The Periodic Multidimensional Schrödinger Operator, Part 3, On the Constructively Determination the Spectral Invariants

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Abstract

In this paper, we constructively determine the spectral invariants of the Multidimensional Schrödinger Operator with periodic potential by the Floquet spectrum.

1 Introduction

We investigate the Schrödinger Operator

$$L(q) = -\Delta + q(x),\ x \in R^d,\ d \geq 2 \tag{1}$$

with a real periodic (relative to the Lattice Ω) potential $q(x) \in W_2^l(F)$, where $l \geq 6(3^d(d+1)^2) + d$, and F is the fundamental domain R^d/Ω of Ω. The spectrum of $L(q)$ is the union of the spectra of the operators $L_t(q)$, for $t \in F^* \equiv R^d/\Gamma$ generated by (1) and the conditions (see [1])

$$u(x+\omega) = e^{i\langle t,\omega\rangle}u(x),\ \forall\omega \in \Omega,$$

where $\Gamma \equiv \{\delta \in R^d : \langle \delta, \omega\rangle \in 2\pi\mathbb{Z}, \forall\omega \in \Omega\}$ is the lattice dual to Ω and $\langle .,. \rangle$ is the inner product in R^d. The eigenvalues $\Lambda_1(t) \leq \Lambda_2(t) \leq$ ...of L_t define functions $\Lambda_n(t)$ $(n = 1, 2, ..)$ of t that are called the band functions or Floquet spectrum of L.

In this paper by the Floquet spectrum of L we constructively determine the following spectral invariants

$$\mu_n^\delta(v), \ J=\int_F \mid q_{\delta,b}(x)\varphi_{n,v}^\delta(\delta,x) \mid^2 dx, \tag{2}$$

for rich value of v, where δ is the maximal element of Γ, that is, δ is the element of Γ of minimal norm belonging to the line δR. The $\mu_n^\delta(v)=\mu_n(v)$ is the eigenvalue satisfying

$$\mid \mu_n(v) - \mid (n+v)\delta \mid^2 \mid \leq M \equiv \sup \mid q(x) \mid, \forall n \in Z, \tag{3}$$

and $\varphi_{n,v}^\delta(s)=\varphi_{n,v}(s)$ is the corresponding normalized eigenfunction of the operator $T_v(Q(s))$ generated by the boundary value problem

$$- \mid \delta \mid^2 y''(s)+Q(s)y(s)=\mu y(s), \ y(s+2\pi)=e^{i2\pi v}y(s),$$

where $s=\langle\delta,x\rangle$, $q^\delta(x)=\sum_{n\in Z} q_{n\delta}e^{in\langle\delta,x\rangle}=Q(s)$ is directional (one dimensional) potential, $\mid \delta \mid$ norm of the vector δ and $q_\gamma \equiv (q(x),e^{i\langle\gamma,x\rangle})$ is the Fourier coefficient of $q(x)$. We assume that $q_0=0$. The function $q_{\delta,b}(x)$ is defined as

$$q_{\delta,b}(x)=\sum_{\gamma\in S(\delta,b)\backslash\delta R} \frac{\gamma}{\langle b,\gamma\rangle} q_\gamma e^{i\langle\gamma,x\rangle}, \tag{4}$$

where $S(\delta,b)=P(\delta,b)\cap\Gamma$, and $P(\delta,b)$ is the plane containing δ, b and 0. Here $b\in\Gamma_\delta$, where Γ_δ is the lattice dual to Ω_δ and Ω_δ is the sublattice $\{d\in\Omega:\langle d,\delta\rangle=0\}$ of Ω in the hyperplane $H_\delta=\{x\in R^d:\langle x,\delta\rangle=0\}$.

In the pioneerian papers [2], [3] about isospectral periodic potential it was proved that if $q(x)\in C^6(F)$, $\omega\in\Omega\backslash 0$ and δ is the maximal element of Γ satisfying $\langle\delta,\omega\rangle=0$ then given Floquet spectrum one may recover $\lambda_0,\lambda_1^-,\lambda_1^+,\lambda_2^-,\lambda_2^+,...$ and

$\int_F \mid Q_\omega(x)\varphi_n^\pm(s) \mid^2 dx$, if $\lambda_n^- < \lambda_n^+$,

or

$\int_F \mid Q_\omega(x) \mid^2 ((\varphi_n^+(s))^2+(\varphi_n^-(s))^2)dx$, if $\lambda_n^-=\lambda_n^+$ where

$$Q_\omega(x)=\sum_{\gamma:\gamma\in\Gamma,\langle\gamma,\omega\rangle\neq 0} \frac{\gamma}{\langle\omega,\gamma\rangle} q_\gamma e^{i\langle\gamma,x\rangle},$$

$\lambda_0 \leq \lambda_2^- \leq \lambda_2^+ ...$ and $\lambda_1^- \leq \lambda_1^+ \leq \lambda_3^- \leq \lambda_3^+ ...$ are the eigenvalues of $T_0(Q(s))$ and $T_{\frac{1}{2}}(Q(s))$ respectively, and $\varphi_n^{\pm}(s)$ is the corresponding eigenfunction.

In section 2 of this paper using the asymptotic formulas for $\Lambda_n(t)$ and Bloch function obtained in Part 1 (that is, in [4]) and in Part 2 (that is, in [5]) we obtain a suitable (for this paper) asymptotic formulas .

In section 3 by using this formulas we constructively determine the spectral invariants (2). Then using the asymptotic decomposition

$$\left|\varphi_{n,v}^{\delta}(s)\right|^2 = A_0 + \frac{A_1(s)}{n} + \frac{A_2(s)}{n^2} + ..., \tag{5}$$

where $A_k(x)$ is expressed via the potential $Q(s)$, we find the invariants

$$J_k = \int |q_{\delta,b}(x)|^2 A_k(x) dx, k = 0, 1, 2, ... \tag{6}$$

Using the asymptotic formulas for solutions of the Sturm-Liouville equation (see Eastham [1], page 63) by direct calculations one can find $A_0(x)$, $A_1(x)$, $A_2(x)$ and the invariant

$$\int |q_{\delta,b}(x)|^2 q^{\delta}(x) dx, \tag{7}$$

The invariant

$$\int \left|q^{\delta}(x)\right|^2 dx, \tag{8}$$

can be obtained from the asymptotic formula (see Eastham [1], page 58) for eigenvalues $\mu_n^{\delta}(v)$. If

$$q^{\delta}(x) = q_{\delta} e^{i\langle \delta, x\rangle} + q_{-\delta} e^{-i\langle \delta, x\rangle} \tag{9}$$

then by direct calculations, we show that (see Appendix 4)

$$\begin{aligned} A_0(x) &= 1, A_1(x) = 0, A_2 = \frac{q^{\delta}(x)}{2} + c_1 \left|q_{\delta}\right|^2, A_3 = c_2 q^{\delta}(x) + c_3 \left|q_{\delta}\right|^2 \quad (10) \\ A_4(x) &= c_4 q^{\delta}(x) + c_5 (q_{\delta}^2 e^{i2\langle \delta, x\rangle} + q_{-\delta}^2 e^{-i2\langle \delta, x\rangle}) + c_6, \end{aligned}$$

where $|q_{\delta}|$ is known by (8) and $c_1, c_2, ..., c_6$ are the known constants. Using (10), (8), and (6) for $k = 2, 4$ we find the invariant (7) in case (9) and

$$\int |q_{\delta,b}(x)|^2 (q_{\delta}^2 e^{i2\langle \delta, x\rangle} + q_{-\delta}^2 e^{-i2\langle \delta, x\rangle}) dx. \tag{11}$$

In next paper (Part 4) we give an algorithm for finding the potential $q(x)$ by the invariants (8), (7) and (11).

2 On the Asymptotic formulas

We use the following statements of [4] and [5].

Stat.1. If $\gamma \in \Gamma$, $t \in F^*$, and δ is the maximal element of Γ then

$$\begin{aligned} \gamma &= \beta + (p - (2\pi)^{-1}\langle \beta, \delta^* \rangle)\delta, \ t = a + \tau + \mid \delta \mid^{-2} \langle t, \delta \rangle \delta, && (12) \\ \gamma + t &= \beta + \tau + (j + v)\delta, && (13) \end{aligned}$$

where $a \in \Gamma_\delta$, $\beta \in \Gamma_\delta$, $\tau \in F_\delta \equiv H_\delta/\Gamma_\delta$, $j, p \in Z$, $v \equiv v(\beta) \equiv v(\beta, t)$ is the fractional part of $\mid \delta \mid^{-2} \langle t, \delta \rangle - (2\pi)^{-1}\langle \beta - a, \delta^* \rangle$, and δ^* is the element of Ω satisfying $\langle \delta^*, \delta \rangle = 2\pi$, (see (40), (44) and Lemma 3(a) of [4]).

Stat.2. The eigenvalues and eigenfunctions of the Schrodinger operator $L_t(q^\delta(x))$ with directional potential $q^\delta(x)$ are (see Lemma 3(b) of [4])

$\lambda_{j,\beta}(v, \tau) = \mu_j(v) + \mid \beta + \tau \mid^2$, $\Phi_{j,\beta}(x) = e^{i\langle \beta + \tau, x \rangle}\varphi_{j,v(\beta,t)}(s)$,

for $j \in Z$, $\beta \in \Gamma_\delta$, $v = v(\beta, t)$, where s, $\mu_j(v)$ and $\varphi_{j,v}(s)$ are defined in introduction. It is proved that for every eigenvalue $\lambda_{j,\beta}(v, \tau) \sim \rho^2$ of the operator $L_t(q^\delta(x))$ such that $\mid j \mid < c_7 \rho^{\alpha_1}$, $v(\beta) \in A(j, \rho)$ there exists an eigenvalue Λ_N of $L_t(q(x))$ satisfying the formulas

$$\Lambda_N = \lambda_{j,\beta} + E_{k-1}(\lambda_{j,\beta}) + O(\rho^{-k\alpha_2}(\ln \rho)^{2k}), \tag{14}$$

where $k = 1, 2, ..., [\frac{1}{9}(p - \frac{1}{2}q(d-1))]$, $p = l - d$, $E_0 = 0$, $\alpha_k = 3^k\alpha$, $\alpha = \frac{1}{q}$,

$q = 4(3^d(d+1)) + 1$ and $A(j, \rho)$ is the set of v satisfying

$$\mid \mu_j(v) - \mu_{j'}(v) \mid > 2(\ln \rho)^{-1}, \ \forall j' \neq j, \tag{15}$$

(see Theorem 10 and Lemma 4(b) of [4]). Farther we denote by c_k the positive constants whose exact values are inessential.

Stat 3. In [5] we constructed the set B_δ (called simple set in resonance domain V_δ) and proved that if $\gamma + t = \beta + \tau + (j + v(\beta))\delta \in B_\delta$, $\mid \gamma \mid \sim \rho$ then there exists a unique eigenvalue Λ_N (this eigenvalue is denoted by $\Lambda_N(\lambda_{j,\beta}(v,\tau))$ and the corresponding index N is denoted by $N(j,\beta)$) satisfying (14). This is a simple eigenvalue and the corresponding eigenfunction $\Psi_N(x) \equiv \Psi_{N(j,\beta)}(x)$ satisfies the asymptotic formula

$$\Psi_N = \Phi_{j,\beta}(x) + \Phi_{k-1}(x) + O(\rho^{-k\alpha_2}(\ln \rho)^{2k}) \tag{16}$$

for $k = 1, 2, ..., n_1 \equiv [\frac{1}{9}(c - \frac{1}{2}q(d-1))]$, where $c = p - dq - 3^d d$, $\Phi_0(x) = 0$, $\Phi_{k-1}(x)$ is linear conbination of $\Phi_{j,\beta}(x)$ and $\Phi_{j',\beta'}(x)$ for $(j', \beta') \in S(k-1)$ with explicitly expressed by $q(x), \lambda_{j',\beta'}, \Phi_{j',\beta'}(x)$ coefficients. Here

$S(n) = \{(j', \beta') : (j', \beta') \neq (j, \beta), \mid \beta - \beta' \mid \leq n\rho^{\alpha}, \mid j'\delta \mid < 10^n h\}$, $h = 10^{-p}\rho^{\frac{1}{2}\alpha_2}$ (see Theorem 4 of [5]). Then we proved that

$$B_\delta \cap (R(2\rho) \backslash R(\rho)) \supset \cup_{j \in S_1, \beta \in S_2} P'(\beta, j), \tag{17}$$

where $S_1(\rho) = \{j \in Z : \mid j \mid < \frac{\rho^{\alpha_1}}{2|\delta|^2} - \frac{3}{2}\}$,

$S_2 = \{\beta \in \Gamma_\delta : \beta \in (R_\delta(2\rho - d_\delta - 1) \backslash R_\delta(\rho + d_\delta + 1)) \backslash (\bigcup_{b \in \Gamma_\delta(\rho^{\alpha_d})} V_b^\delta(\rho^{\frac{1}{2}}))\}$,

$R_\delta(\rho) = \{x \in H_\delta : \mid x \mid < \rho\}$, $\Gamma_\delta(\rho^{\alpha_d}) = \{b \in \Gamma_\delta : 0 < \mid b \mid < \rho^{\alpha_d}\}$,

$V_b^\delta(\rho^{\frac{1}{2}}) = \{x \in H_\delta : \mid\mid x + b \mid^2 - \mid x \mid^2 \mid < \rho^{\frac{1}{2}}\}$, $R(\rho) = \{x \in R^d : \mid x \mid < \rho\}$, $d_\delta \equiv \sup_{\tau, \tau' \in F_\delta} \mid \tau - \tau' \mid$, and

$P'(\beta, j) = \{\beta + \tau + (j + v)\delta : v \in S_3(\beta, j, \rho), \tau \in S_4(\beta, j, v, \rho)\}$,

where $S_3(\beta, j, \rho) = A(j, \rho) \backslash A'(\beta, \rho)$

$$\mu(A'(\beta, \rho)) < c_8 \rho^{-\alpha}, \tag{18}$$

$$\mu(S_4(\beta, j, v, \rho)) = (1 + O(\rho^{-\alpha}))\mu(F_\delta). \tag{19}$$

(see (106), (108), (109) of [5]). Hence if

$$j \in S_1, \beta \in S_2, v \in S_3(\beta, j, \rho), \tau \in S_4(\beta, j, v, \rho), \tag{20}$$

then $\beta + \tau + (j + v)\delta \in B_\delta$ and there exists unique simple eigenvalue $\Lambda_N(\lambda_{j,\beta}(v, \tau))$ satisfying (14). The corresponding eigenfunction satisfies (16).

Note that, the set $A(j, \rho)$ depends on the potential $q(x)$. If the conditions

$$\mid j \mid > \frac{10M}{\mid \delta \mid^2} + 2, v \in (\frac{1}{10}, \frac{4}{10}) \cup (\frac{6}{10}, \frac{9}{10}), \tag{21}$$

where $M = \sup \mid q(x) \mid$, hold then it is not hard to check that

$$\mid\mid (j + v)\delta \mid^2 - \mid (j' + v) \mid^2 \mid > 4M \tag{22}$$

for all $j' \neq j$ and by (3)

$$\mid \quad \mid (j + v)\delta \mid^2 - \mu_{j'}(v) \mid > 3M, \ \forall j' \neq j \tag{23}$$

$$\mid \quad \mu_j(v) - \mu_{j'}(v) \mid > 2M, \ \forall j' \neq j. \tag{24}$$

Therefore $(\frac{1}{10}, \frac{4}{10}) \cup (\frac{6}{10}, \frac{9}{10}) \subset A(j, \rho)$ and

$$((\frac{1}{10}, \frac{4}{10}) \cup (\frac{6}{10}, \frac{9}{10})) \backslash A'(\beta, \rho) \subset S_3(\beta, j, \rho) \tag{25}$$

for $\rho > e^M, \mid j \mid > \frac{10M}{|\delta|^2} + 2.$

Moreover we use the following relations

$$(\Lambda_N - \lambda_{j',\beta'})b(N, j', \beta') = (\Psi_N(x), (q(x) - Q(s))\Phi_{j',\beta'}(x)), \tag{26}$$

where $b(N, j', \beta') = (\Psi_N, \Phi_{j',\beta'})$. If $\mid j'\delta \mid < r$, where $r \geq r_1$, $r_1 = \frac{\rho^{\alpha_1}}{|2\delta|} + \mid 2\delta \mid$, and $\mid \beta' \mid \sim \rho$ then

$$\begin{aligned} & (q(x) - Q(s))\Phi_{j',\beta'}(x) \\ = & \sum_{(j_1,\beta_1)\in Q(\rho^\alpha, 9r)} \overline{A(j', \beta', j' + j_1, \beta' + \beta_1)}\Phi_{j'+j_1,\beta'+\beta_1}(x) + O(\rho^{-p\alpha}), \end{aligned} \tag{27}$$

$$(\Lambda_N - \lambda_{j',\beta'})b(N, j', \beta') = \tag{28}$$

$$\sum_{(j_1,\beta_1)\in Q(\rho^\alpha, 9r)} A(j', \beta', j' + j_1, \beta' + \beta_1)b(N, j' + j_1, \beta' + \beta_1) + O(\rho^{-p\alpha}),$$

where $Q(\rho^\alpha, 9r) = \{(j, \beta) : \mid j\delta \mid < 9r,\ 0 < \mid \beta \mid < \rho^\alpha\}$,

$$\sum_{(j_1,\beta_1)\in Q(\rho^\alpha, 9r)} \mid A(j', \beta', j' + j_1, \beta' + \beta_1) \mid < c_9. \tag{29}$$

Here c_9 does not depend on (j', β') (see (68) and Lemma 6 of [4]).

In this section using these statements we obtain the asymptotic formulas for eigenvalues $\Lambda_N(\lambda_{j,\beta}(v, \tau))$, for Bloch function Ψ_N corresponding to $\Lambda_N(\lambda_{j,\beta}(v, \tau))$ and for the derivatives of $\Lambda_N(\lambda_{j,\beta}(v, \tau))$, when (20), (21), and the following condition holds

$$\beta \in S_2(\rho) \backslash (\bigcup_{b\in\Gamma_\delta(p\rho^\alpha)} V_b^\delta(\rho^a))), \tag{30}$$

where $a = 1 - \alpha_d + \alpha$.

Lemma 1. *a) If the condition (30) holds and* $0 < \mid \beta' - \beta \mid < p\rho^\alpha$, $\mid j'\delta \mid < \rho^{\frac{1}{2}(a-\alpha)}$, $\tau \in F_\delta$ *then*

$$\mid\mid \beta' + \tau \mid^2 - \mid \beta + \tau \mid^2 \mid > \frac{1}{2}\rho^a, \tag{31}$$

and for every eigenvalue Λ_N *satisfying*

$$\mid \Lambda_N - \mid \beta+\tau \mid^2 + \mid (j+v)\delta \mid^2 \mid < 2M, \tag{32}$$

where $j \in S_1$ *and* j, v *satisfy (21), the followings hold*

$$\mid \quad \Lambda_N - \lambda_{j',\beta'} \mid > \frac{1}{3}\rho^a, \tag{33}$$

$$\mid \quad \Lambda_N - \lambda_{j',\beta} \mid > M, \ \forall j' \neq j. \tag{34}$$

If besides (21), (30) the condition (20) holds then the eigenvalue $\Lambda_N(\lambda_{j,\beta}(v,\tau))$ *(see (14)) satisfies (32) and hence (33), (34). Moreover* $\Lambda_N(\lambda_{j,\beta}(v,\tau))$ *and the corresponding eigenfunction* Ψ_N *(see (16)) satisfy the formulas*

$$\begin{aligned} b(N,j',\beta') &= O(\rho^{-a}), && (35)\\ \Lambda_N(\lambda_{j,\beta}(v,\tau)) &= \lambda_{j,\beta}(v,\tau)+O(\rho^{-a}), && (36)\\ b(N,j',\beta) &= O(\rho^{-2a}r^2), \forall j' \neq j, \mid j'\delta \mid < r, \ r_1 \leq r = O(\rho^{\frac{1}{2}\alpha_2}). && (37) \end{aligned}$$

Proof. $a)$ The inequality (31) follows from conditions $\beta \notin \bigcup_{b\in\Gamma_\delta(p\rho^\alpha)} V_b^\delta(\rho^a)$ (see (30)) and $\mid \beta'-\beta \mid < p\rho^\alpha$. Replasing Λ_N by $\mid \beta+\tau \mid^2 + \mid (j+v)\delta \mid^2 + O(1)$ (see (32)) and $\lambda_{j',\beta'}$ by $\mid \beta'+\tau \mid^2 + \mu_{j'}(v)$, using (31) and the relations $\mid (j+v)\delta \mid^2 = O(\rho^{2\alpha_1}) = O(\rho^{a-\alpha})$, $\mu_{j'}(v) = O(\rho^{a-\alpha})$ for $j \in S_1$, $\mid j'\delta \mid < \rho^{\frac{1}{2}(a-\alpha)}$ (see (3)) we get the proof of (33). If (21) holds then (23) with (32) give (34).

It follows from (14) and (3) that $\Lambda_N = \Lambda_N(\lambda_{j,\beta}(v,\tau))$ satisfies (32) and hence (33), (34). The equality (35) follows from (33) and (26).

To prove (36) we use the formula

$$(\Lambda_N - \lambda_{j,\beta})b(N,j,\beta) = O(\rho^{-p\alpha}) \tag{38}$$

$$+ \sum_{(j_1,\beta_1)\in Q(\rho^\alpha,9r_1)} A(j,\beta,j+j_1,\beta+\beta_1)b(N,j+j_1,\beta+\beta_1),$$

obtained from (28) by replacing (j',β') by (j,β), and r by r_1 (since $\mid j\delta \mid < r_1$ for $j \in S_1$), where $\Lambda_N = \Lambda_N(\lambda_{j,\beta}(v,\tau))$. Since

$0 < \mid \beta_1 \mid < \rho^\alpha$, $\mid (j+j_1)\delta \mid < 10r_1 < \rho^{\frac{1}{2}(a-\alpha)}$ (see definition of $Q(\rho^\alpha, 9r_1)$) by (35) we have $b(N,j+j_1,\beta+\beta_1) = O(\rho^{-a})$. This together with (29), (38) and $b(N,j,\beta) = 1+O(\rho^{-\alpha})$ (see (16) for $k=1$) give (36). To prove (37) replacing β' by β in (28) and then iterating it by using (33) and isolating the terms with multiplicand $b(N,j,\beta)$ we get

$$(\Lambda_N - \lambda_{j',\beta})b(N,j',\beta) = O(\rho^{-p\alpha}) + \tag{39}$$

$$\sum_{(j_1,\beta_1),(j_2,\beta_2)} \frac{A(j',\beta,j'+j_1,\beta+\beta_1)A(j'+j_1,\beta+\beta_1,j^2,\beta^2)b(N,j^2,\beta^2)}{\Lambda_N - \lambda_{j'+j_1,\beta+\beta_1}}$$
$$= S_1(j',\Lambda_N)b(N,j,\beta) + S_2 + O(\rho^{-p\alpha})$$

where $j^2 = j' + j_1 + j_2$, $\beta^2 = \beta + \beta_1 + \beta_2$, $j' \neq j$, $\mid j'\delta \mid < r = O(\rho^{\frac{1}{2}\alpha_2})$ $(j_1,\beta_1) \in Q(\rho^\alpha, 9r)$, $(j_2,\beta_2) \in Q(\rho^\alpha, 90r)$,

$$S_1(j',\Lambda_N) = \sum_{(j_1,\beta_1)} \frac{A(j',\beta,j'+j_1,\beta+\beta_1)A(j'+j_1,\beta+\beta_1,j,\beta)}{\Lambda_N - \lambda_{j'+j_1,\beta+\beta_1}}.$$

and $\mid \Lambda_N - \lambda_{j',\beta} \mid > M$ (see 34), $\mid \Lambda_N - \lambda_{j'+j_1,\beta+\beta_1} \mid > \frac{1}{3}\rho^a$ (see (33)) . Using these inequalities and (29) from (39) we include that

$$b(N,j',\beta) = O(\rho^{-a}), \forall j' \neq j, \mid j'\delta \mid = O(\rho^{\frac{1}{2}\alpha_2}). \tag{40}$$

By (29), (33), and (36) we have

$$S_1(j',\Lambda_N) = S_1(j',\lambda_{j,\beta}) + O(\rho^{-3a}). \tag{41}$$

In Appendix 1 we prove that

$$S_1(j',\lambda_{j,\beta}) \equiv \sum_{(j_1,\beta_1)} \frac{A(j',\beta,j'+j_1,\beta+\beta_1)A(j'+j_1,\beta+\beta_1,j,\beta)}{\lambda_{j,\beta} - \lambda_{j'+j_1,\beta+\beta_1}}$$
$$= O(\rho^{-2a}r^2), \forall \mid j'\delta \mid < r, (j_1,\beta_1) \in Q(\rho^\alpha, 9r), j \in S_1. \tag{42}$$

Hence the first sum $S_1(j',\Lambda_N)b(N,j,\beta)$ in the right side of (39) is $O(\rho^{-2a}r^2)$. Since S_2 consist of the terms with multiplicand $b(N,j^2,\beta^2)$ for $(j^2,\beta^2) \neq (j,\beta)$ which is $O(\rho^{-a})$ (see (35), (40)) it follows from (33), (29) that $S_2 = O(\rho^{-2a})$. Thus the right side of (39) is $O(\rho^{-2a}r^2)$. Therefore (34) implies (37) $\diamondsuit$

Lemma 2. *Let $\Lambda_N(\beta+\tau+(j+v)\delta)$, be a simple eigenvalue of L_t satisfying (32), where j,β, satisfy (20) and $\beta+\tau+(j+v)\delta-t\in\Gamma$. Then*

$$|\beta+\tau|\frac{\partial\Lambda_N(t)}{\partial h}=\sum_{j^{'}\in Z,\beta^{'}\in\Gamma_\delta}\langle\beta+\tau,\beta^{'}+\tau\rangle\mid b(N,j^{'},\beta^{'})\mid^2,$$

where $\frac{\partial\Lambda_N(t)}{\partial h}$ is the derivative of $\Lambda_N(t)$ by the direction $h=\frac{\beta+\tau}{|\beta+\tau|}$.

Proof. We calculate the derivative of $\Lambda_N(t)$ by using the formula

$$\frac{\partial\Lambda_N(t)}{\partial t_j}=2t_j-2i(\frac{\partial}{\partial x_j}\Phi_{N,t}(x),\Phi_{N,t}(x)),$$

where $\Phi_{N,t}(x)=e^{-i\langle t,x\rangle}\Psi_{N,t}(x)$, $t=(t_1,t_2,...,t_d)$. Then

$$\frac{\partial\Lambda_N(t)}{\partial h}=\sum_{j=1}^{d}h_j\frac{\partial\Lambda_N(t)}{\partial t_j}=2\langle h,t\rangle-2i\sum_{j=1}^{d}h_j(\frac{\partial}{\partial x_j}\Phi_{N,t}(x),\Phi_{N,t}(x)),\quad(43)$$

Using the decomposition of $\Psi_{N,t}(x)$ over $\{\Psi_{j,\beta}(x):j\in Z,\beta\in\Gamma_\delta\}$, we get

$$\Phi_{N,t}(x)=\sum_{j^{'}\in Z,\beta^{'}\in\Gamma_\delta}b(N,j^{'},\beta^{'})e^{i\langle\beta^{'}+\tau-t,x\rangle}\varphi_{j^{'}}(\langle\delta,x\rangle).\quad(44)$$

If $|\beta^{'}+\tau|\geq 4\rho$ then $|\beta^{'}+\tau-\beta_1-\beta_2-...-\beta_k|^2>\frac{3}{4}|\beta^{'}|^2$, for $|\beta_i|<\rho^\alpha$, and $i,k=0,1,...,p$. This, (32), and $j\in S_1$, $\beta\in S_2$ (see (20)) give

$$|\Lambda_N|<5\rho^2,\ \lambda_{j^{'},\beta^{'}}-\Lambda_N>\frac{1}{5}(|\beta^{'}|^2+|j^{'}\delta|^2),\quad(45)$$

$\lambda_{j^k,\beta^k}-\Lambda_N>\frac{1}{5}|\beta^{'}|^2$, where $j^k=j^{'}+j_1+j_2+...+j_k$, $\beta^k=\beta^{'}+\beta_1+\beta_2+...+\beta_k$, $|j^{'}\delta|<r$, $(j_k,\beta_k)\in Q(\rho^\alpha,10^{k-1}9r)$. Using this and iterating (28) $d+4$ times and then using (29) we get

$$|b(N,j^{'},\beta^{'})|\leq c_{10}\frac{1}{(|\beta^{'}|^2+|j^{'}\delta|^2)|\beta^{'}|^{d+3}},\ \forall\beta^{'}:|\beta^{'}+\tau|\geq 4\rho\quad(46)$$

If $|j^{'}\delta|\geq 9\rho$ and $|\beta^{'}+\tau|<4\rho$ then $\lambda_{j^{'},\beta^{'}}(v)>80\rho^2$ (see (3)) and by the first inequality in (45) we have $|\lambda_{j^{'},\beta^{'}}-\Lambda_N|>\frac{1}{5}|j^{'}\delta|^2$. Therefore (26) implies that

$$|b(N,j^{'},\beta^{'})|\leq c_{11}\frac{1}{|j^{'}\delta|^2},\ \forall|j^{'}\delta|\geq 9\rho,|\beta^{'}+\tau|<4\rho\quad(47)$$

It follows from (46) and (47) that the series in (44) and the series

$$\sum_{j^{'}\in Z,\beta^{'}\in\Gamma_\delta} i\langle\beta^{'}+\tau-t,h\rangle b(N,j^{'},\beta^{'})e^{i\langle\beta^{'}+\tau-t,x\rangle}\varphi_{j^{'}}(\langle\delta,x\rangle)$$

(formally differentiating the series in (44) by direction h and using $\langle\delta,h\rangle=0$ we obtained this series) are mojorized by the convergent series

$$\sum_{|j^{'}\delta|<9\rho,|\beta^{'}+\tau|<4\rho} c_{12}\quad|\quad b(N,j^{'},\beta^{'})\mid+\sum_{|j^{'}\delta|\geq9\rho,|\beta^{'}+\tau|<4\rho} c_{13}\frac{1}{\mid j^{'}\delta\mid^2}+$$

$$\sum_{j^{'}\in Z,|\beta^{'}+\tau|\geq4\rho} c_{14}\frac{1}{(\mid\beta^{'}\mid^2+\mid j^{'}\delta\mid^2)\mid\beta^{'}\mid^{d+3}}$$

and

$$\sum_{|j^{'}\delta|<9\rho,|\beta^{'}+\tau|<4\rho} \rho c_{12}\quad|\quad b(N,j^{'},\beta^{'})\mid+\sum_{|j^{'}\delta|\geq9\rho,|\beta^{'}+\tau|<4\rho} c_{13}\frac{\rho}{\mid j^{'}\delta\mid^2}+$$

$$\sum_{j^{'}\in Z,|\beta^{'}+\tau|\geq2\rho} c_{14}\frac{\mid\beta^{'}\mid}{(\mid\beta^{'}\mid^2+\mid j^{'}\delta\mid^2)\mid\beta^{'}\mid^{d+3}}$$

respectively, which shows that the series in (44) can be differentiated term by term. Therefore

$$\sum_{j=1}^{d} h_j(\frac{\partial}{\partial x_j}\Phi_{N,t}(x))=\sum_{j^{'}\in Z,\beta^{'}\in\Gamma_\delta} i\langle\beta^{'}+\tau-t,h\rangle b(N,j^{'},\beta^{'})e^{i\langle\beta^{'}+\tau-t,x\rangle}\varphi_{j^{'}}(\langle\delta,x\rangle)$$

Substuting this into (43) and using the Parseval equality by direct calculation we obtain the proof of the lemma $\diamondsuit$

Theorem 3. *If the conditions (20), (21), (30) hold then* $\Lambda_N(\lambda_{j,\beta}(v,\tau))$ *and the corresponding eigenfunction* $\Psi_{N(j,\beta)}(x)$ *(see (14), (16)) satisfy*

$$\Lambda_N(\lambda_{j,\beta}(v,\tau))=\lambda_{j,\beta}(v,\tau)+\frac{1}{4}\int_F|f_{\delta,\beta+\tau,\rho}(x)|^2\left|\varphi_{j,v}^{\delta}\right|^2dx+O(\rho^{-2-\varepsilon}),\quad(48)$$

$$\Psi_{N(j,\beta)}(x)=\Phi_{j,\beta}(x)+O(\rho^{-a}),\qquad(49)$$

$$\mid\quad\beta+\tau\mid\frac{\partial\Lambda_N(t)}{\partial h}=\mid\beta+\tau\mid^2+O(\rho^{2-2a}),\qquad(50)$$

where $\varepsilon = 3a - 2\alpha_1 - 2 > 0$, $h = \frac{\beta+\tau}{|\beta+\tau|}$,

$$f_{\delta,\beta+\tau,\rho}(x) = \sum_{\gamma\in\Gamma\backslash\delta R,\ |\gamma|<\rho^{\alpha}} \frac{\gamma}{\langle\beta+\tau,\gamma\rangle} q_\gamma e^{i\langle\gamma,x\rangle}.$$

Proof. Iterating (38) and grouping the terms with multiplicand $b(N,j,\beta)$ we get

$$(\Lambda_N - \lambda_{j,\beta})b(N,j,\beta) = O(\rho^{-p\alpha}) + S_1(j,\Lambda_N)b(N,j,\beta) + \tag{51}$$

$$\sum_{(j_1,\beta_1),(j_2,\beta_2)} \frac{A(j,\beta,j^1,\beta^1)A(j^1,\beta^1,j^2,\beta^2)b(N,j^2,\beta^2)}{\Lambda_N - \lambda_{j+j_1,\beta+\beta_1}},$$

where $S_1(j,\Lambda_N)$ is defined in (39), $\mid j\delta \mid\leq r_1$ (see (20)) $j^k = j+j_1+j_2+...+j_k$, $\beta^k = \beta+\beta_1+\beta_2+...+\beta_k$ and the sum in (51) is taken under conditions $(j_1,\beta_1)\in Q(\rho^\alpha, 9r_1)$, $(j_2,\beta_2)\in Q(\rho^\alpha, 90r_1)$, $(j^2,\beta^2)\neq(j,\beta)$. In (51) instead of $b(N,j^2,\beta^2)$ for $(j^2,\beta^2) = (j^{'},\beta)$, $j^{'}\neq j$ writing $O(\rho^{-2a+2\alpha_1})$ (see (37) and use the relations $\mid j^{'}\delta \mid\leq 100r_1 = O(\rho^{\alpha_1}))$ and instead of $b(N,j^2,\beta^2)$ for $\beta^2\neq\beta$ writing
$\sum_{(j_3,\beta_3)} \frac{A(j^2,\beta^2,j^3,\beta^3)b(N,j^3,\beta^3),}{(\Lambda_N - \lambda_{j^2,\beta^2})}$
(see (28)), using (29) and the obvious relations $\mid \Lambda_N - \lambda_{j^k,\beta^k}\mid > \frac{1}{3}\rho^a$ for $\beta^k\neq\beta$ $(k=1,2)$ (see (33)), and $b(N,j^3,\beta^3) = O(\rho^{-a})$ for $(j^3,\beta^3)\neq(j,\beta)$ (see (35), (37)), and dividing both part of (51) by $b(N,j,\beta)$, where
$b(N,j,\beta) = 1 + O(\rho^{-\alpha})$ (see (16) for $k=1$) we get

$$\Lambda_N = \lambda_{j,\beta} + S_1(j,\Lambda_N) + S_3(\Lambda_N) + O(\rho^{-3a+2\alpha_1}), \tag{52}$$

where $S_3(\Lambda_N) = \sum_{(j_1,\beta_1),(j_2,\beta_2)} \frac{A(j,\beta,j^1,\beta^1)A(j^1,\beta^1,j^2,\beta^2)A(j^2,\beta^2,j,\beta),}{(\Lambda_N - \lambda_{j+j_1,\beta+\beta_1})(\Lambda_N - \lambda_{j^2,\beta^2})}$, $\beta_1\neq 0$, $\beta^2\neq\beta$. By (33), (36), (29) we have $S_3(\Lambda_N) = S_3(\lambda_{j,\beta}) + O(\rho^{-3a})$. The calculations in apendix 3 and in apendix 2 show that

$$S_1(j,\lambda_{j,\beta}) = \frac{1}{4}\int_F |f_{\delta,\beta+\tau,\rho}(x)|^2 \left|\varphi^{\delta}_{j,v}\right|^2 dx + O(\rho^{-2-\varepsilon}) \tag{53}$$

$$S_3(\lambda_{j,\beta}) = \sum_{(j_1,\beta_1),(j_2,\beta_2)} \frac{A(j,\beta,j^1,\beta^1)A(j^1,\beta^1,j^2,\beta^2)A(j^2,\beta^2,j,\beta)}{(\lambda_{j,\beta} - \lambda_{j+j_1,\beta+\beta_1})(\lambda_{j,\beta} - \lambda_{j^2,\beta^2})} \tag{54}$$

$= O(\rho^{-2-\varepsilon})$,

where $(j_1, \beta_1) \in Q(\rho^{\alpha}, 9r), (j_2, \beta_2) \in Q(\rho^{\alpha}, 90r_1), \beta_1 + \beta_2 \neq 0, j \in S_1$. Therefore (48) follows from (52) and (41) for $j^{'} = j$.

To prove (49) we need to show that

$$\sum_{(j^{'}, \beta^{'}) \neq (j, \beta)} \mid b(N(j, \beta), j^{'}, \beta^{'}) \mid^2 = O(\rho^{-2a}). \tag{55}$$

Accoding (16) for $k\alpha_2 > 2a$, it is enough to prove that

$$\sum_{(j^{'}, \beta^{'}) \in S(k-1) \backslash (j, \beta)} \mid b(N, j^{'}, \beta^{'}) \mid^2 = O(\rho^{-2a}). \tag{56}$$

Using (26), (33), definition of $S(k-1)$ and $a - \alpha > \alpha_2$ we have

$$\sum_{(j^{'}, \beta^{'}) \in S(k-1), \beta^{'} \neq \beta} \mid b(N, j^{'}, \beta^{'}) \mid^2 = \sum \frac{\mid (\Psi_N(x)(q(x) - Q(s)), \Phi_{j^{'}, \beta^{'}}(x)) \mid^2}{\mid \Lambda_N - \lambda_{j^{'}, \beta^{'}} \mid^2} = O(\rho^{-2a}).$$

In the case $\beta^{'} = \beta$, $j^{'} \neq j$ using (37) (we can use it since $\mid j^{'} \delta \mid = O(\rho^{\frac{1}{2}\alpha_2})$ for $(j^{'}, \beta^{'}) \in S(k-1)$) we obtain

$$\sum_{(j^{'}, \beta) \in S(k-1), j^{'} \neq j} \mid b(N, j^{'}, \beta) \mid^2 = O(\rho^{-4a+\alpha_2}) K$$

where K is the number of $j^{'}$ satisfying $(j^{'}, \beta) \in S(k-1)$, hence, $K = O(\rho^{\frac{1}{2}\alpha_2})$. Since $\alpha_2 < a$ we obtain the proof of (56). Therefore (49) is proved.

Now it folows from (46), (55), and Parseval equality that

$$\sum_{j^{'} \in Z, \mid \beta^{'} + \tau \mid \geq 4\rho} \langle \beta + \tau, \beta^{'} + \tau \rangle \mid b(N, j^{'}, \beta^{'}) \mid^2 = O(\rho^{2-2a}) \tag{57}$$

$$\sum_{j^{'} \in Z, \mid \beta^{'} + \tau \mid < 4\rho, (j^{'}, \beta^{'}) \neq (j, \beta)} \langle \beta + \tau, \beta^{'} + \tau \rangle \mid b(N, j^{'}, \beta^{'}) \mid^2 = O(\rho^{2-2a}),$$

$$\langle \beta + \tau, \beta + \tau \rangle \mid b(N, j, \beta) \mid^2 = \mid \beta + \tau \mid^2 + O(\rho^{2-2a}).$$

Therefore (50) follows from Lemma 2$\diamondsuit$

3 Finding the Invariants

In this section we constructively determine the invariants (2), (7), (11) from the Floquet spectrum by using Theorem 3 and the following lemma.

Lemma 4. *Suppose the conditions (20), (21), (30) hold. If* $\Lambda_N(\beta+\tau+(j+v)\delta)$ *is a simple eigenvalue of* $L_t(q)$, *where* $\beta+\tau+(j+v)\delta-t\in\Gamma$, *satisfying (32) and* $N\neq N(j,\beta)$ *(for definition of* $N(j,\beta)$ *see Stat.2) then*

$$\mid \beta+\tau \mid \frac{\partial \Lambda_N(t)}{\partial h} <\mid \beta+\tau \mid^2 -\frac{1}{4}\rho^{2\alpha_d}, (h=\frac{\beta+\tau}{\mid \beta+\tau \mid}). \tag{58}$$

Proof. By Lemma 2 we have

$$\mid \beta+\tau \mid \frac{\partial \Lambda_N(t)}{\partial h} = \sum_{j^{'}\in Z,\beta^{'}\in\Gamma_\delta} \langle \beta+\tau,\beta^{'}+\tau\rangle \mid b(N,j^{'},\beta^{'})\mid^2 = \sum_{i=1}^{7} C_i, \tag{59}$$

where

$$C_i=\sum_{\beta^{'}\in A_i}\sum_{j^{'}\in Z}\langle \beta+\tau,\beta^{'}+\tau\rangle \mid b(N,j^{'},\beta^{'})\mid^2 \tag{60}$$

and A_i is defined as follows

$A_1=\{\beta^{'}\in\Gamma_\delta:\beta^{'}+\tau\notin R_\delta(4\rho)\}$,
$A_2=\{\beta^{'}\in\Gamma_\delta:\beta^{'}+\tau\in R_\delta(4\rho)\backslash R_\delta(H+\frac{1}{9}\rho^{a-1})\}$,
$A_3=\{\beta^{'}\in\Gamma_\delta:\beta^{'}+\tau\in (R_\delta(H+\frac{1}{9}\rho^{a-1})\backslash R_\delta(H+\rho^{\alpha_d-1})), \mid\beta-\beta^{'}\mid\geq \rho^{a-2\alpha}\}$,
$A_4=\{\beta^{'}\in\Gamma_\delta:\beta^{'}+\tau\in (R_\delta(H+\frac{1}{9}\rho^{a-1})\backslash R_\delta(H+\rho^{\alpha_d-1})), \mid\beta-\beta^{'}\mid< \rho^{a-2\alpha}\}$,
$A_5=\{\beta^{'}\in\Gamma_\delta:\beta^{'}+\tau\in (R_\delta(H+\rho^{\alpha_d-1})\backslash R_\delta(H-\rho^{2\alpha_d-1})), \mid\beta-\beta^{'}\mid\geq \rho^{\alpha_d}\}$,
$A_6=\{\beta^{'}\in\Gamma_\delta:\beta^{'}+\tau\in (R_\delta(H+\rho^{\alpha_d-1})\backslash R_\delta(H-\rho^{2\alpha_d-1})), \mid\beta-\beta^{'}\mid< \rho^{\alpha_d}\}$
$A_7=\{\beta^{'}\in\Gamma_\delta:\beta^{'}+\tau\in R_\delta(H-\rho^{2\alpha_d-1})\}$, where $H=\mid\beta+\tau\mid$.

First we prove that

$$C_i=O(\rho^{2-2a}),\ \forall i=1,2,4,6 \tag{61}$$

It follows from (46) that (61) holds for $i=1$. For $\beta^{'}\in A_2$ using the obvious relations $\mid\beta^{'}+\tau\mid^2-\mid\beta+\tau\mid^2>\frac{2}{9}\rho^a$, (see the definition of A_2 and take into account that $H =\mid \beta+\tau \mid > \rho$ for $\beta \in S_2$) $\mu_j = O(\rho^{2\alpha_1}) = o(\rho^a)$ (see (3) and take into account that $j \in S_1, a > 2\alpha_1$) and (32) we obtain

$$\Lambda_N =\mid \beta+\tau \mid^2 + O(\rho^{2\alpha_1}), \rho <\mid \beta+\tau \mid = H < 2\rho, \lambda_{j',\beta'} - \Lambda_N(t) > \frac{1}{9}\rho^a \quad (62)$$

This with (26) give the prove of (61) for $i = 2$. To prove (61) for $i = 4$ we write $C_4 < c_{15}\rho^2(C_{4,1} + C_{4,2})$, where

$$C_{4,1} = \sum_{\beta' \in A_4} (\sum_{j' : |j'\delta| \geq \frac{1}{30}\rho^{\frac{a}{2}}} \mid b(N, j', \beta') \mid^2),$$

$$C_{4,2} = \sum_{\beta' \in A_4} (\sum_{j' : |j'\delta| < \frac{1}{30}\rho^{\frac{a}{2}}} \mid b(N, j', \beta') \mid^2),$$

and prove that

$$C_{4,i} = O(\rho^{-2a}), \forall i = 1, 2. \quad (63)$$

For $\beta' \in A_4$, $\mid j'\delta \mid \geq \frac{1}{30}\rho^{\frac{a}{2}}$ the inequality $\lambda_{j',\beta'} - \Lambda_N(t) > c_{16}\rho^a$ holds (see the first relation in (62)). Therefore (26) give the proof of (63) for $i = 1$. Similarly for $\beta' \in A_4, \mid j'\delta \mid < \rho^{\frac{a}{2}}$ the inequality $\lambda_{j',\beta'} - \Lambda_N(t) > c_{17}\rho^{\alpha_d}$ holds, becouse $\alpha_d > 2\alpha_1$. Therefore using the inequality (3) we obtain

$$\sum_{j' : |j'\delta| < \frac{1}{30}\rho^{\frac{a}{2}}} \frac{1}{\mid \Lambda_N(t) - \lambda_{j',\beta'} \mid^2} = O(1), \ \forall \beta' \in A_4. \quad (64)$$

Using this, (26), (27), (29) and denoting $\mid (\Psi_N, (q(x) - Q(s))\Phi_{n(\beta'),\beta'}(x)) \mid =$ $\max_{j' : |j'\delta| < \frac{1}{30}\rho^{\frac{a}{2}}} \mid (\Psi_N, (q(x) - Q(s))\Phi_{j',\beta'}(x)) \mid$ (if max is taken for several indeks $n(\beta')$, then we take one of them) we get

$$\sum_{\beta' \in A_4} (\sum_{j' : |j'\delta| < \frac{1}{30}\rho^{\frac{a}{2}}} \mid b(N', j', \beta') \mid^2 =$$

$$\sum_{\beta' \in A_4} (\sum_{j' : |j'\delta| < \frac{1}{30}\rho^{\frac{a}{2}}} \frac{\mid (\Psi_N(x), (q(x) - Q(s))\Phi_{j',\beta'}(x)) \mid^2}{\mid \Lambda_N(t) - \lambda_{j',\beta'} \mid^2} \leq$$

$$c_{18} \sum_{\beta' \in A_4} \mid (\Psi_N(x), (q(x) - Q(s))\Phi_{n(\beta'),\beta'}(x)) \mid^2 \leq$$

$$c_{19}\rho^{-p\alpha}+c_{20}\sum_{\beta^{'}\in A_4}\mid b(N,n(\beta^{'})+j_1(\beta^{'}),\beta^{'}+\beta_1(\beta^{'}))\mid^2= \tag{65}$$

$$c_{19}\rho^{-p\alpha}+c_{20}\sum_{\beta^{'}\in A_4}\frac{\mid(\Psi_N,(q(x)-Q(s))\Phi_{n(\beta^{'})+j_1(\beta^{'}),\beta^{'}+\beta_1(\beta^{'})}(x))\mid^2}{\mid\Lambda_N-\lambda_{n(\beta^{'})+j_1(\beta^{'}),\beta^{'}+\beta_1(\beta^{'})}\mid^2},$$

where

$\mid b(N,n(\beta^{'})+j_1(\beta^{'}),\beta^{'}+\beta_1(\beta^{'}))\mid=\max_{j_1,\beta_1}\mid b(N,n(\beta^{'})+j_1,\beta^{'}+\beta_1)\mid$ and maximum is taken under conditions $(j_1,\beta_1)\in Q(\rho^{\alpha},9\frac{1}{30}\rho^{\frac{a}{2}})$. This inclusion ant the condition $\mid j^{'}\delta\mid<\frac{1}{30}\rho^{\frac{a}{2}}$ imply that $\mid n(\beta^{'})\delta+j_1(\beta^{'})\delta\mid<\frac{1}{3}\rho^{\frac{a}{2}}$ and $\mid\mu_{n(\beta^{'})+j_1(\beta^{'})}\mid<\frac{1}{8}\rho^{a}$. This, $\mid\mid\beta^{'}+\tau\mid^2-\mid\beta+\tau\mid^2\mid<\frac{1}{2}\rho^{a}$ (see definition of A_4 and use the inequality $H<2\rho$), and

$$\mid\mid\beta^{'}+\tau\mid^2-\mid\beta^{'}+\beta_1+\tau\mid^2\mid>\frac{7}{8}\rho^{a},\forall\beta^{'}\in A_4,\beta_1\in\Gamma_\delta(p\rho^{\alpha}) \tag{66}$$

which can be easily obtained from the conditions $\beta\notin\bigcup_{b\in\Gamma_\delta(p\rho^{\alpha})}V_b^{\delta}(\rho^{a})$ and $\mid\beta-\beta^{'}\mid<\rho^{a-2\alpha}$ (see (30), definition of A_4, and use

$\mid\beta^{'}+\tau\mid^2-\mid\beta^{'}+\beta_1+\tau\mid^2=-2\langle\beta^{'}+\tau,\beta_1\rangle-\mid\beta_1\mid^2=$

$-2\langle\beta,\beta_1\rangle-\mid\beta_1\mid^2-2\langle\beta^{'}-\beta+\tau,\beta_1\rangle=-(\mid\beta+\beta_1\mid^2-\mid\beta\mid^2)+o(\rho^{a})$) imply that $\mid\mid\beta+\tau\mid^2-\mid\beta^{'}+\beta_1+\tau\mid^2\mid>\frac{3}{8}\rho^{a}$ and

$$\mid\Lambda_N-\lambda_{n(\beta^{'})+j_1(\beta^{'}),\beta^{'}+\beta_1(\beta^{'})}\mid>\frac{1}{8}\rho^{a}, \tag{67}$$

(see the first relation in (62)). Note that if $\beta^{'}\in A_4$, $\beta^{''}\in A_4$ and $\beta^{'}\neq\beta^{''}$ then $\beta^{'}+\beta_1(\beta^{'})\neq\beta^{''}+\beta_1(\beta^{''})$ (otherwise we have $\beta^{''}=\beta^{'}+b$, $b\in\Gamma_\delta(2\rho^{\alpha})$, $\mid\mid\beta^{'}+\tau\mid^2-\mid\beta^{''}+\tau\mid^2\mid<\frac{1}{2}\rho^{a}$ which controdicts (66)). Therefore using (67), in (65) and Bessel inequality we obtain the proof of (63) for $i=2$. So (61) is proved for $i=4$.

Now we prove (61) for $i=6$. First we note that $A_6=\{\beta\}$. If $\beta^{'}\neq\beta$ and $\beta^{'}\in A_6$ then we have $\beta^{'}=\beta+b$, where $b\in\Gamma_\delta(\rho^{\alpha_d})$, and from the relations $\beta\notin V_b^{\delta}(\rho^{\frac{1}{2}})$, (see (20) and the definition of S_2) $\mid\beta+\tau\mid=H$, we obtain that $\mid\mid\beta^{'}+\tau\mid^2-H^2\mid>\frac{1}{2}\rho^{\frac{1}{2}}$ which controdicts $\beta^{'}+\tau\in R_\delta(H+\rho^{\alpha_d-1})$). Hence

$$C_6=\sum_{j^{'}\in Z}(\beta+\tau,\beta+\tau)\mid b(N,j^{'},\beta)\mid^2=H^2\sum_{j^{'}\in Z}\mid b(N,j^{'},\beta)\mid^2=H^2\sum_{i=1}^{3}C_{6,i},$$

where $C_{6,1} =\mid b(N,j,\beta) \mid^2$, $C_{6,2} = \sum_{|j^{'}\delta|\geq \frac{1}{30}\rho^{\frac{a}{2}}} \mid b(N,j^{'},\beta) \mid^2$,

$C_{6,3} = \sum_{|j^{'}\delta|< \frac{1}{30}\rho^{\frac{a}{2}}, j^{'}\neq j} \mid b(N,j^{'},\beta) \mid^2$. To prove (61) for $i=6$ we show

$$C_{6,i} = O(\rho^{-2a}),\ \forall i = 1,2,3 \tag{68}$$

By (55) and Parseval equality with respect to the basis $\{\Phi_{j,\beta}(x)\}$ we have $\mid b(N(j,\beta),j,\beta) \mid^2 = 1 + O(\rho^{-2a})$. Therefore (68) for $i=1$ follows from the Parseval equality with respect to the basis $\{\Psi_N(x)\}$. For $\mid j^{'}\delta \mid\geq \frac{1}{30}\rho^{\frac{a}{2}}$ the inequality $\lambda_{j^{'},\beta} - \Lambda_N(t) > c_{21}\rho^a$ holds (see the first relation in (62)). Therefore (26) give the proof of (68) for $i=2$. The proof of (68) for $i=3$ is similar to the proof of (63) for $i=2$. Using (34) and (3) we obtain

$$\sum_{|j^{'}\delta|< \frac{1}{30}\rho^{\frac{a}{2}},, j^{'}\neq j} \frac{1}{\mid \Lambda_N(t) - \lambda_{j^{'},\beta} \mid^2} = O(1) \tag{69}$$

Now we repeat the proof of (63) for $i=2$ as follows. In (65) instead of $\beta^{'}$ and A_4 taking β and instead of (64) using (69) and denoting

$$\mid (\Psi_N, (q(x)-Q(s))\Phi_{n(\beta),\beta}(x)) \mid = \max_{j^{'}\neq j, |j^{'}\delta|<\frac{1}{30}\rho^{\frac{a}{2}}} \mid (\Psi_N, (q(x)-Q(s))\Phi_{j^{'},\beta}) \mid$$

we get

$$C_{6,3} = c_{19}\rho^{-p\alpha} + c_{20}\frac{\mid (\Psi_N, (q(x)-Q(s))\Phi_{n(\beta)+j_1(\beta),\beta+\beta_1(\beta)}(x)) \mid^2}{\mid \Lambda_N - \lambda_{n(\beta)+j_1(\beta),\beta+\beta_1(\beta)} \mid^2} \tag{70}$$

where the indeks $(n(\beta)+j_1(\beta), \beta+\beta_1(\beta))$ is defined from the relation

$\mid b(N, n(\beta)+j_1(\beta), \beta+\beta_1(\beta)) \mid = \max_{j_1,\beta_1} \mid b(N, n(\beta)+j_1, \beta+\beta_1) \mid$. Here maximum is taken under conditions $(j_1,\beta_1) \in Q(\rho^\alpha, 9\frac{1}{30}\rho^{\frac{a}{2}})$. Clearly under these conditions on index $(n(\beta)+j_1(\beta), \beta+\beta_1(\beta))$ we have $\mid n(\beta)\delta \mid < \frac{1}{30}\rho^{\frac{a}{2}}$ (see definition of $n(\beta)$), $\mid n(\beta)+j_1(\beta) \mid < \frac{1}{3}\rho^{\frac{a}{2}}$, $\mid \mu_{n(\beta)+j_1(\beta)} \mid < \frac{1}{8}\rho^a$,

$\mid \beta+\tau \mid^2 - \mid \beta+\beta_1(\beta)+\tau \mid^2 \geq \frac{7}{8}\rho^a$ (see (30)). The last two inequality and the first relation in (62) imply that

$$\mid \Lambda_N - \lambda_{n(\beta)+j_1(\beta),\beta+\beta_1(\beta)} \mid > \frac{1}{2}\rho^a,$$

This with (70) give the proof of (68) for $i=3$. So (61) is proved for $i=6$.

Now we prove that

$$C_i = \sum_{\beta' \in A_i} (\sum_{j' \in Z} \langle \beta+\tau, \beta'+\tau \rangle \mid b(N,j',\beta') \mid^2) \le \qquad (71)$$

$$\sum_{\beta' \in A_i} (\sum_{j' \in Z} \mid b(N,j',\beta') \mid^2 (H^2 - \frac{1}{3}\rho^{2\alpha_d}))$$

for $i = 3,5,7$. Consider the triangle genereted by vectors $\beta+\tau$, $\beta'+\tau$, $\beta-\beta'$. For $\beta' \in A_3$ we have $\mid \beta+\tau \mid = H$, $H + \rho^{\alpha_d - 1} \le \mid \beta'+\tau \mid \le H + \frac{1}{9}\rho^{a-1}$, $\mid \beta - \beta' \mid \ge \rho^{a-2\alpha}$. Let θ be the angle between the vectors $\beta+\tau$, and $\beta'+\tau$. If $\mid \theta \mid \le \frac{\pi}{2}$ then using the cosine theorem we get

$\mid \langle \beta+\tau, \beta'+\tau \rangle \mid = \frac{1}{2}(\mid \beta+\tau \mid^2 + \mid \beta'+\tau \mid^2 - \mid \beta-\beta' \mid^2)$

$\le H^2 - \frac{1}{3}\rho^{2a-4\alpha} < H^2 - \frac{1}{3}\rho^{2\alpha_d}$, since $a - 2\alpha > \alpha_d$. Using this and taking into account that $\langle \beta+\tau, \beta'+\tau \rangle < 0$ for $\frac{\pi}{2} < \mid \theta \mid \le \pi$ we get the proof of (71) for $i = 3$. Similarly if $\beta' \in A_5$, $\mid \theta \mid \le \frac{\pi}{2}$ then $\mid \langle \beta+\tau, \beta'+\tau \rangle \mid \le H^2 - \frac{1}{3}\rho^{2\alpha_d}$ and hence (71) holds for $i = 5$. If $\beta' \in A_7$ then $\mid \beta'+\tau \mid \le H - \rho^{2\alpha_d - 1}$, $\mid \langle \beta+\tau, \beta'+\tau \rangle \mid \le H^2 - \rho^{2\alpha_d}$ and hence (71) holds for $i = 7$ too. Now (71) and Parseval equality imply that $S_3 + S_5 + S_7 \le H^2 - \frac{1}{3}\rho^{2\alpha_d} = \mid \beta+\tau \mid^2 - \frac{1}{3}\rho^{2\alpha_d}$. This, (59), (61) give the proof of (58), since $2 - 2a < 2\alpha_d \diamondsuit$

Let M be a positive constant. Denote by $W_2^l(F,M)$ the set of $q(x) \in W_2^l(F)$ satisfying $\sup \mid q(x) \mid \le M$.

Lemma 5. *Let* $\rho_k = 3^k\rho$, $b \in \Gamma_\delta$

$S_2'(\rho_k) = \{(V_b^\delta(4\rho_k^a) \backslash V_b^\delta(\rho_k^a)) \backslash (D_1(\rho_k^{\frac{1}{2}}) \cup D_2(\rho_k^{a+2\alpha}))\} \cap D_3$, *where*

$D_3 = (R(2\rho_k - d_\delta - 1) \backslash R(\rho_k + d_\delta + 1))$, $D_1(\rho_k^{\frac{1}{2}}) = \bigcup_{b' \in \Gamma_\delta(\rho_k^{\alpha_d})} V_{b'}^\delta(\rho_k^{\frac{1}{2}})$,

$D_2(\rho_k^{a+2\alpha}) = \bigcup_{b' \in \Gamma_\delta(p\rho_k^\alpha) \backslash bR} V_{b'}^\delta(\rho_k^{a+2\alpha})$.

a) For all $k = 1,2,...,$ *the set* $S_2'(\rho_k) \cap \Gamma_\delta$ *is nonempty set.*

b) For $\beta_k \in S_2'(\rho_k) \cap \Gamma_\delta$, *the set* $S_3' = (\frac{1}{10}, \frac{4}{10}) \cup (\frac{6}{10}, \frac{9}{10}) \backslash (\cup_k A'(\beta_k, \rho_k))$ *does not depend on* $q(x) \in W_2^l(F,M)$ *(depends only on* $W_2^l(F,M)$*) and*

$$\mu(S_3') > \frac{1}{2} \qquad (72)$$

c) If $\beta_k \in S_2^{'}(\rho_k)$, $\tau \in F_\delta$ *then*

$$\frac{1}{2}|\rho_k|^a <| \langle \beta_k + \tau, b\rangle |< 3|\rho_k|^a, \tag{73}$$

$$| \langle \beta_k + \tau, \gamma\rangle |> \frac{1}{2}|\rho_k|^a, \forall \gamma \in S(\delta, b)\backslash \delta R \tag{74}$$

$$| \langle \beta_k + \tau, \gamma\rangle |> \frac{1}{2}|\rho_k|^{a+2\alpha}, \forall \gamma \notin S(\delta, b), | \gamma |< |\rho_k|^\alpha \tag{75}$$

$$\int_F |f_{\delta,\beta_k+\tau,\rho_k}(x)|^2 \left|\varphi_{n,v}^{\delta}\right|^2 dx < c_{22}\rho_k^{-2a} \tag{76}$$

where $S(\delta, b)$*is defined in introduction.*

Proof. a) Suppose $\beta + \tau \in D(\rho_k)$, where $\beta \in \Gamma_\delta, \tau \in F_\delta$,

$D(\rho_k) = \{(V_b^\delta(3\rho_k^a)\backslash V_b^\delta(2\rho_k^a))\backslash(D_1(2\rho_k^{\frac{1}{2}}) \cup D_2(2\rho_k^{a+2\alpha}))\} \cap D_4$,

$D_4 = (R_\delta(2\rho_k - 1)\backslash R_\delta(\rho_k + 1))$. Then it is easy to show that $\beta \in S_2^{'}(\rho_k)$. Therefore the sets $\beta + F_\delta$ for $\beta \in S_2^{'}(\rho_k)$ is cover of $D(\rho_k)$. Hence

$$| S_2^{'}(\rho_k) \cap \Gamma_\delta |\geq (\mu(F_\delta))^{-1}\mu(D(\rho_k)), \tag{77}$$

where $| S_2^{'}(\rho_k) \cap \Gamma_\delta |$ is the number of elements of $S_2^{'}(\rho_k) \cap \Gamma_\delta$. Now we calculate $\mu(D(\rho_k))$. It is clear that

$\mu((V_b^\delta(3\rho_k^a)\backslash V_b^\delta(2\rho_k^a)) \cap D_4) > c_{23}\rho^{d-2+a}$ (see Remark 1 of [4]).

To calculate the measure of $V_b^\delta(3\rho_k^a) \cap V_{b'}^\delta(2\rho_k^{\frac{1}{2}}) \cap D_4$ for $b^{'} \in \Gamma_\delta(\rho_k^{\alpha_d})$ we turn the coordinate axes so that the derection of $(1, 0, 0, ..., 0)$ coinsides with the direction $b^{'}$, that is $b^{'} = (| b^{'} |, 0, 0, ..., 0)$ and the plane generated by $b, b^{'}$ coinsides with the plane $(x_1, x_2, 0, ...0)$, that is, $b = (b_1, b_2, 0, ..., 0)$.Then the condition $x \in V_b^\delta(3\rho_k^a) \cap V_{b'}^\delta(2\rho_k^{\frac{1}{2}}) \cap D_4$ imply that

$$\begin{aligned} x_1 \quad | \quad b^{'} &|= O(\rho_k^{\frac{1}{2}}), \\ x_1b_1 + x_2b_2 &= O(\rho_k^a) \\ x_1^2 + x_2^2 + ... + x_{d-1}^2 &= O(\rho_k^2) \end{aligned}$$

Therefore $V_b^\delta(3\rho_k^a) \cap V_{b'}^\delta(2\rho_k^{\frac{1}{2}}) \cap D_4$ is subset of

$[-c_{24}\rho_k^{\frac{1}{2}}, c_{24}\rho_k^{\frac{1}{2}}] \times [-c_{24}\rho_k^{a+\alpha_d}, c_{24}\rho_k^{a+\alpha_d}] \times ([-c_{24}\rho_k, c_{24}\rho_k])^{d-3}$ which has the measure $O(\rho^{d-3+\frac{1}{2}+a+\alpha_d})$. This with $| \Gamma_\delta(\rho^{\alpha_d}) |= O(\rho^{(d-1)\alpha_d})$ give

$\mu((V_b^{\delta}(3\rho_k^a)\cap D_1(2\rho_k^{\frac{1}{2}})\cap D_4)=O(\rho^{d-3+\frac{1}{2}+a+d\alpha_d})=o(\rho^{d-2+a})$ since $d\alpha_d<\frac{1}{2}$.
In the same way we get the estimation
$\mu((V_b^{\delta}(3\rho_k^a)\cap D_2(\rho_k^{a+2\alpha})\cap D_4)=O(\rho^{d-3+a+(d+3)\alpha})=o(\rho^{d-2+a})$,
since $(d+3)\alpha<1$. These with (77) give the proof of a).

The statement b) follows from (18).

c) The inequalities in (73) follow from the inclusion $\beta_k\in V_b^{\delta}(4\rho_k^a)\backslash V_b^{\delta}(\rho_k^a)$, (see definition of $S_2^{'}(\rho_k)$). If $\gamma\in S(\delta,b)\backslash\delta R$ then

$$\gamma=nb+a\delta, n\neq 0, n\in Z, a\in R, \mid\langle\gamma,b\rangle\mid=\mid n\mid\mid b\mid^2\geq\mid b\mid^2. \tag{78}$$

This with the relations $\langle\beta_k+\tau,\delta\rangle=0$, $\beta_k\notin V_b^{\delta}(\rho_k^a)$give $\mid\langle\beta_k+\tau,\gamma\rangle\mid=\mid n\langle\beta_k+\tau,b\rangle\mid>\frac{1}{2}|\rho_k|^a$. So (74) is proved. If $\gamma\notin S(\delta,b), \mid\gamma\mid<|\rho_k|^{\alpha}$ then $\gamma=b^{'}+a\delta$, where $a\in R$, $b^{'}\in\Gamma_{\delta}(\rho_k^{\alpha})\backslash bR$. Therefore the relation $\beta_k\notin V_{b^{'}}^{\delta}(\rho_k^{a+2\alpha})$, (see definition of $S_2^{'}(\rho_k)$) imply that $\mid\langle\beta_k+\tau,\gamma\rangle\mid=\mid\langle\beta_k+\tau,b^{'}\rangle\mid>\frac{1}{2}|\rho_k|^{a+2\alpha}$. Thus (75) is proved. The inequality (76) follows from the definition of $f_{\delta,\beta+\tau,\rho_k}(x)$ (see Theorem 3), (73), (74), (75), and from the obvious relation

$$\sum_{\gamma\in\Gamma}\mid\gamma\mid\mid q_{\gamma}\mid<c_{25},\forall q(x)\in W_2^l(F,M). \tag{79}$$

Theorem 6. *Suppose $q(x)\in W_2^l(F,M)$ and the band functions $\Lambda_1(t),\Lambda_2(t),...$ of $L(q)$ are known. Then the spectral invariants (2), (6), (7), (11) can be constructively determined.*

Proof. Take any integer n satisfying $\mid n\mid>\frac{10M}{|\delta|^2}+2$ (see (21)), β_k from $S_2^{'}(\rho_k)$, v from $S_3^{'}$, τ from F_{δ} and consider the functions $\Lambda_N(\beta_k+\tau+(n+v)\delta)$ for $N=1,2,....$ Let A_k be the set of all $\tau\in F_{\delta}$ for which there is N satisfying the conditions:

Con.1. $\mid\Lambda_N(\beta_k+\tau+(n+v)\delta)-\mid\beta_k+\tau\mid^2-(n+v)^2\mid<M$

Con.2. $\Lambda_N(\beta_k+\tau+(n+v)\delta)$ is a simple eigenvalue

Con.3. $\mid\beta_k+\tau\mid\frac{\partial\Lambda_N(\beta_k+\tau+(n+v)\delta)}{\partial h}=\mid\beta_k+\tau\mid^2+O(\rho_k^{2-2a})$,

where $h=\frac{\beta_k+\tau}{|\beta_k+\tau|}$. For $\tau\in A_k$ take one of eigenvalue $\Lambda_N(\beta_k+\tau+(n+v)\delta)$ satisfying Con.1.–Con.3. and calculate the integral

$$J(A_k)=\frac{1}{\mu(F_{\delta})}\int_{A_k}(\Lambda_N(\beta_k+\tau+(n+v)\delta)-\mid\beta_k+\tau\mid^2)d\tau \tag{80}$$

We write these integral as sum of $J(S_4)$ and $J(A_k\backslash S_4)$. If $\tau\in S_4(\beta_k,j,v,\rho_k)$ then by Theorem 3 $\Lambda_N(\lambda_{j,\beta_k}(v,\tau))$ satisfies the Con.1.-Con.3 and by Lemma

4 the other eigenvalues does not satisfy the conditions Con.1.-Con.3. Hence in integral $J(S_4)$ instead of $\Lambda_N(\beta_k+\tau+(j+v)\delta)$ we must take $\Lambda_N(\lambda_{j,\beta_k}(v,\tau))$. Therefore using (14) and $\mu(S_4(\beta_k,j,v,\rho_k))=\mu(F_\delta)(1+O(\rho_k^{-\alpha}))$ (see (19)) we obtain

$$J(S_4(\beta_k,j,v,\rho_k))=\mu_n(v)+O(\rho_k^{-\alpha}). \tag{81}$$

On the other hand using $\mu(A_k\backslash S_4)\leq\mu(F_\delta\backslash S_4)=O(\rho_k^{-\alpha})$ and the Con.1. we get $J(A_k\backslash S_4)=O(\rho^{-\alpha})$. This and (81) imply that

$$J(A_k)=\mu_n(v)+O(\rho_k^{-\alpha}). \tag{82}$$

Now tending k to ∞ we find the eigenvalue $\mu_n(v)$ of $T_v(Q(s))$ for all $v\in S_3^{'}$ (see (72)). Since the $\mu_n(v)$ for $v\neq 0,\frac{1}{2}$ is simple zero of $F(\mu)=2\cos v$, where the Lyapunov function $F(\mu)$ is entire function, we can determine the $F(\mu)$ and the eigenvalues $\mu_n(v)$ of $T_v(Q(s))$.

Now let B_k be the set of $\tau\in F_\delta$ for which there is N satisfying Con.2., Con.3., and Con.4. : $\mid\Lambda_N(\beta_k+\tau+(n+v)\delta)-\mid\beta_k+\tau\mid^2-\mu_n(v)\mid<2c_{22}\rho_k^{-2a}$ (see (76)). For $\tau\in B_k$ take one of eigenvalue

$\Lambda_N(\beta_k+\tau+(n+v)\delta)$ satisfying Con.2.−Con.4. and calculate the integral

$$J^{'}(B_k)=\frac{\mid\langle\beta_k+\tau,b\rangle\mid^2}{\mu(F_\delta)|b|^4}\int_B(\Lambda_N(\beta_k+\tau+(n+v)\delta)-\mid\beta_k+\tau\mid^2-\mu_n(v))d\tau. \tag{83}$$

We write these integral as sum of $J^{'}(S_4)$ and $J^{'}(B_k\backslash S_4)$. If $\tau\in S_4$ then by Theorem 3 and Lemma 4 only $\Lambda_N(\lambda_{n,\beta_k}(v,\tau))$ satisfies Con.2.-Con.4. Hence in integral $J^{'}(S_4)$ instead of $\Lambda_N(\beta_k+\tau+(n+v)\delta)$ we must take $\Lambda_N(\lambda_{n,\beta_k}(v,\tau))$. Therefore using (48), (73) we get

$$J^{'}(S_4)=\frac{\mid\langle\beta_k+\tau,b\rangle\mid^2}{\mu(F_\delta)|b|^4}\int_{S_4}\int_F\mid f_{\delta,\beta_k+\tau,\rho_k}(x)\varphi_{n,v}^{\delta}\mid^2 dxd\tau+O(\rho_k^{-2+2a-\varepsilon}). \tag{84}$$

Now using Con.4., (73), and $\mu(B_k\backslash S_4)=O(\rho^{-\alpha})$ (see (19)) we obtain

$$J^{'}(B_k\backslash S_4)=O(\rho_k^{-\alpha}). \tag{85}$$

Substituting the decomposition $|\delta|^{-2}\langle\gamma,\delta\rangle\delta+|b|^{-2}\langle\gamma,b\rangle b$ of γ for $\gamma\in S(\delta,b)$, $\mid\gamma\mid<|\rho_k|^{\alpha}$ into the denominator of fraction in $f_{\delta,\beta_k+\tau,\rho_k}(x)$ (for definition of this function see Theorem 3) and using (73), (75) we have

$$\lim_{k\to\infty}|b|^{-2}\langle\beta_k+\tau,b\rangle f_{\delta,\beta_k+\tau,\rho_k}(x)=\sum_{\gamma\in S(\delta,b)\backslash\delta R}\frac{\gamma}{\langle\gamma,b\rangle}q_\gamma e^{\langle\gamma,x\rangle}\equiv q_{\delta,b}(x), \qquad (86)$$

(see (4)). This with (84) and (85) implies that

$$\lim_{k\to\infty}J^{'}(B_k)=\int|q_{\delta,b}(x)|^2\left|\varphi_{n,v}^{\delta}\right|^2dx\equiv J. \qquad (87)$$

The convergence of the series (4) follows from (78) and (79). In (87) tending n to infinity and using (5) we get the invariant J_0 (see (6)). Then we find the invariants $J_1, J_2, ...,$ as follows

$$J_1=\lim_{n\to\infty}(J-J_0)n,\ \ J_2=\lim_{n\to\infty}((J-J_0)n^2-J_1n),...,$$

Thus we find the invariants (6), (7), and (11)$\diamondsuit$

4 Appendix 1. The proof of (42)

Here we prove that $S_1(j^{'},\lambda_{j,\beta})=O(\rho^{-2a}r^2)$, where (see (42))

$$\overline{S_1(j^{'},\lambda_{j,\beta})}\equiv\sum_{\beta_1}(\sum_{j_1}\frac{\overline{A(j^{'},\beta,j^{'}+j_1,\beta+\beta_1)}\,\overline{A(j^{'}+j_1,\beta+\beta_1,j,\beta)}}{\lambda_{j,\beta}-\lambda_{j^{'}+j_1,\beta+\beta_1}} \qquad (88)$$

The conditions on indices $j^{'}$, j_1, j (see (42)) and (3) imply that $\mu_{j^{'}+j_1}=O(r^2)$, $\mu_j=O(r^2)$. These with $\beta\notin V_{\beta_1}^{\delta}(\rho^a)))$ for $\beta_1\in\Gamma_\delta(p\rho^\alpha)$ (see (30)) give

$$\lambda_{j,\beta}-\lambda_{j^{'}+j_1,\beta+\beta_1}=-2\langle\beta,\beta_1\rangle+O(r^2),\ \ |\langle\beta,\beta_1\rangle|>\frac{1}{3}\rho^a \qquad (89)$$

Using (29) and (89) we get

$$\overline{S_1(j^{'},\lambda_{j,\beta})}=\sum_{\beta_1}\frac{S^{'}}{-2\langle\beta,\beta_1\rangle}+O(\rho^{-2a}r^2), \qquad (90)$$

where $S^{'}=\sum_{j_1}\overline{A(j^{'},\beta,j^{'}+j_1,\beta+\beta_1)}\,\overline{A(j^{'}+j_1,\beta+\beta_1,j,\beta)}$. Here (see [4])

$$\overline{A(j',\beta,j'+j_{1,}\beta+\beta_1)} = \sum_{n_1:(n_1,\beta_1)\in\Gamma'(\rho^\alpha)} c(n_1,\beta_1)a(n_1,\beta_1,j',\beta,j'+j_{1,}\beta+\beta_1), \quad (91)$$

$$\overline{A(j'+j_{1,}\beta+\beta_1,j,\beta)} = \sum_{n_2:(n_2,-\beta_1)\in\Gamma'(\rho^\alpha)} c(n_2,-\beta_1)a(n_2,-\beta_1,j'+j_{1,}\beta+\beta_1,j,\beta),$$

where $\Gamma(\rho^\alpha)=\{\gamma\in\Gamma: 0<\mid b\mid<\rho^\alpha\}$,

$\Gamma'(\rho^\alpha)=\{(n_1,\beta_1):\beta_1\in\Gamma_\delta\backslash 0, n_1\in Z, \beta_1+(n_1-\frac{\langle\beta_1,\delta^*\rangle}{2\pi})\delta\in\Gamma(\rho^\alpha)\}$,

$$c(n_1,\beta_1)=q_{\gamma_1}, \gamma_1=\beta_1+(n_1-\frac{\langle\beta_1,\delta^*\rangle}{2\pi})\delta\in\Gamma(\rho^\alpha), \quad (92)$$

$$a(n_1,\beta_1,j',\beta,j'+j_{1,}\beta+\beta_1)=(e^{i(n_1-\frac{\langle\beta_1,\delta^*\rangle}{2\pi})s}\varphi_{j',v(\beta)}(s),\varphi_{j'+j_1,v(\beta+\beta_1)}(s)),$$

$$a(n_2,-\beta_1,j'+j_{1,}\beta+\beta_1,j,\beta)=(e^{i(n_2-\frac{\langle-\beta_1,\delta^*\rangle}{2\pi})s}\varphi_{j'+j_1,v(\beta+\beta_1)},\varphi_{j,v(\beta)}) \quad (93)$$

$$=(\varphi_{j'+j_1,v(\beta+\beta_1)},e^{-i(n_2-\frac{\langle-\beta_1,\delta^*\rangle}{2\pi})s}\varphi_{j,v(\beta)})=\overline{(e^{-i(n_2-\frac{\langle-\beta_1,\delta^*\rangle}{2\pi}s}\varphi_{j,v(\beta)},\varphi_{j'+j_1,v(\beta+\beta_1)})}.$$

To calculate S' we prove that

$$\sum_{j_1} a(n_1,\beta_1,j',\beta,j'+j_{1,}\beta+\beta_1)a(n_2,-\beta_1,j'+j_1,\beta+\beta_1,j,\beta) \quad (94)$$
$$= a(n_1+n_2,0,j',\beta,j,\beta)+O(\rho^{-p\alpha}).$$

By definition $a(n_1+n_2,0,j',\beta,j,\beta)=(e^{i(n_1+n_2)s}\varphi_{j',v(\beta)}(s),\varphi_{j,v(\beta)}(s))=$
$(e^{i(n_1-(2\pi)^{-1}(\beta_1,\delta^*))s}\varphi_{j',v(\beta)}(s),e^{-i(n_2-(2\pi)^{-1}(-\beta_1,\delta^*))s}\varphi_{j,v(\beta)}(s))$

This with the formulas (see Lemma 4 and (57), (65) of [4], and (93))

$$e^{i(n_1-(2\pi)^{-1}(\beta_1,\delta^*))s}\varphi_{j',v(\beta)}(s)$$
$$=\sum_{|j_1\delta|<9r} a(n_1,\beta_1,j',\beta,j'+j_{1,}\beta+\beta_1)\varphi_{j'+j_1,v(\beta+\beta_1)}(s)+O(\rho^{-p\alpha}), \quad (95)$$

$$e^{-i(n_2-(2\pi)^{-1}(-\beta_1,\delta^*))s}\varphi_{j,v(\beta)}(s)$$
$$=\sum_{|j_1\delta|<9r}\overline{a(n_2,-\beta_1,j',\beta,j'+j_{1,}\beta+\beta_1)}\varphi_{j'+j_1,v(\beta+\beta_1)}+O(\rho^{-p\alpha}), \quad (96)$$

$$\sum_{j_1} | a(n_1,\beta_1,j^{'},\beta,j^{'}+j_1,\beta+\beta_1) |= O(1) \qquad (97)$$

give the proof of (94). Now from (88), (90), (91), (94) we obtain

$$\begin{aligned}\overline{S_1(j^{'},\lambda_{j,\beta})} &= \sum_{\beta_1}(\sum_{n_1}(\sum_{n_2}\frac{c(n_1,\beta_1)c(n_2,-\beta_1)a(n_1+n_2,0,j^{'},\beta,j,\beta)}{-2\langle\beta,\beta_1\rangle}\\ &+O(\rho^{-2a}r^2)\end{aligned}$$

Grouping the terms with indices (β_1,n_1,n_2) and $(-\beta_1,n_2,n_1)$ and using $\frac{1}{\langle\beta,\beta_1\rangle}+\frac{1}{\langle\beta,-\beta_1\rangle}=0$ we get $\overline{S_1(j^{'},\lambda_{j,\beta})}=O(\rho^{-2a}r^2)$.

5 Appendix 2. The proof of (54)

Using (91), we obtain

$$\begin{aligned}\overline{S_3(\lambda_{j,\beta})} &= \sum_{\beta_1,\beta_2}(\sum_{n_1,n_2,n_3}(\sum_{j_1}\sum_{j_2}\frac{c(n_1,\beta_1)c(n_2,\beta_2)c(n_3,-\beta_1-\beta_2)}{(\lambda_{j,\beta}-\lambda_{j+j_1,\beta+\beta_1})(\lambda_{j,\beta}-\lambda_{j+j^2,\beta+\beta^2})}\times\\ &a(n_1,\beta_1,j,\beta,j+j_1,\beta+\beta_1)\times\\ &a(n_2,\beta_2,j+j_1,\beta+\beta_1,j+j_1+j_2,\beta+\beta_1+\beta_2)\times\\ &a(n_3,-\beta_1-\beta_2,j+j_1+j_2,\beta+\beta_1+\beta_2,j,\beta),\end{aligned}$$

where $(j_1,\beta_1)\in Q(\rho^\alpha,9r_1)$, $(j_2,\beta_2)\in Q(\rho^\alpha,90r_1)$, $j\in S_1,\beta_1+\beta_2\neq 0$. Applying (94) two times and using (97) we get

$$\begin{aligned}&\sum_{j_1} a(n_1,\beta_1,j,\beta,j+j_1,\beta+\beta_1)\times\\ &(\sum_{j_2} a(n_2,\beta_2,j+j_1,\beta+\beta_1,j+j_1+j_2,\beta+\beta_1+\beta_2)\times\\ &a(n_3,-\beta_1-\beta_2,j+j_1+j_2,\beta+\beta_1+\beta_2,j,\beta))\end{aligned}$$

$=\sum_{j_1} a(n_1,\beta_1,j,\beta,j+j_1,\beta+\beta_1)(a(n_2+n_3,-\beta_1,j+j_1,\beta+\beta_1,j,\beta)+O(\rho^{-p\alpha}))=a(n_1+n_2+n_3,0,j,\beta,j,\beta)+O(\rho^{-p\alpha})$

Using this in above expression for $S_3(\lambda_{j,\beta})$ and taking into account that

$$\begin{aligned}\lambda_{j,\beta}-\lambda_{j+j_1,\beta+\beta_1} &= -2\langle\beta,\beta_1\rangle+O(\rho^{2\alpha_1}), | \langle\beta,\beta_1\rangle |> \frac{1}{3}\rho^a\\ \lambda_{j,\beta}-\lambda_{j+j^2,\beta+\beta^2} &= -2\langle\beta,\beta_1+\beta_2\rangle+O(\rho^{2\alpha_1}), | \langle\beta,\beta_1+\beta_2\rangle |> \frac{1}{3}\rho^a,\end{aligned}$$

which can be proved as (89) we have $S_3(\lambda_{j,\beta}) = O(\rho^{-3a})+$

$$\sum_{\beta_1,\beta_2}(\sum_{n_1,n_2,n_3}\frac{c(n_1,\beta_1)c(n_2,\beta_2)c(n_3,-\beta_1-\beta_2)a(n_1+n_2+n_3,0,j,\beta,j,\beta)}{4\langle\beta,\beta_1\rangle\langle\beta,\beta_1+\beta_2\rangle}$$

Grouping in the last sum terms with the multiplicand

$$c(n_1,\beta_1)c(n_2,\beta_2)c(n_3,-\beta_1-\beta_2)=c(n_2,\beta_2)c(n_1,\beta_1)c(n_3,-\beta_1-\beta_2)=$$

$$c(n_1,\beta_1)c(n_3,-\beta_1-\beta_2)c(n_2,\beta_2)=c(n_2,\beta_2)c(n_3,-\beta_1-\beta_2)c(n_1,\beta_1)=$$

$$c(n_3,-\beta_1-\beta_2)c(n_1,\beta_1)c(n_2,\beta_2)=c(n_3,-\beta_1-\beta_2)c(n_2,\beta_2)c(n_1,\beta_1)$$

and using $\frac{1}{\langle\beta,\beta_1\rangle\langle\beta,\beta_1+\beta_2\rangle}+\frac{1}{\langle\beta,\beta_2\rangle\langle\beta,\beta_2+\beta_1\rangle}+\frac{1}{\langle\beta,\beta_1\rangle\langle\beta,-\beta_2\rangle}$ $\frac{1}{\langle\beta,\beta_2\rangle\langle\beta-,\beta_1\rangle}+\frac{1}{\langle\beta,-\beta_1-\beta_2\rangle\langle\beta,-\beta_2\rangle}+\frac{1}{\langle\beta,-\beta_1-\beta_2\rangle\langle\beta,-\beta_1\rangle}=0$
we see that this sum is zero, that is, $S_3(\lambda_{j,\beta}) = O(\rho^{-3a}) = O(\rho^{-2-\varepsilon})$.

6 Appendix 3. The proof of (53)

Here we prove (53), where

$$\overline{S_1(j,\lambda_{j,\beta})}\equiv\sum_{\beta_1}(\sum_{j_1}\frac{\overline{A(j,\beta,j+j_1,\beta+\beta_1)}\,\overline{A(j+j_1,\beta+\beta_1,j,\beta)}}{\lambda_{j,\beta}-\lambda_{j+j_1,\beta+\beta_1}}, \tag{98}$$

$(j_1,\beta_1)\in Q(\rho^\alpha,9r_1)$, $j\in S_1$, (see (88), (51)) and by (91)

$$\overline{S_1(j,\lambda_{j,\beta})}=\sum_{\beta_1}(\sum_{n_1:(n_1,\beta_1)\in\Gamma'(\rho^\alpha)}(\sum_{n_2:(n_2,-\beta_1)\in\Gamma'(\rho^\alpha)}(\sum_{j_1}\frac{c(n_1,\beta_1)c(n_2,-\beta_1)}{\lambda_{j,\beta}-\lambda_{j+j_1,\beta+\beta_1}}\times \tag{99}$$

$$a(n_1,\beta_1,j,\beta,j+j_1,\beta+\beta_1)a(n_2,-\beta_1,j+j_1,\beta+\beta_1,j,\beta).$$

Replasing $\lambda_{j,\beta}-\lambda_{j+j_1,\beta+\beta_1}$ by $-(2\langle\beta+\tau,\beta_1\rangle+\mid\beta_1\mid^2+\mu_{j+j_1}(v(\beta+\beta_1))-\mu_j(v(\beta)))$ and using (94) for $j^{'}=j$ we have

$$\overline{S_1(j,\lambda_{j,\beta})}=\sum_{\beta_1}(\sum_{n_1}(\sum_{n_2}\frac{c(n_1,\beta_1)c(n_2,-\beta_1)a(n_1+n_2,0,j,\beta,j,\beta)}{-2\langle\beta+\tau,\beta_1\rangle}+$$

$$\sum_{\beta_1}(\sum_{n_1}(\sum_{n_2}(\sum_{j_1}\frac{c(n_1,\beta_1)c(n_2,-\beta_1)a(n_1,\beta_1,j,\beta,j+j_{1,}\beta+\beta_1)}{2\langle\beta+\tau,\beta_1\rangle(2\langle\beta+\tau,\beta_1\rangle+\mid\beta_1\mid^2+\mu_{j+j_1}-\mu_j)}\times$$

$a(n_2,-\beta_1,j+j_1,\beta+\beta_1,j,\beta)(\mid\beta_1\mid^2+\mu_{j+j_1}(v(\beta+\beta_1))-\mu_j(v(\beta))),$

where in the first sum the terms with indices (β_1,n_1,n_2) and $(-\beta_1,n_2,n_1)$ eliminate each other. Thus we need to calculate the second sum. For this we use the following relation

$$\begin{aligned}
& \mu_{j+j_1}(v(\beta+\beta_1))a(n_1,\beta_1,j,\beta,j+j_1,\beta+\beta_1)\\
&= (e^{i(n_1-(2\pi)^{-1}\langle\beta_1,\delta^*\rangle)s}\varphi_{j,v(\beta)}(s),T_v\varphi_{j+j_1,v(\beta+\beta_1)}(s)\\
&= (T_v(e^{i(n_1-(2\pi)^{-1}\langle\beta_1,\delta^*\rangle)s}\varphi_{j,v(\beta)}(s)),\varphi_{j+j_1,v(\beta+\beta_1)}(s)\\
&= ((\mid n_1-(2\pi)^{-1}\langle\beta_1,\delta^*\rangle\mid^2\mid\delta\mid^2+\mu_j(v))\times\\
& (e^{i(n_1-(2\pi)^{-1}\langle\beta_1,\delta^*\rangle)s}\varphi_{j,v(\beta)}(s)),\varphi_{j+j_1,v(\beta+\beta_1)}(s))\\
-2i(n_1-(2\pi)^{-1}\langle\beta_1,\delta^*\rangle) &\mid \delta\mid^2(e^{i(n_1-(2\pi)^{-1}\langle\beta_1,\delta^*\rangle)s}\varphi'_{j,v(\beta)}(s)),\varphi_{j+j_1,v(\beta+\beta_1)}(s))
\end{aligned}$$

¿From this, (94), and the formula

$$\begin{aligned}
&\sum_{j_1}(e^{i(n_1-(2\pi)^{-1}\langle\beta_1,\delta^*\rangle)s}\varphi'_{j,v(\beta)}(s)),\varphi_{j+j_1,v(\beta+\beta_1)}(s))a(n_2,-\beta_1,j+j_1,\beta+\beta_1,j,\beta)\\
&= (e^{i(n_1+n_2)s}\varphi'_{j,v(\beta)}(s)),\varphi_{j,v(\beta)}(s))+O(\rho^{-p\alpha}),
\end{aligned}$$

which can be proved as (94), it follows that

$$\sum_{j_1}\mu_{j+j_1}(v(\beta+\beta_1))a(n_1,\beta_1,j,\beta,j+j_1,\beta+\beta_1)\times \qquad (100)$$
$$a(n_2,-\beta_1,j+j_1,\beta+\beta_1,j,\beta)$$

$=(\mid n_1-(2\pi)^{-1}\langle\beta_1,\delta^*\rangle\mid^2)\mid\delta\mid^2+\mu_j(v))a(n_1+n_2,0,j,\beta,j,\beta)-$
$2i(n_1-(2\pi)^{-1}\langle\beta_1,\delta^*\rangle)\mid\delta\mid^2(e^{i(n_1+n_2)s}\varphi'_{j,v(\beta)}(s)),\varphi_{j,v(\beta)}(s)),$
where the last term can be calculated as follows

$$\begin{aligned}
& \mu_j(v)(\varphi_{j,v(\beta)}(s),e^{i(n_1+n_2)s}\varphi_{j,v(\beta)}(s))\\
&= (\varphi_{j,v(\beta)}(s),T_v(e^{i(n_1+n_2)s}\varphi_{j,v(\beta)}(s)))\\
&= (n_1+n_2)^2\mid\delta\mid^2(\varphi_{j,v(\beta)}(s),e^{i(n_1+n_2)s}\varphi_{j,v(\beta)}(s))+\\
2i(n_1+n_2) &\mid \delta\mid^2(\varphi_{j,v(\beta)}(s),e^{i(n_1+n_2)s}\varphi'_{j,v(\beta)}(s))+\\
& \mu_j(v)(\varphi_{j,v(\beta)}(s),e^{i(n_1+n_2)s}\varphi_{j,v(\beta)}(s)),
\end{aligned}$$

and hence

$$(e^{i(n_1+n_2)s}\varphi_{j,v(\beta)}^{'}(s)),\varphi_{j,v(\beta)}(s))=\frac{n_1+n_2}{2i}(e^{i(n_1+n_2)s}\varphi_{j,v(\beta)}(s)),\varphi_{j,v(\beta)}(s))$$

Using this, (100), and (94) we get

$$\begin{aligned}
&\sum_{j_1}(a(n_1,\beta_1,j,\beta,j+j_{1,}\beta+\beta_1)a(n_2,-\beta_1,j+j_1,\beta+\beta_1,j,\beta))\times\\
(\quad|\quad&\beta_1\ |^2+\mu_{j+j_1}(v(\beta+\beta_1))-\mu_j(v(\beta)))=a(n_1+n_2,0,j,\beta,j,\beta)\times\\
(\quad|\quad&\beta_1\ |^2+|\ n_1-\frac{\langle\beta_1,\delta^*\rangle}{2\pi}\ |^2|\ \delta\ |^2-(n_1-\frac{\langle\beta_1,\delta^*\rangle}{2\pi})\ |\ \delta\ |^2\ (n_1+n_2))\\
=\quad&(|\ \beta_1\ |^2+|\ \delta\ |^2\ (n_1-\frac{\langle\beta_1,\delta^*\rangle}{2\pi})(-n_2-\frac{\langle\beta_1,\delta^*\rangle}{2\pi}))a(n_1+n_2,0,j,\beta,j,\beta),
\end{aligned}$$

Thus $\overline{S_1(j,\lambda_{j,\beta})}=S+O(\rho^{-2-\varepsilon})$, where

$$\begin{aligned}
S\quad=\quad&\sum_{\beta_1,n_1,n_2}\frac{c(n_1,\beta_1)c(n_2,-\beta_1)a(n_1+n_2,0,j,\beta,j,\beta)}{4\ |\ \langle\beta+\tau,\beta_1\rangle\ |^2}\times \qquad (101)\\
(\quad|\quad&\beta_1\ |^2+(n_1-\frac{\langle\beta_1,\delta^*\rangle}{2\pi})(-n_2-\frac{\langle\beta_1,\delta^*\rangle}{2\pi})\ |\ \delta\ |^2)
\end{aligned}$$

Now we consider

$$\int_F|f_{\delta,\beta+\tau,\rho}(x)|^2\left|\varphi_{n,v}^{\delta}(\langle\delta,x\rangle)\right|^2dx=J, \qquad (102)$$

where $f_{\delta,\beta+\tau,\rho}(x)=\sum_{\gamma\in\Gamma\backslash\delta R,\ |\gamma|<\rho^\alpha}\frac{\gamma}{\langle\beta+\tau,\gamma\rangle}q_\gamma e^{i\langle\gamma,x\rangle}$ and by (92)

$$f_{\delta,\beta+\tau,\rho}(x)=\sum_{(n_1,\beta_1)\in\Gamma_\delta^{'}(\rho^\alpha)}\frac{\beta_1+(n_1-\frac{\langle\beta_1,\delta^*\rangle}{2\pi})\delta}{\langle\beta+\tau,\beta_1\rangle}c(n_1,\beta_1)e^{i\langle\beta_1+(n_1-\frac{\langle\beta_1,\delta^*\rangle}{2\pi})\delta,x\rangle}$$

Here for fixed x the $f_{\delta,\beta+\tau,\rho}(x)$ is a vector of R^d and $|f_{\delta,\beta+\tau,\rho}(x)|$ is a norm of this vector, hence using $\langle\beta,\delta\rangle=0$ for $\beta\in\Gamma_\delta$ we obtain

$$\begin{aligned}
|f_{\delta,\beta+\tau,\rho}(x)|^2\quad=\quad&\sum_{(n_1,\beta_1),(n_2,\beta_2)\in\Gamma_\delta^{'}(\rho^\alpha)}\frac{\langle\beta_1,\beta_2\rangle+(n_1-\frac{\langle\beta_1,\delta^*\rangle}{2\pi})(n_2-\frac{\langle\beta_2,\delta^*\rangle}{2\pi})\ |\ \delta\ |^2}{\langle\beta+\tau,\beta_1\rangle\langle\beta+\tau,\beta_2\rangle}\times\\
&c(n_1,\beta_1)c(-n_2,-\beta_2)e^{i\langle\beta_1-\beta_2+(n_1-n_2-(2\pi)^{-1}\langle\beta_1-\beta_2,\delta^*\rangle)\delta,x\rangle}
\end{aligned}$$

Since $\varphi_{j,v}^{\delta}(\langle\delta,x\rangle)$ is a function of $\langle\delta,x\rangle$ we have $\int_F e^{i\langle\beta_1-\beta_2+(n_1-n_2-(2\pi)^{-1}\langle\beta_1-\beta_2,\delta^*\rangle)\delta,x\rangle}\left|\varphi_{j,v}^{\delta}(\langle\delta,x\rangle)\right|^2 dx = 0$ for $\beta_1\neq\beta_2$. Therefore

$$\int_F |f_{\delta,\beta+\tau,\rho}(x)|^2\left|\varphi_{j,v}^{\delta}(\langle\delta,x\rangle)\right|^2 dx = \sum_{\beta_1,n_1,n_2}\frac{c(n_1,\beta_1)c(-n_2,-\beta_1)}{\mid(\beta+\tau,\beta_1)\mid^2}\times$$

$$(\mid\beta_1\mid^2+(n_1-\frac{\langle\beta_1,\delta^*\rangle}{2\pi})(n_2-\frac{\langle\beta_1,\delta^*\rangle}{2\pi})\mid\delta\mid^2 a(n_1-n_2,0,j,\beta,j,\beta).$$

Replasing n_2 by $-n_2$ we get $\int_F |f_{\delta,\beta+\tau,\rho}(x)|^2\left|\varphi_{n,v}^{\delta}(\langle\delta,x\rangle)\right|^2 dx = 4S$ (see (101)). Thus the requared estimation is obtained.

7 Appendix 4. The asymptotic of $\left|\varphi_{n,v}^{\delta}(s)\right|^2$

Here we calculate the asymptotic of eigenfunction $\varphi_{n,v}(s)$ of the operator $T_v(p(s))$ where $v\neq 0,\ \frac{1}{2}$ and

$$p(s)=\sum_{m=1,-1}p_m e^{ims}. \tag{103}$$

The eigenvalues and eigenfunctions of $T_v(0)$ are $(n+v)^2\mid\delta\mid^2$ and $e^{i(n+v)s}$, for $n\in Z$. Since the eigenvalues of $T_v(p)$ are simple for $v\neq 0,\ \frac{1}{2}$ by well-known perturbation formula

$(\varphi_{n,v}(s),\ e^{i(n+v)s})\varphi_{n,v}(s)=e^{i(n+v)s}+$

$$\sum_{k=1,2,\ldots}\frac{(-1)^{k+1}}{2i\pi}\int_C (T_v(0)-\lambda)^{-1}p(x))^k(T_v(0)-\lambda)^{-1}e^{i(n+v)s}d\lambda, \tag{104}$$

where C is contour containing only the eigenvalue $(n+t)^2\mid\delta\mid^2$. Using (103) and

$$(T_v(0)-\lambda)^{-1}e^{i(n+v)s}=\frac{e^{i(n+v)s}}{(n+v)^2\mid\delta\mid^2-\lambda},$$

we see that k-th $(k=1,2,3,4)$ term F_k of series (104) has form

$$F_1=\frac{1}{2i\pi}\int_C\sum_{m=1,-1}\frac{p_m e^{i(n+m+v)s}}{((n+v)^2\mid\delta\mid^2-\lambda)((n+m+v)^2\mid\delta\mid^2-\lambda)}d\lambda$$

$$F_2 = \frac{-1}{2i\pi}\int\limits_C \sum_{m,l=1,-1} \frac{p_m p_l e^{i(n+m+l+v)s}}{((n+v)^2 \mid \delta \mid^2 - \lambda)} \times$$

$$\frac{1}{((n+m+v)^2 \mid \delta \mid^2 - \lambda)((n+m+l+v)^2 \mid \delta \mid^2 - \lambda)} d\lambda,$$

$$F_3 = \frac{1}{2i\pi}\int\limits_C \sum_{m,l,k=1,-1} \frac{p_m p_l p_k e^{i(n+m+l+k+v)s}}{((n+v)^2 \mid \delta \mid^2 - \lambda)((n+m+v)^2 \mid \delta \mid^2 - \lambda)} \times$$

$$\frac{1}{((n+m+l+v)^2 \mid \delta \mid^2 - \lambda)((n+m+l+k+v)^2 \mid \delta \mid^2 - \lambda)} d\lambda,$$

$$\begin{aligned} F_4 &= \frac{-1}{2i\pi}\int\limits_C \sum_{m,l,k,r=1,-1} \frac{p_m p_l p_k p_r e^{i(n+m+l+k+r+v)s}}{((n+m+l+k+r+v)^2 \mid \delta \mid^2 - \lambda)} \times \\ & \frac{1}{((n+m+v)^2 \mid \delta \mid^2 - \lambda)((n+m+l+v)^2 \mid \delta \mid^2 - \lambda)} \times \\ & \frac{1}{((n+m+l+k+v)^2 \mid \delta \mid^2 - \lambda)((n+v)^2 \mid \delta \mid^2 - \lambda)} d\lambda. \end{aligned}$$

Since the distance between $(n+v)^2 \mid \delta \mid^2$ and $(n+m+v)^2 \mid \delta \mid^2$ more than $2c_{26}n$, we can choose the contour C such that the modulus of all $k+1$ multiplicants $((s+v)^2 \mid \delta \mid^2 - \lambda)$ in expression of F_k more than $c_{27}n$, and the length of C less than c_{28}. Therefore

$$\sum_{k=5,6,\ldots} F_k = O(\frac{1}{n^5}).$$

Now we calculate the integrals in F_1, F_2, F_3, F_4 by Cauchy integral formula and then decompose the obtained expression in power of $\frac{1}{n}$. Thus

$$F_1 = e^{i(n+v)s}((p_1 e^{is} - p_{-1}e^{-is})\frac{1}{\mid \delta \mid^2}(\frac{-1}{2n} + \frac{v}{2n^2} - \frac{4v^2+1}{8n^3} + O(\frac{1}{n^4})) +$$

$$(p_1e^{is}+p_{-1}e^{-is})\frac{1}{\mid\delta\mid^2}(\frac{v}{4n^2}-\frac{v}{2n^3}+\frac{12v^2+1}{16n^4}+O(\frac{1}{n^5})))$$

We see that $F_2=F_{2,1}+F_{2,2}$, where $F_{2,1}$ and $F_{2,2}$ are the sum of terms in F_2 for which respectively $m+l=\pm2$ and $m+l=0$. Thus

$$F_{2,1}=e^{i(n+v)s}(((p_1)^2e^{2is}+(p_{-1})^2e^{-2is})\frac{1}{\mid\delta\mid^4}(\frac{-1}{8n^2}+\frac{-s}{4n^3}-\frac{12s^2+7}{32n^4}+O(\frac{1}{n^5}))+$$

$$((p_1)^2e^{2is}-(p_{-1})^2e^{-2is})\frac{1}{\mid\delta\mid^4}(\frac{-3}{16n^3}+O(\frac{1}{n^4})))$$

and

$$F_{2,2}=e^{i(n+v)s}\left|p_1\right|^2(\frac{c_{29}}{n^2}++\frac{c_{30}}{n^3}+\frac{c_{31}}{n^4}+O(\frac{1}{n^5})),$$

where $c_{29},c_{30},...$,are knowh constants. As above we write $F_3=F_{3,1}+F_{3,2}$, where $F_{3,1}$ and $F_{3,2}$ are the sum of terms in F_3 for which respectively $m+l+k=\pm3$ and $m+l+k=\pm1$. So

$$F_{3,1}=e^{i(n+v)s}((p_1^3e^{3is}-p_{-1}^3e^{-is})\frac{1}{\mid\delta\mid^6}(\frac{-1}{48n^3}+O(\frac{1}{n^4}))+$$

$$(p_1^3e^{3is}+p_{-1}^3e^{-3is})\frac{1}{\mid\delta\mid^6}(\frac{1}{16n^4}+O(\frac{1}{n^5})))$$

and

$$F_{3,2}=e^{i(n+v)s}((p_1e^{is}-p_{-1}e^{-is})\left|p_1\right|^2(\frac{c_{32}}{n^3}+\frac{c_{33}}{n^4}+O(\frac{1}{n^5}))+$$

$$(p_1e^{is}+p_{-1}e^{-is})\left|p_1\right|^2(\frac{c_{34}}{n^4}+O(\frac{1}{n^5})))$$

Similarly we can write $F_4=F_{4,1}+F_{4,2}+F_{4,3}$, where $F_{4,1}$, $F_{4,2}$, $F_{4,3}$ are the sum of terms in F_4 for which respectively $m+l+k+r=\pm4$, $m+l+k+r=\pm2$, $m+l+k+r=0$. Hence

$$F_{4,1}=e^{i(n+v)s}(p_1^4e^{4is}+p_{-1}^4e^{-4is})\frac{1}{\mid\delta\mid^8}(\frac{1}{384n^4}+O(\frac{1}{n^5})),$$

$$F_{4,2}=e^{i(n+v)s}(p_1^2e^{2is}+p_{-1}^2e^{-2is})\,|p_1|^2\,(\frac{c_{35}}{n^4}+O(\frac{1}{n^5}))),$$

$$F_{4,3}=e^{i(n+v)s}\,|p_1|^4\,(\frac{c_{36}}{n^4}+O(\frac{1}{n^5}))).$$

Since $p_{-1}^ke^{-iks}$ is conjugate of $p_1^ke^{iks}$ the real and imaginary parts of $F_ke^{-i(n+v)s}$ consist respectively of terms with multiplicant $p_1^ke^{iks}+p_{-1}^ke^{-kis}$ and $p_1^ke^{iks}-p_{-1}^ke^{-iks}$. Taking into account it, and using the above calculation we get

$$\begin{aligned}|(\varphi_{n,v}(s),e^{i(n+v)s})\varphi_{n,v}(s)|^2 &= 2(\sum_{k=1,2,3,4}\mathbf{Re}F_k+\mathbf{Re}F_1F_2+\mathbf{Re}F_1F_3)\\ &\quad+|F_1|^2+|F_2|^2+O(n^{-5})\end{aligned}$$

$$\begin{aligned}= & \ 1+\frac{1}{2n^2}\frac{1}{\mid\delta\mid^2}(p_1e^{is}+p_{-1}e^{-is}+c_{37}|p_1|^2)+\frac{1}{n^3}((p_1e^{is}+p_{-1}e^{-is})c_{38}\\ & +c_{39}|p_1|^2)+\frac{1}{n^4}((p_1e^{is}+p_{-1}e^{-is})c_{40}+c_{41}|p_1|^2+c_{42}|p_1|^4\\ & +c_{43}|p_1|^2(p_1e^{is}+p_{-1}e^{-is})+(c_{44}+c_{45}|p_1|^2)(p_1^2e^{2is}+p_{-1}^2e^{-2is}))+O(\frac{1}{n^5}).\end{aligned}$$

On the other hand

$$|(\varphi_{n,v}(s),e^{i(n+v)s})|^2=(c_{46}\frac{1}{n^2}+c_{47}\frac{1}{n^3}+c_{48}\frac{1}{n^4})|p_1|^2+c_{49}\frac{1}{n^4}|p_1|^4+O(\frac{1}{n^5}).$$

So the formula (10) is proved ■

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