussion in [7]). We therefore could not expect that an *infinite* number of gaps occur.

Before we prove Theorem 3.1, we need two lemmas. The idea is to compare the θ-periodic eigenvalues on M_ε with Dirichlet and Neumann eigenvalues on X_ε. The crucial point is, that the corresponding θ-periodic eigenfunctions on M_ε do not concentrate on A_ε, i.e., on the cylindrical ends. This will be shown in the second lemma. First we need to compare the density of the $(d-1)$-dimensional volume of $A^i_{\varepsilon,s}$ with the volume of the sphere of radius $r_\varepsilon(s)$:

Lemma 3.2. *There exists a constant $c \ge 1$ such that*

$$\frac{1}{c}\, r_\varepsilon(s)^{d-1} \le (\det h^i_{\varepsilon,s})^{\frac{1}{2}} \le c\, r_\varepsilon(s)^{d-1}$$

for all $0 \le s \le \varepsilon_0$ and all i.

Proof. The metric $g = \mathrm{d}s^2 + h^i_s$ on $B^i_{\varepsilon_0}$ can be compared with the flat metric $\mathrm{d}s^2 + s^2\mathrm{d}\sigma^2$ (pointwise in the sense of sesquilinear forms), i.e., there exists a constant $c' \ge 1$ such that

$$\frac{1}{c'}\, s^2 \mathrm{d}\sigma \le h^i_s \le c'\, s^2 \mathrm{d}\sigma.$$

By our assumptions on r_ε the same estimate is true with s^2 replaced by $r_\varepsilon(s)^2$ and h^i_s replaced by the convexe combination (3.2). The result follows from the monotonicity of det. $\square$

Now we prove the non-concentration of the eigenfunctions on the cylindrical ends as $\varepsilon \to 0$:

Lemma 3.3. *There exists a positive function $\omega(\varepsilon)$ converging to 0 as $\varepsilon \to 0$ such that*

$$\int_{A_\varepsilon} |u|^2 \le \omega(\varepsilon) \int_{M_\varepsilon} \big(|u|^2 + |\mathrm{d}u|^2\big), \tag{3.3}$$

for all u out of the domain of the quadratic form with θ-periodic boundary conditions on M_ε.

Note that $\omega(\varepsilon)$ only depends on the geometry of X near x_i.

Proof. Without loss of generality, we can assume that $u \in C^\infty(M_\varepsilon)$. Suppose furthermore that $u(\varepsilon_0, \sigma) = 0$ for all $\sigma \in \mathbb{S}^{d-1}$. First we show an L_2-estimate over $A^i_{\varepsilon,s}$ with its induced metric $h^i_{\varepsilon,s}$.

Applying the Cauchy-Schwarz Inequality and Lemma 3.2 yields

$$\begin{aligned} |u(s,\sigma)|^2 &= \Big| \int_s^{\varepsilon_0} \partial_t u(t,\sigma)\,\mathrm{d}t \Big|^2 \\ &\le c \int_s^{\varepsilon_0} r_\varepsilon(t)^{1-d} \mathrm{d}t \int_s^{\varepsilon_0} |\partial_t u(t,\sigma)|^2 (\det h^i_{\varepsilon,t}(\sigma))^{\frac{1}{2}} \mathrm{d}t. \end{aligned}$$

If we integrate over $\sigma \in \mathbb{S}^{d-1}$ and apply Lemma 3.2 once more we obtain

$$\begin{aligned} \int_{A^i_{\varepsilon,s}} |u|^2 &= \int_{\mathbb{S}^{d-1}} |u(s,\sigma)|^2 (\det h^i_{\varepsilon,s}(\sigma))^{\frac{1}{2}} \\ &\le c^2 r_\varepsilon(s)^{d-1} \int_s^{\varepsilon_0} r_\varepsilon(t)^{1-d} \mathrm{d}t \int_{M_\varepsilon} |\mathrm{d}u|^2_{T^*M_\varepsilon}. \end{aligned} \tag{3.4}$$

If $0 \le s \le 2\varepsilon$ we have $r(s)^{d-1} \le (2\varepsilon)^{d-1}$. Furthermore, the integral over t can be split into an integral over $0 \le t \le 2\varepsilon$ and $2\varepsilon \le t \le \varepsilon_0$. The first integral can be estimated by ε^{2-d}, the second by $\int_{2\varepsilon}^{\varepsilon_0} t^{1-d} \mathrm{d}t$. Therefore we have an estimate of the order $O(\varepsilon)$ if $d \ge 3$ resp. $O(\varepsilon|\ln \varepsilon|)$ if $d = 2$. Finally, if we integrate the integral on the LHS of (3.4) over $s \in [0, 2\varepsilon]$ we obtain the desired Estimate (3.3). If $u(\varepsilon_0, \sigma) \ne 0$ we choose a cut-off function. □

The argument in the proof is due to [1]. The following lemma is proven in [6] resp. [1].

Lemma 3.4. *We have* $\lambda_k^{\mathrm{D}}(X_\varepsilon) \to \lambda_k(X)$ *resp.* $\lambda_k^{\mathrm{N}}(X_\varepsilon) \to \lambda_k(X)$.

Now we can show Theorem 3.1:

Proof. From the Min-max Principle (2.1) we conclude

$$\lambda_k^\theta(M_\varepsilon) \le \lambda_k^{\mathrm{D}}(X_\varepsilon) \tag{3.5}$$

since the domains of the quadratic forms obey the opposite inclusions. In particular, $\lambda_k^\theta(M_\varepsilon)$ is bounded in θ and ε by some constant $c_k > 0$. To prove the opposite inequality we apply our Main Lemma 2.2 with $\mathcal{H}_\varepsilon := L_2(M_\varepsilon)$, $q_\varepsilon := q^\theta_{M_\varepsilon}$, $\mathcal{H}'_\varepsilon := L_2(X_\varepsilon)$, $q'_\varepsilon := q^{\mathrm{N}}_{X_\varepsilon}$ and $\Phi_\varepsilon u := u\restriction_{X_\varepsilon}$ being the restriction operator.

Condition (2.2) is satisfied by Lemma 3.3, the inequality in Condition (2.3) is trivially satisfied. Finally, Condition (2.4) is satisfied by the upper bound (3.5) and Lemma 3.4. The Main Lemma therefore yields

$$\lambda_k^{\mathrm{N}}(X_\varepsilon) \le \lambda_k^\theta(M_\varepsilon) + \delta_k(\varepsilon)$$

with $\delta_k(\varepsilon) \to 0$ as $\varepsilon \to 0$. Together with Estimate (3.5) and Lemma 3.4 we are done. □

Remark 3.5. If we assume that the metric g of the original manifold X is flat on $B^i_{\varepsilon_0}$, i.e., $h^i_s = s^2 \mathrm{d}\sigma^2$ in Equation (3.1), we can calculate the sectional curvature of the cylindrical end A_ε of M_ε. Let ∂_s, $\partial_{\sigma_1}, \dots, \partial_{\sigma_{d-1}}$ be orthonormal tangent vectors corresponding to (s, σ) on A^i_ε. Then the sectional curvature is

$$K(\partial_s, \partial_{\sigma_j}) = -\frac{\ddot{r}_\varepsilon}{r_\varepsilon} \qquad \text{and} \qquad K(\partial_{\sigma_j}, \partial_{\sigma_k}) = \frac{1 - \dot{r}_\varepsilon^2}{r_\varepsilon^2}$$

for $j \ne k$. With our assumptions on r_ε, $K(\partial_s, \partial_{\sigma_j})$ is a negative number of order $O(\varepsilon^{-2})$ and $K(\partial_{\sigma_j}, \partial_{\sigma_k})$ is a positive number of the same orderas $\varepsilon \to 0$. In particular, the Ricci and the scalar curvature are also of order $O(\varepsilon^{-2})$. Therefore, all curvature terms are unbounded in ε as Green conjectured in [14].

Periodic manifold joined by cylinders. In the previous example $\mathcal{M}_\varepsilon$ one may think that it is essential for obtaining spectral gaps that the parts which break down reduce to a point. To avoid this impression we will confront another periodic manifold with spectral gaps where the collapsing parts reduce to intervals instead of points. It only seems to be important that the period cells are seperated by very short closed geodesics (in dimension 2) or more generally by very small submanifolds of codimension 1. The construction in this section is closely related to the work of Anné [1].

For simplicitywe will only assume that $\Gamma = \mathbb{Z}$. Let I be the interval $[0, L]$ and let C_ε be the cylinder $I \times \mathbb{S}^{d-1}$ of radius ε and length $L > 0$ with metric $\mathrm{d}s^2 + \varepsilon^2 \mathrm{d}\sigma^2$ (if $L = 0$ we are in the case of the previous section) . We denote by N_ε the period cell M_ε where we have glued the cylinder C_ε by identifying $\{0\} \times \mathbb{S}^{d-1}$ with ∂A^1_ε. In the same way as above we obtain a periodic manifold $\mathcal{N}_\varepsilon$ with period cell N_ε by glueing together $\mathbb{Z}$ copies of N_ε. As above we can prove:

Theorem 3.6. *As $\varepsilon \to 0$ we have $\lambda^\theta_k(N_\varepsilon) \to \lambda^{\mathrm{D}}_k(X \dot\cup I)$ uniformly in $\theta \in \hat\Gamma$.*

Here, $\lambda^{\mathrm{D}}_k(X \dot\cup I)$ denotes the eigenvalues of the operator $\Delta_X \oplus \Delta^{\mathrm{D}}_I$ written in increasing order and repeated according to multiplicity. Note that $\operatorname{spec}(\Delta_X \oplus \Delta^{\mathrm{D}}_I) = \operatorname{spec} \Delta_X \cup \operatorname{spec} \Delta_I$, i.e., $\lambda^{\mathrm{D}}_k(X \dot\cup I)$ is a reordering of $\lambda^{\mathrm{D}}_{k_1}(X)$ and $\lambda^{\mathrm{D}}_{k_2}(I)$. Again, by Floquet Theory Theorem 1.2 follows.

Proof. We only sketch the proof since it is similar to the proof of Theorem 3.1. Again, we have an upper estimate of $\lambda^\theta_k(N_\varepsilon)$ by the k-th Dirichlet eigenvalue $\lambda^{\mathrm{D}}_k(X_\varepsilon \dot\cup C_\varepsilon)$ on $X_\varepsilon \dot\cup C_\varepsilon$. For the estimate from below we apply the Main Lemma once more with $\mathcal{H}_\varepsilon := L_2(N_\varepsilon)$, $q_\varepsilon := q^\theta_{N_\varepsilon}$, $\mathcal{H}'_\varepsilon := L_2(X_\varepsilon) \oplus L_2(C_\varepsilon)$ and $q'_\varepsilon := q^{\mathrm{N}}_{X_\varepsilon} \oplus q^{\mathrm{D}}_{C_\varepsilon}$. Furthermore, for $u \in \operatorname{dom} q^\theta_{N_\varepsilon}$, we set

$$\Phi_\varepsilon u := u{\restriction}_{X_\varepsilon} \oplus (u{\restriction}_{C_\varepsilon} - h)$$

where $h = h_\varepsilon$ is the (unique) function satisfying

$$\Delta_{C_\varepsilon} h = 0 \qquad \text{and} \qquad h{\restriction}_{\partial C_\varepsilon} = u{\restriction}_{\partial C_\varepsilon}$$

(see [1] and [24]). To verify Condition 2.2, we estimate

$$\begin{aligned}\big| \|\Phi_\varepsilon u\|^2_{X_\varepsilon \dot\cup C_\varepsilon} - \|u\|^2_{N_\varepsilon} \big| &\le \int_{A_\varepsilon} |u|^2 + \int_{C_\varepsilon} \big(|u-h|^2 - |u|^2\big) \\ &\le \|u\|^2_{A_\varepsilon} + 2\|u\|_{C_\varepsilon} \|h\|_{C_\varepsilon} + \|h\|^2_{C_\varepsilon}.\end{aligned}$$

By Lemma 3.3 we only need to show that the harmonic extenstion h converges to 0 in $L_2(C_\varepsilon)$ uniformly in θ (note that $h = h^\theta_\varepsilon$ depends also on θ since $u = u^\theta_\varepsilon \in \operatorname{dom} q^\theta_{M_\varepsilon}$ does). The convergence does not seem to be very surprising since the harmonic function h is given on the boundary ∂C_ε by u which is small (in L_2-sense) by Lemma 3.3. Nevertheless a little more work is necessary which will be omitted here (see [1] and [24]).

Condition 2.3 is satisfied since

$$q_\varepsilon(u) = \int_{N_\varepsilon} |\mathrm{d}u|^2 \ge \int_{X_\varepsilon} |\mathrm{d}u|^2 + \int_{C_\varepsilon} |\mathrm{d}h|^2 = q'_\varepsilon(\Phi_\varepsilon u).$$

Note that the harmonic function h minimizes the energy integral $q_{C_\varepsilon}(u)$. Since

$$\int_{C_\varepsilon} \langle \mathrm{d}(u-h), \mathrm{d}h \rangle = 0 \tag{3.6}$$

by the Gauss-Green Formula we have $q_{C_\varepsilon}(u) = q_{C_\varepsilon}(h) + q_{C_\varepsilon}(u-h) \geq q_{C_\varepsilon}(h)$. Note that $u-h$ satisfies Dirichlet boundary conditions.

From the Main Lemma we obtain a lower bound (up to an error term) given by the k-th eigenvalue $\lambda_k^{\mathrm{N,D}}(X_\varepsilon \dot\cup C_\varepsilon)$ with Neumann boundary conditions on X_ε and Dirichlet boundary conditions on C_ε. We only have to add on that $\lambda_k^{\mathrm{N,D}}(X_\varepsilon \dot\cup C_\varepsilon)$ resp. $\lambda_k^{\mathrm{D}}(X_\varepsilon \dot\cup C_\varepsilon)$ converge to $\lambda_k^{\mathrm{D}}(X \dot\cup I)$ if the cylinder C_ε collapes to the interval I as $\varepsilon \to 0$. □

4. Conformal Deformation

Suppose $\mathcal{M}$ is a Γ-periodic Riemannian manifold of dimension $d \geq 2$ with metric g. Let $X \subset \mathcal{M}$ be a compact subset with smooth boundary such that $\gamma X \cap X \neq \emptyset$ implies $\gamma = 0$. Then a period cell M with $\mathrm{dist}(\partial X, \partial M) > 0$ exists. We introduce *normal* or *Fermi coordinates* (r, y) with respect to $Y := \partial X$ (for details cf. [5]). Here, $r \in (-r_0, r_0)$ parametrises the normal direction and $y \in Y$ parametrises the tangential direction; $r < 0$ corresponds to the interior of X and $r = 0$ corresponds to Y.

Furthermore, we assume that normal coordinates also exist on $\overline{M \setminus X}$, i.e., we suppose that $\overline{M \setminus X}$ can be parametrised by (r, y) with $r \in I_y$ and $y \in Y$. Here, I_y is a compact subset of $\mathbb{R}$ containing $[0, r_0]$. The existence of normal coordinates on $\overline{M \setminus X}$ is a geometrical restriction on X. In some situations this condition means that X is close enough to ∂M. For example, this condition is satisfied for a centered ball in a cube.

Suppose we have for each $\varepsilon > 0$ a smooth Γ-periodic function $\rho_\varepsilon \colon \mathcal{M} \longrightarrow (0, 1]$ with the following properties:

$$\rho_\varepsilon(x) = 1 \qquad \text{for all } x \in X, \tag{4.1}$$

$$\rho_\varepsilon(x) = \varepsilon \qquad \text{for all } x \in M \text{ with } \mathrm{dist}(x, X) \geq \varepsilon^d. \tag{4.2}$$

For simplicity, we also assume that $\rho_\varepsilon(x)$ is only a function on r in normal coordinates on $\overline{M \setminus X}$. Note that the function ρ_ε converges pointwise to the indicator function of the set $\mathcal{X} = \bigcup_{\gamma \in \Gamma} X$. We define $g_\varepsilon := \rho_\varepsilon^2 g$ and denote the resulting Riemannian manifolds with metric g_ε by $\mathcal{M}_\varepsilon$ resp. M_ε. We therefore obtain a conformally deformed Γ-periodic manifold $\mathcal{M}_\varepsilon$ with periodic metric g_ε and period cell M_ε. In particular, the squared norm and the quadratic form on the deformed manifold M_ε are given by

$$\|u\|^2_{M_\varepsilon} = \int_M |u|^2 \rho_\varepsilon^d \qquad \text{and} \qquad q_{M_\varepsilon}(u) = \int_M |\mathrm{d}u|^2_{T^*M}\, \rho_\varepsilon^{d-2}. \tag{4.3}$$

Here we can see that the case $d = 2$ is in some sense particular, since the quadratic form does not depend on ε any more (but the norm does).

First, let us calculate the curvature of the conformally deformed case:

Remark 4.1. We denote the sectional curvatures of M_ε by the subscript ε and the sectional curvatures of M without subscript. Let $\partial_r, \partial_{y_1}, \dots, \partial_{y_{d-1}}$ be orthonormal basis tangent vectors corresponding to the coordinates (r, y) near $Y = \partial X$. Then the sectional curvatures are given by

$$\begin{aligned} K_\varepsilon(\partial_r, \partial_{y_j}) &= \rho_\varepsilon^5 \big(-\ddot{\rho}_\varepsilon + \partial_r g_{y_j y_j} \dot{\rho}_\varepsilon / 2 + \rho_\varepsilon K(\partial_r, \partial_{y_j}) \big) \\ K_\varepsilon(\partial_{y_j}, \partial_{y_k}) &= \rho_\varepsilon^6 \big(-\dot{\rho}_\varepsilon^2 - (\partial_r g_{y_j y_j} + \partial_r g_{y_k y_k}) \dot{\rho}_\varepsilon \rho_\varepsilon / 2 + \rho_\varepsilon^2 K(\partial_{y_j}, \partial_{y_k}) \big) \end{aligned}$$

for $j \neq k$. With our assumptions on ρ_ε, $K(\partial_r, \partial_{y_j}) = O(\varepsilon^{-2d-5})$ with changing sign and $K(\partial_{\sigma_j}, \partial_{\sigma_k}) = O(\varepsilon^{-2d-6})$ provided $\varepsilon > 0$ is small enough. Calculating the Ricci and scalar curvature one can see that all curvature terms are not bounded in ε as Green conjectured in [14].

The Higher Dimensional Case. Now we are able to state the following theorem (again, Theorem 1.3 follows via Floquet Theory):

Theorem 4.2. *Suppose that $d \geq 3$. Then $\lambda_k^\theta(M_\varepsilon) \to \lambda_k^{\mathrm{N}}(X)$ as $\varepsilon \to 0$ uniformly in $\theta \in \hat{\Gamma}$.*

Again, the k-th band $B_k(\mathcal{M}_\varepsilon)$ reduces to the point $\{\lambda_k^{\mathrm{N}}(X)\}$ as $\varepsilon \to 0$ and the convergence is *not* uniform in k.

As in the previous section, we need the following lemma which shows that the θ-periodic eigenfunctions on M_ε do not concentrate on the metrically shrunken set $M_\varepsilon \setminus X$ as $\varepsilon \to 0$:

Lemma 4.3. *There exists a positive function $\omega(\varepsilon)$ converging to 0 as $\varepsilon \to 0$ such that*

$$\int_{M_\varepsilon \setminus X} |u|^2 \leq \omega(\varepsilon) \int_M \big(|u|^2 + |\mathrm{d}u|^2_{T^* M_\varepsilon} \big), \tag{4.4}$$

for all u out of the domain of the quadratic form with θ-periodic boundary conditions on M_ε.

Again, $\omega(\varepsilon)$ only depends on the geometry of $M \setminus X$.

Proof. We proceed in the same way as in the proof of Lemma 3.3. We introduce normal coordinates. For notational simplicity only, we assume that $I_y = [0, r_y]$ for some number $r_0 \leq r_y$. Suppose that $u \in C^\infty(M_\varepsilon)$ with $u(r, y) = 0$ for all $y \in Y$ and $r \leq -r_0$. As in (3.1) we have the orthogonal splitting

$$g_\varepsilon = \rho_\varepsilon^2 \, g_\varepsilon = \rho_\varepsilon^2 \, (\mathrm{d}r^2 + h_r)$$

in normal coordinates where h_r is a parameter-dependent metric on Y. By the Cauchy-Schwarz Inequality we have

$$|u(s,y)|^2 = \Big| \int_{-r_0}^{s} \partial_r u(r,y)\,\mathrm{d}r \Big|^2 \\ \le \int_{-r_0}^{s} \big(\det g(r,y)\big)^{-\frac12}\,\mathrm{d}r \cdot \int_{-r_0}^{s} \big(|\partial_r u|^2 (\det g)^{\frac12}\big)(r,y)\,\mathrm{d}r$$

for $0 \le s \le r_y$. Since Y is compact we can estimate the first integral by $c > 0$. Therefore integrating over $s \in I_y$ and $y \in Y$ yields

$$\int_{M_\varepsilon \setminus X} |u|^2 = \int_{y\in Y} \int_{s=0}^{r_y} \big(|u|^2 (\det g)^{\frac12} \rho_\varepsilon^d\big)(s,y)\ \mathrm{d}s\,\mathrm{d}y \\ \le c \int_{y\in Y} \int_{s=0}^{r_y} (\det g)^{\frac12}(s,y) \rho_\varepsilon^d(s) \int_{r=-r_0}^{s} \big(|\partial_r u|^2 (\det g)^{\frac12}\big)(r,y)\ \mathrm{d}r\,\mathrm{d}s\,\mathrm{d}y.$$

We can estimate the s-dependent terms as follows: for $\varepsilon^d \le s \le r_y$ we have $\rho_\varepsilon(s) = \varepsilon$ by Assumption (4.2). Furthermore, there exists a constant $c' > 0$ such that $(\det g)^{\frac12}(s,y) \le c'$ for all $y \in Y$ and $0 \le s \le r_y$ since $\overline{M \setminus X}$ is compact. Therefore the integral over $0 \le s \le \varepsilon^d$ and $\varepsilon^d \le s \le r_y$ can be estimated by $c'\varepsilon^d$. We conclude

$$\int_{M_\varepsilon \setminus X} |u|^2 \le c\,c'\varepsilon^d \int_{y\in Y} \int_{r=-r_0}^{r_y} \big(|\partial_r u|^2 (\det g)^{\frac12}\big)(r,y)\ \mathrm{d}r\,\mathrm{d}y \\ \le c\,c'\varepsilon^2 \int_{y\in Y} \int_{r=-r_0}^{r_y} \big(|\partial_r u|^2 (\det g)^{\frac12} \rho_\varepsilon^{d-2}\big)(r,y)\ \mathrm{d}r\,\mathrm{d}y \\ \le c\,c'\varepsilon^2 \int_M |\mathrm{d}u|^2_{T^*M_\varepsilon}$$

where we have used $\rho_\varepsilon \ge \varepsilon$ in the second line.

If $u(r,y) \ne 0$ for some $y \in Y$ and $r < -r_0$ we multiply u with a cut-off function χ such that $\chi(r) = 1$ for $r \ge -r_0/2$ and $\chi(r) = 0$ for $r \le -r_0$. Note that $\operatorname{supp}\chi \subset X$, i.e., on $\operatorname{supp}\chi$, there is no conformal deformation. If $u \in \operatorname{dom} q^\theta_{M_\varepsilon}$ we apply an approximation argument. □

In the same way we have proven Theorem 3.1 we can show Theorem 4.2:

Proof. First, we prove an upper bound. For this we apply the Main Lemma 2.2 with $\mathcal{H} := L_2(X)$, $q := q^{\mathrm{N}}_X$ (not depending on ε), $\mathcal{H}'_\varepsilon := L_2(M_\varepsilon)$ and $q'_\varepsilon := q^\theta_{M_\varepsilon}$. Furthermore, let Φu be an extension of $u \in \operatorname{dom} q^{\mathrm{N}}_X$ onto M such that $u = 0$ in a neighbourhood of ∂M. In particular, $u \in \operatorname{dom} q^\theta_{M_\varepsilon}$. The inequality of Condition (2.2) is trivially satisfied, Condition (2.3) follows because of

$$\big|q'_\varepsilon(\Phi u) - q(u)\big| = \int_{M_\varepsilon \setminus X} |\mathrm{d}\Phi u|^2_{T^*M_\varepsilon} = \int_{M_\varepsilon \setminus X} |\mathrm{d}\Phi u|^2_{T^*M_\varepsilon} \rho_\varepsilon^{d-2} \to 0$$

by the Lebesgue convergence theorem and Assumption (4.2). Here, the assumption $d \geq 3$ is essential. In the ε-independent case, Condition (2.4) is obsolete. The Main Lemma yields

$$\lambda_k^\theta(M_\varepsilon) \leq \lambda_k^{\mathrm{N}}(X) + \delta_k(\varepsilon) \tag{4.5}$$

with $\delta_k(\varepsilon) \to 0$ as $\varepsilon \to 0$.

To prove the opposite inequality we apply the Main Lemma once more, this time with $\mathcal{H}_\varepsilon := L_2(M_\varepsilon)$, $q_\varepsilon := q^\theta_{M_\varepsilon}$, $\mathcal{H}' := L_2(X)$, $q' := q_X^{\mathrm{N}}$ and $\Phi_\varepsilon u := u\restriction_X$ being the restriction operator.

Again, Condition (2.2) is satisfied by Lemma 4.3, and the inequality in Condition (2.3) is trivially satisfied. Finally, Condition (2.4) is satisfied by the upper bound (4.5). The Main Lemma therefore yields

$$\lambda_k^{\mathrm{N}}(X) \leq \lambda_k^\theta(M_\varepsilon) + \delta_k(\varepsilon)$$

with $\delta_k(\varepsilon) \to 0$ as $\varepsilon \to 0$. Together with Estimate (4.5) we are done. $\square$

The Two-Dimensional Case. In dimension 2, the special form of the quotient $q^\theta_{M_\varepsilon}(u)/\|u\|^2_{M_\varepsilon}$ causes a different behaviour. The θ-periodic eigenvalue of the Laplacian on a period cell M_ε still converges, but the limit depends on θ, i.e., the k-th band $B_k(\mathcal{M}_\varepsilon)$ in general does not reduce to a point. In particular, if we assume that ρ_ε is monotonely decreasing (as $\varepsilon \searrow 0$), the first θ-periodic eigenvalue is monotonely increasing (as $\varepsilon \searrow 0$) since we have

$$\lambda_k^\theta(M_\varepsilon) = \inf_u \frac{q^\theta_{M_\varepsilon}(u)}{\|u\|^2_{M_\varepsilon}} = \inf_u \frac{\int_M |\mathrm{d}u|^2}{\int_M |u|^2 \rho_\varepsilon^2 s} \tag{4.6}$$

due to the Min-max Principle and (4.3). Here, the infimum is taken over all $u \in \operatorname{dom} q^\theta_{M_\varepsilon}$ such that $u \neq 0$. Note that $\operatorname{dom} q^\theta_{M_\varepsilon}$ is independent of ε as vector space. Since the first band $B_1(\mathcal{M}_\varepsilon)$ of a connected periodic manifold $\mathcal{M}_\varepsilon$ has always nontrivial interior (see Remark 2.1), $B_1(\mathcal{M}_\varepsilon)$ cannot reduce to a point as $\varepsilon \to 0$.

As in the higher dimensional case, the norm on M_ε converges to the norm on X, i.e., the limit quadratic form lives in the Hilbert space $L_2(X)$. But there is no reason why the limit form should only be an integral over X since the quadratic form $q^\theta_{M_\varepsilon}$ does not depend on ε any more (see Equation (4.6)).

Indeed, the following candidate for the limit form is the right one (a more detailed motivation can be found in [24], note that harmonic functions minimize the energy integral, i.e., the integral over $|\mathrm{d}u|^2$). For $u \in \operatorname{dom} q_X^{\mathrm{N}}$ let $h = H^\theta u \in \operatorname{dom} q^\theta_M$ be the θ-periodic harmonic extension of u on M. In particular, $u = h$ on X and $\Delta_{M \setminus X} h = 0$ on $M \setminus X$ such that h and $\mathrm{d}h$ are θ-periodic, i.e., $h(x) = \theta(\gamma) h(\gamma x)$ resp. $\mathrm{d}h(x) = \theta(\gamma)\mathrm{d}h(\gamma x)$ for all $x \in M$ and $\gamma \in \Gamma$ such that $\gamma x \in M$. Then we set

$$q_0^\theta(u) := \int_M |\mathrm{d}(H^\theta u)|^2$$

for all $u \in \operatorname{dom} q_0^\theta := \operatorname{dom} q_X^{\mathrm{N}}$. Note that $q_X^{\mathrm{N}}(u) \le q_0^\theta(u)$ for all u. In particular, the corresponding operator to q_0^θ has purely discrete spectrum denoted by $\lambda_k^\theta(0)$ (written in increasing order and repeated according to multiplicity). Furthermore, since $\operatorname{dom} q_0^\theta \supset \operatorname{dom} q_X^{\mathrm{D}}$ the Min-max Principle yields

$$\lambda_k^{\mathrm{N}}(X) \le \lambda_k^\theta(0) \le \lambda_k^{\mathrm{D}}(X).$$

Now we show the convergence of the θ-periodic eigenvalues on M_ε to the eigenvalues $\lambda_k^\theta(0)$:

Theorem 4.4. *Suppose that $d = 2$. Then $\lambda_k^\theta(M_\varepsilon) \to \lambda_k^\theta(0)$ as $\varepsilon \to 0$ for all $\theta \in \hat{\Gamma}$.*

Proof. First, we prove an upper bound. For this note that $\|u\|_X \le \|H^\theta u\|_{M_\varepsilon}$ and $q_0^\theta(u) = q_{M_\varepsilon}^\theta(H^\theta u)$. By the Min-max Principle (or formally one could also apply the Main Lemma) we obtain

$$\lambda_k^\theta(M_\varepsilon) \le \lambda_k^\theta(0). \tag{4.7}$$

To prove the opposite inequality we apply the Main Lemma 2.2 once more, this time with $\mathcal{H}_\varepsilon := L_2(M_\varepsilon)$, $q_\varepsilon := q_{M_\varepsilon}^\theta$, $\mathcal{H}' := L_2(X)$, $q' := q_0^\theta$ and $\Phi_\varepsilon u := u\restriction_X$ being the restriction operator.

Again, Condition (2.2) is satisfied by Lemma 4.3. The inequality in Condition (2.3) is satisfied since

$$\begin{aligned} q_\varepsilon(u) &= \int_X |\mathrm{d}u|^2 + \int_{M\setminus X} |\mathrm{d}u|^2 \\ &\ge \int_X |\mathrm{d}u|^2 + \int_{M\setminus X} |\mathrm{d}(H^\theta(u\restriction_X))|^2 = q_0^\theta(\Phi_\varepsilon u). \end{aligned}$$

Note that the harmonic function $H^\theta(u\restriction_X)$ minimizes the second integral, see Equation (3.6). Finally, Condition (2.4) is satisfied since $\lambda_k^\theta(M_\varepsilon) \le \lambda_k^{\mathrm{D}}(X)$. The Main Lemma therefore yields

$$\lambda_k^\theta(0) \le \lambda_k^\theta(M_\varepsilon) + \delta_k(\varepsilon)$$

with $\delta_k(\varepsilon) \to 0$ as $\varepsilon \to 0$. With regard to Estimate (4.7) the proof is finished. ☐

Next we characerise the domain of the operator Q_0^θ corresponding to q_0^θ. Note that $\operatorname{dom} Q_0^\theta$ consists of those $u \in \operatorname{dom} q_0^\theta$ such that there exists a (unique) element $v \in L_2(X)$ satisfying

$$q_0^\theta(u, w) = \langle v, w \rangle$$

for all $w \in \operatorname{dom} q_0^\theta$. In particular, $Q_0^\theta u = v$ (see e.g. [19, Theorem VI.2.1]). Here, we can give a more explicit characterisation of the limit operator:

Lemma 4.5. *The domain of the operator Q_0^θ corresponding to the limit quadratic form q_0^θ is given by*

$$\operatorname{dom} Q_0^\theta = \bigl\{ u \in \mathcal{H}^2(X) \bigm| \partial_{\mathrm{n}} u = \partial_{\mathrm{n}} H^\theta u \text{ on } \partial X \bigr\}. \tag{4.8}$$

Furthermore, $Q_0^\theta u = \Delta_X u$ *for* $u \in \operatorname{dom} Q_0^\theta$.

Here, $\partial_{\mathrm{n}} u$ denotes the normal (outer) derivate with respect to X. Furthermore, $\mathcal{H}^2(X)$ denotes the Sobolev space of square integrable weak derivatives up to second order.

Proof. The lemma follows from the Gauss-Green Formula and the characterisation of $\operatorname{dom} Q_0^\theta$. Note that the integral over ∂M vanishes since $h = H^\theta u$ and $\mathrm{d}h$ are both θ-periodic. Furthermore, $u, \Delta u \in L_2(X)$ imply $u \in \mathcal{H}^2(X)$ by regularity theory. □

Since the limit operator is quite complicated, we are only able to construct an example of a conformally deformed 2-dimensional manifold with gaps in the spectrum of its Laplacian:

Example 4.6. Let $\mathcal{M} := \mathbb{R} \times \mathbb{S}^1$ be a cylinder with $\Gamma = \mathbb{Z}$ acting on $\mathcal{M}$ by $\gamma(x, \sigma) = (\gamma + x, \sigma)$. The periodic metric is given by $g = \mathrm{d}x^2 + r^2 \mathrm{d}\sigma^2$ for some fixed $r > 0$. We choose $M = [0, 1] \times \mathbb{S}^1$ as period cell.

Let $0 < a < b < 1$ and let $X = [a, b] \times \mathbb{S}^1$ be the undisturbed region of M. Note that normal coordinates exist on $\overline{M \setminus X}$. Let $\theta \in \hat{\Gamma} \cong \mathbb{T}^1$. In this context we prefer to view θ as $\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{T}^1$.

We first have to calculate the θ-periodic harmonic extension $h = H^\theta u$ of a function $u \in C^\infty(X)$ given by $u(x, \sigma) = v(x) \mathrm{e}^{\mathrm{i}n\sigma}$ for some $n \in \mathbb{Z}$, i.e., we have to solve the boundary value problem

$$\Delta_{M \setminus X} h = -\partial_{xx} h - \frac{1}{r^2} \partial_{\sigma\sigma} h = 0, \qquad \text{with} \qquad \begin{aligned} h(a, \cdot) &= u(a, \cdot) \\ h(b, \cdot) &= u(b, \cdot) \\ h(1, \cdot) &= \mathrm{e}^{i\theta} h(0, \cdot) \\ \partial_x h(1, \cdot) &= \mathrm{e}^{i\theta} \partial_x h(0, \cdot) \end{aligned}$$

which has a unique solution. Next, we search for eigenvalues $\lambda = \lambda_k^\theta(0) \geq 0$ with eigenfunctions u. Again, by separating the variables we can calculate them expicitely. Since eigenfunctions u have to be in the domain of Q_0^θ, the normal derivatives of u and the harmonic extension h agree on ∂X by the preceding lemma. Therefore we have to solve

$$\Delta_X u = -\partial_{xx} u - \frac{1}{r^2} \partial_{\sigma\sigma} u = \lambda u, \qquad \text{with} \qquad \begin{aligned} \partial_x u(a, \cdot) &= \partial_x h(a, \cdot) \\ \partial_x u(b, \cdot) &= \partial_x h(b, \cdot). \end{aligned}$$

This gives a restriction on the possible values of λ. A long, but straightforward calculation yields

$$2\big(\cos(L\omega) - \cos\theta\big) = \ell\omega \sin(L\omega) \tag{4.9}$$

$$\Big(\omega^2 - \frac{n^2}{r^2}\Big) \sin(L\omega) = 2\omega \frac{n}{r} \cdot \frac{\cosh(\ell \frac{n}{r}) \cos(L\omega) - \cos\theta}{\sinh(\ell \frac{n}{r})} \tag{4.10}$$

with $\lambda = \omega^2 + n^2/r^2$ and $\ell := 1 - L = 1 - b + a$ where the first equation is valid for $n = 0$ and $\omega \geq 0$ and the second equation for $n \neq 0$ and $\omega > 0$ (see [24]). Note that ℓ is the length of the perturbed cylinder $M \setminus X$ and that L is the length of the unperturbed cylinder X.

If $n = 0$ we obtain smooth functions $\theta \mapsto \omega_m(\theta)$ for each $m \in \mathbb{N}_0$ solving Equation (4.9) (see Figure 3). In this case, we have $\lambda = \sqrt{\omega_m(\theta)}$. Furthermore,

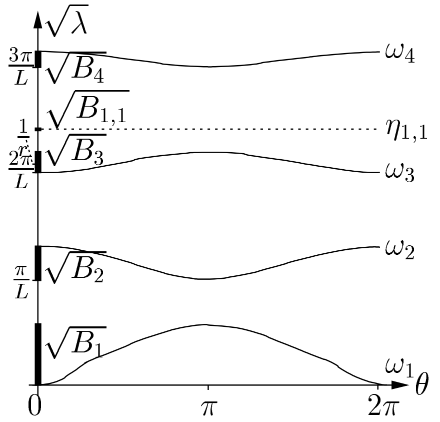

FIGURE 3. The square root of the eigenvalues of the limit operator Q_0^θ plotted for $L = 0.5$ and $r = 1/13$. Here, at least $m = 2$ gaps occure.

note that the compact intervals $B_m := \{\omega_m^2(\theta);\ 0 \le \theta \le 2\pi\}$ are all disjoint: One can prove that

$$\inf B_m = \Big(\frac{m\pi}{L}\Big)^2 \qquad \text{and} \qquad \sup B_m = \Big(\frac{(m+1)\pi}{L}\Big)^2 - \varepsilon_0$$

for all $m \in \mathbb{N}_0$ if $\varepsilon_0 = \varepsilon_0(L)$ is small enough. Finally note that $(m\pi/L)^2$ are the Neumann eigenvalues of the interval $[0, L]$.

If $n \neq 0$ we have

$$\eta := \sqrt{\lambda} = \sqrt{\omega^2 + n^2/r^2} > \frac{|n|}{r} \ge \frac{1}{r}.$$

If we replace ω by $\sqrt{\eta^2 - n^2/r^2}$ in Equation (4.10) we obtain solutions $\theta \mapsto \eta_{n,p}(\theta)$ for $n, p \in \mathbb{N}$ (see Figure 3). Note that we do not expect that the intervals $B_{n,p} := \{\eta_{n,p}^2(\theta);\ 0 \le \theta \le 2\pi\}$ are disjoint, we rather expect that the intervals $B_{n,p}$ cover the gaps between the intervals B_m when m, n or p are large. But we still have $\inf B_{n,p} \ge 1/r^2$, i.e., if $r \le \frac{L}{m\pi}$, the intervals $B_0, \dots, B_m$ remain disjoint. Therefore we have proven the following:

Theorem 4.7. *Suppose that $\mathcal{M}_\varepsilon$ is a conformally perturbed cylinder of radius $r > 0$ with conformal factor satisfying Conditions* (4.1) *and* (4.2)*. Suppose further that the unperturbed area is a cylinder X of length $0 < L < 1$ (periodically continued with period 1). Then the corresponding Laplacian has at least m gaps if $r \le \frac{L}{m\pi}$ and if $\varepsilon > 0$ is small enough.*

5. Conclusions and outlook

So far we have proven the existence of two different classes of periodic manifolds with spectral gaps. But we still have not found a satisfying answer (in terms of geometrical properties) whether a given periodic manifold has gaps or not. We define the ν-isoperimetric constant of a periodic manifold $\mathcal{M}$ as

$$I_\nu(\mathcal{M}) := \inf_M \inf_\Omega \frac{(\mathrm{vol}_{d-1} \partial\Omega)^\nu}{(\mathrm{vol}_d \Omega)^{\nu-1}}$$

where vol_k denotes k-dimensional Riemannian volume and $\nu > 1$. The infimum is taken over all period cells M of $\mathcal{M}$ and all open submanifolds $\Omega \subset M$ with $\partial M \cap \partial\Omega = \emptyset$. In both classes of examples given in Sections 3 and 4 we have $\mathrm{vol}_{d-1}(\partial M_\varepsilon) \to 0$ whereas $\mathrm{vol}_d(M_\varepsilon)$ is bounded from below by some positive constant. Therefore $I_\nu(\mathcal{M}_\varepsilon) \to 0$ as $\varepsilon \to 0$ for all $\nu > 1$. Are the isoperimetric constants some hint for the analytic decoupling? Here, by analytic decoupling we mean that the k-th θ-periodic eigenvalue of M_ε converges uniformly to some θ-independent constant λ_k (as in Theorems 3.1, 3.6 and 4.2). Note, for example that the isoperimetric constants do not all converge to 0 for a straight cylinder $\mathbb{S}^{d-1} \times \mathbb{R}$ of diameter of order ε and length of a period cell of order ε^α, $\alpha > 0$. Clearly, the spectrum of the straight cylinder has no gaps.

What rôle does the curvature play? In the construction in Section 3 as well as the conformal deformation in Section 4 the curvatures are neither bounded from below nor bounded from above. There is a result of Li [21] showing that the essential spectrum of a complete non-compact Riemannian manifold $\mathcal{M}$ with non-negative Ricci curvature and a pole is equal to $[0, \infty)$. (A pole is a point x_0 where the exponential map is a diffeomorphism from $T_{x_0}\mathcal{M}$ onto $\mathcal{M}$, which is a strong geometric condition.) If furthermore, $\mathcal{M}$ is periodic then the bound on the curvature prevents the existence of spectral gaps. In contrast, we would expect to have a great number of gaps if the curvature has great absolute values.

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