Massive Scalar Field in an One-Dimensional Oscillating Region

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June 27, 2002

Abstract

The classical scalar massive field satisfying the Klein-Gordon equation in a finite one-dimensional space interval of periodically varying length with Dirichlet boundary conditions is studied. For the sufficiently small mass, the energy can exponentially grow with time under the same conditions as for the massless case. The proofs are based on estimates of exactly given mass-induced corrections to the massless case.

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1 Introduction

The classical massless scalar field satisfying d'Alembert equation in 1+1 dimensional space-time restricted to finite space interval with one end-point fixed and other end-point periodically oscillating was studied in several papers [1] - [7]. At the end-points, Dirichlet boundary conditions are assumed. Results for Neumann boundary conditions are also known [5] but the physical condition on the moving boundary obtained from the static Neumann one by Lorentz transformation is different and the results for this condition are more similar to those for Dirichlet boundary condition. Various regimes of the energy time evolution are possible - the energy can be unlimited or bounded, with limit or without limit at time infinity. In particular, there are cases where energy $E(t)$ is exponentially growing with time t in the sense that

$$Ae^{\gamma t} \leq E(t) \leq Be^{\gamma t}$$

for all $t > 0$ with some $A, B, \gamma > 0$ (the energy is not monotone in time for a periodic wall motion of course).

Similar model in quantum field theory was also studied in [8] and references therein.

In the present paper we consider classical massive scalar field satisfying Klein-Gordon equation with Dirichlet boundary conditions at the end-points of finite one-dimensional space interval, one end-point periodically moving. We prove that the simplest sufficient conditions for the exponential growth of the energy are the same as for the d'Alembert equation provided that the mass is small enough. This shows that the mass can be considered as a small perturbation.

2 The Model

Let φ be a real function defined on the space-time region

$$\Omega = \{(t, x) \in \mathbb{R}^2 | 0 \leq x \leq a(t), t \geq 0\} \tag{1}$$

where $a : \mathbb{R} \to \mathbb{R}$ is a strictly positive periodic C^2-function with a period $T > 0$. We assume that $|\dot{a}(t)| < 1$ (subluminal velocity of the end-point motion). We restrict ourselves to the fields φ which are in $\mathrm{C}^2(\mathring{\Omega}) \cap \mathrm{C}^1(\Omega)$ and assume that Klein-Gordon equation

$$\frac{\partial^2\varphi}{\partial t^2} - \frac{\partial^2\varphi}{\partial x^2} + m^2\varphi = 0 \tag{2}$$

is satisfied in the interior $\overset{\circ}{\Omega}$. The constant mass m is non-negative. On the boundary, the conditions

$$\varphi(0,x) = \varphi_0(x), \quad \frac{\partial\varphi(0,x)}{\partial t} = \varphi_1(x) \quad (0 < x < a(0)) \tag{3}$$

$$\varphi(t,0) = \varphi(t,a(t)) = 0 \quad (t \geq 0) \tag{4}$$

are required.

Let us define

$$h = \mathrm{Id} - a, \quad k = \mathrm{Id} + a, \quad F = k \circ h^{-1} \tag{5}$$

as in [5]. Then $h, k, F : \mathbb{R} \to \mathbb{R}$ are increasing C^2-functions. The identities

$$\begin{aligned} F &= \mathrm{Id} + 2a \circ h^{-1} = 2h^{-1} - \mathrm{Id} \quad , \\ F^{-1} &= \mathrm{Id} - 2a \circ k^{-1} = 2k^{-1} - \mathrm{Id} \quad . \end{aligned} \tag{6}$$

are useful in some calculations.

We shall rewrite the problem in new variables

$$\xi = t + x, \quad \eta = t - x \quad . \tag{7}$$

It is not difficult to see that Ω is transformed into the set $\tilde{\Omega}$ described by inequalities

$$|\eta| \leq \xi \leq F(\eta), \quad \eta \geq -a(0) \tag{8}$$

or equivalently

$$\max\{-\xi, F^{-1}(\xi)\} \leq \eta \leq \xi, \quad \xi \geq 0 \tag{9}$$

the last inequalities in (8) and (9) following automatically from the first ones in fact. The inequalities describing $\tilde{\Omega}$ can be also written distinguishing two cases

$$0 \leq \xi \leq a(0), \quad -\xi \leq \eta \leq \xi \quad , \tag{10}$$

$$\xi \geq a(0), \quad F^{-1}(\xi) \leq \eta \leq \xi \quad . \tag{11}$$

The transformed field

$$\tilde{\varphi}(\xi,\eta) = \varphi(t,x) \tag{12}$$

satisfies the equation

$$\frac{\partial^2 \tilde{\varphi}}{\partial\xi\partial\eta} = -\frac{m^2}{4}\tilde{\varphi} \tag{13}$$

in $\overset{\circ}{\tilde{\Omega}}$ with boundary conditions given by the transformation of equations (3-4).

The energy of the field

$$\begin{aligned} E_m(t) &= \frac{1}{2}\int_0^{a(t)} \left[\left(\frac{\partial\varphi(t,x)}{\partial t}\right)^2 + \left(\frac{\partial\varphi(t,x)}{\partial x}\right)^2 + m^2\varphi(t,x)^2 \right] dx \qquad (14) \\ &= \int_0^{a(t)} \left[\left(\frac{\partial\tilde{\varphi}(\xi,\eta)}{\partial\xi}\right)^2 + \left(\frac{\partial\tilde{\varphi}(\xi,\eta)}{\partial\eta}\right)^2 + \frac{1}{2}m^2\tilde{\varphi}(\xi,\eta)^2 \right] dx \quad . \end{aligned}$$

3 Solution of Inhomogeneous Equation

As a preliminary step, let us consider the equation

$$\frac{\partial^2 \tilde{\varphi}}{\partial\xi\partial\eta} = f(\xi,\eta) \tag{15}$$

with given function $f \in \mathrm{C}^1(\tilde{\Omega})$ for an unknown function $\tilde{\varphi}$ satisfying the same boundary conditions as the required solution of Klein-Gordon equation. We shall put $f = -\frac{1}{4}m^2\tilde{\varphi}$ after some calculations. The results of this and the next section do not require the periodicity of function a so it is not assumed here.

We put Equation (15) into an integral form integrating twice. The integration bounds must be taken in such a way that the integration domain is included in $\tilde{\Omega}$. A possible choice is

$$\frac{\partial\tilde{\varphi}(\xi,\eta)}{\partial\xi} = -\int_\eta^\xi f(\xi,z)dz + H_1(\xi) \tag{16}$$

with an arbitrary function H_1 (which must be in C^1 if $\tilde{\varphi}$ should be in C^2 of course). It is clear from (9) that if $(\xi,\eta) \in \tilde{\Omega}$ then $(\xi,z) \in \tilde{\Omega}$ for all $\eta \leq z \leq \xi$ so the choice of integration bounds is possible.

Similarly,

$$\tilde{\varphi}(\xi,\eta) = \int_{|\eta|}^{\xi} \frac{\partial \tilde{\varphi}(y,\eta)}{\partial y} dy + G_1(\eta) \tag{17}$$

taking into account (8). Here G_1 is again an arbitrary function with continuous second derivative (possibly with exception of the point $\eta = 0$). The last two formulas give

$$\tilde{\varphi}(\xi,\eta) = -\int_{|\eta|}^{\xi} dy \int_{\eta}^{y} dz f(y,z) + H(\xi) + G(\eta) \tag{18}$$

where H and G are up to now arbitrary functions (such that $\tilde{\varphi}$ is in C^2) which has to be determined form the boundary conditions and Cauchy data.

For $t = 0$, i.e. $\xi = -\eta = x \in [0, a(0)]$, Eqs. (3) read

$$\varphi_0(\xi) = H(\xi) + G(-\xi) \tag{19}$$

$$\varphi_1(\xi) = -2\int_{-\xi}^{\xi} f(\xi,z)dz + H'(\xi) + G'(-\xi) \tag{20}$$

Dirichlet boundary condition at $x = 0$, i.e. $\xi = \eta = t \geq 0$, gives

$$H(\xi) = -G(\xi) \tag{21}$$

and that at $x = a(t)$, i.e. $\xi = F(\eta)$, $\eta \geq -a(0)$, reads

$$0 = -\int_{|\eta|}^{F(\eta)} dy \int_{\eta}^{y} dz f(y,z) + H(F(\eta)) + G(\eta) \quad . \tag{22}$$

By (21) we can use function G only,

$$\tilde{\varphi}(\xi,\eta) = -\int_{|\eta|}^{\xi} dy \int_{\eta}^{y} dz f(y,z) + G(\eta) - G(\xi) \quad . \tag{23}$$

Equations (19-20) give

$$G(\eta) = -\frac{1}{2}\varphi_0(|\eta|)\mathrm{sgn}(\eta) - \frac{1}{2}\int_0^{|\eta|} \varphi_1(x)dx - \int_0^{|\eta|} dy \int_{-y}^{y} dz f(y,z) + c \tag{24}$$

for $\eta \in [-a(0), a(0)]$. Here c is an arbitrary constant which we can choose as zero since $\tilde{\varphi}$ is independent of it. Relation (22) gives a prolongation formula for G,

$$G(F(\eta)) = G(\eta) - \int_{|\eta|}^{F(\eta)} dy \int_{\eta}^{y} dz f(y,z) \tag{25}$$

for $\eta \geq -a(0)$. By Eqs. (23-25), $\tilde{\varphi}$ is determined in the whole domain $\tilde{\Omega}$.

It remains to prove that the above relations really determine a C^2-function $\tilde{\varphi}$. The only points where continuity of $\tilde{\varphi}$ and its derivatives requires a special check are $\eta = 0$, $\eta = a(0)$ (or $\xi = a(0)$) and their images by the function F (where the continuity then follows automatically). This can be done by a straightforward but a little tedious calculations which reveal a sufficient and necessary conditions on the Cauchy data φ_0, φ_1 and the right-hand side function f. The necessity of these conditions can be also easily seen from the boundary conditions and their derivatives. It is seen that G and G' are continuous in $[-a(0), \infty)$ while G'' is discontinuous at $\eta = 0$ and continuous in $[-a(0), 0) \cup (0, \infty)$. We summarize the obtained consistency conditions in the following Proposition. The derivatives in closed sets are considered as derivatives with respect to these sets here.

Proposition 1 *Let* $a \in \mathrm{C}^2([0,\infty))$, $\inf a > 0$, $|a'| < 1$, $f \in \mathrm{C}^1(\tilde{\Omega})$, $\varphi_0 \in \mathrm{C}^2([0, \mathrm{a}(0)])$, $\varphi_1 \in \mathrm{C}^1([0, \mathrm{a}(0)])$ *and the following relations are satisfied:*

$$\varphi_0(0) = 0, \quad \varphi_0(a(0)) = 0, \tag{26}$$

$$\varphi_1(0) = 0, \quad \varphi_1(a(0)) + a'(0)\varphi_0'(a(0)) = 0, \tag{27}$$

$$\varphi_0''(0) = -4f(0,0), \tag{28}$$

$$\begin{aligned}(1 + a'(0)^2)\varphi_0''(a(0)) + a''(0)\varphi_0'(a(0)) + 2a'(0)\varphi_1'(a(0))& \\ = -4f(a(0), -a(0)).&\end{aligned} \tag{29}$$

Then there exists a unique $\tilde{\varphi} \in \mathrm{C}^2(\tilde{\Omega})$ *satisfying the equation (15) in* $\tilde{\Omega}$ *and boundary conditions corresponding to the transformed relations (3-4). The solution* $\tilde{\varphi}$ *is given by Eqs. (23-25).*

□

4 Existence and Unicity of the Solution

We use the results of previous section to write down the integral form of the Klein-Gordon equation in our case. Let us start with some notations. For $(\xi, \eta) \in \tilde{\Omega}$ let us denote the corresponding time t as

$$T(\xi, \eta) = \frac{\xi + \eta}{2} \quad , \tag{30}$$

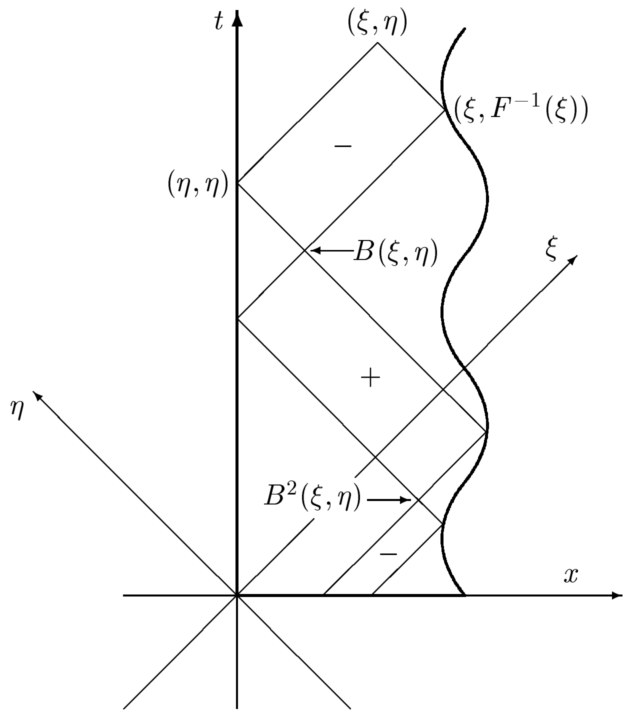

Figure 1: *The set $M(\xi,\eta)$ bounded by backward characteristics and the signs of function $\vartheta(\xi,\eta,y,z)$. The coordinates of the points are indicated in variables ξ and η. The number $N(\xi,\eta)=2$ for the displayed case.*

$$Q(\xi,\eta)=\{(y,z)\in\mathbb{R}^2\,|\,\eta\le y\le\xi, F^{-1}(\xi)\le z\le\eta\}$$
$$=[\eta,\xi]\times[F^{-1}(\xi),\eta] \tag{31}$$

the rectangle bounded by backward characteristics starting from (ξ,η) (see Fig. 1) and

$$B(\xi,\eta)=(\eta,F^{-1}(\xi)) \tag{32}$$

its lowest vertex. It is easy to verify that the written formulas correspond to Fig. 1. The following trivial facts are easily seen.

Lemma 1 *Let $(\xi,\eta)\in\tilde{\Omega}$. Then*

$$(y,z)\in Q(\xi,\eta)\Longrightarrow T(B(\xi,\eta))\le T(y,z)\le T(\xi,\eta)\quad , \tag{33}$$

$$Q(\xi,\eta) \subset \tilde{\Omega} \Longleftrightarrow B(\xi,\eta) \in \tilde{\Omega} \Longleftrightarrow T(B(\xi,\eta)) \geq 0 \quad , \tag{34}$$

$$Q(\xi,\eta) \cap \tilde{\Omega} = \{(y,z) \in Q(\xi,\eta) | T(y,z) \geq 0\}$$
$$= \{(y,z) \in \mathbb{R}^2 | \, |\eta| \leq y \leq \xi, \max(F^{-1}(\xi), -y) \leq z \leq \eta\} \quad , \tag{35}$$

$$T(\xi,\eta) - T(B(\xi,\eta)) = a \circ k^{-1}(\xi) \geq \inf a \quad . \tag{36}$$

□

Lemma 2 *(a) Let $\tilde{\varphi} \in C^2(\tilde{\Omega})$ satisfies Eq. (13) and the Dirichlet boundary conditions corresponding to (4). Then*

$$\tilde{\varphi}(\xi,\eta) = -\tilde{\varphi}(B(\xi,\eta)) - \frac{m^2}{4} \int_{Q(\xi,\eta)} \tilde{\varphi}(y,z) \, dy \, dz \tag{37}$$

for every $(\xi,\eta) \in \tilde{\Omega}$ such that $T(\xi,\eta) \geq 0$.

(b) Let $(\xi_0,\eta_0) \in \tilde{\Omega}$ be such that $T(B(\xi_0,\eta_0)) > 0$ and $\tilde{\varphi} \in C^0(\tilde{\Omega})$ has continuous first derivatives in a neighborhood of boundary $\partial Q(\xi_0,\eta_0)$. If $\tilde{\varphi}$ is of class C^2 in a neighborhood of point $B(\xi_0,\eta_0)$, satisfies Eq. (13) there, and satisfies Eq. (37) for (ξ,η) in a neighborhood of (ξ_0,η_0) then $\tilde{\varphi}$ is of class C^2 in a neighborhood of point (ξ_0,η_0) and satisfies Eq. (13) there.

Proof. Notice that $Q(\xi,\eta) \subset \tilde{\Omega}$ for $T(B(\xi,\eta)) \geq 0$ according to (34). For the statement (a), integrate Eq. (13) over (31) and use the Dirichlet boundary conditions at the vertices $(\xi_1 = \eta, \eta)$ and $(\xi, \eta_1 = F^{-1}(\xi))$ (notation as in Fig. 1). To see the statement (b), use Eqs. (31), (32) and differentiate (37). In particular,

$$\frac{\partial^2 \tilde{\varphi}(\xi,\eta)}{\partial\xi\partial\eta} = -\left[\frac{\partial^2 \tilde{\varphi}}{\partial\xi\partial\eta} + \frac{m^2}{4}\tilde{\varphi}\right]_{|B(\xi,\eta)} \frac{d}{d\xi} F^{-1}(\xi) - \frac{m^2}{4}\tilde{\varphi}(\xi,\eta) \, .$$

□

Remark. For $m = 0$ the Lemma is a special case of the well known relation of four values at the vertices of rectangle bounded by characteristics for the solution of the wave equation (e.g. Eq. (1.22) of Chapter 8 in [9]).

Lemma 3 *Let $(\xi,\eta) \in \tilde{\Omega}$ be such that $T(B(\xi,\eta)) \leq 0$, $\tilde{\varphi}$ satisfies Eq. (13) with boundary conditions corresponding to (3), (4) and $\tilde{\varphi}^{(0)}$ satisfies Eq. (13) with m replaced by zero and the same boundary conditions (3), (4) as $\tilde{\varphi}$. Then*

$$\tilde{\varphi}(\xi,\eta) = \tilde{\varphi}^{(0)}(\xi,\eta) - \frac{m^2}{4} \int_{Q(\xi,\eta)\cap\tilde{\Omega}} \tilde{\varphi}(y,z) \, dy \, dz \quad . \tag{38}$$

If $\tilde\varphi^{(0)} \in C^2(\tilde\Omega)$ and $\tilde\varphi \in C^0(\tilde\Omega)$ satisfies the relation (38) then $\tilde\varphi$ satisfies the boundary conditions corresponding to (3), (4) and Eq. (13). Further, $\tilde\varphi$ has continuous derivatives up to the second order in the considered part of $\tilde\Omega$.

Proof. Let us first realize that for (ξ,η) on the boundary of $\tilde\Omega$ the Lebesgue measure of the set $Q(\xi,\eta) \cap \tilde\Omega$ is zero so the first initial condition (3) and boundary conditions (4) are satisfied if Eq. (38) holds.

We shall use equations (23-25) for the several ranges of variables ξ, η. We denote as G_0 the function (24-25) corresponding to the d'Alembert equation solution $\tilde\varphi^{(0)}$ with the zero mass. As for $T(B(\xi,\eta)) \le 0$

$$F^{-1}(\xi) \le -\eta \le a(0) \tag{39}$$

(see (30) and (8)), $\xi \le F(a(0))$. From (9) and (39) we see that $F^{-1}(\xi) \le \eta \le -F^{-1}(\xi)$, i.e.

$$F^{-1}(\xi) \le 0 \quad , \quad |\eta| \le |F^{-1}(\xi)| \quad . \tag{40}$$

Further it is clear that ξ and η cannot be both simultaneously greater than $a(0)$ as $T(B(\xi,\eta))$ would be positive in this case (remember that $F^{-1}(a(0)) = -a(0)$). Taking into account (9) we see that always $\eta \le a(0)$ under the assumptions of Lemma. We have to distinguish two cases now.

1) $\xi \le a(0)$, $\eta \le a(0)$
Now equation (24) can be used in (23) and we obtain

$$\begin{aligned}
&\tilde\varphi(\xi,\eta) = \\
&\quad \tilde\varphi^{(0)}(\xi,\eta) + \frac{m^2}{4}\left[\int_{|\eta|}^{\xi} dy \int_{\eta}^{y} dz + \int_{0}^{|\eta|} dy \int_{-y}^{y} dz - \int_{0}^{\xi} dy \int_{-y}^{y} dz\right]\tilde\varphi(y,z) \\
&\quad = \tilde\varphi^{(0)}(\xi,\eta) + \frac{m^2}{4}\left[\int_{|\eta|}^{\xi} dy \int_{\eta}^{y} dz - \int_{|\eta|}^{\xi} dy \int_{-y}^{y} dz\right]\tilde\varphi(y,z) \\
&\quad = \tilde\varphi^{(0)}(\xi,\eta) - \frac{m^2}{4}\int_{|\eta|}^{\xi} dy \int_{-y}^{\eta} dz\, \tilde\varphi(y,z)
\end{aligned}$$

which is (38) due to (35) as here $F^{-1}(\xi) \le -a(0) \le -\xi \le -y$. On the contrary, differentiating the last equation and using the already proved boundary conditions we can verify that $\tilde\varphi$ has continuous second derivatives and satisfies (13) if the integral relation (38) holds. The initial condition for the time derivative in (3) is also seen.

2) $a(0) < \xi \leq F(a(0))$*,* $\eta \leq a(0)$
Now formula (24) holds for $G(\eta)$ while (25) gives

$$
\begin{aligned}
G(\xi) &= G(F^{-1}(\xi)) + \frac{m^2}{4}\int_{|F^{-1}(\xi)|}^{\xi} dy \int_{F^{-1}(\xi)}^{y} dz\, \tilde{\varphi}(y,z) \\
&= G_0(\xi) + \frac{m^2}{4}\left[\int_0^{|F^{-1}(\xi)|} dy \int_{-y}^{y} dz + \int_{|F^{-1}(\xi)|}^{\xi} dy \int_{F^{-1}(\xi)}^{y} dz\right] \tilde{\varphi}(y,z)
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{\varphi}(\xi,\eta) = \tilde{\varphi}^{(0)}(\xi,\eta) + \frac{m^2}{4}\Bigg[&\int_{|\eta|}^{\xi} dy \int_{\eta}^{y} dz + \int_0^{|\eta|} dy \int_{-y}^{y} dz \\
&- \int_0^{|F^{-1}(\xi)|} dy \int_{-y}^{y} dz - \int_{|F^{-1}(\xi)|}^{\xi} dy \int_{F^{-1}(\xi)}^{y} dz\Bigg] \tilde{\varphi}(y,z) \quad .
\end{aligned}
$$

With the help of (40) we combine the second and third integral and the first and fourth integral separating the first one into two parts. Then

$$
\begin{aligned}
\tilde{\varphi}(\xi,\eta) = \tilde{\varphi}^{(0)}(\xi,\eta) + \frac{m^2}{4}\Bigg[&\int_{|\eta|}^{|F^{-1}(\xi)|} dy \int_{\eta}^{y} dz \\
&- \int_{|F^{-1}(\xi)|}^{\xi} dy \int_{F^{-1}(\xi)}^{\eta} dz - \int_{|\eta|}^{|F^{-1}(\xi)|} dy \int_{-y}^{y} dz\Bigg] \tilde{\varphi}(y,z) \\
= \ \tilde{\varphi}^{(0)}(\xi,\eta) + \frac{m^2}{4}\Bigg[&- \int_{|\eta|}^{|F^{-1}(\xi)|} dy \int_{-y}^{\eta} dz - \int_{|F^{-1}(\xi)|}^{\xi} dy \int_{F^{-1}(\xi)}^{\eta} dz\Bigg] \tilde{\varphi}(y,z) \\
= \ \tilde{\varphi}^{(0)}(\xi,\eta) - \frac{m^2}{4}&\int_{|\eta|}^{\xi} dy \int_{\max(F^{-1}(\xi),-y)}^{\eta} dz\, \tilde{\varphi}(y,z)
\end{aligned}
$$

which is (38). On the contrary, differentiating the next-to-last equation and using the already proved boundary conditions we can verify that $\tilde{\varphi}$ has continuous second derivatives and satisfies (13) if the integral relation (38) holds.

The continuity of the first and second derivatives at $\xi = a(0)$ can be also seen.

$\square$

Let us denote the iterations of the map B as

$$B^0(\xi,\eta) = (\xi,\eta),\ B^1(\xi,\eta) = B(\xi,\eta),\ B^2(\xi,\eta) = B(B(\xi,\eta)),\ \ldots$$

and let

$$N(\xi,\eta) = \max\{n \in \mathbb{Z} |\, B^n(\xi,\eta) \in \tilde{\Omega}\} \quad , \tag{41}$$

$$M(\xi,\eta) = \bigcup_{n=0}^{N(\xi,\eta)-1} Q(B^n(\xi,\eta)) \cup (Q(B^{N(\xi,\eta)}(\xi,\eta)) \cap \tilde{\Omega}) \tag{42}$$

(see Fig. 1),

$$\vartheta(\xi,\eta,y,z) = (-1)^{n+1} \tag{43}$$

for $(y,z) \in Q(B^n(\xi,\eta))$, $n \in \mathbb{Z}$.

Theorem 1 *Let $\tilde{\varphi}^{(0)}$ satisfies Eq. (13) with m replaced by zero and boundary conditions corresponding to (3), (4). If $\tilde{\varphi} \in C^2(\tilde{\Omega})$ satisfies Eq. (13) with boundary conditions corresponding to (3), (4) then*

$$\tilde{\varphi}(\xi,\eta) = \tilde{\varphi}^{(0)}(\xi,\eta) + \frac{m^2}{4}\int_{M(\xi,\eta)} \vartheta(\xi,\eta,y,z)\tilde{\varphi}(y,z)\,dy\,dz \quad . \tag{44}$$

Conversely, if $\tilde{\varphi} \in C^0(\tilde{\Omega})$ satisfies Eq. (44) then $\tilde{\varphi} \in C^2(\tilde{\Omega})$, $\tilde{\varphi}$ satisfies Eq. (13) and the boundary conditions corresponding to (3-4).

Proof. Let $(\xi,\eta) \in \tilde{\Omega}$. By Eq. (36), $N(\xi,\eta)$ defined by (41) exists since $\inf\{a(t) \mid 0 \le t \le T(\xi,\eta)\} > 0$. With the help of Lemmas 2 and 3, the Theorem follows by induction with respect to $N(\xi,\eta)$. The continuity of $\tilde{\varphi}$ and its derivatives up to the second order at the curve $T(B(\xi,\eta)) = 0$ separating the regions of validity of the two lemmas can be also checked.

□

Lemma 4 *Let $a_{max} := \sup a < \infty$ and $(\xi,\eta) \in \tilde{\Omega}$. Then*

$$M(\xi,\eta) \subset \{(y,z) \in \mathbb{R}^2\,|0 \le y \le \xi,\ y - 2a_{max} \le z \le y\} \tag{45}$$

and the Lebesgue measure of the set $M(\xi,\eta)$ verifies

$$0 \le \int_{M(\xi,\eta)} dy\,dz\ \le 2a_{max}T(\xi,\eta) \le\ 2a_{max}\xi \quad . \tag{46}$$

If (ξ,η) is on the boundary of $\tilde{\Omega}$ then the Lebesgue measure of $M(\xi,\eta)$ is zero.

Proof. Let $(y,z) \in M(\xi,\eta)$. By Eq. (31), $y \le \xi$ for $(y,z) \in Q(\xi,\eta)$. Looking at Eqs. (42), (32) and (9) the inequality $y \le \xi$ is seen for all $(y,z) \in M(\xi,\eta)$. Further $y \ge 0$ as $M(\xi,\eta) \subset \tilde{\Omega}$ (see again (9)). By (31), $z \le \eta \le y$ and

$$z \ge F^{-1}(\xi) \ge F^{-1}(y) = y - 2a \circ k^{-1}(y) \ge y - 2a_{max}$$

for arbitrary $(y,z) \in Q(\xi,\eta)$, $(\xi,\eta) \in \tilde{\Omega}$. The estimate (46) now immediately follows using variables t and x to calculate the first bound.

The boundary of $\tilde{\Omega}$ consists of points for which

(i) $\xi = -\eta$, i.e. $t = 0$,

(ii) $\xi = \eta$, i.e. $x = 0$,

(iii) $\xi = F(\eta)$, i.e. $x = a(t)$.

In the case (i), $M(\xi,\eta)$ is one-point and therefore zero-measure. In the cases (ii) and (iii), the measure of $Q(\xi,\eta)$ is zero by (31) and $B(\xi,\eta)$ is also on the boundary of $\tilde{\Omega}$ if still $B(\xi,\eta) \in \tilde{\Omega}$. So the measure of $M(\xi,\eta)$ is zero according to (42).

$\square$

Theorem 2 *Let* $a \in \mathrm{C}^2([0,\infty))$, $0 < \inf a \le \sup a < \infty$, $|a'| < 1$, $\varphi_0 \in \mathrm{C}^2([0,\mathrm{a}(0)])$, $\varphi_1 \in \mathrm{C}^1([0,\mathrm{a}(0)])$ *and the following relations are satisfied:*

$$\varphi_0(0) = 0, \quad \varphi_0(a(0)) = 0, \tag{47}$$

$$\varphi_1(0) = 0, \quad \varphi_1(a(0)) + a'(0)\varphi_0'(a(0)) = 0, \tag{48}$$

$$\varphi_0''(0) = 0, \tag{49}$$

$$(1 + a'(0)^2)\varphi_0''(a(0)) + a''(0)\varphi_0'(a(0)) + 2a'(0)\varphi_1'(a(0)) = 0. \tag{50}$$

Then there exists unique $\tilde{\varphi} \in \mathrm{C}^2(\tilde{\Omega})$ *satisfying the equation (13) in* $\tilde{\Omega}$ *and boundary conditions corresponding to the transformed relations (3-4). The solution satisfies the estimate*

$$|\tilde{\varphi}(\xi,\eta)| \le c e^{\frac{1}{2}a_{max} m^2 \xi} \tag{51}$$

with a constant $0 < c < \infty$ *independent of* m *(but dependent on* φ_0 *and* φ_1*).*

Proof. We iterate the Equation (44) denoting

$$\tilde{\varphi}^{(n)}(\xi,\eta) = \tilde{\varphi}^{(0)}(\xi,\eta) + \frac{m^2}{4} \int_{M(\xi,\eta)} \vartheta(\xi,\eta,y,z) \tilde{\varphi}^{(n-1)}(y,z)\, dy\, dz\,, \tag{52}$$

$$\varepsilon^{(n-1)}(\xi,\eta) = \tilde{\varphi}^{(n)}(\xi,\eta) - \tilde{\varphi}^{(n-1)}(\xi,\eta) \tag{53}$$

for $n = 1, 2, 3, \ldots$. Now

$$\varepsilon^{(n)}(\xi, \eta) = \frac{m^2}{4} \int_{M(\xi,\eta)} \vartheta(\xi, \eta, y, z) \varepsilon^{(n-1)}(y, z)\, dy\, dz \quad . \tag{54}$$

Under our assumptions, all the functions $\tilde{\varphi}^{(n)}$ and $\varepsilon^{(n)}$ are continuous. Then the estimate

$$\left|\varepsilon^{(n)}(\xi, \eta)\right| \leq \frac{a_{max}^n m^{2n} \xi^n}{2^n\, n!} \left\|\varepsilon^{(0)}\right\|_{L^\infty(M_0)} \tag{55}$$

follows by induction using Lemma 1 and Lemma 4 for

$$(\xi, \eta) \in M_0 := \{(y, z) \in \tilde{\Omega} \,|\, T(y, z) \leq T_0\}$$

with arbitrary given $T_0 > 0$. So the sequence

$$\tilde{\varphi}^{(n+1)} = \sum_{k=0}^{n} \varepsilon^{(k)} + \tilde{\varphi}^{(0)} \tag{56}$$

is uniformly convergent in any compact subset of $\tilde{\Omega}$, its limit $\tilde{\varphi}$ is continuous, satisfies Equation (44) and the required boundary and initial conditions. Therefore $\tilde{\varphi}$ satisfies also Equation (13).

Assume that we have two solutions of Equation (13) satisfying the required boundary and initial conditions. Then they satisfy also Equation (44) and the estimate like (55) holds for their difference with any n. So the two solutions must be identical and the uniqueness is proved.

The solution $\tilde{\varphi}^{(0)}$ of d'Alembert equation is known to be bounded in $\tilde{\Omega}$ (see also (23) and (25) for $f = 0$). Equations (52) then leads to

$$|\tilde{\varphi}(\xi, \eta)| \leq \|\tilde{\varphi}^{(0)}\|_\infty e^{\frac{1}{2} a_{max} m^2 \xi} \tag{57}$$

and (51) is proved.

$\square$

5 Energy Large-Time Behavior

In this section we are going to prove that the energy can be exponentially increasing (up to non-monotone evolution within the period of the end-point motion) for sufficiently small mass under the same assumptions as for the massless case. Let us first write a formula for the function G by iterations of relation (25). Let G_0 be the corresponding function for $m = 0$. For any $\eta \in [-a(0), \infty)$ there exists just one non-negative integer $n(\eta)$ such that

$$\eta \in F^{n(\eta)}([-a(0), a(0))) \quad . \tag{58}$$

We shall use also integer

$$K(t) = n(t + a_{max}) \geq n(\xi) \geq n(\eta) \quad . \tag{59}$$

By induction with respect to $n(\eta)$ and comparison of the relations with $m = 0$ and $m \neq 0$ we obtain

$$\begin{aligned} G(\eta) = G_0(\eta) \quad &+ \quad \frac{m^2}{4} \int_0^{|F^{-n(\eta)}(\eta)|} dy \int_{-y}^{y} dz \, \tilde{\varphi}(y, z) \\ &+ \quad \frac{m^2}{4} \sum_{j=1}^{n(\eta)} \int_{|F^{-j}(\eta)|}^{F^{1-j}(\eta)} dy \int_{F^{-j}(\eta)}^{y} dz \, \tilde{\varphi}(y, z) \, . \end{aligned} \tag{60}$$

To calculate the energy density, we need also derivatives of the function $\tilde{\varphi}$ and therefore G. Taking into account that $F^{-j}(\eta)$ can be negative only for $j = n(\eta)$ and excluding a discrete set of values of η we obtain

$$\begin{aligned} G'(\eta) \quad = \quad & G_0'(\eta) \\ + \quad & \frac{m^2}{4} \Bigg\{ \frac{dF^{-n(\eta)}(\eta)}{d\eta} \mathrm{sgn}\left(F^{-n(\eta)}(\eta)\right) \cdot \\ \cdot \quad & \int_{-|F^{-n(\eta)}(\eta)|}^{\delta_{n(\eta),0}|F^{-n(\eta)}(\eta)| + (1-\delta_{n(\eta),0}) F^{-n(\eta)}(\eta)} \tilde{\varphi}(|F^{-n(\eta)}(\eta)|, z) \, dz \\ + \quad & \sum_{j=1}^{n(\eta)} \frac{dF^{1-j}(\eta)}{d\eta} \int_{F^{-j}(\eta)}^{F^{1-j}(\eta)} \tilde{\varphi}(F^{1-j}(\eta), z) \, dz \\ - \quad & \sum_{j=1}^{n(\eta)} \frac{dF^{-j}(\eta)}{d\eta} \int_{|F^{-j}(\eta)|}^{F^{1-j}(\eta)} \tilde{\varphi}(y, F^{-j}(\eta)) \, dy \Bigg\} \quad . \end{aligned} \tag{61}$$

This formula has the form

$$G'(\eta) = G_0'(\eta) + \frac{m^2}{4} \sum_{j=0}^{n(\eta)} B_j(\eta) \frac{dF^{-j}(\eta)}{d\eta} \tag{62}$$

where $B_j(\eta)$ are seen above. Using (51) and the relation

$$\eta - F^{-1}(\eta) = 2a \circ k^{-1}(\eta) \leq 2a_{max} \tag{63}$$

following from (6), the estimate

$$|B_j(\eta)| \leq c_1(m) e^{\frac{1}{2} a_{max} m^2 |F^{-j}(\eta)|} \tag{64}$$

is shown for $j = 0, \ldots, n(\eta) > 0$ with

$$c_1(m) = 2a_{max} c \left(1 + e^{a_{max}^2 m^2}\right) \quad . \tag{65}$$

We indicated here the dependence of constant c_1 on the mass m but we do not indicate the automatically assumed dependence on the initial data φ_0, φ_1 (see (51) and (57)) and the function a. We shall keep such notation for further constants in estimates below as we finally want to have an m-independent estimate over the range of mass values. By (58) and (63),

$$|F^{-j}(\eta)| \leq [2n(\eta) - 2j + 1] a_{max} \tag{66}$$

for $j = 0, \ldots, n(\eta)$ and therefore

$$|B_j(\eta)| \leq c_1(m) e^{(n(\eta) - j + \frac{1}{2}) a_{max}^2 m^2} \quad . \tag{67}$$

Using the above formulas together with Eq. (23) we obtain for $\omega = \xi, \eta$

$$\frac{\partial \tilde{\varphi}(\xi,\eta)}{\partial \omega} = \frac{\partial \tilde{\varphi}^{(0)}(\xi,\eta)}{\partial \omega} + \frac{m^2}{4} \sum_{j=0}^{n(\omega)} A_j^{(\omega)}(\xi,\eta) \frac{dF^{-j}(\omega)}{d\omega} \tag{68}$$

where

$$A_0^{(\xi)}(\xi,\eta) = \int_\eta^\xi \tilde{\varphi}(\xi,z)\, dz - B_0(\xi) \quad , \quad A_j^{(\xi)}(\xi,\eta) = -B_j(\xi) \quad , \tag{69}$$

$$A_0^{(\eta)}(\xi,\eta) = -\mathrm{sgn}(\eta) \int_\eta^{|\eta|} \tilde{\varphi}(|\eta|,z)\, dz - \int_{|\eta|}^\xi \tilde{\varphi}(y,\eta)\, dy + B_0(\eta) \ , \tag{70}$$

$$A_j^{(\eta)}(\xi,\eta) = B_j(\eta) \tag{71}$$

for $j = 1, \ldots, n(\omega)$. The upper estimate of $2ca_{max}e^{\frac{1}{2}a_{max}m^2\xi}$ for integral terms in both formulas for $A_0^{(\omega)}(\xi,\eta)$ can be seen from (51). Realizing that

$n(\eta) \leq n(\xi)$ as $\eta \leq \xi$ and that $\xi < (2n(\xi)+1)a_{max}$ according to (66), we arrive at

$$|A_j^{(\omega)}(\xi,\eta)| \leq c_2(m) e^{(n(\xi)-j)a_{max}^2 m^2} \quad , \tag{72}$$

$$c_2(m) = 2a_{max} c \left(2 + e^{a_{max}^2 m^2}\right) e^{\frac{1}{2}a_{max}^2 m^2} \tag{73}$$

for $j = 0, \ldots, n(\omega)$ and $\omega = \xi, \eta$.

To estimate the contributions of terms in (68) to the energy (14) we shall use the results for the d'Alembert equation [1, 5]. Let us first remind them in a form suitable for that. Function F defined in (5) is an increasing diffeomorphism of the line $\mathbb{R}$ satisfying the relation $F(t+T) = F(t) + T$ for $t \in R$, i.e. a covering of a diffeomorphism of the circle of length T. The notions of the rotation number and periodic point used below are defined e.g. in [10], a brief review is given also in [5]. We shall use the notation

$$F^n = F \circ \ldots \circ F$$

for the composition of the function F n-times with itself, $(f)^n$ for the n-th power of the function f, i.e. for the function with values

$$f(x)^n = f(x) \cdot \ldots \cdot f(x) \quad ,$$

$DF = F'$ for the derivative of function F, and χ_I for the characteristic function of the interval I with the value equal 1 in I and 0 outside I.

Lemma 5 *Let function F has the rotation number $\rho(F) = \frac{p}{q}T$ where $p \in \mathbb{N}^* = \{1, 2, \ldots\}$ and $q \in \mathbb{N}^*$ are relatively prime, a_1 be an attracting periodic point of F, the function F has in $[a_1, F(a_1))$ a finite number of periodic points of period q from which $a_1 < a_2 < \ldots < a_N$ are attracting with*

$$DF^q(a_i) < 1 \quad \text{for } i = 1, \ldots, N$$

and other periodic points in $[a_1, F(a_1))$ are repelling. Let us denote as b_{i-1}, b_i the nearest repelling periodic points to a_i such that $b_{i-1} < a_i < b_i$, and as $J_1 = [a_1, b_1) \cup (b_N, F(a_1))$, $J_i = (b_{i-1}, b_i)$ for $i = 2, \ldots, N$. Let $f \in L^2((a_1, F(a_1)))$ be a real function, $\|f\| > 0$. Then

$$\int_{a_1}^{F(a_1)} \frac{f(x)^2}{DF^{nq}(x)}\, dx = \sum_{i=1}^{N} A_i [DF^q(a_i)]^{-n} + R_n \tag{74}$$

where $0 \leq A_i < \infty$, $A_i > 0$ *if* $\|f\|_{L^2(J_i)} > 0$,

$$A_i = \|\sqrt{l^{(i)}} f\|^2_{L^2(J_i)} \quad , \quad l^{(i)}(x) = \prod_{k=0}^{\infty} \frac{DF^q(a_i)}{DF^q \circ F^{kq}(x)} \tag{75}$$

for $x \in J_i$ *and* $i = 1, \ldots, N$. *The remainder*

$$R_n = o([DF^q(a_{i_0})]^{-n}) \tag{76}$$

as $n \to \infty$, i_0 *being defined by the relation*

$$DF^q(a_{i_0}) = \min\{DF^q(a_i) | \, i = 1, \ldots, N \text{ and } A_i > 0\} \quad . \tag{77}$$

Proof. The set of periodic points b satisfying

$$F^q(b) = b + pT \quad ,$$

the set of attracting periodic points, and the set of repelling periodic points are invariant under the action of function F. Further

$$DF^q(b) = DF^q(F^n(b))$$

for any periodic point b and $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$. Under our assumptions, necessarily

$$a_1 < b_1 < a_2 < \ldots < b_{N-1} < a_N < b_N < F(a_1)$$

are just the all periodic points in $[a_1, F(a_1)]$. Writing

$$\int_{a_1}^{F(a_1)} \frac{(f)^2}{DF^{nq}} \, dx = \sum_{i=1}^{N} \int_{J_i} \frac{(f)^2}{DF^{nq}} \, dx \quad , \tag{78}$$

all the terms can be treated as in the proof of Th. 3.25 of [5] applied to the function F^q. We know from [5] that the sequence of functions

$$l_n^{(i)} = \prod_{j=0}^{n-1} \frac{DF^q(a_i)}{DF^q \circ F^{jq}}$$

is uniformly bounded and pointwise convergent in J_i to the strictly positive function $l^{(i)}$ as $n \to \infty$. So the limit of each term in (78) multiplied by

$[DF^q(a_i)]^n$ can be calculated taking the limit under the integral and the formulas (74-76) are obtained. As $\|f\|_{L^2((a_1,F(a_1)))} > 0$ an index i_0 surely exists and the leading term is nontrivial.

$\square$

Remark. The validity of Lemma 5 was mentioned in [5] but only the case of integer rotation number ($\rho(F) = pT$ where $p \in \mathbb{N}^*$) and two periodic points in $[-a(0), a(0))$ was explicitly written for simplicity. However, the assumption of only two periodic points then leads to $p = 1$ as can be seen using the invariance of the set of periodic points under F and under the translation by period T. We have overlooked this constraint in [5].

Let us now repeat some assumptions and formulate some new ones for the purpose of the main theorem.

Assumptions. Let $a \in C^2(\mathbb{R})$ be a strictly positive periodic function with a period $T > 0$, satisfying $|a'| < 1$. Let the function F defined by relations (5) has the rotation number

$$\rho(F) = \frac{p}{q} \tag{79}$$

where $p, q \in \mathbb{N}^* = \{1, 2, \ldots\}$ are relatively prime. Let the function F has a finite number of periodic points in the interval $I_0 = [-a(0), a(0))$, of them $a_1, \ldots, a_N$ being attracting ($N \in \mathbb{N}^*$) and other periodic points being repelling. Let us denote as $b_0, \ldots, b_N$ the repelling periodic points such that

$$b_0 < a_1 < b_1 < \ldots < b_{N-1} < a_N < b_N = F(b_0) \tag{80}$$

and there are no other periodic points in (b_0, b_N). Notice that these repelling periodic points necessarily exist, (80) is the only possible ordering of periodic points, and just one of the two periodic points b_0, b_N lies in I_0. As the initial data determine the function G directly in the interval I_0 by Eq. (24) while the whole interval (b_0, b_1) resp. (b_{N-1}, b_N) need not be a part of I_0 we have to map the outreaching part into I_0 by function F resp. F^{-1} if necessary. Let us therefore denote

$$\begin{aligned} J_1 &= (\max\{b_0, -a(0)\}, b_1) \cup (\min\{b_N, a(0)\}, a(0)) \quad , \\ J_i &= (b_{i-1}, b_i) \quad \text{for} \quad i = 2, \ldots, N-1 \quad , \\ J_N &= (b_{N-1}, \min\{b_N, a(0)\}) \cup (-a(0), \max\{b_0, -a(0)\}) \end{aligned} \tag{81}$$

where notation $(x, y) = \emptyset$ for $x \geq y$ is used. These formulas can be also written in a more compact and for the further use clearer way as

$$J_i = \overset{\circ}{I}_0 \bigcap \bigcup_{j=-1}^{1} F^j((b_{i-1}, b_i)) \quad \text{for} \quad i = 1, \ldots, N \ .$$

Let the all attracting periodic points be such that

$$DF^q(a_i) < 1 \quad \text{for} \quad i = 1, \ldots, N \tag{82}$$

and let index i_0 be such that

$$DF^q(a_{i_0}) = \min\{DF^q(a_i) |\, i = 1, \ldots, N\} \quad . \tag{83}$$

Let us denote

$$J = \cup\{J_i |\, DF^q(a_i) = DF^q(a_{i_0})\} \quad . \tag{84}$$

We are ready to formulate the main statement now.

Theorem 3 *Let functions a, φ_0 and φ_1 be as in Theorem 2 and satisfy the Assumptions given above. Let the function $\varphi_0'(|x|) + \varphi_1(|x|)\,\mathrm{sgn}(x)$ has a strictly positive norm in $L^2(J)$. Then there exist strictly positive finite constants m_0, A , B (dependent on φ_0 and φ_1) such that for masses $0 \leq m \leq m_0$ and time $t \geq 0$ the energy $E_m(t)$ satisfies the inequality*

$$Ae^{\gamma t} \leq E_m(t) \leq Be^{\gamma t} \tag{85}$$

where

$$\gamma = -\frac{1}{pT} \ln\left(DF^q(a_{i_0})\right) \ > 0 \quad . \tag{86}$$

Proof. Preliminarily, let us show that the terms defining the energy do not change more than by a constant factor if time undergoes a constant translation. This will enable us to use a special sequence of times only. Let us consider two times t_1 and t_2 satisfying the inequalities

$$h(t_1) \leq h(t_2) \leq k(t_1) \quad , \tag{87}$$

i.e. $t_1 \leq t_2 \leq h^{-1} \circ k(t_1)$. The last inequality is clearly satisfied if

$$t_1 \leq t_2 \leq t_1 + 2a_{min} \tag{88}$$

as the second inequality (87) reads $t_2 \leq t_1 + a(t_1) + a(t_2)$. Similar calculations as in the proof of Lemma 2.17 of [5] lead to the estimate for the energy of massless field

$$\frac{1}{F'_{max}} E_0(t_1) \leq E_0(t_2) \leq \frac{1}{F'_{min}} E_0(t_1) \quad . \tag{89}$$

Analogously for

$$S_j(t) := \int_{h(t)}^{k(t)} \left(DF^{-j}(y)\right)^2 \, dy \tag{90}$$

where $j \in \mathbb{N}$ we can write

$$\begin{aligned} S_j(t_1) &= \left(\int_{h(t_1)}^{h(t_2)} + \int_{h(t_2)}^{k(t_1)} \right) DF^{-j}(y)^2 \, dy \\ &= \int_{h(t_2)}^{k(t_1)} DF^{-j}(y)^2 \, dy + \int_{k(t_1)}^{k(t_2)} \left(DF^{-j} \circ F^{-1}(y)\right)^2 DF^{-1}(y) \, dy \\ &= \int_{h(t_2)}^{k(t_1)} DF^{-j}(y)^2 \, dy + \int_{k(t_1)}^{k(t_2)} DF^{-j}(y)^2 \frac{\left(DF^{-1} \circ F^{-j}(y)\right)^2}{DF^{-1}(y)} \, dy \quad . \end{aligned}$$

Estimating the fraction in the last integrand in terms of $F'_{min} < 1 < F'_{max}$ and the factor 1 in the first integral by the same value we obtain

$$\frac{F'^2_{min}}{F'_{max}} S_j(t_1) \leq S_j(t_2) \leq \frac{F'^2_{max}}{F'_{min}} S_j(t_1) \quad . \tag{91}$$

Let us denote

$$t_0 = h^{-1}(a_{i_0}) \quad . \tag{92}$$

Combining inequalities (88), (89) and (91) we see that for

$$npT + t_0 \leq t \leq (n+1)pT + t_0 \tag{93}$$

with any $n \in \mathbb{N}$ the estimates

$$L_1 E_0(npT + t_0) \leq E_0(t) \leq L_2 E_0(npT + t_0) \quad , \tag{94}$$

$$M_1 S_j(npT + t_0) \leq S_j(t) \leq M_2 S_j(npT + t_0) \tag{95}$$

hold where

$$\begin{aligned} L_1 &= F'^{-n_0}_{max} \, , \; L_2 = F'^{-n_0}_{min} \, , \; M_1 = \left(\frac{F'^2_{min}}{F'_{max}} \right)^{n_0} \, , \; M_2 = \left(\frac{F'^2_{max}}{F'_{min}} \right)^{n_0} \, , \\ n_0 &= \left[\frac{pT}{2a_{min}} \right] + 1 \end{aligned} \tag{96}$$

the square brackets denoting the entire part here.

Let us denote as

$$\tilde{E}(t) = \sum_{\omega=\xi,\eta} \left\| \frac{\partial \tilde{\varphi}}{\partial \omega} \right\|^2_{L^2((0,a(t)),dx)} \tag{97}$$

and let us write Eq. (68) as

$$\frac{\partial \tilde{\varphi}}{\partial \omega} = \frac{\partial \tilde{\varphi}^{(0)}}{\partial \omega} + \kappa_\omega \quad . \tag{98}$$

Now

$$\sqrt{E_0(t)} - \sqrt{\sum_{\omega=\xi,\eta} \|\kappa_\omega\|^2} \le \sqrt{\tilde{E}(t)} \le \sqrt{E_0(t)} + \sqrt{\sum_{\omega=\xi,\eta} \|\kappa_\omega\|^2} \tag{99}$$

by the triangle inequality. By estimates (72),

$$\begin{aligned}
\sqrt{\sum_{\omega=\xi,\eta} \|\kappa_\omega\|^2} \quad &\le \quad \frac{m^2}{4} c_2(m) \sum_{j=0}^{K(t)} e^{(K(t)-j)a^2_{max}m^2} \|DF^{-j}\|_{L^2(h(t),k(t))} \\
&\le \quad \frac{m^2}{4} c_2(m) \sqrt{M_2} \sum_{j=0}^{K(t)} e^{(K(t)-j)a^2_{max}m^2} \sqrt{S_j(npT+t_0)}
\end{aligned} \tag{100}$$

for t satisfying (93). For $j = iq + r$ with $i \in \mathbb{N}$ and $r = 0, \dots, q-1$ let us calculate

$$\begin{aligned}
S_j(npT+t_0) &= \int_{npT+a_{i_0}}^{npT+F(a_{i_0})} DF^{-j}(y)^2 \, dy = \int_{a_{i_0}}^{F(a_{i_0})} DF^{-j}(y)^2 \, dy \\
&= \int_{F^{-j}(a_{i_0})}^{F^{1-j}(a_{i_0})} DF^{-j} \circ F^j(y) \, dy = \int_{F^{-r}(a_{i_0})}^{F^{1-r}(a_{i_0})} \frac{dy}{DF^j(y)} \\
&= \int_{a_{i_0}}^{F(a_{i_0})} \frac{DF^{-r}(y)^2}{DF^{j-r}(y)} \, dy \le F'^{-2r}_{min} \int_{a_{i_0}}^{F(a_{i_0})} \frac{dy}{DF^{iq}(y)} \quad .
\end{aligned} \tag{101}$$

By Lemma 5 there exists a finite constant $c_3 > 0$ such that

$$0 < \int_{a_{i_0}}^{F(a_{i_0})} \frac{dy}{DF^{iq}(y)} < c_3 [DF^q(a_{i_0})]^{-i} \le c_3 [DF^q(a_{i_0})]^{-\frac{j}{q}} \quad . \tag{102}$$

Estimate (100) now gives

$$\sqrt{\sum_{\omega=\xi,\eta} \|\kappa_\omega\|^2} \leq$$

$$\frac{m^2}{4} c_2(m) \sqrt{c_3 M_2} F'^{-(q-1)}_{min} e^{K(t) a^2_{max} m^2} \frac{\left(DF^q(a_{i_0})^{-\frac{1}{2q}} e^{-a^2_{max} m^2} \right)^{K(t)+1} - 1}{DF^q(a_{i_0})^{-\frac{1}{2q}} e^{-a^2_{max} m^2} - 1} . \quad (103)$$

Let us now choose $m_1 > 0$ such that

$$m_1^2 < -\frac{1}{2q a^2_{max}} \ln\left(DF^q(a_{i_0}) \right) \quad (104)$$

and consider only mass values

$$0 \leq m \leq m_1 \quad . \quad (105)$$

Now we can estimate

$$\sqrt{\sum_{\omega=\xi,\eta} \|\kappa_\omega\|^2} \leq c_4 m^2 \left(DF^q(a_{i_0}) \right)^{-\frac{K(t)+1}{2q}} \quad (106)$$

where we denoted

$$c_4 = \frac{\frac{1}{4} c_2(m_1) \sqrt{c_3 M_2} F'^{-(q-1)}_{min}}{DF^q(a_{i_0})^{-\frac{1}{2q}} e^{-a^2_{max} m_1^2} - 1} e^{-a^2_{max} m_1^2} \quad . \quad (107)$$

The energy of the massless field is given by function G_0 defined by Eqs. (24-25) with $f = 0$. For the considered sequence of times it reads

$$\begin{aligned} E_0(npT + t_0) &= \int_{npT + a_{i_0}}^{npT + F(a_{i_0})} G_0'(y)^2 \, dy \\ &= \int_{F^{nq}(a_{i_0})}^{F^{nq+1}(a_{i_0})} G_0'(y)^2 \, dy = \int_{a_{i_0}}^{F(a_{i_0})} \frac{G_0'(y)^2}{DF^{nq}(y)} \, dy \ . \quad (108) \end{aligned}$$

Now Lemma 5 and inequality (94) show the existence of constants $0 < D_1 < D_2 < \infty$ such that

$$D_1 \left(DF^q(a_{i_0}) \right)^{-n} \leq E_0(t) \leq D_2 \left(DF^q(a_{i_0}) \right)^{-n} \quad (109)$$

for t satisfying (93) as it is clear from the prolongation formula for G_0, (25) with $f = 0$, that the assumed nontriviality of $\varphi_0(|x|) + \varphi_1'(|x|)\,\mathrm{sgn}(x)$ in J leads to the nontriviality of G_0' in a suitable neighborhood of a suitable attracting periodic point of F in $[a_{i_0}, F(a_{i_0}))$.

Let us relate the numbers n in (93) and $K(t)$ defined by (59) and (58). As (93) gives

$$t + a_{max} \geq k(t) = F(h(t)) \geq F(npT + a_{i_0}) \geq F^{nq+1}(-a(0))$$

we have

$$K(t) \geq nq + 1 \quad . \tag{110}$$

Similarly (93) implies

$$\begin{aligned} t + a_{max} &= h(t) + a(t) + a_{max} \leq h(t) + 2a_{max} \\ &\leq a_{i_0} + (n+1)pT + 2a_{max} < a_{i_0} + \left(n + \left[\frac{2a_{max}}{pT}\right] + 2\right) pT \\ &< F^{\left(n+\left[\frac{2a_{max}}{pT}\right]+2\right)q}(a(0)) \end{aligned}$$

where square brackets denotes the entire part and

$$K(t) \leq \left(n + \left[\frac{2a_{max}}{pT}\right] + 2\right) q \quad . \tag{111}$$

Estimate (109) can be written as

$$D_1'[DF^q(a_{i_0})]^{-\frac{K(t)}{q}} \leq E_0(t) \leq D_2'[DF^q(a_{i_0})]^{-\frac{K(t)}{q}} \tag{112}$$

with

$$D_1' = D_1\,[DF^q(a_{i_0})]^{\left[\frac{2a_{max}}{pT}\right]+2} \quad , \quad D_2' = D_2\,[DF^q(a_{i_0})]^{\frac{1}{q}} \quad . \tag{113}$$

Let us choose m_2 such that

$$0 < m_2 < c_4^{-\frac{1}{2}} D_1'^{\frac{1}{4}}\,[DF^q(a_{i_0})]^{\frac{1}{4q}} \tag{114}$$

and denote

$$\begin{aligned} C_1 &= \left(\sqrt{D_1'} - c_4 m_2^2\,[DF^q(a_{i_0})]^{-\frac{1}{2q}}\right)^2 \quad , \\ C_2 &= \left(\sqrt{D_2'} + c_4 m_2^2\,[DF^q(a_{i_0})]^{-\frac{1}{2q}}\right)^2 \quad . \end{aligned} \tag{115}$$

Then by (99)

$$C_1\,[DF^q(a_{i_0})]^{-\frac{K(t)}{q}} \le \tilde{E}(t) \le C_2\,[DF^q(a_{i_0})]^{-\frac{K(t)}{q}} \tag{116}$$

for any $t \ge 0$ and $0 \le m \le \min(m_1, m_2)$.

Let us relate $K(t)$ to t. For $t \ge 0$,

$$t + a_{max} \ge a(0) = F(-a(0))$$

and therefore $K(t) \ge 1$. For any $x \in \mathbb{R}$ and $n \in \mathbb{N}^*$ the relation

$$-\frac{T}{n} < \frac{F^n(x) - x}{n} - \rho(F) < \frac{T}{n} \tag{117}$$

holds by Proposition II.2.3 of Ref. [10] (the same relation was used in the proof of Lemma 2.17 in [5]). Putting here $n = K(t)$, $x = F^{-n}(t + a_{max})$, $\rho(F) = \frac{p}{q}T$ and taking into account definitions (59), (58) we obtain

$$\frac{t}{pT} + \frac{a_{max} - a(0) - T}{pT} < \frac{K(t)}{q} < \frac{t}{pT} + \frac{a_{max} + a(0) + T}{pT} \quad . \tag{118}$$

Now (116) gives

$$C_1' e^{\gamma t} \le \tilde{E}(t) \le C_2' e^{\gamma t} \tag{119}$$

with

$$C_1' = C_1\,[DF^q(a_{i_0})]^{\frac{T + a(0) - a_{max}}{pT}} \;\;,\; C_2' = C_2\,[DF^q(a_{i_0})]^{-\frac{T + a(0) + a_{max}}{pT}} \quad . \tag{120}$$

Let us estimate the contribution of the mass term to the energy (14). By the estimate (51),

$$\int_0^{a(t)} \tilde{\varphi}(\xi, \eta)^2\, dx \le c^2 a_{max} e^{a_{max} m^2 (t + a_{max})} \; . \tag{121}$$

If we now denote

$$m_0 = \min\left(m_1, m_2, \sqrt{\frac{\gamma}{a_{max}}}\right) \quad , \tag{122}$$

$$A = C_1' \quad , \quad B = C_2' + \frac{1}{2} c^2 a_{max} m_0^2 e^{a_{max}^2 m_0^2} \tag{123}$$

the estimate (85) follows for every $0 \le m \le m_0$ and $t \ge 0$.

□

Remark. If the condition $m \le \sqrt{\frac{\gamma}{a_{max}}}$ is relaxed from (122) we have still exponential lower and upper bounds for the energy time-development but the upper exponent may be higher than that for the massless case. However, we cannot claim that such bound would be saturated as we do not know whether the estimate (51) can be improved substantially. In particular, we do not know whether the field φ is bounded as in the massless case since we were able to prove the exponential estimate only.

Acknowledgments. The authors are indebted to Prof. J. Cooper and Dr. N. Gonzalez for discussions. The stages of J.D. at CPT CNRS and PhyMat UTV and P.D. at NPI ASCR are thankfully acknowledged. The work is partly supported by ASCR grant No. IAA1048101.

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