THE SPECTRAL MEASURE OF A JACOBI MATRIX IN TERMS OF THE FOURIER TRANSFORM OF THE PERTURBATION

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ABSTRACT. We study the spectral properties of Jabobi matrices. By combining Killip's technique [13] with the technique of Killip and Simon [12] we obtain a result relating the properties of the elements of Jacobi matrices and the corresponding spectral measures. This theorem is a natural extension of a recent result of Laptev-Naboko-Safronov [17].

The author contemplated this paper as a part of an updated version of a different article. Therefore the present text will probably exist only as a preprint or will be published as a part of another paper.

0. INTRODUCTION

Let S be the shift operator in $l^2(\mathbb{N})$ whose action on the canonical orthonormal basis $\{e_n\}_{n=0}^{\infty}$ is given by $Se_n = e_{n+1}$. Let A, B be selfadjoint diagonal operators: $Ae_n = \alpha_n e_n$, $Be_n = \beta_n e_n$, $\alpha_n > -1$, $\beta \in \mathbb{R}$. We study the spectrum of the operator

$$J = S + S^* + Q, \quad \text{where } Q = SA + AS^* + B.$$

Such an operator can be identified with the following Jacobi matrix:

$$J = \begin{pmatrix} \beta_0 & 1+\alpha_0 & 0 & 0 & \cdots \\ 1+\alpha_0 & \beta_1 & 1+\alpha_1 & 0 & \cdots \\ 0 & 1+\alpha_1 & \beta_2 & 1+\alpha_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{0.1}$$

If the entries of this matrix are bounded then J is a bounded operator in $l^2(\mathbb{N})$. To J we associate the following measure μ given by

$$m_\mu(z) := (e_0, (J-z)^{-1}e_0) = \int \frac{d\mu(t)}{t-z}, \qquad z \in \mathbb{C}. \tag{0.2}$$

The spectral significance of this function is represented by the relation

$$\mu(\delta) = (E_J(\delta)e_0, e_0), \tag{0.3}$$

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where E_J denotes the spectral measure of J and $\delta \subset \mathbb{R}$ is a Borel set.

Conversely, to each measure μ with a compact support containing infinitely many points there is a standard procedure of constructing a Jacobi matrix via the corresponding orthogonal polynomials (see [12] for historical references and bibliography). Therefore similar problems were intensively studied by specialists in this field [21], [1].

Since there is a one-to-one correspondence between Jacobi matrices and probability measures, it is natural to ask how the properties of entries of Jacobi matrices are related to the properties of probability measures. We are interested in a class of matrices J "close" to the "free" matrix J_0 for which $\alpha_n = 0$ and $\beta_n = 0$, $n = 0, 1, \dots$.

Obviously,

$$\mu(\mathbb{R}) = \|e_0\|^2 = 1.$$

Denote by μ' the derivative of the absolutely continuous part of the measure μ. Most of our questions will be related to convergence of the integral of $\log(1/\mu'(2\cos\theta))$ with respect to the weight

$$\rho(e^{i\theta})\sin^2(\theta);$$

$$\rho(k) = (W(k))^2, \quad W(k) = \sum_{n=-n_0}^{n_0} w_n k^n \tag{0.4}$$

where $W(e^{i\theta})$ is real valued and n_0 is finite. In order to state our main result we denote by $\mathfrak{S}_p$ the standard Shatten classes of compact operators:

$$\mathfrak{S}_p = \{\, T : \ \ ||T||^p_{\mathfrak{S}_p} = \mathrm{tr}\,(T^*T)^{p/2} < \infty \,\}.$$

Theorem 0.1. *Let J be a Jacobi matrix, μ the corresponding measure* (0.3) *and the weight ρ be given by* (0.4) *with $W(1)W(-1) \neq 0$. Assume that the operator $Q = J - J_0$ satisfies*

$$Q \in \mathfrak{S}_3. \tag{0.5}$$

Let also $\hat{A}(\theta)$ and $\hat{B}(\theta)$ be the sums of the Fourier series $\sum e^{in\theta}\alpha_n$ and $\sum e^{in\theta}\beta_n$. Then

$$\int_{-\pi}^{\pi} |2e^{i\theta}\hat{A}(2\theta) + \hat{B}(2\theta)|^2 \rho(e^{i\theta})\, d\theta < \infty \tag{0.6}$$

if and only if μ satisfies the following three properties:

(1) *The support of μ is $[-2,2] \cup \{E_j^+\}_{j=1}^{N_+} \cup \{E_j^-\}_{j=1}^{N_-}$, where $\pm E_j^{\pm} > 2$, $0 \le N_\pm \le \infty$.*

(2) (Quasi-Szegö Condition)

$$\int_{-\pi}^{\pi} \log\left[\frac{1}{\mu'(2\cos\theta)}\right] \rho(e^{i\theta})\sin^2(\theta)\, d\theta < \infty.$$

(3) (Lieb–Thirring Bound)

$$\sum_{j=1}^{N_+} |E_j^+ - 2|^{3/2} + \sum_{j=1}^{N_-} |E_j^- + 2|^{3/2} < \infty.$$

Remark. One can generalize this result for the case $W(1) = W(-1) = 0$, where instead of the Lieb-Thirring bound with the power $3/2$ we would have the one with the power $5/2$. In this case Condition (3) can be simply omitted.

Although this theorem is a natural generalization of a recent result of Killip-Simon, it has some disadvantages. Namely, for a given measure μ we are not able to check in advance whether the condition (0.5) is fulfilled for the corresponding Jacobi matrix J. However, in the cases $\rho(e^{i\theta}) = 1$ [12] and $\rho(e^{i\theta}) = \cos^2(\theta)$ [17] one is able to avoid this obstacle and obtain stronger results where one does not have the *a priori* condition (0.5).

The main idea of this paper is to modify the known trace formulae. For J's with $Q = J - J_0$ of finite rank , the standard trace formulae are due to Case [2],[3]. Recently, Killip-Simon [12] found how to exploit these sum rules as a spectral tool. In particular, Killip-Simon have shown the importance of extension of sum rules to a larger class of Q's. In its turn the impressive paper [12] was motivated by work on Schrödinger operators by Deift-Killip [6] and Denisov [7], [8]. Finally, the author would like to mention the paper [26] the results of which are used in this work.

1. Preliminaries

For each $T \in \mathfrak{S}_1$, one can define a complex-valued function $\det(1+T)$, so that

$$|\det(1+T)| \le \exp(\|T\|_{\mathfrak{S}_1})$$

For $T \in \mathfrak{S}_3$ one defines

$$\det_3(1+T) = \det((1+T)e^{-T+T^2/2}). \tag{1.1}$$

Then

$$|\det_3(1+T)| \le \exp(c\|T\|^3_{\mathfrak{S}_3}), \quad c > 0. \tag{1.2}$$

Lemma 1.1. *Let A, B be of finite rank. There is a constant C such that one of the conditions*

$$|\text{Im } z| > 1 \;\text{ or }\; |\text{Re } z| > 3 + ||Q|| \tag{1.3}$$

imply

$$|\log\det_3(I + Q(J_0 - z)^{-1})| \le C||Q||^3_{\mathfrak{S}_3} \tag{1.4}$$

Proof. Let $z = \lambda + i\epsilon$, where λ and ϵ are real. One can repeat the arguments of Proposition 5.2 [13] to show that

$$
\begin{aligned}
&|\frac{d}{d\epsilon}\log\det{}_3(I+Q(J_0-z)^{-1})| = \\
&|\mathrm{tr}\,(i(J_0-z)^{-1}Q(J_0-z)^{-1}Q(J-z)^{-1}(J_0-z)^{-1}Q)| \leq \\
&||(J_0-z)^{-1}||^3||(J-z)^{-1}||||Q||^3_{\mathfrak{S}_3}.
\end{aligned}
\tag{1.5}
$$

On the other hand,

$$\lim_{\epsilon\to\infty}\det{}_3(I+Q(J_0-z)^{-1}) = 1$$

Therefore the estimate (1.2) follows from (1.5) by the Fundamental Theorem of Calculus. □

2. Trace formulae

Let A and B be of finite rank. By H_0 we denote the operator on $l^2(\mathbb{Z})$ defined on the standard basis $\{e_n\}_{-\infty}^{\infty}$ by

$$H_0 e_n = e_{n+1} + e_{n-1}, \quad n \in \mathbb{Z}.$$

It is convenient to replace m_μ by

$$M_\mu(k) = -m_\mu\big(z(k)\big) = -m_\mu\big(k+k^{-1}\big) = \int \frac{k\,d\mu(t)}{1-tk+k^2}, \qquad |k|<1. \tag{2.1}$$

It is known (see [22]) that the limit

$$M(e^{i\theta}) = \lim_{r\to 1} M_\mu(re^{i\theta}), \;\; r<1,$$

exists almost everywhere on the unit circle and

$$\frac{\mathrm{Im}\, M(e^{i\theta})}{\pi} = \mu'(2\cos\theta), \quad \theta\in[0,\pi].$$

We borrow from [12] one important relation which expresses the spectral measure in terms of the perturbation determinant

$$|L(e^{i\theta})|^2 \mathrm{Im}\, M(e^{i\theta}) = \sin(\theta)\det(1+A)^2,$$

where

$$L(z) = \det(1+Q(J_0-zI)^{-1}).$$

Therefore for $z = 2\cos\theta$

$$
\begin{aligned}
&\frac{1}{2}\log\Big[\frac{\sin(\theta)}{\mathrm{Im}\, M(e^{i\theta})}\Big] = -\log|\det(I+A)| + \mathrm{Re\ tr}\, Q(J_0-z)^{-1} - \\
&-\frac{1}{2}\mathrm{Re\ tr}\,(Q(J_0-z)^{-1})^2 + \log|\det{}_3(I+Q(J_0-z)^{-1})| =: \\
&g(\theta) + \log|\det{}_3(I+Q(J_0-z)^{-1})|
\end{aligned}
\tag{2.2}
$$

The function $d_3(k) = \det_3(I + Q(J_0 - z)^{-1})$ vanishes when $k + 1/k$ is an eigenvalue of J. Let $\{\varkappa_n\}$ be the zeros of $d_3(k)$ lying in the domain $|k| < 1$. We introduce the Blaschke product

$$G = \prod_n \frac{\varkappa_n - k}{1 - \varkappa_n k} \frac{\varkappa_n}{|\varkappa_n|}, \qquad (\overline{\varkappa_n} = \varkappa_n).$$

Clearly $|G| = 1$ when $k = e^{i\theta}$. Choose $a > 0$ so small that every point $z = k+1/k$ with $|k| = a$ satisfies the condition (1.3). Obviously $a < \min_n |\varkappa_n|$. Thus, by using the Cauchy theorem we find

$$\frac{2}{\pi}\int_{-\pi}^{\pi} \log|d_3(k)|\rho(e^{i\theta})\sin^2\theta\, d\theta =$$

$$\operatorname{Re}\frac{i}{2\pi}\int_{|k|=a} \log(d_3(k)/G)\rho(k)\frac{(k^2-1)^2}{k^3}\,dk =$$

$$\operatorname{Re}\frac{i}{2\pi}\int_{|k|=a} \log(d_3(k))w(k)\,\frac{dk}{k} - \sum_n f(\varkappa_n) \tag{2.3}$$

where

$$w(k) = \rho(k)\frac{(k^2-1)^2}{k^2}.$$

and

$$f(\varkappa) = \operatorname{Re}\frac{i}{2\pi}\int_{|k|=a} \log\Big(\frac{\varkappa - k}{1 - \varkappa k}\frac{\varkappa}{|\varkappa|}\Big)w(k)\,\frac{dk}{k}.$$

The latter integral can be computed by the residue calculus, in particular one can show that $f(\varkappa)$ is a finite linear combination of powers $\varkappa^{\pm n}$ and $\log(|\varkappa|)$. However, instead of calculating it explicitly we prove that

$$f(t) = \frac{4\rho(\mp 1)}{3}|t \pm 1|^3 + o(|t \pm 1|^3), \quad \text{as } t \to \mp 1,\ |t| < 1. \tag{2.4}$$

Indeed, since $\overline{w(k)} = w(1/\overline{k})$ we observe that

$$\begin{aligned} 2f(\varkappa) = \operatorname{Re}\frac{i}{2\pi}\int_{|k|=a} \log\Big(\frac{\varkappa - k}{1 - \varkappa k}\frac{\varkappa}{|\varkappa|}\Big)w(k)\,\frac{dk}{k} - \\ \operatorname{Re}\frac{i}{2\pi}\int_{|k|=1/a} \log\Big(\frac{\varkappa - k}{1 - \varkappa k}\frac{\varkappa}{|\varkappa|}\Big)w(k)\,\frac{dk}{k}. \end{aligned} \tag{2.5}$$

Now assume that $\varkappa = 1 - s$, where $s \to 0$. Then one can expand the following logarithm into a power series

$$\log\Big(\frac{\varkappa - k}{1 - \varkappa k}\frac{\varkappa}{|\varkappa|}\Big) = -\sum_n \frac{s^n}{n(1-k)^n} + \sum_n \frac{(-sk)^n}{n(1-k)^n}. \tag{2.6}$$

Then one can integrate each term in the expansion (2.6) with the weight $w(k)$ to calculate the difference of the integrals in (2.5). Since $w(k) =$

$4\rho(1)(k-1)^2 + O((k-1)^3)$, as $k \to 1$, we come to (2.4) for the case $t \to 1$. The other case is similar.

By the inequality (1.4) the first term in (2.3) is bounded by a constant $C = C(a, \rho, ||Q||)$ (depending on a, ρ and $||Q||$) times $||Q||^3_{\mathfrak{S}_3}$:

$$\Big| \int_{|k|=a} \log(d_3(k)w(k)\,\frac{dk}{k}\Big| \leq C||Q||^3_{\mathfrak{S}_3}, \quad C = C(a, \rho, ||Q||). \tag{2.7}$$

Now, since the kernel of the resolvent $R(z) = (H_0 - zI)^{-1}$ is of the form

$$G_2(n,m;z) = \frac{k}{k^2-1}k^{|n-m|}, \quad z = k + k^{-1},\ 0 < |k| < 1, \tag{2.8}$$

we obtain

Lemma 2.1. *For* $k = e^{i\theta}$

$$\text{Re tr } R(z)QR(z)Q = \frac{-1}{4\sin^2(\theta)}|2e^{i\theta}\hat{A}(2\theta) + \hat{B}(2\theta)|^2 + \text{tr } A^2.$$

Proof. By using (2.8) we compute

$$\text{Re tr } R(z)BR(z)B = \frac{-1}{4\sin^2(\theta)}|\hat{B}(2\theta)|^2, \quad z = 2\cos\theta,$$

Furthermore

$$\text{Re tr } R(z)SAR(z)SA = \text{Re tr } R(z)AS^*R(z)AS^* =$$
$$= \frac{-1}{4\sin^2(\theta)}(|\hat{A}(2\theta)|^2 - (2\sin^2(\theta))\text{tr } A^2),$$

and

$$\text{Re tr } R(z)SAR(z)AS^* = \text{Re tr } R(z)AS^*R(z)SA = \frac{-|\hat{A}(2\theta)|^2}{4\sin^2(\theta)}.$$

Finally

$$\text{Re tr } R(z)SAR(z)B = \text{Re tr } R(z)AS^*R(z)B = \frac{-1}{4\sin^2(\theta)}\text{Re }\{e^{i\theta}\hat{A}(2\theta)\overline{\hat{B}(2\theta)}\}$$

which completes the proof. $\square$

We will need the following relation which is valid for $k = e^{i\theta}$

$$\text{Re tr } R(z)Q = \text{tr } A. \tag{2.9}$$

The kernel for the resolvent $(J_0 - zI)^{-1}$ of J_0 is of the following form:

$$G(n,m;z) = \frac{k}{1-k^2}(k^{n+m+2} - k^{|n-m|}) =$$
$$= G_1(n,m;z) + G_2(n,m;z).$$

The first term here does not play any role because its contribution involves only finite number of entries of J. Indeed we can prove the following result

Proposition 2.1. *Let $\phi(\theta)$ be a boundary value of an analytic function in the unit disc except for a pole of order j at the point zero. Then the function*

$$\begin{aligned}\Phi(Q) = \operatorname{tr} \int_{-\pi}^{\pi} \big(QR(z) - Q(J_0 - z)^{-1}\big)\phi(\theta)\, d\theta - \\ -\frac{1}{2} \operatorname{tr} \int_{-\pi}^{\pi} \big((QR(z))^2 - (Q(J_0 - z)^{-1})^2\big)\phi(\theta)\, d\theta\end{aligned} \tag{2.10}$$

is a second order polynomial only of a finite number of elements α_n and β_n.

Proof. Let Γ_k be the operator in $l^2(\mathbb{N})$ with the kernel $G_1(n, m; z)$, $z = k + k^{-1}$ and P be the orthogonal projection in $l^2(\mathbb{Z})$ onto $l^2(\mathbb{N})$. Then $\Gamma_k = (J_0 - z)^{-1} - PR(z)P$. Therefore the integrated function in (2.10) can be rewritten in terms of tr $Q\Gamma_k$, tr $(Q\Gamma_k)^2$ and tr $Q\Gamma_k QR(z)$. The functions $(k^2 - 1)$tr $Q\Gamma_k$, $(k^2 - 1)^2$tr $(Q\Gamma_k)^2$ and $(k^2 - 1)^2$tr $Q\Gamma_k QR(z)$ are analytic in the disc and the coefficients of the Taylor series about zero depend on finite number of entries α_n and β_n. For example

$$(k^2 - 1)^2 \operatorname{tr} B\Gamma_k BR(z) = k^2 \sum_{n=0}^{\infty} k^{2n} \big(\beta_n^2 + 2\beta_n \sum_{m=0}^{n-1} \beta_m\big). \quad \square$$

From (2.2), (2.3), (2.7) and (2.9) we obtain

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \log\Big[\frac{\sin(\theta)}{\operatorname{Im} M(e^{i\theta})}\Big] \rho(e^{i\theta}) \sin^2\theta\, d\theta =$$
$$\frac{2}{\pi} \int_{-\pi}^{\pi} g(\theta)\rho(e^{i\theta}) \sin^2\theta\, d\theta - \sum_n f(\varkappa_n) + \Psi(Q)$$

where $\Psi(Q)$ is finite even if $Q \in \mathfrak{S}_3$ and can be estimates by a function of $||Q||_{\mathfrak{S}_3}$. From Lemma 2.1 we see that

$$\begin{aligned}g(\theta) = -\log|\det{}_3(1 + A)| + \\ \frac{1}{8\sin^2(\theta)} |2e^{i\theta}\hat{A}(2\theta) + \hat{B}(2\theta)|^2 + \operatorname{Re} \Phi_0(\theta),\end{aligned} \tag{2.11}$$

where Φ_0 is the integrand in (2.10) with $\phi = 1$. Thus we obtain the trace formulae:

$$\begin{aligned}\sum_n f(\varkappa_n) + \frac{1}{\pi} \int_{-\pi}^{\pi} \log\Big[\frac{\sin(\theta)}{\operatorname{Im} M(e^{i\theta})}\Big] \rho(e^{i\theta}) \sin^2\theta\, d\theta = \\ \frac{1}{4\pi} \int_{-\pi}^{\pi} |2e^{i\theta}\hat{A}(2\theta) + \hat{B}(2\theta)|^2 \rho(e^{i\theta})\, d\theta + \Lambda(Q),\end{aligned} \tag{2.12}$$

where the absolute value of $\Lambda(Q)$ is bounded by a certain function of $||Q||_{\mathfrak{S}_3}$.

3. Perturbations of infinite rank

Given a perturbation of class $\mathfrak{S}_3$ there is a beautiful technique which enables one to prove that if one of the sides in (2.12) is finite, then so is the other side. This technique is due to Killip and Simon [12] and uses the following facts:

$$-\int_{-\pi}^{\pi} \log\Big[\frac{\sin(\theta)}{\text{Im}\, M(e^{i\theta})}\Big]\rho(e^{i\theta})\sin^2\theta\, d\theta \le C, \quad C = C(\rho),$$

and

(3.1)
$$\int_{-\pi}^{\pi} |2e^{i\theta}\hat{A}(2\theta) + \hat{B}(2\theta)|^2\rho(e^{i\theta})\, d\theta = 2\pi \sum_n \Big|\sum_j 2w_{n-2j-1}\alpha_j + w_{n-2j}\beta_j\Big|^2$$

However, since we are forced to deal with the sums appearing in the r.h.s. of (3.1) the better reference is the one on the paper [17] where the technique is more adjusted to the special case of the trace formulae (2.12). Bellow we denote by D the unit disc in $\mathbb{C}$.

Let $J^{(N)}$, $Q^{(N)}$ be operators whose realizations in the standard basis $\{e_n\}_{n=0}^\infty$ are given by

$$J^{(N)} = \begin{pmatrix} \beta_{N+1} & 1+\alpha_{N+1} & 0 & \dots \\ 1+\alpha_{N+1} & \beta_{N+2} & 1+\alpha_{N+2} & \dots \\ 0 & 1+\alpha_{N+2} & \beta_{N+3} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}, \tag{3.2}$$

$$Q^{(N)} = \begin{pmatrix} \beta_{N+1} & \alpha_{N+1} & 0 & \dots \\ \alpha_{N+1} & \beta_{N+2} & \alpha_{N+2} & \dots \\ 0 & \alpha_{N+2} & \beta_{N+3} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix} \tag{3.3}$$

and let

$$\Lambda_\rho(J) := \frac{1}{4\pi}\int_{-\pi}^{\pi} |2e^{i\theta}\hat{A}(2\theta) + \hat{B}(2\theta)|^2\rho(e^{i\theta})\, d\theta.$$

We introduce

$$P_{N,\rho}(Q) := \Lambda_\rho(J) - \Lambda_\rho(J^{(N)})$$

The "tails" in the sums $\Lambda_\rho(J)$ and $\Lambda_\rho(J^{(N)})$ cancel each other, so that the elements of the matrices B and A do not enter in this difference $P_{N,\rho}(Q)$ starting from a certain index. Thus $P_{N,\rho}(Q)$ is a continuous function of a finite number of elements of the matrices B and A and can be extended to arbitrary matrices B, A. Below $P_{N,\rho}(Q)$ is extended for any B and A. Denote

$$\Phi_\rho(\mu) = \frac{1}{\pi}\int_{-\pi}^{\pi} \log\Big|\frac{\sin\theta}{\text{Im}\, M_\mu}\Big|\rho(e^{i\theta})\sin^2\theta\, d\theta + \sum_n f(\varkappa_n). \tag{3.4}$$

It is important for us to find conditions when

$$\Phi_\rho(\mu) - \Phi_\rho(J^{(N)}) = P_{N,\rho}(Q) + \Lambda(Q) - \Lambda(Q^{(N)}). \tag{3.5}$$

The identity (3.5) is valid if Q is of a finite rank. The case when the function M_μ meromorphic in the neighborhood of the unit disc follows from Proposition 4.3 and Theorem 4.4 in [12] and the fact that

$$\rho(k) = (\sum_{n=-n_0}^{n_0} w_n k^n)^2$$

is a finite linear combination of powers $k^{\pm m}$.

Notice that $y \mapsto -\log(y)$ is convex. Assume that

$$\int_{-\pi}^{\pi} \rho(e^{i\theta}) \sin^2\theta \, d\theta = 1$$

Employing Jensen's inequality we find

$$\int_{-\pi}^{\pi} \log\left[\frac{\sin(\theta)}{\text{Im } M_\mu(e^{i\theta})}\right] \rho(e^{i\theta}) \sin^2\theta \, d\theta = -2\int_0^\pi \log\left[\frac{\text{Im } M_\mu}{\sin\theta}\right] \sin^2(\theta)\, \rho \, d\theta$$

$$\geq -\log\left[2\int_0^\pi (\text{Im } M_\mu) \sin(\theta)\, d\theta\right] - 2\int_0^\pi \log\left[\rho(e^{i\theta})\right] \sin^2\theta \, \rho \, d\theta$$

$$= -\log[\pi\mu_{\text{ac}}(-2,2)] - 2\int_0^\pi \log\left[\rho(e^{i\theta})\right] \sin^2\theta\, \rho\, d\theta \geq$$

$$-\log(\pi) - 2\int_0^\pi \log\left[\rho(e^{i\theta})\right] \sin^2\theta \, \rho\, d\theta =: C(\rho), \tag{3.6}$$

where we use that $\mu_{\text{ac}}(-2,2) \leq 1$. Formulae (3.5) and (3.6) imply

$$P_{N,\rho}(Q) \leq \Phi_\rho(\mu) - C(\rho) + F(||Q||_{\mathfrak{S}_3}, \rho),$$

where the quantity $F(||Q||_{\mathfrak{S}_3}, \rho)$ depends only on $||Q||_{\mathfrak{S}_3}$ and ρ. The latter inequality was obtained for M_μ meromorphic in the neighborhood of the unit disc. However, this inequality can also be extended to arbitrary measures μ satisfying Conditions (1)-(3) of Theorem 0.1.

Indeed, assume that Condition (2) of Theorem 0.1 is fulfilled According to Remark 1 following after Theorem 2.1 in [26]

$$\lim_{r\uparrow 1} \int \log[\text{Im } M(re^{i\theta})]\, \rho(e^{i\theta}) \sin^2\theta\, d\theta = \int \log[\text{Im } M(e^{i\theta})]\, \rho(e^{i\theta}) \sin^2\theta\, d\theta \tag{3.7}$$

Now, given any J and associated to it M-function $M(z)$, there is a natural approximating family of M-functions meromorphic in a neighborhood of the closure of the unit disc $\bar{D}$. The next result is proved in [12], Lemma 8.3.

Lemma 3.1. *Let M_μ be the M-function of a probability measure μ obeying Condition (1) of Theorem 0.1. Define*

$$M^{(r)}(z) = r^{-1}M_\mu(rz) \tag{3.8}$$

for $0 < r < 1$. Then, there is a family of probability measures $\mu^{(r)}$ such that $M^{(r)} = M_{\mu^{(r)}}$.

The poles of $M_{\mu^{(r)}}$ are given by

$$\varkappa_j(\mu^{(r)}) = \frac{\varkappa_j}{r},$$

where we consider only those j for which $|\varkappa_j| < r$. Thus if Condition (3) of Theorem 0.1 is satisfied then $\sum f(\varkappa_j(\mu^{(r)}))$ is a continuous function of r whose limit is equal to $\sum f(\varkappa_j)$, as $r \uparrow 1$. Moreover, the convergence $M_{\mu^{(r)}}(z) \to M_\mu(z)$ is uniform on compact subsets of upper half of D, which means that the coefficients of Jacobi matrices must converge. Thus for any N

$$P_{N,\rho}(Q) = \lim_{r\uparrow 1} P_{N,\rho}(Q(r))$$

$$\leq \lim_{r\uparrow 1} \Phi_\rho(\mu^{(r)}) - C(\rho) + F(||Q||_{\mathfrak{S}_3}, \rho) = \Phi_\rho(\mu) - C(\rho) + F(||Q||_{\mathfrak{S}_3}, \rho).$$

Therefore $\sup_N P_{N,\rho}(Q) < \infty$ which guarantees (0.6).

2. Conversely, suppose that the conditions (0.6) and (0.5) are fulfilled. We would like to establish that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \log\left[\frac{\sin(\theta)}{\operatorname{Im} M_\mu(e^{i\theta})}\right] \sin^2(\theta)\rho\, d\theta < \infty. \tag{3.9}$$

Definition. Let ν, μ be finite Borel measures on a compact Hausdorff space X. The entropy $S(\nu \mid \mu)$ of ν relative to μ is defined by

$$S(\nu \mid \mu) = \begin{cases} -\infty & \text{if } \nu \text{ is not } \mu\text{-ac}, \\ -\int \log(\frac{d\nu}{d\mu}) d\nu & \text{if } \nu \text{ is } \mu\text{-ac}. \end{cases} \tag{3.10}$$

The following result is proved in the paper of Simon and Killip, [12], Corollary 5.3.

Lemma 3.2. *$S(\nu \mid \mu)$ is weakly upper semi-continuous in μ, that is, if $\mu_n \xrightarrow{w} \mu$, then*

$$S(\nu \mid \mu) \geq \limsup_{n\to\infty} S(\nu \mid \mu_n).$$

Let us use the fact that the formulae (2.12) are valid at least for finite rank operators A, B. Suppose now that A, B are arbitrary compact selfadjoint operators such that (0.5) and (0.6) hold. It is clear then that the right hand side of (2.12) is finite. Now let the sequences of operators A_n, B_n converge to A and B in $\mathfrak{S}_3$ so that

$$\int_{-\pi}^{\pi} |2e^{i\theta}\hat{A}_n(2\theta)+\hat{B}_n(2\theta)|^2\rho(e^{i\theta})\,d\theta \;\to\; \int_{-\pi}^{\pi} |2e^{i\theta}\hat{A}(2\theta)+\hat{B}(2\theta)|^2\rho(e^{i\theta})\,d\theta$$

as $n \to \infty$. Let $Q_n = SA_n + A_nS^* + B_n$, $J_n = S + S^* + Q_n$ and $\mu_n(\delta) = (E_{J_n}(\delta)e_0, e_0)$, where δ is an arbitrary Borel set. Since $(J_n - z)^{-1}$ converges to $(J-z)^{-1}$ uniformly on compact subsets of the upper half-plane we obtain that μ_n is weakly convergent to μ,

$$\mu_n \xrightarrow{w} \mu, \quad \text{as} \quad n \to \infty.$$

Applying Lemma 3.2 we obtain that if $d\nu = \rho(e^{i\theta})\sin^2\theta\,d\theta$ and μ is the spectral measure of J, then

$$S(\nu \mid \mu) > -\infty.$$

This is exactly what is needed for (3.9).

In order to complete the proof we only have to show that (0.6) and (0.5) imply Condition (3) of Theorem 0.1. This can be done by a simple trick (see [17]) whose essence is that one has to consider first a finite sum $\sum_{j=1}^{p} f(\varkappa_j)$, prove that this sum can be approximated by the corresponding sum for A_n, B_n and then let $p \to \infty$.

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