# Glauber versus Kawasaki for spectral gap and logarithmic Sobolev inequalities of some unbounded conservative spin systems

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**Abstract**

Inspired by the recent results of C. Landim, G. Panizo and H.-T. Yau [LPY00] on spectral gap and logarithmic Sobolev inequalities for unbounded conservative spin systems, we study uniform bounds in these inequalities for Glauber dynamics of Hamiltonian of the form

$$\sum_{i=1}^{n} V(x_i) + V\left(M - \sum_{i=1}^{n} x_i\right), \quad (x_1, \dots, x_n) \in \mathbb{R}^n$$

Specifically, we examine the case $V$ is strictly convex (or small perturbation of strictly convex) and, following [LPY00], the case $V$ is a bounded perturbation of a quadratic potential. By a simple path counting argument for the standard random walk, uniform bounds for the Glauber dynamics yields, in a transparent way, the classical $L^{-2}$ decay for the Kawasaki dynamics on $d$-dimensional cubes of length $L$. The arguments of proofs however closely follow and make heavy use of the conservative approach and estimates of [LPY00], relying in particular on the Lu-Yau martingale decomposition and clever partitionings of the conditional measure.

# Introduction

Let $Q$ be a probability measure on $\mathbb{R}^n$. In the sequel, we denote by $\mathbf{E}_Q(f)$ the expectation of $f$ with respect to $Q$, $\mathbf{Var}_Q(f) := \mathbf{E}_Q(f^2) - \mathbf{E}_Q(f)^2$ the variance of $f$ for $Q$, and $\mathbf{Ent}_Q(f)$ the entropy of a non negative measurable function $f$ with respect to $Q$, defined by

$$\mathbf{Ent}_Q(f) := \int f \log f dQ - \int f dQ \ \log \int f dQ.$$

We say that $Q$ satisfies a Poincaré inequality if there exists a positive constant $\mathcal{P}$ such that for any smooth function $f : \mathbb{R}^n \to \mathbb{R}$,

$$\mathbf{Var}_Q(f) \leqslant \mathcal{P}\, \mathbf{E}_Q\big(|\nabla f|^2\big)\,, \tag{1}$$

where $|\nabla f|^2 := \sum_{i=1}^n |\partial_i f|^2$. Similarly, we say that $Q$ satisfies a logarithmic Sobolev inequality if there exists a positive constant $\mathcal{L}$ such that for any smooth function $f$,

$$\mathbf{Ent}_Q\big(f^2\big) \leqslant \mathcal{L}\,\mathbf{E}_Q\big(|\nabla f|^2\big)\,. \tag{2}$$

This inequality strengthen the Poincaré inequality (1) since for $\varepsilon$ small enough,

$$\mathbf{Ent}_Q\big((1+\varepsilon f)^2\big) = 2\varepsilon^2 \mathbf{Var}_Q(f) + \mathcal{O}(\varepsilon^3),$$

which gives $2\mathcal{P} \leqslant \mathcal{L}$. Let $\mathcal{H} \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$ such that

$$Z_{\mathcal{H}} := \int_{\mathbb{R}^n} e^{-\mathcal{H}(x)}\,dx < +\infty.$$

The probability measure $Q$ defined by $dQ(x) = (Z_{\mathcal{H}})^{-1} \exp\left(-\mathcal{H}(x)\right) dx$ is the symmetric invariant measure of the diffusion process $(X_t)_{t\geqslant 0}$ on $\mathbb{R}^n$ driven by the S.D.E.

$$dX_t = \sqrt{2}\,dB_t - \nabla\mathcal{H}(X_t)\,dt,$$

where $(B_t)_{t\geqslant 0}$ is a standard brownian motion on $\mathbb{R}^n$. In this context, we say that the probability measure $Q$ is associated with the "Hamiltonian" $\mathcal{H}$. It is well known that $Q$ satisfies the Poincaré inequality (1) with a constant $\mathcal{P}$ if and only if the infinitesimal generator $\mathbf{L} := \boldsymbol{\Delta} - \nabla\mathcal{H}\cdot\nabla$ possesses a spectral gap greater than $\mathcal{P}^{-1}$. In the other hand, a famous Theorem of Gross states that $Q$ satisfies the logarithmic Sobolev inequality (2) if and only if the diffusion semi-group generated by $\mathbf{L}$ is hyper-contractive. A celebrated result of Bakry and Émery ensures that when there exists a constant $\rho > 0$ such that for any $x \in \mathbb{R}^n$,

$$\mathrm{Hess}(\mathcal{H})(x) \geqslant \rho\,\mathbf{I}_p$$

as quadratic forms on $\mathbb{R}^n$, i.e. $\mathcal{H}$ is uniformly strictly convex or $Q$ is log-concave, then $Q$ satisfies to (1) and (2) with constants $\mathcal{P} = \rho^{-1}$ and $\mathcal{L} = 2\,\rho^{-1}$ respectively. Moreover, $Q$ satisfies to (1) with a constant $\mathcal{P}$ if and only if

$$\mathcal{P}\mathbf{E}_Q\big((\mathbf{L}f)^2\big) \geqslant \mathbf{E}_Q\big(|\nabla f|^2\big)$$

for any smooth function $f$. The reader may find an introduction to logarithmic Sobolev inequalities and related fields in [ABC$^+$00].

We are interested in the present work to particular "Hamiltonians" $H \in \mathcal{C}^2(\mathbb{R}^{n+1}, \mathbb{R})$. Let $M \in \mathbb{R}$ and define $H_M \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$ by

$$H_M(x_1, \ldots, x_n) := H\left(x_1, \ldots, x_n, M - \sum_{i=1}^n x_i\right). \tag{3}$$

Assume that $Z_{H_M} < \infty$. Our aim is to establish Poincaré and logarithmic Sobolev inequalities for probability measures on $\mathbb{R}^n$ of the form

$$d\sigma_M(x_1, \ldots, x_n) := (Z_{H_M})^{-1} \exp\left(-H_M(x)\right) dx_1 \cdots dx_n, \tag{4}$$

with constants $\mathcal{P}$ and $\mathcal{L}$ which does not depend on $n$ and $M$. This investivation is motivated by the study of certain conditional probability measures. Namely, if the probability measure $\mu$ on $\mathbb{R}^{n+1}$ given by

$$d\mu(x) := (Z_H)^{-1} \exp\left(-H(x_1, \dots, x_{n+1})\right) dx_1 \cdots dx_{n+1} \tag{5}$$

is well-defined, i.e. $Z_H < +\infty$, then for any $M \in \mathbb{R}$, one can define the conditional probability measure $\mu_M$ by

$$\mu_M := \mu\Big( \cdot \Big| \sum_{i=1}^{n+1} x_i = M \Big), \tag{6}$$

and we get, for any $f \in \mathcal{C}_b(\mathbb{R}^{n+1}, \mathbb{R})$,

$$\mathbf{E}_{\mu_M}(f) = \int_{\mathbb{R}^n} f\Big( x_1, \dots, x_n, M - \sum_{i=1}^{n} x_i \Big) \, d\sigma_M(x_1, \dots, x_n). \tag{7}$$

Thus, $\sigma_M$ can be viewed as the translation of the conditional probability measure $\mu_M$ under the affine hyperplane of $\mathbb{R}^{n+1}$ of equation $x_1 + \cdots + x_{n+1} = M$. Alternatively, and following Caputo in [Cap01], the conditional probability measure $\mu_M$ can be defined from the probability measure $\mu$ given in (5) by adding an infinite potential outside of the affine constraint $x_1 + \cdots + x_{n+1} = M$. Namely for any bounded continuous function $f : \mathbb{R}^{n+1} \to \mathbb{R}$

$$\mathbf{E}_{\mu_M}(f) = \lim_{\beta \to +\infty} \mathbf{E}_{\mu_{M,\beta}}(f) \, .$$

where $\mu_{M,\beta}$ denotes the probability measure on $\mathbb{R}^{n+1}$ defined by

$$d\mu_{M,\beta}(x) := \mathcal{Z}_{\mu_{M,\beta}}^{-1} \, e^{-\beta\,(M - x_1 - \cdots - x_{n+1})^2} \, d\mu(x).$$

A simple change of variable in $\mathbf{E}_{\mu_{M,\beta}}(f)$ gives that

$$\lim_{\beta \to +\infty} \mathbf{E}_{\mu_{M,\beta}}(f) = \int_{\mathbb{R}^n} f\Big( x_1, \dots, x_n, M - \sum_{i=1}^{n} x_i \Big) \, d\sigma_M(x_1, \dots, x_n),$$

This weak limit definition of $\mu_M$ was used by Caputo in [Cap01] in order to study the case of a convex Hamiltonian $H$. We do not use it in our approach. Notice that if $f \in \mathcal{C}_b(\mathbb{R}^n, \mathbb{R})$, we get from (7) that

$$\mathbf{E}_{\sigma_M}(f) = \int_{\mathbb{R}^{n+1}} f(x_1, \dots, x_n) \, d\mu_M(x_1, \dots, x_{n+1}). \tag{8}$$

Observe that (7) gives $\mathbf{E}_{\mu_M}(x_1 + \cdots + x_{n+1}) = M$. Thus, when $H$ is a symmetric function, $\sigma_M$ and $\mu_M$ are exchangeable measures, i.e. invariant by any permutation of the coordinates. This holds for example when $H(x) = V(x_1) + \cdots + V(x_{n+1})$. Moreover, $M - \sum_{i=1}^{n} x_i$ and $x_j$ have then the same law under $\sigma_M$ for any $j$ in $\{1, \dots, n\}$ and we get

$$\mathbf{E}_{\mu_M}(x_1) = \cdots = \mathbf{E}_{\mu_M}(x_{n+1}) = \mathbf{E}_{\sigma_M}(x_1) = \cdots = \mathbf{E}_{\sigma_M}(x_n) = \frac{M}{n+1}. \tag{9}$$

Thus, the mean of $\mu_M$ and $\sigma_M$ does not depend on $H$ in this case.

Let us see now how to translate (1) and (2) for $\sigma_M$ in terms of $\mu_M$. One can observe that for any $i \in \{1, \dots, n\}$

$$\partial_i\left(f\left(x_1, \dots, x_n, M - \sum_{i=1}^n x_i\right)\right)$$
$$= (\partial_i f)\left(x_1, \dots, x_n, M - \sum_{i=1}^n x_i\right) - (\partial_{n+1} f)\left(x_1, \dots, x_n, M - \sum_{i=1}^n x_i\right).$$

By replacing the coordinate $x_{n+1}$ by any of the $x_1, \dots, x_n$ in (7), we obtain the following proposition

**Proposition 0.1.** *Let $H : \mathbb{R}^{n+1} \to \mathbb{R}$ and assume that for any permutation $\pi$ of the coordinates, the probability measure $\sigma_M^\pi$ on $\mathbb{R}^n$ defined by (4) and associated to $H \circ \pi$ satisfies to Poincaré (resp. logarithmic Sobolev) inequality with a constant $\mathcal{P}$ (resp. $\mathcal{L}$) which does not depend on $n$, $M$ and $\pi$. Then, if $\mu_M$ is the associated conditional probability measure defined by (6), we get for any smooth $f : \mathbb{R}^{n+1} \to \mathbb{R}$*

$$\mathbf{Var}_{\mu_M}(f) \leqslant \frac{\mathcal{P}}{n+1}\,\mathbf{E}_{\mu_M}\left(\sum_{1 \leqslant i,j \leqslant n+1} |\partial_i f - \partial_j f|^2\right), \tag{10}$$

*and respectively*

$$\mathbf{Ent}_{\mu_M}(f^2) \leqslant \frac{\mathcal{L}}{n+1}\,\mathbf{E}_{\mu_M}\left(\sum_{1 \leqslant i,j \leqslant n+1} |\partial_i f - \partial_j f|^2\right). \tag{11}$$

These inequalities leads to constants in $L^2$ for the "Kawasaki dynamics" associated to $\mu_M$. Namely, consider a finite box $\Lambda := \{1, \dots, L\}^d \subset \mathbb{Z}^d$ on the lattice $\mathbb{Z}^d$ and $n$ such that $\mathbb{R}^\Lambda \simeq \mathbb{R}^{n+1}$ (i.e. $n+1 = |\Lambda| = L^d$). There exists a constant $C > 0$ depending only on $d$ such that for any $a \in \mathbb{R}^\Lambda$

$$\frac{1}{|\Lambda|}\sum_{i,j \in \Lambda}(a_i - a_j)^2 \leqslant C\,L^2 \sum_{\substack{i,j \in \Lambda \\ |i-j|=1}} (a_i - a_j)^2, \tag{12}$$

Therefore, it is straightforward to deduce from (10) and (11) that for a constant $C > 0$ which does not depend on $n$ and $M$, one have

$$\mathbf{Var}_{\mu_M}(f) \leqslant C\,L^2 \sum_{\substack{k,l \in \Lambda \\ |k-l|=1}} \mathbf{E}_{\mu_M}\left(|\partial_i f - \partial_j f|^2\right), \tag{13}$$

and

$$\mathbf{Ent}_{\mu_M}(f^2) \leqslant C\,L^2 \sum_{\substack{k,l \in \Lambda \\ |k-l|=1}} \mathbf{E}_{\mu_M}\left(|\partial_i f - \partial_j f|^2\right). \tag{14}$$

Inequality (12) follows from a classical path counting argument (see for example section 4.2 of [SC97]). However, let us gives briefly a proof. For any $i, j$ in $\Lambda$, consider the path $\Gamma_{ij}$ inside $\Lambda$ joining $i$ and $j$ obtained by adjusting the $d$ coordinates one after the other. We have $|\Gamma_{ij}| \leqslant dL$ and for each $k, l$ in $\Lambda$ with $|k-l| = 1$, the number of such paths containing the edge $(k, l)$ is bounded above by $c_d\, L^{d+1}$ where $c_d > 0$ is a constant depending only on $d$. Now by Cauchy-Schwarz's inequality

$$(a_i - a_j)^2 = \left[ \sum_{(k,l)\in\Gamma_{i,j},\, |k-l|=1} (a_k - a_l) \right]^2 \leqslant dL \sum_{(k,l)\in\Gamma_{i,j},\, |k-l|=1} (a_k - a_l)^2,$$

and therefore

$$\sum_{i,j\in\Lambda} (a_i - a_j)^2 \leqslant dL \sum_{\substack{|k-l|=1 \\ k,l\in\Lambda}} (a_k - a_l)^2 \sum_{\substack{i,j\in\Lambda \\ \Gamma_{i,j}\ni(k,l)}} 1 \leqslant dc_d L^{d+2} \sum_{\substack{|k-l|=1 \\ k,l\in\Lambda}} (a_k - a_l)^2,$$

which gives the desired result (12).

A simple example is given by uniformly strictly convex $H$ with a constant $\rho$ in $\mathbb{R}^{n+1}$. Namely, if there exists a constant $\rho > 0$ such that for any $x \in \mathbb{R}^{n+1}$, $\mathrm{Hess}(H)(x) \geqslant \rho \mathbf{I}_{n+1}$ as quadratic forms on $\mathbb{R}^{n+1}$, then, an easy calculus gives for any $x \in \mathbb{R}^n$ and $h \in \mathbb{R}^n$

$$\langle (\mathrm{Hess}(H_M))(x)h, h\rangle_{\mathbb{R}^n} \geqslant \rho \sum_{i=1}^n h_i^2 + \rho \left( - \sum_{i=1}^n h_i \right)^2 \geqslant \rho \sum_{i=1}^n h_i^2.$$

Thus, $H_M$ is uniformly strictly convex with the same constant $\rho$, and therefore, by the Bakry-Émery criterion, $\sigma_M$ satisfies to Poincaré and logarithmic Sobolev inequalities with a constant $\rho^{-1}$ and $2\,\rho^{-1}$ respectively, which does not depend on $n$ and $M$. The hypotheses of Proposition 0.1 are full-filled since by the same calculus, $(H \circ \pi)_M$ is also uniformly strictly convex with a constant $\rho$. A more simple example is given by

$$H(x) = V(x_1) + \cdots + V(x_{n+1})$$

where $V$ is in $\mathcal{C}^2(\mathbb{R}, \mathbb{R})$ with $V'' > \rho > 0$. Let us consider now another convex Hamiltonian example on $\mathbb{R}^{n+1}$ defined by

$$H(x) := \frac{1}{2(n+1)} \sum_{i,j=1}^{n+1} V_{\{i,j\}}(x_i - x_j),$$

where $V_{\{i,j\}}$ are in $\mathcal{C}^2(\mathbb{R}, \mathbb{R})$ and even. This is a so called mean-field Hamiltonian when all the $V_{\{i,j\}}$ are equal. We have for any $i, j$ in $\{1, \ldots, n+1\}$

$$(n+1)\, \partial_{ij}^2 H(x) = \begin{cases} \sum_{\substack{k=1 \\ k\neq i}}^{n+1} V''_{\{i,k\}}(x_i - x_k) & \text{if } i = j \\ -V''_{\{i,j\}}(x_i - x_j) & \text{if } i \neq j \end{cases}$$

Therefore, if $V''_{\{i,j\}}(u) \geqslant 0$ for any $u \in \mathbb{R}$ and any $i,j \in \{1,\dots,n+1\}$, i.e. $V_{\{i,j\}}$ is convex, the Gershgorin-Hadamard theorem implies that for any $x \in \mathbb{R}^{n+1}$, $\mathrm{Hess}(H)(x) \geqslant 0$ as a quadratic form, and thus $H$ is convex on $\mathbb{R}^{n+1}$. Unfortunately, since $\sum_{j=1}^{n+1} \partial^2_{ij} H(x) = 0$ for any $i \in \{1,\dots,n+1\}$, the null space of $\mathrm{Hess}(H)$ contains $1_{n+1}$ and therefore, the measure $\mu$ on $\mathbb{R}^{n+1}$ defined by $d\mu(x) := \exp(-H(x))\,dx$ cannot be normalised into a probability measure since $Z_\mu := \mu(\mathbb{R}^{n+1}) = +\infty$. Nevertheless, suppose that there exists a constant $\rho > 0$ such that $V''_{\{i,j\}}(u) \geqslant \rho$ for any $u \in \mathbb{R}$ and any $i,j \in \{1,\dots,n\}$. Then, $u \in \mathbb{R} \mapsto V_{\{i,j\}}(u) - \rho u^2/2$ is convex and the latter implies that

$$\mathrm{Hess}(H)(x) \geqslant \rho\,\mathbf{I}_{n+1} - (n+1)^{-1}\rho\,\mathbf{1}_{n+1},$$

as quadratic forms. Thus, by writing $\mathbb{R}^{n+1} = \mathbb{R}1_{n+1} \overset{\perp}{\oplus} \mathcal{H}_n$ where $\mathcal{H}_n$ is the hyperplane of equation $h_1 + \cdots + h_{n+1} = 0$, we get that the spectrum of $\mathrm{Hess}(H)(x)$ is of the form

$$\{0 = \lambda_1(x) < \lambda_2(x) \leqslant \cdots \leqslant \lambda_{n+1}(x)\}$$

with $\lambda_2(x) \geqslant n(n+1)^{-1}\rho$. Hence, one can define the probability measure $\sigma_M$ on $\mathbb{R}^n$ as in (4) for any $M$ in $\mathbb{R}$. Moreover $\sigma_M$ is uniformly log-concave with a constant $n(n+1)^{-1}\rho$ and therefore the conditional measure $\mu_M$ can be defined from $\sigma_M$ as a probability measure by equation (7), despite the fact that $\mu$ is not a probability measure on $\mathbb{R}^{n+1}$. The particular case $V_{\{i,j\}} = V$ with $V$ even and uniformly convex is considered for example in [Mal01], in terms of the associated S.D.E., in order to study the granular media equation.

As we seen, when $H$ is uniformly strictly convex with a constant $\rho > 0$, the hypotheses of Proposition 0.1 are full-filled and hence, inequalities (13) and (14) hold. It is quite natural to ask if (13) and (14) remains true for symmetric but non convex Hamiltonians $H$. In this direction, the Bakry-Émery criterion allows the following perturbative statement due to Ivan Gentil. The proof, prototype of which can be found in [Led01], is taken from [BH99] and is postponed to section 1.

**Proposition 0.2 (Perturbative result).** *Let $H(x) = V(x_1) + \cdots + V(x_{n+1})$ with*

$$V(u) = \frac{u^2}{2} + F(u)$$

*where $F : \mathbb{R} \to \mathbb{R}$, and let $\sigma_M$ be the probability measure on $\mathbb{R}^n$ defined by (4), namely*

$$\sigma_M(dx_1,\dots,dx_n) = (Z_{\sigma_M})^{-1} \int_{\mathbb{R}^n} \exp\left(-\sum_{i=1}^n V(x_i) - V(M - \sum_{i=1}^n x_i)\right) dx_1 \cdots dx_n.$$

*Then, for $\|F\|_\infty$ small enough, there exists a positive constant $\mathcal{P}$ depending only on $\|F\|_\infty$ such that for any $n$, any $M$ and any smooth $f : \mathbb{R}^n \to \mathbb{R}$,*

$$\mathbf{Var}_{\sigma_M}(f) \leqslant \mathcal{P}\,\mathbf{E}_{\sigma_M}\left(|\nabla f|^2\right). \tag{15}$$

Proposition 0.2 remains valid if we replace, in the definition of $\sigma_M$, the square function $u \mapsto u^2/2$ by a smooth convex function $u \mapsto \Phi(u)$, provided that there exists real constants

$\alpha$ and $\beta$ such that $0 < \alpha \leqslant \beta \leqslant 2\alpha$ and $\alpha \leqslant \Phi''(u) \leqslant \beta$ for every $u \in \mathbb{R}$. The constant $\mathcal{P}$ becomes in this case $e^{2\,\mathrm{osc}(F)}/\big(2\alpha\, e^{-2\,\mathrm{osc}(F)} - \beta\big)$ for $\mathrm{osc}(F) < \log\sqrt{2\alpha/\beta}$.

The exchangeability of the underlying measure $\mu_M$ indicates that the perturbative approach by mean of Helffer's method which sees $\sigma_M$ as a quasi-product measure with small interactions is not relevant : any reduction of $F$ in the interaction term

$$V\left(M - \sum_{i=1}^{n} x_i\right)$$

affects the product term $\sum_{i=1}^{n} V(x_i)$. Helffer's method was essentially developed for spins systems with *boundary conditions* for which the measure is not exchangeable. For our measure $\sigma_M$, one can expect in contrast that the symmetries of $H_M$ induces a stronger result, as for many mean field models. In this direction, Landim, Panizo and Yau have recently established in [LPY00] that $\mu_M$ satisfies inequalities (13) and (14) when $H$ is of the form $H(x) = V(x_1) + \cdots + V(x_{n+1})$ where $V(u) = u^2/2 + F(u)$ with $F$ and $F'$ bounded and Lipschitz. A simple example is given by $F(x) = P(\sin(Q(x)))$ where $P$ and $Q$ are fixed polynomials in $\mathbb{R}[X]$. Their proof relies on Lu-Yau's markovian decomposition [LY93] and on Local Central Limit Theorem estimates [KL99].

Following closely [LPY00], we are actually able to show that measure $\sigma_M$ itself satisfies to (1) and (2) with a constants which does not depend on $n$ and $M$, as stated in our main result, which follows.

**Theorem 0.3.** *Let $H(x) = V(x_1) + \cdots + V(x_{n+1})$ with $V(u) = u^2/2 + F(u)$ and let $\sigma_M$ be the probability measure on $\mathbb{R}^n$ defined by (4), namely*

$$\sigma_M(dx_1, \ldots, dx_n) = (Z_{\sigma_M})^{-1} \int_{\mathbb{R}^n} \exp\left(-\sum_{i=1}^{n} V(x_i) - V(M - \sum_{i=1}^{n} x_i)\right) dx_1 \cdots dx_n.$$

*Then, if $F$ is bounded and Lipschitz, there exists a positive constant $\mathcal{P}$ depending only on $\|F\|_\infty$ and $\|F'\|_\infty$ such that for any $n$ and $M$ and any smooth $f : \mathbb{R}^n \to \mathbb{R}$,*

$$\mathbf{Var}_{\sigma_M}(f) \leqslant \mathcal{P}\, \mathbf{E}_{\sigma_M}\left(|\nabla f|^2\right). \tag{16}$$

*Moreover, if $F''$ is also bounded, there exists a positive constant $\mathcal{L}$ depending only on $\|F\|_\infty$, $\|F'\|_\infty$ and $\|F''\|_\infty$ such that for any $n$ and $M$ and any smooth $f : \mathbb{R}^n \to \mathbb{R}$,*

$$\mathbf{Ent}_{\sigma_M}\left(f^2\right) \leqslant \mathcal{L}\, \mathbf{E}_{\sigma_M}\left(|\nabla f|^2\right). \tag{17}$$

As a Corollary, we recover from Proposition 0.1 and (12) the $L^2$ factor for the Kawasaki dynamics (cf. (13) and (14)) obtained by [LPY00].

The rest of the paper is divided as follows. The first section gives the proof of Proposition 0.2, which relies only on the Bakry-Émery criterion. In Section 2, we give some preliminaries to the proof of Theorem 0.3. Lemma 2.1 gives some covariance bounds taken from [LPY00]. This Lemma allows us to derive the "one spin Lemma" 2.2 by a simple application of the Bakry-Émery criterion. Section 3 is devoted to the derivation of the Poincaré inequality (16) and section 4 to the derivation of the logarithmic Sobolev counterpart (16). The proofs make heavy use of the LCLT based estimates of [LPY00] throughout Lemmas 3.1 and 4.1, but our induction in $n$ is quite different.

It is quite natural to ask if Theorem 0.3 remains valid if we replace the quadratic potential $u^2/2$ by a uniformly strictly convex potential $\Phi$. We believe that this is true, at least for $\Phi$ such that $0 < \alpha \leqslant \Phi''(u) \leqslant \beta$. In an other direction, one can ask if our method remains valid for discrete spins systems similar to those presented in [Mar99]. It is not clear at all for us.

# 1 Proof of Proposition 0.2

We give here a proof of Proposition 0.2, which relies only on the Bakry-Émery criterion. Let $\sigma_M^*$ the probability measure on $\mathbb{R}^n$ defined by

$$(Z_{\sigma_M^*})^{-1}\,\exp\left(-\sum_{i=1}^n V(x)-\frac{1}{2}\left(M-\sum_{i=1}^n x_i\right)^2\right)dx_1\cdots dx_n.$$

If $\sigma_M^*$ satisfies a Poincaré inequality with a constant $c>0$, then $\sigma_M$ satisfies a Poincaré inequality with a constant $c\exp(2\operatorname{osc}(F))$. Now, for any smooth function $f:\mathbb{R}^n\to\mathbb{R}$,

$$\mathbf{E}_{\sigma_M^*}\left((\mathbf{L}f)^2\right)=\sum_{i,j=1}^n\mathbf{E}_{\sigma_M^*}\left(\left|\partial_{ij}^2 f\right|^2\right)+\mathbf{E}_{\sigma_M^*}\left(\sum_{i=1}^n(1+F''(x_i))\left|\partial_i f\right|^2\right)+\mathbf{E}_{\sigma_M^*}\left(\left(\sum_{i=1}^n\partial_i f\right)^2\right).$$

In another hand, for any $i\in\{1,\dots,n\}$ and any $x_1,\dots,x_{i-1},x_{i+1},\dots,x_n$, the Bakry-Émery criterion gives that the one dimensional probability measure

$$\rho_i(dx_i):=(Z_{\rho_i})^{-1}\,\exp\left(-V(x_i)-\frac{1}{2}\left(M-\sum_{i=1}^n x_i\right)^2\right)dx_i$$

satisfies a Poincaré inequality with a constant $(1/2)\,\exp(2\operatorname{osc}(F))$, hence, by the Bakry-Émery criterion applied reversely, we get for any smooth function $f:\mathbb{R}^n\to\mathbb{R}$, by summing over $i$

$$\sum_{i=1}^n\mathbf{E}_{\rho_i}\left(\left|\partial_{ii}^2 f\right|^2\right)+\sum_{i=1}^n\mathbf{E}_{\rho_i}\left((2+F''(x_i))\left|\partial_i f\right|^2\right)\geqslant 2\,e^{-2\operatorname{osc}(F)}\sum_{i=1}^n\mathbf{E}_{\rho_i}\left(\left|\partial_i f\right|^2\right).$$

Notice that $\rho_i=\mathrm{Law}_{\sigma_M^*}(x_i\,|\,x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)$. Therefore, by taking the expectation with respect to $\sigma_M^*$, we get

$$\begin{aligned}\mathbf{E}_{\sigma_M^*}\left((\mathbf{L}f)^2\right)&\geqslant\left(2\,e^{-2\operatorname{osc}(F)}-1\right)\sum_{i=1}^n\mathbf{E}_{\sigma_M^*}\left(\left|\partial_i f\right|^2\right)\\&=:\left(2\,e^{-2\operatorname{osc}(F)}-1\right)\mathbf{E}_{\sigma_M^*}\left(\left|\nabla f\right|^2\right).\end{aligned}$$

Thus, for $\operatorname{osc}(F)$ sufficiently small ($<\log\sqrt{2}$), one can take

$$\mathcal{P}=\frac{e^{2\operatorname{osc}(F)}}{2\,e^{-2\operatorname{osc}(F)}-1},$$

which is optimal when $F\equiv 0$ (pure Gaussian case).

## 2 Preliminaries to the proof of Theorem 0.3

Let $\gamma_{n,M}$ the Gaussian measure of mean $M/(n+1)$ and covariance matrix $(\mathbf{I}_n + \mathbf{1}_n)^{-1}$. If $B(x) := \sum_{i=1}^n F(x_i) + F\Bigg(M - x_1 - \cdots - x_n\Bigg)$, on can write

$$d\sigma_M(x_1, \dots, x_n) = (Z_{n,M})^{-1} \exp\left(-B(x)\right) d\gamma_{n,M}(x_1, \dots, x_n).$$

Thus, $\sigma_M$ is a bounded perturbation of $\gamma_{n,M}$, which is log-concave with a constant $\rho$ equal to 1, and therefore, $\sigma_M$ satisfies to Poincaré and logarithmic Sobolev inequalities with constants depending only on $\|B\|_\infty$ (i.e. $\|F\|_\infty$ and $n$). Our goal is to show that the dependence in $n$ can be dropped by taking into account $\|F'\|_\infty$ and $\|F''\|_\infty$. The presence of the bounded part $F$ in $V$ and the non-product nature of $\sigma_M$ does not allow any direct approach based on the Bakry-Émery criterion.

Observe that $\mathbf{Cov}_{\gamma_{n,M}}(x_1, x_2) = -(n+1)^{-1}$, and we can then expect the same decrease in $n$ for $\mathbf{Cov}_{\sigma_M}(V'(x_1), V'(x_2))$. This is actually the case, as stated in the following Lemma. Notice that since $\sigma_M$ is exchangeable and since $M - \sum_{i=1}^n x_i$ and $x_i$ have the same law under $\sigma_M$, we have $\mathbf{Var}_{\sigma_M}(x_1) = -n\mathbf{Cov}_{\sigma_M}(x_1, x_2)$, as for $\gamma_{n,M}$.

**Lemma 2.1.** *Let $\sigma_M$ be the probability measure on $\mathbb{R}^n$ ($n > 2$) defined in Theorem 0.3 and $\mu_M$ the associated conditional measure defined by* (6). *Assume that $F$ and $F'$ are bounded, then there exists a constant $C > 0$ depending only on $\|F\|_\infty$ and $\|F'\|_\infty$ such that for any $M \in \mathbb{R}$*

$$|\mathbf{Cov}_{\sigma_M}(V'(x_1), V'(x_2))| = |\mathbf{Cov}_{\mu_M}(V'(x_1), V'(x_2))| \leqslant \frac{C}{n}, \tag{18}$$

*and for any $\alpha \in \mathbb{R}$*

$$\begin{aligned} \mathbf{Var}_{\sigma_M}\left(\sum_{i=1}^n V'(x_i) + V'(M - \sum_{i=1}^n x_i)\right) &= \mathbf{Var}_{\sigma_M}\left(\sum_{i=1}^n F'(x_i) + F'(M - \sum_{i=1}^n x_i)\right) \\ &= \mathbf{Var}_{\mu_M}\left(\sum_{i=1}^{n+1} F'(x_i)\right) \leqslant nC. \end{aligned} \tag{19}$$

*Proof.* Inequality (19) follows from (7) and [LPY00, Corollary 5.4]. For (18), just write

$$\mathbf{Cov}_{\sigma_M}(V'(x_1), V'(x_2)) = \mathbf{Cov}_{\sigma_M}(x_1, x_2) + 2\,\mathbf{Cov}_{\sigma_M}(x_1, F'(x_2)) + \mathbf{Cov}_{\sigma_M}(F'(x_1), F'(x_2)),$$

and use (7) and [LPY00, Corollary 5.3] to estimate each term. Actually, one can derive the estimates of $\mathbf{Cov}_{\sigma_M}(x_1, F'(x_2))$ and $\mathbf{Cov}_{\sigma_M}(x_1, x_2)$ directly by using the symmetries of $\sigma_M$. □

Inequality (18) of Lemma 2.1 allows us to establish the following one spin result, which is the first step in our proof of Poincaré and logarithmic Sobolev inequalities for $\sigma_M$ by induction on $n$ by mean of the Lu-Yau Markovian decomposition. In the other hand, inequality (19) will be usefull, as we will see in sections 3 and 4, for the induction itself.

**Lemma 2.2 (One spin Lemma).** *Let $\sigma_M$ be the probability measure on $\mathbb{R}^n$ defined in Theorem 0.3. If $F$ is bounded and Lipschitz, there exists a constant $A > 0$ depending only on $\|F\|_\infty$ and $\|F'\|_\infty$ and not on $n$ and $M$ such that for any $n$ and $M$ and any smooth $f : \mathbb{R} \to \mathbb{R}$,*

$$\mathbf{Ent}_{\sigma_M}\left(f(x_1)^2\right) \leqslant 2A\,\mathbf{E}_{\sigma_M}\left(f'(x_1)^2\right),$$

*and*

$$\mathbf{Var}_{\sigma_M}(f(x_1)) \leqslant A\,\mathbf{E}_{\sigma_M}\left(f'(x_1)^2\right).$$

*Proof of Lemma 2.2.* As we already noticed, it is clear that the desired inequalities are true with a constant depending on $n$ and $\|F\|_\infty$, so we just have to see what happens for large values of $n$. We have in mind the use of the Bakry-Émery criterion. The Hamiltonian of the probability measure in $x_1$ is given by

$$\varphi_{M,n}(x_1) := V(x_1) + \log Z_{M,n} - \log \int \exp\left(-\sum_{i=2}^{n} V(x_i) - V(M - \sum_{i=1}^{n} x_i)\right) dx_2 \cdots dx_n.$$

We first observe that we can forget the $F(x_1)$ part in $V(x_1)$, which is payed by a factor $\exp(2\operatorname{osc}(F))$ in $A$. Hence, we simply have, after an integration by parts

$$\varphi''_{M,n}(x_1) = 1 - \mathbf{Cov}_{\sigma_{M-x_1}(dx_2,\dots,dx_n)}(V'(x_2), V'(x_3)).$$

Now, (18) gives $\varphi''_{M,n}(x_1) \geqslant 1 - Cn^{-1}$, where $C$ is a positive constant depending only on $\|F\|_\infty$ and $\|F'\|_\infty$ and not on $n$ and $M$. Thus, we are able to apply the Bakry-Émery criterion for large values of $n$. Hence, the proof is completed, with a constant $A$ depending only on $\|F\|_\infty$ and $\|F'\|_\infty$ and not on $M$ and $n$. $\square$

Obviously, one can replace $x_1$ in $f$ and $f'$ by $M - x_1 - \cdots - x_n$ or by any $x_i$ for $i \in \{1, \dots, n\}$. Moreover, according to (8), on can replace $\mathbf{E}_{\sigma_M}$ by $\mathbf{E}_{\mu_M}$.

# 3 Derivation of the Poincaré inequality

This section is devoted to the derivation of inequality (16) of Theorem 0.3. The proof relies on the one spin Lemma 2.2 and on the crucial Lemma 3.1 which allows us to use the Lu-Yau Markovian decomposition.

*Proof of* (16). As we already noticed, the result is true with a constant depending on $n$, so that if we denote by $\mathcal{P}_n$ the maximum of best Poincaré constants in dimension less than or equal to $n$, we just have to show that the non decreasing sequence of constants $(\mathcal{P}_n)_{n\geqslant 1}$ is bounded.

Let us denote by $\sigma$ the measure $\sigma_M$ and by $\sigma^{(k)}$ the measure $\sigma_M$ given $x_1, \dots, x_k$ for $k \in \{0, \dots, n\}$ and by $f_k$ the conditional expectation

$$\mathbf{E}_\sigma(f|x_1, \dots, x_k) = \mathbf{E}_{\sigma^{(k)}}(f).$$

Notice that $\sigma^{(k)}$ is nothing else but $\sigma_{M-x_1-\cdots-x_k}(dx_{k+1},\ldots,dx_n)$. Moreover, $f_n = f$ and by convention $\sigma^{(0)} := \sigma$ and thus $f_0 = \mathbf{E}_\mu(f)$. For a *fixed* function $f$, we can always choose the order of the coordinates $x_1,\ldots,x_n$ such that $\mathbf{E}_\sigma\big(|\partial_k f|^2\big)$ becomes a *non increasing* sequence in $k\in\{1,\ldots,n\}$. This gives

$$\sum_{i=k+1}^{n}\frac{1}{n-k}\,\mathbf{E}_\sigma\big(|\partial_i f|^2\big)\leqslant \mathbf{E}_\sigma\big(|\partial_{k+1} f|^2\big)\,.$$

Following Lu-Yau [LY93], we have the following Markovian decomposition of the variance

$$\mathbf{Var}_\sigma(f) := \mathbf{E}_\sigma\big(f^2\big)-\mathbf{E}_\sigma(f)^2=\sum_{k=1}^{n}\mathbf{E}_\sigma\big((f_k)^2-(f_{k-1})^2\big)=\sum_{k=1}^{n}\mathbf{E}_\sigma(\mathbf{Var}_{\sigma^{(k-1)}}(f_k))\,.$$

Since measure $\sigma^{(k-1)}$ integrates coordinates $x_k,\ldots,x_n$ and function $f_k$ depends only on coordinates $x_1,\ldots,x_k$, the quantity $\mathbf{Var}_{\sigma^{(k-1)}}(f_k)$ is actually a variance for a one spin function. Therefore, by the one spin Lemma 2.2, there exists a constant $A>0$ depending on $\|F\|_\infty$ and $\|F'\|_\infty$ but not on $n$ and $M$ such that

$$\mathbf{Var}_\sigma(f)\leqslant A\sum_{k=1}^{n}\mathbf{E}_\sigma\big(|\partial_k f_k|^2\big)\,.$$

Our aim is to express the right hand side of the previous inequality in terms of $|\partial_k f|^2$. Notice that the $k=n$ term in the sum is trivial since $f_n = f$. By definition of $f_k$, we get for any $k\in\{1,\ldots,n-1\}$

$$\partial_k f_k = \mathbf{E}_{\sigma^{(k)}}(\partial_k f)-\mathbf{Cov}_{\sigma^{(k)}}\Big(f,V'(M-\sum_{i=1}^{n}x_i)\Big).$$

At this stage, we notice that by $n-k$ integrations by parts, we have

$$\mathbf{Cov}_{\sigma^{(k)}}\Big(f,V'(M-\sum_{i=1}^{n}x_i)\Big)=\frac{1}{n-k}\sum_{i=k+1}^{n}\mathbf{Cov}_{\sigma^{(k)}}(f,V'(x_i))-\frac{1}{n-k}\sum_{i=k+1}^{n}\mathbf{E}_{\sigma^{(k)}}(\partial_i f)\,.$$

Therefore, we can write by denoting $S_k := \sum_{i=k+1}^{n}V'(x_i)+V'(M-\sum_{i=1}^{n}x_i)$

$$\partial_k f_k = \mathbf{E}_{\sigma^{(k)}}(\partial_k f)-\frac{1}{n-k+1}\mathbf{Cov}_{\sigma^{(k)}}(f,S_k)+\frac{1}{n-k+1}\sum_{i=k+1}^{n}\mathbf{E}_{\sigma^{(k)}}(\partial_i f)\,.$$

Now, by the Cauchy-Schwarz inequality

$$|\partial_k f_k|^2\leqslant 3\,\mathbf{E}_{\sigma^{(k)}}\big(|\partial_k f|^2\big)+\frac{3}{(n-k)^2}\mathbf{Cov}_{\sigma^{(k)}}(f,S_k)^2+\frac{3}{n-k}\sum_{i=k+1}^{n}\mathbf{E}_{\sigma^{(k)}}\big(|\partial_i f|^2\big)\,.$$

This gives by summing over all $k$ in $\{1,\dots,n-1\}$ (the case $k=n$ is trivial)

$$\sum_{k=1}^{n-1}\mathbf{E}_\sigma\big(|\partial_k f_k|^2\big)\leqslant 3\,\mathbf{E}_\sigma\big(|\nabla f|^2\big)+3\sum_{k=1}^{n-1}\frac{1}{(n-k)^2}\,\mathbf{E}_\sigma\big(\mathbf{Cov}_{\sigma^{(k)}}(f,S_k)^2\big)$$
$$+\,3\sum_{k=1}^{n-1}\frac{1}{n-k}\sum_{i=k+1}^{n}\mathbf{E}_\sigma\big(|\partial_i f|^2\big)\,.$$

The monotonicity of $\mathbf{E}_\sigma\big(|\partial_i f|^2\big)$ yields

$$\sum_{k=1}^{n-1}\mathbf{E}_\sigma\big(|\partial_k f_k|^2\big)\leqslant\ 6\,\mathbf{E}_\sigma\big(|\nabla f|^2\big)+3\sum_{k=1}^{n-1}\frac{1}{(n-k)^2}\,\mathbf{E}_\sigma\big(\mathbf{Cov}_{\sigma^{(k)}}(f,S_k)^2\big)\,.$$

By inequality (21) of Lemma 3.1, there exists a positive constant $C$ depending only on $\|F\|_\infty$ and $\|F'\|_\infty$ such that for any $\varepsilon>0$, there exists a positive constant $C_\varepsilon$ depending only on $\|F\|_\infty$, $\|F'\|_\infty$ and $\varepsilon$ such that for any $k\in\{1,\dots,n-1\}$

$$\mathbf{Cov}_{\sigma^{(k)}}(f,S_k)^2\leqslant(C_\varepsilon+\varepsilon(n-k)C)\,\mathbf{Var}_{\sigma^{(k)}}(f)+(n-k)C_\varepsilon\sum_{i=k+1}^{n}\mathbf{E}_{\sigma^{(k)}}\big(|\partial_i f|^2\big)\,.$$

Therefore, by the monotonicity of $\mathbf{E}_\sigma\big(|\partial_i f|^2\big)$ again

$$\begin{aligned}\sum_{k=1}^{n-1}\mathbf{E}_\sigma\big(|\partial_k f_k|^2\big)&\leqslant C'_\varepsilon\,\mathbf{E}_\sigma\big(|\nabla f|^2\big)+C'_\varepsilon\sum_{k=1}^{n-1}\frac{1}{(n-k)^2}\,\mathbf{E}_\sigma(\mathbf{Var}_{\sigma^{(k)}}(f))\\&\quad+\varepsilon C'\sum_{k=1}^{n-1}\frac{1}{n-k}\,\mathbf{E}_\sigma(\mathbf{Var}_{\sigma^{(k)}}(f))\,.\end{aligned}\tag{20}$$

Recall that $\mathcal{P}_n$ is the maximum of best Poincaré constants in dimension less than or equal to $n$. The last sum of the right hand side (RHS) of (20) can be bounded above as follows

$$\sum_{k=1}^{n-1}\frac{1}{n-k}\mathbf{E}_\sigma(\mathbf{Var}_{\sigma^{(k)}}(f))\leqslant\mathcal{P}_{n-1}\mathbf{E}_\sigma\big(|\nabla f|^2\big)\,.$$

It remains to examine the first sum of the RHS of (20). The Jensen inequality yields

$$\mathbf{E}_\sigma(\mathbf{Var}_{\sigma^{(k)}}(f))\leqslant\mathbf{Var}_\sigma(f)\,,$$

and therefore, we get for any $p\in\{1,\dots,n-1\}$

$$\begin{aligned}\sum_{k=1}^{n-1}\frac{1}{(n-k)^2}\,\mathbf{E}_\sigma(\mathbf{Var}_{\sigma^{(k)}}(f))&=\mathbf{Var}_\sigma(f)\sum_{k=1}^{n-p-1}\frac{1}{(n-k)^2}+\sum_{k=n-p}^{n-1}\frac{1}{(n-k)^2}\,\mathbf{E}_\sigma(\mathbf{Var}_{\sigma^{(k)}}(f))\\&\leqslant\mathbf{Var}_\sigma(f)\sum_{k=p+1}^{n-1}\frac{1}{k^2}+\sum_{k=1}^{p}\frac{1}{k^2}\,\mathbf{E}_\sigma(\mathbf{Var}_{\sigma^{(n-k)}}(f))\,.\end{aligned}$$

At this stage, we observe that for every $k$ in $\{1,\dots,p\}$,

$$\mathbf{E}_\sigma(\mathbf{Var}_{\sigma^{(n-k)}}(f)) \leqslant \mathcal{P}_p \sum_{i=n-k+1}^{n} \mathbf{E}_\sigma\left(|\partial_i f|^2\right) \leqslant p\mathcal{P}_p\, \mathbf{E}_\sigma\left(|\partial_{n-p+1} f|^2\right).$$

We are now able to collect our estimates of the RHS of (20). Putting all together, we have obtained that

$$\sum_{k=1}^{n-1} \mathbf{E}_\sigma\left(|\partial_k f_k|^2\right) \leqslant (C'_\varepsilon + p\pi^2\mathcal{P}_p C'_\varepsilon + \varepsilon C'\mathcal{P}_{n-1})\, \mathbf{E}_\sigma\left(|\nabla f|^2\right) + (C'_\varepsilon R_p)\mathbf{Var}_\sigma(f)\,,$$

where $R_p := \sum_{k=p+1}^{n-1} k^{-2}$. Therefore, for somme $C''_{p,\varepsilon} > 0$,

$$(1 - AC'_\varepsilon R_p)\, \mathbf{Var}_\sigma(f) \leqslant (C''_{p,\varepsilon} + \varepsilon AC'\mathcal{P}_{n-1})\, \mathbf{E}_\sigma\left(|\nabla f|^2\right).$$

Now, we may choose $\varepsilon < 1/(AC')$ and then $p$ large enough (always possible when $n$ is sufficiently large) to ensure that

$$R_p < \min\left(\frac{1}{AC'_\varepsilon}, \frac{1-\varepsilon AC'}{AC'_\varepsilon}\right).$$

This gives two positive constants $\alpha$ and $\beta$ with $\beta < 1$ depending only on $\|F\|_\infty$ and $\|F'\|_\infty$ such that for large values of $n$, one has $\mathcal{P}_n \leqslant \alpha + \beta\,\mathcal{P}_{n-1}$, and therefore $\sup_n \mathcal{P}_n < +\infty$. □

Let us give now the crucial Lemma which allows us to use the Markovian decomposition of Lu-Yau, by splitting the covariance term into a variance term and a gradient term. The proof makes heavy use of estimates taken from [LPY00].

**Lemma 3.1.** *Let $\sigma_M$ be the probability measure on $\mathbb{R}^n$ defined in Theorem 0.3. Assume that $F$ is bounded and Lipschitz, then there exists a positive constant $C$ depending only on $\|F\|_\infty$ and $\|F'\|_\infty$ such that for any $\varepsilon > 0$, there exists a positive constant $C_\varepsilon$ depending only on $\|F\|_\infty$, $\|F'\|_\infty$ and $\varepsilon$ such that for any $n \in \mathbb{N}^*$, any $M \in \mathbb{R}$ and any smooth function $f : \mathbb{R}^n \to \mathbb{R}$*

$$\mathbf{Cov}_{\sigma_M}(f,S)^2 \leqslant (C_\varepsilon + \varepsilon nC)\, \mathbf{Var}_{\sigma_M}(f) + nC_\varepsilon\, \mathbf{E}_{\sigma_M}\left(|\nabla f|^2\right), \tag{21}$$

*where $S := \sum_{i=1}^n V'(x_i) + V'(M - \sum_{i=1}^n x_i)$.*

*Proof of Lemma 3.1.* Notice that we just have to study what happens for small values of $\varepsilon$ and large values of $n$, since for any $\varepsilon > 0$ and any $n \leqslant n_\varepsilon$, we get by the Cauchy-Schwarz inequality and (19) that

$$\mathbf{Cov}_{\sigma_M}(f,S)^2 \leqslant n_\varepsilon C\, \mathbf{Var}_{\sigma_M}(f) =: C_\varepsilon\, \mathbf{Var}_{\sigma_M}(f)\,.$$

We have in mind the use of the partitioning result of [LPY00]. If $\mu_M$ denotes the conditional measure on $\mathbb{R}^{n+1}$ associated to $\sigma_M$ as in (7), we have

$$\mathbf{Cov}_{\sigma_M}(f, S(x_1,\dots,x_n))^2 = \mathbf{Cov}_{\mu_M}\left(f, \sum_{i=1}^{n+1} F'(x_i)\right)^2.$$

Now, for $n$ large enough, one can then subdivide the set $\{1, \dots, n+1\}$ into $\ell$ adjacent subsets $I_i$ of size $K$ or $K+1$. We have in mind to take $K^{-1} \leqslant \varepsilon$, which is always possible when $n$ is large enough. We can write with this decomposition

$$\mathbf{Cov}_{\mu_M}\Big(f, \sum_{i=1}^{n+1} F'(x_i)\Big) = \mathbf{Cov}_{\mu_M}\Big(f, \sum_{i=1}^{\ell} \sum_{k \in I_i} F'(x_k)\Big).$$

For any $(i, x) \in \{1, \dots, \ell\} \times \mathbb{R}^{n+1}$, we define the "total spin on $I_i$" by $M_i(x) := \sum_{k \in I_i} x_k$. On $\mathbb{R}^{I_i}$, one can define the conditional measure $\mu_{M_i}$ with total spin $M_i$, as in (6). To lighten the notations, we denote this measure by $\mu_{(i)}$. We get from the latter by the Cauchy-Schwarz inequality

$$\begin{aligned} \mathbf{Cov}_{\mu_M}\Big(f, \sum_{i=1}^{n+1} F'(x_i)\Big)^2 &\leqslant 2\,\mathbf{Cov}_{\mu_M}\Big(f, \sum_{i=1}^{\ell} \sum_{k \in I_i} \big(F'(x_k) - \mathbf{E}_{\mu_{(i)}}(F')\big)\Big)^2 \\ &\quad + 2\,\mathbf{Cov}_{\mu_M}\Big(f, \sum_{i=1}^{\ell} |I_i| \mathbf{E}_{\mu_{(i)}}(F')\Big)^2 \end{aligned} \tag{22}$$

By the Cauchy-Schwarz inequality again and by (8), the second term of the RHS of (22) can be bounded above by

$$\mathbf{Var}_{\sigma_M}(f)\,\mathbf{Var}_{\mu_M}\Big(\sum_{i=1}^{\ell} |I_i| \mathbf{E}_{\mu_{(i)}}(F')\Big).$$

Now, according to [LPY00, ineq. (3.10)], the last variance in the RHS is bounded above by $nC/K$ for $n$ sufficiently large, which can be rewritten as $\varepsilon n C$. We turn now to the control of the first term of the RHS of (22). Since $\mathbf{E}_{\mu_M} = \mathbf{E}_{\mu_M} \circ \mathbf{E}_{\mu_{(i)}}$, we get

$$\mathbf{Cov}_{\mu_M}\Big(f, \sum_{i=1}^{\ell} \sum_{k \in I_i} \big(F'(x_k) - \mathbf{E}_{\mu_{(i)}}(F')\big)\Big) = \sum_{i=1}^{\ell} \mathbf{E}_{\mu_M}\Big(\mathbf{Cov}_{\mu_{(i)}}\Big(f, \sum_{k \in I_i} F'(x_k)\Big)\Big).$$

Thus, the Cauchy-Schwarz inequality yields

$$\mathbf{Cov}_{\mu_M}\Big(f, \sum_{i=1}^{\ell} \sum_{k \in I_i} \big(F'(x_k) - \mathbf{E}_{\mu_{(i)}}(F')\big)\Big)^2 \leqslant \ell \sum_{i=1}^{\ell} \mathbf{E}_{\mu_M}\Big(\mathbf{Cov}_{\mu_{(i)}}\Big(f, \sum_{k \in I_i} F'(x_k)\Big)^2\Big).$$

Again by the Cauchy-Schwarz inequality, we get

$$\mathbf{Cov}_{\mu_{(i)}}\Big(f, \sum_{k \in I_i} F'(x_k)\Big)^2 \leqslant \mathbf{Var}_{\mu_{(i)}}(f)\,\mathbf{Var}_{\mu_{(i)}}\Big(\sum_{k \in I_i} F'(x_k)\Big).$$

By virtue of (19) applied to $\mu_{(i)}$, we obtain

$$\mathbf{Cov}_{\mu_{(i)}}\Big(f, \sum_{k \in I_i} F'(x_k)\Big)^2 \leqslant C\,|I_i|\,\mathbf{Var}_{\mu_{(i)}}(f)\,.$$

Now, for any $i$, let $r_i = \max\{k,\, k \in I_i\}$ and $J_i := I_i \backslash \{r_i\}$ and $\sigma_{(i)}$ the probability measure on $\mathbb{R}^{J_i}$ associated with the Hamiltonian

$$\sum_{k \in J_i} V(x_k) + V(M_i - \sum_{k \in J_i} x_k).$$

Equation (8) simply gives

$$\mathbf{Var}_{\mu_{(i)}}(f) = \mathbf{Var}_{\sigma_{(i)}}(f(\varphi_i(x)))\,,$$

where $\varphi_i : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is defined by

$$(\varphi_i(x))_k := \begin{cases} x_k & \text{if } k \neq r_i \\ M_i - \sum_{l \in J_i} x_l & \text{if } k = r_i \end{cases}$$

Recall that $\mathcal{P}_K$ is the maximum of the best Poincaré constants for $\sigma_M$ in dimensions less than or equal to $K$. We get by definition of $\mathcal{P}_K$ that

$$\begin{aligned} \mathbf{Var}_{\sigma_{(i)}}(f) &\leqslant \mathcal{P}_K \sum_{k \in J_i} \mathbf{E}_{\sigma_{(i)}}\left(|(\partial_k f)(\varphi_i) - (\partial_{r_i} f)(\varphi_i)|^2\right) \\ &= \mathcal{P}_K \sum_{k \in J_i} \mathbf{E}_{\mu_{(i)}}\left(|\partial_k f - \partial_{r_i} f|^2\right). \end{aligned}$$

Hence, by the Cauchy-Schwarz inequality, we get

$$\mathbf{Var}_{\sigma_{(i)}}(f) \leqslant 2\mathcal{P}_K\, \mathbf{E}_{\mu_{(i)}}\left(\sum_{k \in J_i} |\partial_k f|^2\right) + 2(|I_i| - 1)\mathcal{P}_K\, \mathbf{E}_{\mu_{(i)}}\left(|\partial_{r_i} f|^2\right),$$

Summarising, since $\mathcal{P}_K$ depends only on $K$, $\|F\|_\infty$, $\|F'\|_\infty$, we have obtained that the first term of the right hand side of (22) is bounded above by

$$nC_K\mathcal{P}_K\, \mathbf{E}_{\mu_M}\left(|\nabla f|^2\right),$$

which can be rewritten by virtue of (8) as $n\, C'_\varepsilon\, \mathbf{E}_{\sigma_M}\left(|\nabla f|^2\right)$. This concludes the proof of (21) and Lemma 3.1. □

# 4 Derivation of the Logarithmic Sobolev inequality

This section is devoted to the derivation of inequality (17) of Theorem 0.3. As for the Poincaré inequality (16), the proof relies on the one spin Lemma 2.2 and on a crucial Lemma 4.1 which allows us to use the Lu-Yau Markovian decomposition.

*Proof of the logarithmic Sobolev inequality* (17) *of Theorem 0.3.* We follow here the same scheme used for the Poincaré inequality. For any smooth non negative function $g : \mathbb{R}^n \to$

$\mathbb{R}^+$, we have the following decomposition of the entropy

$$\begin{aligned}\mathbf{Ent}_\sigma(g) :&= \mathbf{E}_\sigma(g\log g) - \mathbf{E}_\sigma(g)\log\mathbf{E}_\sigma(g)\\ &= \sum_{k=1}^n \mathbf{E}_\sigma(g_k\log g_k - g_{k-1}\log g_{k-1})\\ &= \sum_{k=1}^n \mathbf{E}_\sigma(\mathbf{Ent}_{\sigma^{(k-1)}}(g_k)).\end{aligned}$$

Alike for the variance, measure $\sigma^{(k-1)}$ integrates on $x_k,\dots,x_n$ and function $f_k$ depends only on $x_1,\dots,x_k$, so that $\mathbf{Ent}_{\sigma^{(k-1)}}(g_k)$ is actually an entropy for a one spin function. Therefore, by the one spin Lemma 2.2, there exists a positive constant $A$ depending on $\|F\|_\infty$ and $\|F'\|_\infty$ but not on $n$ and $M$ such that

$$\mathbf{Ent}_\sigma(g) \leqslant 2A\sum_{k=1}^n \mathbf{E}_\sigma\bigg(\frac{|\partial_k g_k|^2}{4g_k}\bigg),$$

By taking $g = f^2$ for a smooth function $f:\mathbb{R}^n\to\mathbb{R}$, we get

$$\mathbf{Ent}_\sigma\big(f^2\big) \leqslant 2A\sum_{k=1}^n \mathbf{E}_\sigma\bigg(\frac{|\partial_k (f^2)_k|^2}{4(f^2)_k}\bigg).$$

By imitating the method used for the Poincaré inequality, we get that

$$\frac{|\partial_k (f^2)_k|^2}{4(f^2)_k} \leqslant 3\,\frac{|\mathbf{E}_{\sigma^{(k)}}(f\partial_k f)|^2}{\mathbf{E}_{\sigma^{(k)}}(f^2)} + \frac{6}{(n-k)^2}\,\frac{\mathbf{Cov}_{\sigma^{(k)}}(f^2,S_k)^2}{\mathbf{E}_{\sigma^{(k)}}(f^2)} + \frac{3}{n-k}\sum_{i=k+1}^n \frac{|\mathbf{E}_{\sigma^{(k)}}(f\partial_i f)|^2}{\mathbf{E}_{\sigma^{(k)}}(f^2)}.$$

The Cauchy-Schwarz inequality yields

$$\frac{|\mathbf{E}_{\sigma^{(k)}}(f\partial_k f)|^2}{\mathbf{E}_{\sigma^{(k)}}(f^2)} \leqslant \mathbf{E}_{\sigma^{(k)}}\big(|\partial_k f|^2\big).$$

Therefore, the Jensen inequality and the monotonicity of $\mathbf{E}_\sigma\big(|\partial_i f|^2\big)$ yield

$$\sum_{k=1}^{n-1}\mathbf{E}_\sigma\left(\frac{|\partial_k (f^2)_k|^2}{4(f^2)_k}\right) \leqslant\ 6\,\mathbf{E}_\sigma\big(|\nabla f|^2\big) + 6\sum_{k=1}^{n-1}\frac{1}{(n-k)^2}\,\mathbf{E}_\sigma\left(\frac{\mathbf{Cov}_{\sigma^{(k)}}(f^2,S_k)^2}{\mathbf{E}_{\sigma^{(k)}}(f^2)}\right).$$

By inequality (23) of Lemma (4.1), there exists a positive constant $C$ depending only on $\|F\|_\infty$, $\|F'\|_\infty$ and $\|F''\|_\infty$ such that for any $\varepsilon>0$, there exists a positive constant $C_\varepsilon$ depending only on $\|F\|_\infty$, $\|F'\|_\infty$, $\|F''\|_\infty$ and $\varepsilon$ such that for any $n$ and $M$

$$\frac{\mathbf{Cov}_{\sigma^{(k)}}(f^2,S_k)^2}{\mathbf{E}_{\sigma^{(k)}}(f^2)} \leqslant (C_\varepsilon + \varepsilon(n-k)C)\,\mathbf{Ent}_{\sigma^{(k)}}\big(f^2\big) + (n-k)C_\varepsilon\sum_{i=k+1}^n \mathbf{E}_{\sigma^{(k)}}\big(|\partial_i f|^2\big).$$

Hence, we are now able to proceed as the same way as for the Poincaré inequality. $\square$

As for the derivation of the Poincaré inequality, we give now the crucial Lemma which allows us to use the Markovian decomposition of Lu-Yau.

**Lemma 4.1.** *Let $\sigma_M$ be the probability measure on $\mathbb{R}^n$ defined in Theorem 0.3. Assume that $F$, $F'$ and $F''$ are bounded, then there exists a positive constant $C$ depending only on $\|F\|_\infty$, $\|F'\|_\infty$ and $\|F''\|_\infty$ such that for any $\varepsilon > 0$, there exists a positive constant $C_\varepsilon$ depending only on $\|F\|_\infty$, $\|F'\|_\infty$, $\|F''\|_\infty$ and $\varepsilon$ such that for any $n \in \mathbb{N}^*$, any $M \in \mathbb{R}$ and any smooth function $f : \mathbb{R}^n \to \mathbb{R}$ such that $\mathbf{E}_{\sigma_M}(f^2) = 1$*

$$\mathbf{Cov}_{\sigma_M}\left(f^2, S\right)^2 \leqslant (C_\varepsilon + \varepsilon n C)\,\mathbf{Ent}_{\sigma_M}\left(f^2\right) + nC_\varepsilon\,\mathbf{E}_{\sigma_M}\left(|\nabla f|^2\right), \tag{23}$$

*where $S(x) := \sum_{i=1}^n V'(x_i) + V'(M - \sum_{i=1}^n x_i)$.*

*Proof of Lemma 4.1.* We follow the same scheme as for (21), by replacing the Cauchy-Schwarz inequality by the entropy inequality. Since $f^2$ is a density with respect to $\sigma_M$, we can write

$$\mathbf{Cov}_{\sigma_M}\left(f^2, S\right) = \mathbf{E}_{\sigma_M}\left(\left(S - \mathbf{E}_{\sigma_M}(S)\right) f^2\right),$$

and hence, we get by the entropy inequality that for any $\beta > 0$

$$\mathbf{Cov}_{\sigma_M}\left(f^2, S\right) \leqslant \beta^{-1} \log \mathbf{E}_{\sigma_M}(\exp\left(\beta(S - \mathbf{E}_{\sigma_M}(S))\right)) + \beta^{-1}\,\mathbf{E}_{\sigma_M}\left(f^2 \log f^2\right).$$

By (7) and [LPY00, Lemma 6.1], the first term of the right hand side is bounded above by $nC\beta$ where $C$ depends only on $\|F\|_\infty$, and $\|F''\|_\infty$. This yields by considering the minimum in $\beta > 0$

$$\mathbf{Cov}_{\sigma_M}\left(f^2, S\right)^2 \leqslant nC\,\mathbf{E}_{\sigma_M}\left(f^2 \log f^2\right).$$

Thus, for any fixed $\varepsilon > 0$, we just have to study what happens for large values of $n$ since $nC \leqslant n_\varepsilon C =: C_\varepsilon$ for $n \leqslant n_\varepsilon$. After rewritting (23) in terms of $\mu_M$, we get by Cauchy-Schwarz's inequality

$$\begin{aligned}\mathbf{Cov}_{\mu_M}\left(f^2, \sum_{i=1}^{n+1} F'(x_i)\right)^2 &\leqslant 2\,\mathbf{Cov}_{\mu_M}\left(f^2, \sum_{i=1}^{\ell}\sum_{k\in I_i}\left(F'(x_k) - \mathbf{E}_{\mu_{(i)}}(F')\right)\right)^2 \\ &\quad + 2\,\mathbf{Cov}_{\mu_M}\left(f^2, \sum_{i=1}^{\ell} |I_i|\mathbf{E}_{\mu_{(i)}}(F')\right)^2.\end{aligned} \tag{24}$$

Let us treat the first term of the right hand side of (24). It can be rewritten as

$$2\sum_{i=1}^{\ell} \mathbf{E}_{\mu_M}\left(\mathbf{E}_{\mu_{(i)}}\left(f^2\right)\mathbf{Cov}_{\mu_{(i)}}\left(f_i^2, \sum_{k\in I_i} F'(x_k)\right)\right),$$

where $f_i^2 := f^2/\mathbf{E}_{\mu_{(i)}}(f^2)$. Thus, by the Cauchy-Schwarz inequality, the first term of the RHS of (24) is bounded above by

$$2\,\ell\sum_{i=1}^{\ell} \mathbf{E}_{\mu_M}\left(\mathbf{E}_{\mu_{(i)}}\left(f^2\right)\mathbf{Cov}_{\mu_{(i)}}\left(f_i^2, \sum_{k\in I_i} F'(x_k)\right)^2\right),$$

where we used the Jensen inequality with respect to the density $\mathbf{E}_{\mu_{(i)}}(f^2)$. Now, by the entropy inequality and by [LPY00, Lemma 6.1]

$$\mathbf{E}_{\mu_{(i)}}\left(f^2\right)\mathbf{Cov}_{\mu_{(i)}}\left(f_i^2, \sum_{k\in I_i} F'(x_k)\right)^2 \leqslant C\,|I_i|\,\mathbf{Ent}_{\mu_{(i)}}\left(f^2\right).$$

At this stage, the argument used for the Poincaré inequality can be rewritten exactly in the same way, by replacing the variance by the entropy and $\mathcal{P}_K$ by $\mathcal{L}_K$. It gives finally that the first term of the RHS of (24) is bounded above by

$$nC_K\mathcal{L}_K\,\mathbf{E}_{\mu_M}\left(|\nabla f|^2\right).$$

The latter can be rewritten by virtue of (8) as $n\,C'_\varepsilon\,\mathbf{E}_{\sigma_M}\left(|\nabla f|^2\right)$. It remains to bound the last term of the RHS of (24). Let $\beta_0$ as in [LPY00, Lemma 6.5] and $\delta \in (0,2)$. By a simple rewriting of [LPY00, Lemma 4.5], one gets that if $\mathbf{Ent}_{\mu_M}(f^2) \leqslant \delta(n+1)\beta_0^2$ with $n$ and $K$ large enough

$$\mathbf{Cov}_{\mu_M}\left(f^2, \sum_{i=1}^{\ell} |I_i|\mathbf{E}_{\mu_{(i)}}(F')\right)^2 \leqslant \delta nC\,\mathbf{Ent}_{\mu_M}\left(f^2\right).$$

In another hand, if $\mathbf{Ent}_{\mu_M}(f^2) \geqslant \delta(n+1)\beta_0^2$, one gets

$$\mathbf{Cov}_{\mu_M}\left(f^2, \sum_{i=1}^{\ell} |I_i|\mathbf{E}_{\mu_{(i)}}(F')\right)^2 \leqslant \delta nC_K\,\mathbf{Ent}_{\mu_M}\left(f^2\right) + C_{K,\delta}\,n\,\mathbf{E}_{\mu_M}\left(|\nabla f|^2\right).$$

This last estimate is based on a simple rewriting of [LPY00, Lemma 4.5] together with the following straightforward but essential version of [LPY00, Lemma 4.6] :

$$\mathbf{E}_{\nu_{I_i\cup I_j,M}}\left((m_i-m_j)^2 f^2\right) \leqslant C_1(K)\,\mathbf{E}_{\nu_{I_i\cup I_j,M}}\left(f^2\right) + C_2(K)\,\mathcal{L}_{2K}\,\mathbf{E}_{\nu_{I_i\cup I_j,M}}\left(\sum_{k\in I_i\cup I_j} |\partial_k f|^2\right),$$

where $\nu_{I_i\cup I_j,M}$ is the conditional measure on $I_i\cup I_j$, $m_i = |I_i|^{-1}\sum_{k\in I_i}$, and $C_1(K)\to 0$ when $K\to+\infty$.

Summarising, we get that for any $\delta\in(0,2)$ and for $n$ and $K$ large enough, the last term of the RHS of (24) is bounded above as follows

$$\mathbf{Cov}_{\mu_M}\left(f^2, \sum_{i=1}^{\ell} |I_i|\mathbf{E}_{\mu_{(i)}}(F')\right)^2 \leqslant n\delta C_K\,\mathbf{Ent}_{\mu_M}\left(f^2\right) + nC_{K,\delta}\mathcal{L}_{2K}\,\mathbf{E}_{\mu_M}\left(|\nabla f|^2\right),$$

which can be rewritten by virtue of (8) as $\varepsilon nC'\,\mathbf{Ent}_{\sigma_M}(f^2) + nC''_\varepsilon\,\mathbf{E}_{\sigma_M}\left(|\nabla f|^2\right)$. This achieves the proof of (23) and Lemma 4.1. $\square$

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