CRITICAL $(\mathbf{\Phi}^4)_{3,\,\epsilon}$

D. C. Brydges[1,*], **P. K. Mitter**[2,**], **B. Scoppola**[3,***]

[1]Department of Mathematics, University of British Columbia
121-1984 Mathematics Rd, Vancouver, B.C, Canada V6T 1Z2
e-mail: db5d@math.ubc.ca

[2]Laboratoire de Physique mathématique, Université Montpellier 2
Place E. Bataillon, Case 070, 34095 Montpellier Cedex 05 France
e-mail: mitter@LPM.univ-montp2.fr

[3]Dipartimento di Matematica, Universitá "La Sapienza" di Roma
Piazzale A. Moro 2, 00185 Roma, Italy
e-mail: benedetto.scoppola@roma1.infn.it

ABSTRACT

The Euclidean $(\phi^4)_{3,\,\epsilon}$ model in $\mathbf{R}^3$ corresponds to a perturbation by a ϕ^4 interaction of a Gaussian measure on scalar fields with a covariance depending on a real parameter ε in the range $0 \le \varepsilon \le 1$. For $\varepsilon = 1$ one recovers the covariance of a massless scalar field in $\mathbf{R}^3$. For $\varepsilon = 0$ ϕ^4 is a marginal interaction. For $0 \le \varepsilon < 1$ the covariance continues to be Osterwalder-Schrader and pointwise positive. After introducing cutoffs we prove that for $\varepsilon > 0$, sufficiently small, there exists a non-gaussian fixed point (with one unstable direction) of the Renormalization Group iterations. These iterations converge to the fixed point on its stable (critical) manifold which is constructed.

* Partially supported by NSERC of Canada
** Laboratoire Associé au CNRS, UMR 5825
*** Partially supported by CNR, G.N.F.M. and MURST

1. INTRODUCTION, MODEL, RG TRANSFORMATION

1.1 Introduction

Let ϕ be a mean zero Gaussian scalar random field on $\mathbf{R}^3$ with covariance

$$E(\phi(x)\phi(y)) = \frac{\text{const}}{|x-y|^{(3-\epsilon)/2}}$$

$$= \int \frac{d^3p}{(2\pi)^3} e^{ip\cdot(x-y)}(p^2)^{-(3+\epsilon)/4} \tag{1.1}$$

Here $p^2 = |p|^2$ is the standard Euclidean norm in $\mathbf{R}^3$ and $p \cdot x$ is the standard scalar product. ε is a real parameter which we take in the region $0 \leq \epsilon < 1$. Note that for $\epsilon = 1$ we have the standard massless free scalar field in $\mathbf{R}^3$.

This covariance $(-\Delta)^{-(3+\epsilon)/4}$ has interesting physical properties. For example, Osterwalder- Schrader positivity plays no role in the present paper, but it is of interest because scaling limits of these theories will be Euclidean quantum field theories. $(-\Delta)^{-(3+\epsilon)/4}$ is Osterwalder- Schrader positive as well as being pointwise positive not only for $\epsilon = 1$ but also in the range that we consider, namely $0 \leq \epsilon < 1$. In the latter range we have the convergent integral representation

$$(-\Delta)^{-(3+\epsilon)/4}(x-y) = 1/c_\epsilon \int_0^\infty ds \;\; s^{-(3+\epsilon)/4}(-\Delta+s)^{-1}(x-y) \tag{1.2}$$

where

$$c_\epsilon = \int_0^\infty ds \;\; s^{-(3+\epsilon)/4}(1+s)^{-1} \tag{1.3}$$

The Osterwalder-Schrader and pointwise positivities now follow from those of $(-\Delta+s)^{-1}(x-y)$.

Furthermore $(-\Delta)^{-(3+\epsilon)/4}$ is the Green's function (or potential) for a stable Lévy process in $\mathbf{R}^3$ with parameters (α, β) in the Lévy-Khintchine notation [KG], with the characteristic exponent $\alpha = (3+\epsilon)/2$, and $\beta = 0$. This process has jumps and diffuses very fast. This property also plays no role in the present paper but we expect that self-avoiding Levy processes are accessible to the methods of this paper.

These properties make the study of the above gaussian random field and its non-linear perturbations worthwhile. In particular we will study here the critical properties of a model corresponding to the partition function

$$\mathbf{Z} = \int d\mu_C(\phi) \;\; e^{-V_0(\phi)} \tag{1.4}$$

where $d\mu_C$ is the Gaussian measure with covariance C and C is $(-\Delta)^{-(3+\epsilon)/4}$ with an ultraviolet cutoff described below and

$$V_0(\phi) = V(\phi, C, g_0, \mu_0) = g_0 \int d^3x :\phi^4:_C(x) + \mu_0 \int d^3x :\phi^2:_C(x) \tag{1.5}$$

and the coupling constant g is held strictly positive. Moreover in order to define the model completely we must also introduce a volume cutoff. These cutoffs will be introduced presently when we give a precise definition of the covariance C in(1.5) as well as that of the model.
From(1.1) we can read off the canonical scaling dimension $[\phi]$ of ϕ

$$[\phi] = (3-\epsilon)/4 \tag{1.6}$$

This, together with(1.4), implies that we can assign to g, μ the dimensions

$$[g] = \epsilon, \quad [\mu] = (3+\epsilon)/2 \tag{1.7}$$

Note that we have not put in a term

$$\frac{z}{2}\int d^3x |\nabla\phi(x)|^2$$

in (1.5). This is because the dimension $[z] = -(1-\varepsilon)/2$. Hence for $\varepsilon < 1$ this is a candidate for an irrelevant (stable) direction.

We thus expect from Wilson's theory of critical phenomena [KW] that for $\epsilon > 0$ the critical (infra-red) properties of the model are dominated by a non-Gaussian fixed point of Renormalization Group iterations with $g = g_* = O(\epsilon)$ provided the unstable parameter μ is fine tuned to a critical function $\mu_c(g)$ which determines the stable (critical) manifold of the fixed point. In the present work we will prove the existence of the non-Gaussian fixed point for $\epsilon > 0$ held sufficiently small and, on the way, construct the stable manifold.

The mathematical analysis of Renormalization Group (henceforth denoted RG) transformations has by now a long history [F,BG]. Our particular line of attack is influenced by a series of works which started with [BY], developed further in [DH1,DH2, BDH-est,BDH-eps], with more recent developments in [MS]. We shall be concerned here with these latter developments. In [MS] fluctuation covariances of finite range were exploited, and this simplifies considerably the RG analysis. In particular the analysis of the fluctuation integration becomes a matter of geometry and one no longer needs the cluster expansion and analyticity norms. In the continuum approach of [MS] that is also adopted here the existence of fluctuation covariances with finite range follows easily from a judicious choice of a class of ultraviolet cutoffs. This raises the general question of which gaussian random fields can be decomposed into sums of fluctuation fields with covariances with finite range. A partial answer which includes the standard massless Euclidean field with lattice cutoff will be given in [Gu]. Only the existence of multiscale decompositions with the above finite range property (together with some regularity and positivity properties) is required in what follows.

The present work borrows many technical considerations introduced in [BDH-est]. Although these are independent of the manner of treating the fluctuation step, we repeat some of them because the simpler norms in this paper allow easier proofs. We also borrow some ideas from [BDH-eps] where a related model (which however does not possess the physical properties mentioned earlier) was considered and the existence of a non-Gaussian fixed point was proved. We use this paper as an opportunity to improve previous arguments. In particular, in Section 4, there are much simpler formulas for the remainder after approximating the RG step by second order perturbation theory.

We plan to study in a subsequent work critical properties of Self-Avoiding Lévy Flights in $\mathbf{Z}^3$ for the Lévy-Khintchine parameters given above with $\alpha > \alpha_c$ and $\alpha - \alpha_c$ very small. Here $\alpha_c = 3/2$ at which value we expect (heuristically) mean field behaviour.

1.2 Multiscale decomposition

We introduce a special type of ultraviolet cutoff as follows: Let g be a non-negative, C^∞, $O(3)$ invariant function of compact support in $\mathbf{R^3}$ such that $g(x)$ vanishes for $|x| \geq 1/2$. Define $u = g * g$. Thus u is positive, C^∞, and of compact support: $u(x) = 0$ for $|x| \geq 1$. First we note note that

$$\frac{\text{const}}{|x-y|^{(3-\epsilon)/2}} = \int_0^\infty \frac{dl}{l} \quad l^{-(3-\epsilon)/2} \quad u(\frac{x-y}{l})$$

by scaling the variable of integration. Define

$$C(x-y)=\int_1^{\infty}\frac{dl}{l}\;\; l^{-(3-\epsilon)/2}\;\; u(\frac{x-y}{l}) \tag{1.8}$$

Because the lower limit is 1 and not 0 this C is pointwise positive and C^{∞}.

Remark
We can exhibit C in the traditional form with an ultraviolet cutoff function. Let $\hat{u}$ be the Fourier transform of u. Because of $O(3)$ invariance we can write $\hat{u}(p)=v(p^2)$. Then it is easy to see from the above that

$$C(x-y)=\int\frac{d^3p}{(2\pi)^3}e^{ip.(x-y)}F(p^2)(p^2)^{-(3+\epsilon)/4} \tag{1.9}$$

where

$$F(p^2)=1/2\int_{p^2}^{\infty}\frac{ds}{s}\;\; s^{(3+\epsilon)/4}\;\; v(s) \tag{1.10}$$

From this we see that F is positive, continuous and of fast decrease (since $v(p^2)=|\hat{g}(p)|^2$ and g is of compact support) and can be thus identified as the ultraviolet cutoff function.

Let $L\geq 2$ be an integer. Let

$$\Gamma(x-y)=\int_1^{L}\frac{dl}{l}\;\; l^{-(3-\epsilon)/2}\;\; u(\frac{x-y}{l}) \tag{1.11}$$

Clearly Γ is C^{∞}, pointwise positive and of finite range:

$$\Gamma(x-y)=0:\quad |x-y|\geq L \tag{1.12}$$

Γ is our *fluctuation covariance* and it satisfies

$$C(x-y)=\Gamma(x-y)+L^{-(3-\epsilon)/2}C(\frac{x-y}{L}). \tag{1.13}$$

Moreover because of our choice of the form of u, namely $u=g*g$, Γ and C are generalized positive definite. Now it can be shown (see e.g. Section 1.1 of [MS]) that under these conditions for any compact set Λ_N in $\mathbf{R^3}$ there exists a Gaussian measure $d\mu_C$ of mean 0 and covariance C supported on the Sobolev space $H_s(\Lambda_N)$ for any $s>0$. Choosing $s>3/2+2$ is sufficient for our purpose. Then, by Sobolev embedding, sample paths are at least twice differentiable.

Define the compact set

$$\Lambda_N=[-\frac{1}{2}L^N,\frac{1}{2}L^N]^3\subset\mathbf{R^3}. \tag{1.14}$$

Our model with an ultraviolet cutoff and volume cutoff is now defined by the partition function

$$\mathbf{Z}=\int d\mu_C(\phi)\;\; \mathcal{Z}_0(\Lambda_N,\phi) \tag{1.15}$$

where

$$\mathcal{Z}_0(\Lambda_N,\phi)=e^{-V(\Lambda_N,\phi)} \tag{1.16}$$

and the potential has now been restricted to the volume Λ_N. The Wick ordering in the potential is with respect to the infinite volume covariance C and this is well defined because the Wick constants are finite.

From(1.13), if we now define the rescaled field

$$\mathcal{R}\phi(x) = \phi_{L^{-1}}(x) = L^{-(3-\epsilon)/4}\phi(x/L) \tag{1.17}$$

we get

$$\int d\mu_C(\phi)\mathcal{Z}_0(\Lambda_N,\phi) = \int d\mu_C(\phi)\mathcal{Z}_1(\Lambda_{N-1},\phi) \tag{1.18}$$

where

$$\mathcal{Z}_1(\Lambda_{N-1},\phi) = \int d\mu_\Gamma(\zeta)\mathcal{Z}_0(\Lambda_N,\zeta+\phi_{L^{-1}}) \tag{1.19}$$

The iteration of(1.19) will constitute our RG transformations. After n steps we have

$$\int d\mu_C(\phi)\mathcal{Z}_0(\Lambda_N,\phi) = \int d\mu_C(\phi)\mathcal{Z}_n(\Lambda_{N-n},\phi) \tag{1.20}$$

where

$$\mathcal{Z}_n(\Lambda_{N-n},\phi) = \int d\mu_\Gamma(\zeta)\mathcal{Z}_{n-1}(\Lambda_{N-n+1},\zeta+\phi_{L^{-1}}) \tag{1.21}$$

After $N-1$ steps we arrive at $\mathcal{Z}_{N-1}(\Lambda_1,\phi)$ where Λ_1 is the L- block $[-\frac{1}{2}L,\frac{1}{2}L]^3$.

In order to analyse the RG transformations, it is convenient to write the partition function density in a *polymer gas* representation. Pave $\mathbf{R}^3$ with closed unit blocks denoted henceforth Δ. Now let $\Lambda\subset\mathbf{R}^3$ be the volume after a certain number of RG steps. We take Λ with the induced paving. A *connected* polymer X is a connected union of a subset of these closed unit blocks and is thus closed. A polymer activity $K(X,\phi)$ is a map $X,\phi\to\mathbf{R}$ where the fields ϕ depend only on the points of X. We shall only consider polymer activities supported on connected polymers. This notion will be preserved by the RG operations. We then write, suppressing the dependence on ϕ,

$$\mathcal{Z}(\Lambda) = \sum_{N=0}^{\infty}\frac{1}{N!}e^{-V(\Lambda\setminus X)}\sum_{X_1,..,X_N}\prod_{j=1}^{N}K(X_j) \tag{1.22}$$

where the connected polymers X_j are disjoint, $X=\cup_1^N X_j$ and $V(Y)=V(Y,\phi,C,g,\mu)$ is given by(1.5) with parameters g,μ and integration over Y. Initially of course K is absent, but they are naturally generated by the RG operations and the above form is stable.

It is possible to rewrite the above in a more compact form if we extend the polymer algebra by *cells* as done in [BY]. A *cell* may be the interior of a block, an open face, an open edge or a vertex. A polymer, which is a union of closed blocks, can be uniquely written as a disjoint union of cells, but the point is that all other sets generated by our manipulations, such as complements of polymers, can also be uniquely written as disjoint unions of cells. We define a commutative product, denoted $\circ$, on functions of sets (unions of cells), in the following way $(F_1\circ F_2)(X)=\sum_{Y,Z:Y\overset{\circ}{\cup}Z=X}F_1(Y)F_2(Z)$ where $X=Y\overset{\circ}{\cup}Z$ iff $X=Y\cup Z$ and $Y\cap Z=\emptyset$. The $\circ$ identity $\mathcal{I}$ is defined by $\mathcal{I}(X)=1$ if $X=\emptyset$ and otherwise vanishes. Finally we can define an $\mathcal{E}$xponential operation on functions of unions of cells as the usual power series based on the $\circ$ product and the definition of the $\circ$ identity $\mathcal{I}$. This has the properties of the usual exponential. We also define a *space filling* function $\square$ by $\square(X)=1$ if X is a cell and otherwise vanishes. We then have $\mathcal{E}xp\,\square=1$ and it easy to see that(1.22) can be rewritten as

$$\mathcal{Z}(\Lambda) = [\mathcal{E}xp(\square\, e^{-V}+K)](\Lambda) \tag{1.23}$$

The Formal Infinite Volume Limit

By writing the integrands in the form (1.23), the ith RG transformation induces a map

$$f_{N-i} : (K_{i-1}, V_{i-1}) \mapsto (K_i, V_i)$$

which will be described in detail in the next section. The subscript $N-i$ is there because this map has dependence on the region Λ_{N-i}. For any set X,

$$V_i(X) = \sum_{\Delta \subset X} V_i(\Delta)$$

so V_i is determined by $\{V_i(\Delta) : \Delta \subset \Lambda_{N-i}\}$. Our formulas for the RG map will show that for any fixed set $X \subset \Lambda_{N-i}$, the polymer activity $K_i(X,\phi)$ is independent of N for all N large enough so that Λ_{N-i} contains X and likewise $V_i(\Delta)$ is independent of N for all N large enough so that Λ_{N-i} contains Δ and a neighborhood of Δ. A detailed look at our formulas, particularly the section on extraction, shows that the neighborhood has diameter 8. $\lim_{N\to\infty} K_i(X)$ *and* $\lim_{N\to\infty} K_i(\Delta)$ *exist and, in this pointwise sense, the infinite volume limit* $f = \lim_{N\to\infty} f_{N-i}$ *exists.* In this paper we prove that f has a fixed point in a Banach space of polymer activities K.

The finite volume RG could also be studied by these methods by including in V a surface integral over the boundary of Λ_{N-i} which fits naturally in this scheme as an object associated to $D-1$ cells on the boundary.

2. REGULATORS, DERIVATIVES AND NORMS

2.1 Regulators

We first introduce a *large field regulator* which measures the growth of polymer activities in the fields ϕ, actually in $\partial\phi$:

$$G_\kappa(X,\phi) = e^{\kappa \|\phi\|^2_{X,1,\sigma}} \tag{2.1}$$

where

$$\|\phi\|^2_{X,1,\sigma} = \sum_{1\le|\alpha|\le\sigma} \|\partial^\alpha \phi\|^2_X \tag{2.2}$$

Here $\|\phi\|_X$ is the L^2 norm and α is a multi-index. We take $\sigma > 3/2+2$ so that this norm can be used in Sobolev inequalities to control ϕ and its first two derivatives pointwise. After the function u is fixed, the parameter $\kappa > 0$ is fixed, for the whole paper, by a choice depending only on u, so that for all $L \ge 1$ the large field regulator satisfies the *stability property*

$$\int d\mu_\Gamma(\zeta)\, G_\kappa(X,\zeta+\phi) \le 2^{|X|} G_{2\kappa}(X,\phi) \tag{2.3}$$

where X is a polymer and $|X|$ is the number of unit blocks in X. This can be shown in the same way as in the proof of the stability property of the large field regulator in Section 2.4 of [BDH-est].
Now hold L sufficiently large and recall that $\epsilon < 1$. Then we get after rescaling

$$\int d\mu_\Gamma(\zeta)\, G_\kappa(X,\zeta+\phi_{L^{-1}}) \le 2^{|X|} G_\kappa(L^{-1}X,\phi) \tag{2.4}$$

This follows easily from the scaling property(1.17) of the fields ϕ which gives

$$\|\phi_{L^{-1}}\|^2_{X,1,\sigma} \le L^{-(1-\epsilon)/2} \|\phi\|^2_{L^{-1}X,1,\sigma}$$

Next we introduce a *large set regulator*. Let X be a connected polymer. We define

$$\mathcal{A}_p(X) = 2^{p|X|} L^{(D+2)|X|} \tag{2.5}$$

where for us the dimension of space $D = 3$, and $p \geq 0$ is an integer.

Call a connected polymer *small* if $|X| \leq 2^D$. A connected polymer which is not small is called *large*. Let $\bar{X}^L$ be the L-closure of X. This is the smallest union of L-blocks containing X. Let L be sufficiently large. Then we have from Lemma 1 of [BDH-est] the following two facts:
For any connected polymer X and for any integer $p \geq 0$

$$\mathcal{A}(L^{-1}\bar{X}^L) \leq c_p \mathcal{A}_{-p}(X) \tag{2.6}$$

For X a *large* connected polymer,

$$\mathcal{A}(L^{-1}\bar{X}^L) \leq c_p L^{-D-1} \mathcal{A}_{-p}(X) \tag{2.7}$$

Here $c_p = O(1)$ is a constant independent of L.

2.2 Derivatives in fields and polymer activity norms

The following definitions are motivated by those in [BDH-est] the main difference being that we will need only a finite number of field derivatives and for them only *natural* norms. The kernel norms defined below are different.

For a polymer activity $K(X,\phi)$ define the n-th derivative in the direction $(f_1, ..., f_n)$ as:

$$(D^n K)(X, \phi; f^{\times n}) = \prod_{j=1}^{n} (\partial/\partial s_j)\, K(X, \phi + \sum s_j f_j)\, |_{s_j=0\,\forall j} \tag{2.8}$$

where the shorthand notation $f^{\times n}$ stands for $f_1, f_2, ..., f_n$. The functions f_j belong to $C^2(X)$ and we assume that such derivatives exist for $n \leq n_0$ for some n_0.
We will measure the size of such derivatives, which are multilinear functionals on $C^2(X)$, by the norm:

$$\|(D^n K)(X,\phi)\| = \sup_{\|f_j\|_{C^2(X)} \leq 1\ \forall j} |(D^n K)(X, \phi; f^{\times n})| \tag{2.9}$$

Let $h > 0$ be a real parameter. We define the following set of norms:

$$\|K(X,\phi)\|_h = \sum_{j=0}^{n_0} \frac{h^j}{j!} \|(D^j K)(X,\phi)\| \tag{2.10}$$

and then

$$\|(K(X)\|_{h,G_\kappa} = \sup_{\phi \in C^2(X)} \|K(X,\phi)\|_h\, G_\kappa^{-1}(X,\phi) \tag{2.11}$$

These norms differ from norms used in [BDH-est] in having the supremum over ϕ outside the sum over derivatives, as well as involving only finitely many derivatives. They are easier to use and retain the product property

$$\|K_1(X_1,\phi)K_2(X_2,\phi)\|_h \leq \|K_1(X_1,\phi)\|_h \|K_2(X_2,\phi)\|_h.$$

which was the basis for proofs in [BDH-est]. We assume as earlier that the activity K is supported on connected polymers. We then define

$$\|K\|_{h,G_\kappa,\mathcal{A}} = \sup_{\Delta} \sum_{X\supset\Delta} \|(K(X)\|_{h,G_\kappa}\mathcal{A}(X) \tag{2.12}$$

In addition we define *kernel norms*:

$$|K(X)|_{h'} = \sum_{j=0}^{n_0} \frac{h'}{j!}\|(D^j K)(X,0)\| \tag{2.13}$$

where h' is a real parameter and $h' \geq 0$. We define

$$|K|_{h',\mathcal{A}} = \sup_{\Delta} \sum_{X\supset\Delta} |K(X)|_{h'}\mathcal{A}(X) \tag{2.14}$$

These definitions are also different from the kernel norms in [BDH-est,BDH-eps]), but retain the product property. When using these norms for our model we will choose $n_0 = 9$, $h = \varepsilon^{-1/4}$ and either $h' = h$ or $h' = h_*$, where

$$h_* = L^{(3-2[\phi])/2} = L^{(3+\varepsilon)/4}$$

The kernel norms with $h' = h_*$ will be useful for controlling flow coefficients.

3. RG STEP

In this section we describe the RG step. This consists of two parts: Fluctuation integration and rescaling, followed by the extraction of relevant parts.

3.1 Fluctuation integration and rescaling

The integration over the fluctuation field exploits the independence of $\zeta(x)$ and $\zeta(y)$ when $|x-y| \geq L$. To do this we pave Λ by blocks of side L called $L-$blocks so that each $L-$block is a union of the original $1-$blocks. Let $\bar{X}^L$ denote the $L-$closure of a set X, namely the smallest union of closed $L-$blocks containing X. The polymers will be combined into larger $L-$polymers which by definition are closed, connected and unions of $L-$blocks. The combination is performed in such a way that the new polymers are associated to independent functionals of ζ.

We start with the representation (1.25) of Section 1.3.

$$\mathcal{Z}(\Lambda) = [\mathcal{E}xp(\Box\, e^{-V} + K)](\Lambda) \tag{3.1}$$

with $K(X) = 0$ unless X is closed and connected. By definition we have:

$$\mathcal{E}xp(\Box\, e^{-V} + K)(\Lambda) = \sum_N \frac{1}{N!}\sum\nolimits_{(X_j)} e^{-V(\Lambda\setminus\cup X_j)}\prod\nolimits_{j=1}^N K(X_j) \tag{3.2}$$

where the sum is over sequences of disjoint polymers $(X_j) = (X_1 .....X_N)$.

Let us define: $\cup X_j = X$ and $\Lambda\setminus X = X_c$. X_c is an open set. We denote by $\bar{X}_c$ its closure. Obviously, $V(X_c) = V(\bar{X}_c)$, since $V(X_c)$ is given by a Lebesgue integral over X_c. Hence we can write:

$$e^{-V(X_c,\zeta+\phi)} = \prod\nolimits_{\Delta\subset\bar{X}_c} e^{-V(\Delta,\zeta+\phi)}$$

Define the polymer activity P, supported on closed unit blocks, as:

$$P(\Delta,\zeta,\phi)=e^{-V(\Delta,\zeta+\phi)}-e^{-\tilde{V}(\Delta,\phi)} \tag{3.3}$$

with $\tilde{V}$ to be chosen. In the following V,K has field argument $\zeta+\phi$ whereas $\tilde{V}$ depends only on ϕ. The dependence of P on ζ,ϕ is as defined above.

Now write:

$$e^{-V(X_c)}=\prod\nolimits_{\Delta\subset\tilde{X}_c}[e^{-\tilde{V}(\Delta)}+P(\Delta)]$$

then expand the product and insert the expansion into (3.2):

$$\mathcal{E}xp(\square\, e^{-V}+K)(\Lambda)=\sum\frac{1}{N!M!}\sum\nolimits_{(X_j),(\Delta_i)}e^{-\tilde{V}(X_0)}\prod\nolimits_{j=1}^{N}K(X_j)\prod\nolimits_{i=1}^{M}P(\Delta_i) \tag{3.4}$$

where $X_0=\Lambda\setminus(\cup X_j)\cup(\cup\Delta_i)$. Let Y be the $L-$closure of $(\cup X_j)\cup(\cup\Delta_i)$ and let $Y_1,\dots,Y_P$ be the connected components of Y. These are $L-$polymers. Let f be the function that maps $\pi:=(X_j),(\Delta_i)$ into $\{Y_1,\dots,Y_P\}$. Now we perform the sum over $(X_j),(\Delta_i)$ in (3.4) by summing over $\pi\in f^{-1}(\{Y_1,\dots,Y_P\})$ and then $\{Y_1,\dots,Y_P\}$. The result is:

$$\mathcal{E}xp(\square\, e^{-V}+K)(\Lambda)=\mathcal{E}xp_L(\square\, e^{-\tilde{V}}+\mathcal{B}K)(\Lambda) \tag{3.5}$$

where the subscript on $\mathcal{E}xp_L$ indicates that that the domain is functionals of $L-$polymers and

$$(\mathcal{B}K)(Y)=\sum_{N+M\geq 1}\frac{1}{N!M!}\sum\nolimits_{(X_j),(\Delta_i)\to\{Y\}}e^{-\tilde{V}(X_0)}\prod\nolimits_{j=1}^{N}K(X_j)\prod\nolimits_{i=1}^{M}P(\Delta_i) \tag{3.6}$$

$$X_0=Y\setminus(\cup X_j)\cup(\cup\Delta_i)$$

where the $\to$ is the map f. This representation (3.5) is a sum over products of polymer activities $(\mathcal{B}\hat{K})(Z_j)$ where the closed disjoint polymers Z_j are necessarily separated by a distance $\geq L$ and the spaces between the polymers are filled with $e^{-\tilde{V}}$ which are independent of the fluctuation field ζ. The covariance $\Gamma(x-y)=0$ if $|x-y|\geq L$. So the fluctuation integral factorises and we obtain:

$$\int d\mu_\Gamma(\zeta)[\mathcal{E}xp(\square\, e^{-V}+K)](\Lambda,\zeta+\phi)=\mathcal{E}xp_L(\square_L\, e^{-\tilde{V}}+(\mathcal{B}K^\sharp)(\Lambda,\phi) \tag{3.7}$$

where the superscript $\sharp$ ("sharp") denotes integration with the measure $d\mu_\Gamma(\zeta)$.
Now we perform the rescaling. This is accomplished by replacing ϕ by the rescaled field $\mathcal{R}\phi$ where

$$(\mathcal{R}\phi)(x)=\phi_{L^{-1}}(x)=L^{-[\phi]}\phi(x/L)$$

$$(\mathcal{R}K)(L^{-1}X,\phi)=K(X,\phi_{L^{-1}})$$

If we now define $\mathcal{S}=\mathcal{R}\mathcal{B}$ then we have:

$$\int d\mu_\Gamma(\zeta)[\mathcal{E}xp(\square\, e^{-V}+K)](\Lambda,\zeta+\phi_{L^{-1}})=\mathcal{E}xp(\square\, e^{-\tilde{V}_L}+(\mathcal{S}K)^\natural)(L^{-1}\Lambda,\phi) \tag{3.8}$$

where $\tilde{V}_L(\Delta,\phi)=\tilde{V}(L\Delta,\phi_{L^{-1}})$ and the superscript $\natural$ ("natural") denotes integration with the measure $d\mu_{\Gamma_L}(\zeta)$. Here:

$$\Gamma_L(x-y)=L^{2[\phi]}\Gamma(L(x-y))$$

We have returned to a functional of the form $\mathcal{E}xp(\square\, e^{-V}+K)(\Lambda)$ with $V\to\tilde V$ and $K\to(\mathcal{S}K)^\natural$. Thus the operation is an evolution of the interaction described in coordinates V,K.

We will refer in the future to (3.8) as the *fluctuation step.* The RG step will be completed by removing relevant parts from $(\mathcal{S}\hat K)^\natural$ and compensating by a new local potential, this operation being called *Extraction.*

3.2 Extraction

The objective is to cancel parts Fe^{-V} of K in $\mathcal{E}xp(\square e^{-V}+K)(\Lambda)$ by a change in V, adding terms V_F to V. This will be possible for functionals F of a special form which we first describe.

Let $\mathcal{P}$ and $\mathcal{P}_j$ be polynomials. For Y, any union of cells, V has the form:

$$V(Y)=\int_Y dx\ \mathcal{P}(\phi(x))$$

We define polymer activities

$$F(X)=\sum_{\Delta\subset X}F(X,\Delta) \tag{3.9}$$

where $F(X,\Delta)$ has the form

$$F(X,\Delta)=\sum_j\int_\Delta dx\ \alpha_j(X,\Delta,x)\mathcal{P}_j(\phi(x)) \tag{3.10}$$

and $F(X,\Delta)=0$ for $\Delta\not\subset\Lambda$. We also define:

$$V_F(\Delta)=\sum_{X\supset\Delta}F(X,\Delta) \tag{3.11}$$

Then

$$V_F(\Delta)=\sum_j\int_\Delta dx\ \alpha_j(\Delta,x)\mathcal{P}_j(\phi(x))=\sum_j\int dx\ \alpha_j(x)\mathcal{P}_j(\phi(x))$$

where $\alpha(x)=\alpha(\Delta,x)$ with $\Delta\ni x$ and

$$\alpha_j(\Delta,x)=\sum_{X\supset\Delta}\alpha_j(X,\Delta,x) \tag{3.12}$$

and for any polymer X

$$V_F(X)=\sum_{\Delta\subset X}V_F(\Delta)=\sum_j\int_X dx\ \alpha_j(x)\mathcal{P}_j(\phi(x)) \tag{3.13}$$

We define $V_F(X_c)$ by the last member of (3.13), replacing X by X_c.

Following [BDH-est] Section 4.2,

Extraction Formula:

$$\mathcal{E}xp(\square\, e^{-V}+K)(\Lambda)=\mathcal{E}xp(\square\, e^{-V'}+\mathcal{E}(K,F)](\Lambda) \tag{3.14}$$

where

$$
\begin{aligned}
&V' = V - V_F \\
&\mathcal{E}_1(K,F) = K - Fe^{-V}
\end{aligned}
\tag{3.15}
$$

where $\mathcal{E}_1(K,F)$ is the linearization of $\mathcal{E}(K,F)$ in K and F.

The complete formula for $\mathcal{E}(K,F)$ is described at the end of this section, but in fact we will only need a crude estimate on the nonlinear part of it which will be quoted from [BDH-est].

We will need a variant of the extraction formula in which vacuum energy is factored out completely as follows. Suppose F has an additive piece F_0 which is field independent, i.e. if it is of the form

$$
F = F_1 + F_0 \tag{3.16}
$$

we have

$$
V'_F = V - V_F = V'_{F_1} - V_{F_0}
$$

and then

$$
\mathcal{E}xp(\square\, e^{-V'_F} + \mathcal{E}(K,F)(\Lambda) = e^{V_{F_0}(\Lambda)}\mathcal{E}xp(\square\, e^{-V'_{F_1}} + \mathcal{E}(K,F_0,F_1)(\Lambda) \tag{3.17}
$$

where

$$
\mathcal{E}(K,F_0,F_1) = e^{-V_{F_0}}[\mathcal{E}(K,F)] \tag{3.18}
$$

The extraction formula (3.17) is applied to (3.8) so that the combination of integrating out the fluctuation field and extracting returns $\mathcal{E}xp(\square\, e^{-V} + K)(\Lambda)$ to a functional of the same form with $V \to \tilde{V}'_{F_1}$ and $K \to \mathcal{E}((\mathcal{S}K)^\natural, F_0, F_1)$. This is a complete RG step .

Formulas for $\mathcal{E}(K,F)$:

$$
\mathcal{E}(K,F)(W) = \sum_{M,N} \frac{1}{M!}\frac{1}{N!} \sum_{(Z_j),(Y_k)\to W} e^{-V'(W\setminus Y)}
$$

$$
\prod_{j=1}^{M} (e^{-F(Z_j, Z_j\cap \bar{Y}_c)} - 1) \prod_{k=1}^{N} \tilde{K}(Y_k) \tag{3.19}
$$

where $N \geq 1,\quad M \geq 0$ and $Y = \cup_{k=1}^N Y_k$, the Y_k are disjoint and:

1) the polymers $(Z_j),(Y_k)$ are connected and such that $W = (\cup Z_j) \cup (\cup Y_k)$

2) for every j, $Z_j \not\subset Y_c, Z_j \not\subset Y$

and

$$
F(Z, Z\cap \bar{Y}_c) = \sum_{\Delta\subset Z\cap\bar{Y}_c} F(Z,\Delta) \tag{3.20}
$$

$$
\tilde{K}(X) = K(X) - e^{-V(X)}(e^F - 1)^+(X) \tag{3.21}
$$

$$
J^+(X) = \sum_{N>1} \frac{1}{N!} \sum_{(X_j)\to X} \prod J(X_j)
$$

where $(X_j) \to X$ iff $X = \cup X_j$ and X_j are distinct (as opposed to disjoint) sets. $J(X) = e^{F(X)} - 1$. J^+ is a polymer activity (vanishes when X is not closed and connected).

Remark: The proof of (3.19) is step by step the same as Sec. 4.2 of [BDH-est] with appropriate changes to take into account the use of closed polymers instead of open polymers.

Appendix: For convenient later reference we collect here the notations associated with rescaling that will be used in this paper. Some of them refer to objects not yet introduced.

$$\mathcal{R}\phi(x) = \phi_{L^{-1}}(x) = L^{-[\phi]}\phi(x/L) \tag{3.22}$$

$$(\mathcal{R}K)(L^{-1}X,\phi) = K_L(L^{-1}X,\phi) = K(X,\phi_{L^{-1}})$$

$$V_L(\Delta,\phi) = V(L\Delta,\phi_{L^{-1}})$$

$$\Gamma_L(x-y) = L^{2[\phi]}\Gamma(L(x-y))$$

For an integral kernel $u(x-y)$, e.g. a covariance, we define a rescaled version

$$u_L(x-y) = L^{2[\phi]}u(L(x-y)) = L^{\frac{3-\varepsilon}{2}}u(L(x-y)) \tag{3.23}$$

$$\begin{aligned}\tilde{V}_L(\Delta,g,\mu) =& V(\Delta,\phi,C,g_L,\mu_L)\\ g_L = L^{\varepsilon}g,& \quad \mu_L = L^{\frac{3+\varepsilon}{2}}\mu\end{aligned} \tag{3.24}$$

A set X is said to be *small* if X is connected and $|X| \le 2^D$. In the present case the dimension $D=3$. $\bar{X}^L$ is the L-closure of X, which by definition is smallest union of closed L-blocks containing X.

The superscript $\natural$ ("natural") denotes integration with the measure $d\mu_{\Gamma_L}(\zeta)$ and the superscript $\sharp$ ("sharp") denotes integration with the measure $d\mu_\Gamma(\zeta)$.

4. APPLICATION OF RENORMALIZATION GROUP STEP

In this section we specify a particular RG map by making choices for $\tilde{V}, F$ in (3.3) and (3.15). We define the second order approximation to the RG map and derive formula for the error after second order.
Recall:

$$V(\Delta,\phi) = V(\Delta,\phi,C,g,\mu) = g\int_\Delta d^3x :\phi^4:_C(x) + \mu\int_\Delta d^3x :\phi^2:_C(x) \tag{4.1}$$

We define $\tilde{V}$ by

$$\tilde{V}(\Delta,\phi) = V(\Delta,\phi,C_{L^{-1}},g,\mu) = g\int_\Delta d^3x :\phi^4:_{C_{L^{-1}}}(x) + \mu\int_\Delta d^3x :\phi^2:_{C_{L^{-1}}}(x) \tag{4.2}$$

Recall that the RG acts on functionals written in the form $\mathcal{E}xp(\square e^{-V}+K)(\Lambda)$. In this section we refine this description by writing

$$K = Qe^{-V} + R \tag{4.3}$$

where Q is an explicit polymer activity which we will call the *"second order polymer activity"*. It will be derived from second order perturbation theory in powers of g and is defined as follows: Q is supported on connected polymers X, $|X|\le 2$. We write

$$Q(X,\phi) = Q(X,\phi;C,\mathbf{w},g) = g^2\sum_{j=1}^{3} n_j Q^{(j,j)}(\tilde{X},\phi;C,w^{(4-j)}) \tag{4.4}$$

where $(n_1,n_2,n_3) = (48,36,8)$ and $\mathbf{w} = (w^{(1)},w^{(2)},w^{(3)})$ is a triple of integral kernels to be obtained inductively and

$$\tilde{X} = \begin{cases} \Delta\times\Delta & \text{if } X=\Delta \\ (\Delta_1\times\Delta_2)\cup(\Delta_2\times\Delta_1) & \text{if } X=\Delta_1\cup\Delta_2 \\ 0 & \text{otherwise} \end{cases} \tag{4.5}$$

$$\begin{aligned} Q^{(m,m)}(\tilde{X},\phi;C,w^{(4-m)}) &= \frac{1}{2}\int_{\tilde{X}} d^3x d^3y :(\phi^m(x)-\phi^m(y))^2:_C w^{(4-m)}(x-y) \quad \text{for } m=1,2 \\ Q^{(3,3)}(\tilde{X},\phi;C,w^{(1)}) &= \frac{1}{2}\int_{\tilde{X}} d^3x d^3y :\phi^3(x)\phi^3(y):_C w^{(1)}(x-y) \end{aligned} \tag{4.6}$$

Next we define the second order approximation to the RG map. We say that an activity $p(X)$ is supported on (closed/open) unit blocks if $p(X)=0$ for X not a (closed/open) unit block. A block is closed by default. Let p be the activity supported on unit blocks defined by

$$p(\Delta,\zeta,\phi) = V(\Delta,\zeta+\phi) - \tilde{V}(\Delta,\phi) = p_g + p_\mu \tag{4.7}$$

where

$$\begin{aligned} p_g = g\int_\Delta d^3x \big(&:\zeta^4:_\Gamma(x) + 4\phi(x):\zeta^3:_\Gamma(x) + 6:\phi^2:_{C_{L^{-1}}}(x):\zeta^2:_\Gamma(x) + \\ &4:\phi^3:_{C_{L^{-1}}}(x)\zeta(x)\big) \\ p_\mu = \mu\int_\Delta d^3x &\left(2\phi(x)\zeta(x) + :\zeta^2:_\Gamma(x)\right) \end{aligned} \tag{4.8}$$

We insert a parameter λ into our previous definitions in such a way that (i) at $\lambda=1$ our λ dependent objects correspond with the previous definitions. (ii) The expansion through order λ^2 is second order perturbation theory in g counting $\mu=O(g^2)$. (iii) Powers of λ are determined so as to correspond with leading powers of g buried inside polymer activities. (iv) All functions will turn out to be norm analytic in λ. Thus

$$P(\lambda) = e^{-\tilde{V}}\big(-\lambda p_g - \lambda^2 p_\mu + \frac{1}{2}\lambda^2 p_g^2\big) + \lambda^3 r_1 \tag{4.9}$$

where r_1 is defined by the condition $P(\lambda=1) = P = e^{-V} - e^{-\tilde{V}}$. Similarly, we define

$$K(\lambda) = \lambda^2 e^{-\tilde{V}} Q + \lambda^3\big([e^{-V} - e^{-\tilde{V}}]Q + R\big) \tag{4.10}$$

which, for $\lambda=1$ coincides with $K = e^{-V}Q + R$, where Q will be an explicit polymer activity obtained by second order perturbative calculation and R will be the remainder after second order perturbation theory. Corresponding to (3.6) we define

$$\mathcal{B}(\lambda,K)(Y) = \sum_{N+M\geq 1}\frac{1}{N!M!}\sum\nolimits_{(X_j),(\Delta_i)\to\{Y\}} e^{-\tilde{V}(X_0)}\prod\nolimits_{j=1}^{N} K(\lambda,X_j)\prod\nolimits_{i=1}^{M} P(\lambda,\Delta_i) \tag{4.11}$$

$$X_0 = Y\setminus(\cup X_j)\cup(\cup\Delta_i) \tag{4.12}$$

Let $\mathcal{S}(\lambda,K) = \mathcal{R}\circ\mathcal{B}(\lambda,K)$, where $\mathcal{R}$ is the rescaling defined in the last section. The RG evolution for K with parameter λ is $f_K : K \mapsto \mathcal{E}(\mathcal{S}(\lambda,K)^\natural, F(\lambda))$, where

$$F(\lambda) = \lambda^2 F_Q + \lambda^3 F_R \tag{4.13}$$

will be specified later, and $\natural$ is the rescaled integration over ζ, and, as usual, $F(\lambda) = F$, when $\lambda = 1$. Given a function $f(\lambda)$ let

$$T_\lambda f = f(0) + f'(0) + \frac{1}{2} f''(0) \tag{4.14}$$

be the Taylor expansion to second order evaluated at $\lambda = 1$. Then the second order approximation to the RG map is $f^{(\leq 2)} = (f_K^{(\leq 2)}, f_V^{(\leq 2)})$ with

$$f_K^{(\leq 2)}(K,V) = T_\lambda \mathcal{E}(\mathcal{S}(\lambda, K)^\natural, F(\lambda)) = \mathcal{E}_1(T_\lambda \mathcal{S}(\lambda, K)^\natural, F_Q) \tag{4.15}$$

$$f_V^{(\leq 2)}(K,V) = V'_F \tag{4.16}$$

Note also that only the linearized $\mathcal{E}_1$ intervenes, because it will turn out that the nonlinear part of extraction generates terms only at order λ^3 or higher.

Proposition 4.1

There is a choice of F_Q such that the form of Q remains invariant under the RG evolution at second order. In more detail, $f^{(\leq 2)}(V, Qe^{-V}) = (V'_{F_Q,(\leq 2)}, Q'_{(\leq 2)} e^{-\tilde{V}_L})$ where the parameters in $V'_{(\leq 2)}$ evolved according to

$$\begin{aligned} V'_{(\leq 2)}(\Delta) &= V(\Delta, C, g'_{(\leq 2)}, \mu'_{(\leq 2)}) \\ g'_{(\leq 2)} &= L^\varepsilon g(1 - L^\varepsilon a g) \\ \mu'_{(\leq 2)} &= L^{\frac{3+\varepsilon}{2}} \mu - L^{2\varepsilon} b g^2 \end{aligned} \tag{4.17}$$

The parameters in $Q'_{(\leq 2)}$ evolved according to

$$\begin{aligned} &Q'_{(\leq 2)} = Q(C, \mathbf{w}', g_L) \\ &\mathbf{w}' = \mathbf{v} + \mathbf{w}_L \\ &v^{(1)} = \Gamma_L, \qquad v^{(p)} = (C_L)^p - (C)^p \qquad 2 \leq p \leq 4 \end{aligned} \tag{4.18}$$

Proof: We define a polymer activity $\hat{Q}$ supported on connected polymers X with $|X| \leq 2$

$$\hat{Q}(X, \zeta, \phi) = \begin{cases} \frac{1}{2}(p_\lambda(\Delta, \zeta, \phi))^2 & |X| = \Delta \\ \frac{1}{2} \sum\limits_{\substack{\Delta_1, \Delta_2 \\ \Delta_1 \cup \Delta_2 = X}} p_\lambda(\Delta_1, \zeta, \phi) p_\lambda(\Delta_2, \zeta, \phi) & |X| = 2, \text{ connected} \end{cases} \tag{4.19}$$

It is easy to check that

$$T_\lambda \mathcal{S}(K, \lambda) = -p_L e^{-\tilde{V}_L} + \left\{ e^{-\tilde{V}_L} \hat{Q}_L + e^{-\tilde{V}_L} Q_L \right\} \tag{4.20}$$

Let

$$\tilde{Q}(C, \mathbf{v}, g_L) = g_L^2 \sum_{j=1}^{4} n_j \tilde{Q}^{(j,j)}(\tilde{X}, \phi; C, v^{(4-j)}) \tag{4.21}$$

where $(n_1, n_2, n_3, n_4) = (48, 36, 8, 12)$ and

$$\tilde{Q}^{(m,n)}(\tilde{X}, \phi; C, u) = \frac{1}{2} \int_{\tilde{X}} d^3x d^3y : \phi^m(x) \phi^n(y) :_C u(x-y) \tag{4.22}$$

We then have

$$T_\lambda \mathcal{S}(K,\lambda)^\natural = e^{-\tilde V_L}\left(\tilde Q(C,\mathbf{v},g_L)+Q(C,\mathbf{w}_L,g_L)\right) \tag{4.23}$$

Define

$$F_Q=\tilde Q(C,\mathbf{v},g_L)-Q(C,\mathbf{v},g_L) \tag{4.24}$$

Then we have from (4.23) and (4.24)

$$\mathcal{E}_1\left(T_\lambda\mathcal{S}(\lambda,K)^\natural,F\right)=T_\lambda\mathcal{S}(\lambda,K)^\natural-F_Qe^{-\tilde V_L}=e^{-\tilde V_L}Q(C,\mathbf{w}^{(\le 2)},g_L) \tag{4.25}$$

We write

$$F_Q(X,\phi)=F_{1,Q}(X,\phi)+F_{0,Q}(X) \tag{4.26}$$

where $F_{0,Q}$ is the field independent part of F_Q. Then

$$\begin{aligned}F_{1,Q}=&g_L^2\left\{36\ \tilde Q^{(4,0)}(C,v^{(2)})+48\ \tilde Q^{(2,0)}(C,v^{(3)})\right\}\\ F_{0,Q}=&12g_L^2\tilde Q^{(0,0)}(C,v^{(4)})\end{aligned} \tag{4.27}$$

$F_{1,Q}(X)$ can be written as:

$$F_{1,Q}(X)=\sum_{\Delta\subset X}F_{1,Q}(X,\Delta) \tag{4.28}$$

where

$$F_{1,Q}(X,\Delta)=36g_L^2F^{(4)}_{1,Q}(X,\Delta)+48g_L^2F^{(2)}_{1,Q}(X,\Delta) \tag{4.29}$$

and

$$F^{(m)}_{1,Q}(X,\Delta)=\int_\Delta d^3x:\phi^m:_C(x)f^{(m)}_{1,Q}(x,X,\Delta) \tag{4.30}$$

with

$$f^{(m)}_{1,Q}(x,X,\Delta)=\begin{cases}\int_\Delta d^3y v^{(m')}(x-y) & X=\Delta\\ \int_{\Delta'} d^3y v^{(m')}(x-y) & X=\Delta\cup\Delta',\ \text{connected}\end{cases} \tag{4.31}$$

and

$$m'=4-m/2$$

$$V(F_{1,Q},\Delta)=\sum_{X\supset\Delta}F_{1,Q}(X,\Delta)=36g_L^2\sum_{X\supset\Delta}F^{(4)}_{1,Q}(X,\Delta)+48g_L^2\sum_{X\supset\Delta}F^{(2)}_{1,Q}(X,\Delta) \tag{4.32}$$

In the following we will use the:

Claim:
$v^{(j)},\ 1\le j\le 4$ are C^∞, positive, and have support $v^{(j)}(x-y)=0$ for $|x-y|\ge 1$.

Proof: That they are C^∞ follows from their definition (4.25), Γ is C^∞ and C is C^∞ . For the support property, this is obvious for $v^{(1)}=\Gamma_L$, and for $p\ge 2$: $v^{(p)}=C_L^p-C^p$ with pointwise multiplication. The latter has Γ_L as a factor because $C_L=C+\Gamma_L$ and Γ_L has the required support property. The positivity follows from that of Γ_L and C. This proves the claim.

Now return to (4.32).

$$\sum_{X\supset\Delta}F^{(m)}_{1,Q}(X,\Delta)=\int_\Delta d^3x:\phi^m:_C(x)\left[\int_\Delta d^3y v^{(m')}(x-y)+\sum_{\substack{\Delta'\neq\Delta\\(\Delta,\Delta')\ \text{connected}}}\int_{\Delta'}d^3y v^{(m')}(x-y)\right]$$

On the r.h.s. use $v^{(m')}(x-y)=0$ for $|x-y|\geq 1$ to extend the sum on Δ' to *all* the $\Delta'\neq\Delta$. We then get

$$\sum_{X\supset\Delta} F^{(m)}_{1,Q}(X,\Delta)=\int_\Delta d^3x :\phi^m :_C (x)\left[\int d^3y v^{(m')}(x-y)\right]$$

Hence from (4.32) and above we get

$$V(F_{1,Q},\Delta)=ag_L^2\int_\Delta d^3x :\phi^4 :_C (x)+bg_L^2\int_\Delta d^3x :\phi^2 :_C (x) \tag{4.33}$$

where

$$\begin{aligned} a=&36\int d^3y\ v^{(2)}(y)>0=O(log\ L)>0\\ b=&48\int d^3y\ v^{(3)}(y)>0=O(L^{3/2})>0 \end{aligned} \tag{4.34}$$

That a and b are well defined and positive is easy to see using the claim above. That $a=O(log\ L), b=O(L^{3/2})$ is proved later in Section 5 (see Lemma 5.12). End of proof.

The exact RG evolution for $K=Qe^{-V}+R$.

The exact map

$$K\mapsto K'=f_K(\lambda,K,V)|_{\lambda=1}=\mathcal{E}(\mathcal{S}(\lambda,K)^\natural,F(\lambda))|_{\lambda=1} \tag{4.35}$$

induces an evolution of the remainder R which is studied by Taylor series around $\lambda=0$ with remainder written using the Cauchy formula:

$$f_K(\lambda=1)=\sum_{j=0}^{3}\frac{f_K^{(j)}(0)}{j!}+\frac{1}{2\pi i}\oint_\gamma\frac{d\lambda}{\lambda^4(\lambda-1)}f_K(\lambda)$$

The terms $j=0,1,2$ are the second order part $f_K^{(\leq 2)}$. In the $j=3$ term there are no terms mixing R with Q,P because of the λ^3 in front of R. Therefore it splits

$$\frac{f_K^{(3)}(0)}{3!}=R_1+R_2$$

into the third order derivative at $R=0$, which we write using the Cauchy formula as

$$R_1\equiv R_{\rm main}=\frac{1}{2\pi i}\oint_\gamma\frac{d\lambda}{\lambda^4}\mathcal{E}\Big(\mathcal{S}(\lambda,Qe^{-V})^\natural,F_Q(\lambda)\Big) \tag{4.36}$$

and terms linear in R:

$$\begin{aligned} R_2\equiv R_{\rm linear}&=\big(\mathcal{S}_1R\big)^\natural-F_Re^{-\tilde V_L}\\ \mathcal{S}_1R(Z)&=\sum_{X:L^{-1}\bar X^L=Z}e^{-\tilde V_L(Z\backslash L^{-1}X)}R_L(L^{-1}X) \end{aligned} \tag{4.37}$$

The remainder term in the Taylor expansion is

$$R_3=\frac{1}{2\pi i}\oint_\gamma\frac{d\lambda}{\lambda^4(\lambda-1)}\mathcal{E}(\mathcal{S}(\lambda,K)^\natural,F(\lambda)) \tag{4.38}$$

In Proposition 4.1 the coupling constant in Q is not the same as the coupling constant in $V^{(\leq 2)}$. Furthermore, the coupling constant in $V^{(\leq 2)}$ will further change because of the contribution from F_R. To take this into account we introduce

$$
\begin{aligned}
V'(\Delta) &= V(\Delta, C, g', \mu') \\
g' &= L^{\varepsilon} g(1 - L^{\varepsilon} a g) + \xi_R \\
\mu' &= L^{\frac{3+\varepsilon}{2}} \mu - L^{2\varepsilon} b g^2 + \rho_R
\end{aligned} \tag{4.39}
$$

where the remainders ξ_R, ρ_R anticipate the effects of a yet-to-be-specified F_R. Then we set

$$
R_4 = e^{-V'} Q(C, \mathbf{w}', g') - e^{-\tilde{V}_L} Q(C, \mathbf{w}', g_L) \tag{4.40}
$$

With these definitions we have written the RG as a map

$$
\mathcal{E}xp(\square\, e^{-V} + Qe^{-V} + R)(\Lambda) \mapsto \mathcal{E}xp(\square\, e^{-V'} + Q' e^{-V'} + R')(L^{-1}\Lambda)
$$

with $Q' = Q(C, \mathbf{w}', g')$ and the RG map f_K has induced a map f_R

$$
R' \equiv f_R(V, R) = R_{\mathrm{main}} + R_{\mathrm{linear}} + R_3 + R_4 \tag{4.41}
$$

Construction of F_R

To complete the RG step we must specify the relevant part from the remainder F_R. The goal is to choose F_R so that the map $R \to R_{\mathrm{linear}}$ will be contractive. As we will later prove, this will be the case provided the small set part of R_{linear} is normalized so that certain derivatives with respect to ϕ vanish at $\phi = 0$. We say that a functional J is normalized if, for all small sets X,

$$
\begin{aligned}
&J(X, 0) = 0 \\
&D^2 J(X, 0; 1, 1) = 0 \\
&D^2 J(X, 0; 1, x_\mu) = 0 \\
&D^4 J(X, 0; 1, 1, 1, 1) = 0
\end{aligned} \tag{4.42}
$$

Define

$$
\tilde{F}_R(X, \phi) = \sum_P \int_X d^3x \; \tilde{\alpha}_P(X) P(\phi(x), \partial\phi(x)) \tag{4.43}
$$

where P runs over the *relevant* monomials, which , in this model are

$$
P = 1,\ \phi^2,\ \phi^4,\ \phi\partial_\mu\phi \text{ with } \mu = 1, 2, 3,
$$

Choose the coefficients $\tilde{\alpha}_P$ so that

$$
J = R^\sharp - \tilde{F}_R e^{-\tilde{V}} \tag{4.44}
$$

is normalized (details are given below). We define the relevant part, a functional supported on small sets, by

$$
F_R(Z) = \sum_{\substack{X:\text{small sets} \\ L^{-1}\bar{X}^L = Z}} \tilde{F}_{R,L}(L^{-1}X) \tag{4.45}
$$

Then, from the definition of R_{linear} in (4.37),

$$
R_{\mathrm{linear}}(Z) = \sum_{\substack{X:\text{small sets} \\ L^{-1}\bar{X}^L = Z}} e^{-\tilde{V}_L(Z \setminus L^{-1}X)} J_L(L^{-1}X) + \sum_{\substack{X:\text{large sets} \\ L^{-1}\bar{X}^L = Z}} e^{-\tilde{V}_L(Z \setminus L^{-1}X)} J_L(L^{-1}X) \tag{4.46}
$$

Therefore the first sum in $R_{\rm linear}$ is also normalized because normalization is preserved under multiplication by smooth functionals of ϕ and rescaling.

Having defined F_R, we must show that it has the form (3.9), (3.10) required for the extraction operation. Define $\tilde F_R(X,Y)$ by replacing the integral over X by integration over $X\cap Y$ in (4.43). Let

$$F_R(Z,\Delta,\phi) = \sum_{\substack{X \text{ small set}\\ \overline{L^{-1}X}^L = Z}} \tilde F_R(X, L\Delta\cap X, \phi_{L^{-1}}) \tag{4.47}$$

It is clear that (3.9) holds. For each monomial $\alpha_P(X)P$ in (4.43) define

$$\alpha_P(Z,\Delta,x) = \sum_{\substack{X \text{ small set}\\ \overline{L^{-1}X}^L = Z}} \tilde\alpha_P(X) L^{-[P]+3} 1_{\Delta\cap L^{-1}X}(x) \tag{4.48}$$

where $[P]$ is the dimension of the monomial P, ($n[\phi]$ for ϕ^n and $2[\phi]+1$ for $\phi\partial\phi$). Then

$$F_R(Z,\Delta,\phi) = \sum_P \int d^3x\ \alpha_P(Z,\Delta,x) P(\phi(x),\partial\phi(x)) \tag{4.49}$$

which shows that (3.10) holds.

In order to compute V_{F_R}, note that by translation invariance,

$$\begin{aligned}\sum_{Z\supset\Delta} \alpha_P(Z,\Delta,x) &= \sum_{Z\supset\Delta} \sum_{\substack{X \text{ small set}\\ \overline{L^{-1}X}^L = Z}} \alpha_P(X) L^{-[P]+3} 1_{\Delta\cap L^{-1}X}(x)\\ &= \sum_{\substack{X \text{ small set}\\ \overline{L^{-1}X}^L \supset\Delta}} \alpha_P(X) L^{-[P]+3} 1_{\Delta\cap L^{-1}X}(x)\\ &= \sum_{X \text{ small set}} \alpha_P(X) L^{-[P]+3} 1_{\Delta\cap L^{-1}X}(x)\\ &=: \alpha_P \qquad \text{a.e}\end{aligned} \tag{4.50}$$

is independent of x,Δ, (equality a.s. because the equality fails for x in boundaries of blocks). This can be written more simply as

$$\alpha_P = L^{-[P]+3} \sum_{X \text{ small set}\supset\Delta(x)} \alpha_P(X) \tag{4.51}$$

where $\Delta(x)$ is the block that contains x, excluding x in a boundary. Therefore, with coefficients derived as in (4.51),

$$\begin{aligned}V(F_R,\Delta) &= \sum_{Z\supset\Delta} F_R(Z,\Delta) = \sum_P \alpha_P \int_\Delta d^3x\, P(\phi)\\ &=: \int_\Delta d^3x \Big\{\alpha_0 + \alpha_{2,0}\phi^2 + \alpha_{4,0}\phi^4\Big\}\end{aligned} \tag{4.52}$$

where the term $\alpha_{2,1}\phi(x)\partial_\mu\phi(x)$ is absent because $\alpha_{2,1}=0$ by reflection invariance. We have to rewrite this in a C Wick ordered basis in order to compute V'

$$V(F_R,\Delta) = \int_\Delta d^3x \Big\{\beta_0 + \rho_R :\phi^2:_C(x) + \xi_R :\phi^4:_C(x)\Big\} \tag{4.53}$$

where

$$
\begin{aligned}
\beta_0 &= \alpha_0 + C(0)\alpha_{2,0} + 3C^2(0)\alpha_{4,0} \\
\rho_R &= \alpha_{2,0} + 6C(0)\alpha_{4,0} \\
\xi_R &= \alpha_{4,0}
\end{aligned}
\tag{4.54}
$$

which are formulas for the error terms in (4.17).

Determining coefficients from (4.42)

Note that the odd derivatives $D^j J(X, 0; f^{\times j})$, j=odd integer, vanish identically by $\phi \rightleftharpoons -\phi$ symmetry. Taking derivatives of (4.44) we get

$$
f_K(\lambda = 1) = \sum_{j=0}^{3} \frac{f_K^{(j)}(0)}{j!} + \frac{1}{2\pi i} \oint_\gamma \frac{d\lambda}{\lambda^4(\lambda - 1)} f_K(\lambda)
$$

The terms $j = 0, 1, 2$ are the second order part $f_K^{(\leq 2)}$. In the $j = 3$ term there are no terms mixing R with Q, P because of the λ^3 in front of R. Therefore it splits

$$
\frac{f_K^{(3)}(0)}{3!} = R_1 + R_2
$$

into the third order derivative at $R = 0$, which we write using the Cauchy formula as

$$
R_1 \equiv R_{\text{main}} = \frac{1}{2\pi i} \oint_\gamma \frac{d\lambda}{\lambda^4} \mathcal{E}\Big(\mathcal{S}(\lambda, Qe^{-V})^\natural, F_Q(\lambda)\Big) \tag{4.55}
$$

and terms linear in R:

$$
\begin{aligned}
R_2 \equiv R_{\text{linear}} &= \big(\mathcal{S}_1 R\big)^\natural - F_R e^{-,1}(X, \mu) \\
= \frac{1}{|X|} e^{-\tilde{V}(X,0)} &\Big[D^2 R^\sharp(X, 0; 1, x_\mu) + R^\sharp(X, 0) D^2 \tilde{V}(X, 0; 1, x_\mu)\Big] \\
\tilde{\alpha}_4(X) &= \frac{1}{24} \frac{1}{|X|} e^{-\tilde{V}(X,0)} \big[D^4 R^\sharp(X, 0; 1, 1, 1, 1) \\
&+ D^2 \tilde{V}(X, 0; 1, 1) \Big([D^2 R^\sharp(X, 0; 1, 1) + R^\sharp(X, 0) D^2 \tilde{V}(X, 0; 1, 1)\Big) \\
&+ R^\sharp(X, 0) \Big(D^4 \tilde{V}(X, 0; 1, 1, 1, 1) - 3(D^2 \tilde{V}(X, 0; 1, 1))^2\Big)\Big]
\end{aligned}
\tag{4.56}
$$

Note that the leading contributions to the $\tilde{\alpha}_P(X)$ are obtained by setting $\tilde{V} = 0$ in the above formulae.

Resume

We have thus produced at the end of the RG step the promised map :

$$
(V, Q, R) \to (V', Q', R')
$$

V' is the same as V with evolved coupling $g \to g'$ $\mu \to \mu'$ given by the flow ((4.39)), with a, b given in (4.34) and ξ_R, ρ_R in (4.54). Q' is the same as Q with the change $\mathbf{w} \to \mathbf{w}'$, (4.18), and R' is given by (4.41) with intervening quantities defined earlier.

5. ESTIMATES.

We will assume L large but fixed and then ε sufficiently small depending on L . $O(1)$ denotes a constant independent of L and ε. Constants C are independent of ε but may depend on L. These constants may change from line to line. It will not be necessary to keep track of these changes.

Throughout we will assume that $\mathbf{w}$ *at a generic step has been obtained by successive iterations (4.18) with initial* $\mathbf{w_0} = 0$.

We make in this section the following hypothesis in terms of the norms introduced in Section 2.2.

Hypothesis

$$|g - \bar{g}| \le \varepsilon^{3/2}, \quad |\mu| \le \varepsilon^{2-\delta} \tag{5.1}$$

$$\|R\|_{h,G_\kappa,\mathcal{A}} \le \varepsilon^{3/4-\eta} \tag{5.2}$$

$$|R|_{h_*,\mathcal{A}} \le \varepsilon^{11/4-\eta} \tag{5.3}$$

where $\delta, \eta = O(1) > 0$ are very small fixed numbers, say $1/64$, and $h = c\varepsilon^{-1/4}$ with $c = O(1)$ a very small number. Further more we take $h_* = L^{(3+\varepsilon)/4}$ and choose $n_0 = 9$. Moreover $\bar{g}$ is the approximate fixed point in the flow of the coupling constant g obtained from the first equation in (4.39) by ignoring the remainder $\xi_{\hat{R}}$. Namely,

$$\bar{g} = \frac{L^\varepsilon - 1}{L^\varepsilon a} = O(\varepsilon) > 0 \tag{5.4}$$

for ε sufficiently small depending on L. We have used the estimate $a = O(log\ \ L) > 0$ which is proved below in Lemma 5.12 (independent of the Hypothesis). Note that the Hypothesis now implies that $g = O(\varepsilon)$ for ε sufficiently small.

Recall the definitions of ρ_R and ξ_R from (4.54). We will prove in this section the following

Theorem 1
Given the Hypothesis above we have for L sufficiently large and then ε sufficiently small

$$|\xi_R| \le L^\varepsilon \varepsilon^{11/4-\eta} \tag{5.5}$$

$$|\rho_R| \le L^{(3+\varepsilon)/2} \varepsilon^{11/4-\eta} \tag{5.6}$$

$$|g' - \bar{g}| \le \varepsilon^{3/2}, \quad |\mu'| \le C\varepsilon^{2-\delta} \tag{5.7}$$

$$|g' - g| \le \varepsilon^2 \tag{5.8}$$

$$\|R'\|_{h,G_\kappa,\mathcal{A}} \le L^{-1/4}\varepsilon^{3/4-\eta} \tag{5.9}$$

$$|R'|_{h_*,\mathcal{A}} \le L^{-1/4}\varepsilon^{11/4-\eta} \tag{5.10}$$

The following long series of lemmas, *except* Lemmas 5.1-5.4 and 5.14, are proved under the Hypothesis above, and will serve to prove Theorem 1. Lemmas 5.21, 5.22, 5.23 and 5.27 are the major parts of the program. R_{main} is bounded in Lemma 5.21 and this result determines the qualitative form of the bound on the remainder. R_3 and R_4 are seen, in Lemmas 5.22, 5.23 to be negligible in comparison. R_{linear} is the crux of the program and it is bounded in Lemma 5.27. The remaining Lemmas are auxiliary results on the way to these Lemmas.

These auxiliary lemmas implement some the following principles: in bounds by $G, h, \mathcal{A}$ norms, a fluctuation field $\zeta(x)$ contributes C_L and a field ϕ contributes $O(1)g^{-1/4}$. In bounds by the $1, \mathcal{A}$ norms, fluctuation fields and ϕ fields contribute $O(1)$.

Lemma 5.1
Let Z be a 1-polymer, Y be a L^{-1}-polymer or $\emptyset$, $Y \subset Z$ and vol$(Z\backslash Y) \geq \frac{1}{2}$. Choose $\gamma = O(1) > 0$, $\kappa = O(1) > 0$. Let σ sufficiently large. Then for any $x \in Z$, there exists a constant $O(1)$ depending on κ, γ, j such that

$$\|\phi\|^j_{C^2(Z)} \leq O(1) g^{-\frac{j}{4}} e^{\gamma g \int_{Z\backslash Y} d^3y|\phi(y)|^4} G_\kappa(Z,\phi) \tag{5.11}$$

Proof
This is a simple variant of an analogous lemma in [BDH-est]. Write

$$\phi(x) = \frac{1}{\text{vol}(Z\backslash Y)} \int_{Z\backslash Y} d^3y(\phi(y) + \phi(x) - \phi(y))$$

and bound

$$\begin{aligned}|\phi(x)| \leq & \frac{1}{\text{vol}(Z\backslash Y)} \int_{Z\backslash Y} d^3y|\phi(y)| + \frac{1}{\text{vol}(Z\backslash Y)} \int_{Z\backslash Y} d^3y|\phi(x) - \phi(y)| \\ \leq & O(1) \left(\|\phi\|_{L^4(Z\backslash Y)} + \|\phi\|_{Z,1,\sigma}\right)\end{aligned}$$

where the first term was bounded using the Hölder inequality. The second term was bounded by writing the difference as the integral of $\nabla\phi$ and using

$$|\nabla\phi(x)| \leq O(1)\|\phi\|_{\Delta,1,\sigma}, \qquad \text{for } \Delta \ni x$$

which is the Sobolev embedding theorem, valid for $\sigma > 3/2 + 2$. We also have under the same condition on σ

$$\|\nabla^2\phi\|_{C(Z)} \leq O(1)\|\phi\|_{Z,1,\sigma}$$

Hence

$$\begin{aligned}\|\phi\|^j_{C^2(Z)} \leq & O(1) \left(\|\phi\|^j_{L^4(Z\backslash Y)} + \|\phi\|^j_{Z,1,\sigma}\right) \\ \leq & O(1) g^{-j/4} e^{\gamma g \int_{Z\backslash Y} d^3y|\phi(y)|^4} G_\kappa(Z,\phi)\end{aligned}$$

where $O(1)$ depends on κ, γ, j. This proves the lemma.
For fluctuation fields ζ, we will have occasion to use the following lemmas
Define

$$\tilde{G}_{\kappa,\alpha}(X,\zeta) = e^{\alpha\|\zeta\|^2_{L^2(X)}} G_\kappa(X,\zeta), \quad \alpha, \kappa > 0 \tag{5.12}$$

κ is O(1) and will be held sufficiently small. The choice of α is dictated by Lemma 5.3 below.

Lemma 5.2
For any $x \in X$

$$|\zeta(x)|^j \leq C_{\alpha,j} \tilde{G}_{\kappa,\alpha}(X,\zeta) \tag{5.13}$$

where

$$C_{\alpha,j} = (\alpha)^{-(j/2)} O(1)$$

and $O(1)$ depends on j and κ. We have isolated out the α dependance in the bound.

Lemma 5.3

$$\int d\mu_\Gamma(\zeta)\tilde{G}_{\kappa,\alpha}(X,\zeta) \le 2^{|X|} \tag{5.14}$$

for $\alpha = \alpha(L) > 0$ sufficiently small and $\kappa = O(1) > 0$ sufficiently small. Here

$$\alpha(L) = L^{-(3-2[\phi])}\kappa' = L^{-(3+\varepsilon)/2}\kappa'$$

and $\kappa' = O(1) > 0$ is held sufficiently small.

The proof of Lemma 5.2 follows the lines of Lemma 5.1 except that we replace the L^4 norm there by the L^2 norm in the appropriate places and $Y = \emptyset$ and $Z = \Delta \ni x$. The proof of Lemma 5.3 is the same as the one referenced for (2.3).

It is convenient, for the control of norms of our polymer activities in intermediate steps, to introduce some new regulators and some intermediate norms in the following way.

Define

$$\hat{G}_{\kappa,\alpha}(X,\zeta,\phi) = G_\kappa(X,\zeta+\phi)G_\kappa(X,\phi)\tilde{G}_{\kappa,\alpha}(X,\zeta) \tag{5.15}$$

$\hat{G}_{\kappa,\alpha}$ is a regulator.

Lemma 5.4

$$\int d\mu_\Gamma(\zeta)\hat{G}_{\kappa,\alpha}(X,\zeta,\phi) \le 2^{|X|}G_{3\kappa}(X,\phi) \tag{5.16}$$

for $\alpha = \alpha(L) > 0$ sufficiently small and $\kappa = O(1) > 0$ sufficiently small.

Proof: use Cauchy-Schwartz, stability of G_κ, (2.3) and Lemma 5.3.

For polymer activity $K(X,\zeta,\phi)$ define the norms

$$\|K(X)\|_{h,\hat{G}_{\kappa,\alpha}} = \sup_{\phi,\zeta}\|K(X,\zeta,\phi)\|_h\hat{G}^{-1}_{\kappa,\alpha}(X,\zeta,\phi) \tag{5.17}$$

$$\|K(X)\|_{h_*,\tilde{G}_{\kappa,\alpha}} = \sup_{\zeta}\|K(X,\zeta,0)\|_{h_*}\tilde{G}^{-1}_{\kappa,\alpha}(X,\zeta) \tag{5.18}$$

where in (5.17) and (5.18) the functional derivatives in $\|K(X,\zeta,\phi)\|_h$ and in $\|K(X,\zeta,0)\|_{h_*}$ are computed with respect to the field ϕ.

The norms above are useful because before fluctuation integration we will encounter activities $K(X,\zeta,\phi)$ which are not just functions of $\zeta+\phi$.

The following lemma is a variant of Theorem 1 [BDH-est] adapted to our purposes.

Lemma 5.5

For $V(Y,\phi,\zeta) = V(Y,\phi+\zeta,C,g,\mu)$ or $V(Y,\phi+\zeta,C_{L^{-1}},g,\mu)$,

$$\|e^{-V(Y,\phi+\zeta)}\|_h \le 2^{|Y|}e^{-g/4\int_Y d^3x(\phi+\zeta)^4(x)} \tag{5.19}$$

$$\|e^{-V(Y,\phi+\zeta)}\|_{h_*} \le 2^{|Y|} \tag{5.20}$$

for $\varepsilon > 0$ sufficiently small, and, for the second bound, depending on L.

Proof:

$$\tilde{V}(\Delta,\phi) = V(\Delta,\phi,C_{L^{-1}},g,\mu) = g\int_\Delta d^3x :\phi^4:_{C_{L^{-1}}}(x) + \mu\int_\Delta d^3x :\phi^2:_{C_{L^{-1}}}(x)$$

Undo the Wick ordering. Wick constants are finite and $O(1)$. Recall from the initial Hypothesis that $g = O(\varepsilon)$ and $|\mu| \le O(\varepsilon^2)$, and $h = c\varepsilon^{-1/4}$, with c small enough. Hence

$$\tilde V(\Delta,\phi)) = g \int_\Delta d^3x\ \phi^4(x) - O(1) g \int_\Delta d^3x\ \phi^2(x) - O(\varepsilon)$$

$$\tilde V(\Delta,\phi) - \frac{g}{2}\int_\Delta d^3x\ \phi^4(x) = \frac{g}{2}\int_\Delta d^3x\ \phi^4(x) - O(1) g \int_\Delta d^3x\ \phi^2(x) - O(\varepsilon) =$$

$$= \frac{g}{2}\int_\Delta d^3x\ \left((\phi^2(x) - O(1))^2 - O(1)\right) \ge -O(\varepsilon)$$

Hence

$$e^{-\tilde V(\Delta,\phi)} \le (1 + O(\varepsilon)) e^{-g/2 \int_\Delta d^3x \phi^4(x)}$$

Compute now the derivatives $D^k\ e^{-\tilde V}$. We get for $1 \le k \le n_0$

$$\frac{h^k}{k!}\left\|(D^k\ e^{-\tilde V})(\Delta,\phi)\right\| \le c^k \sum_{j=1}^k \frac{1}{j!} \sum_{\substack{1\le l_i \le 4\\ \sum l_i = k}} \prod_{i=1}^j \frac{(\varepsilon^{-1/4})^{l_i}}{l_i!} \left\|D^{l_i}\tilde V(\Delta,\phi)\right\| e^{-\tilde V(\Delta,\phi)}$$

$$\le c^k \sum_{j=1}^k \frac{1}{j!}\Big(\sum_{1\le l\le 4} \frac{(\varepsilon^{-1/4})^l}{l!}\left\|D^l \tilde V(\Delta,\phi)\right\|\Big)^j e^{-\tilde V(\Delta,\phi)}$$

$$\le O(1) c^k e^{-\frac{g}{2}\int_\Delta d^3x \phi^4(x)} e^{\sum_{1\le l\le 4} \frac{(\varepsilon^{1/4})^l}{l!}\|D^l \tilde V(\Delta,\phi)\|}$$

Take the expression for $\tilde V$ where the Wick ordering has been undone. Then it is easy to see that

$$\sum_{1\le l\le 4} \frac{(\varepsilon^{-1/4})^l}{l!}\left\|D^l \tilde V(\Delta,\phi)\right\| \le O(1)\int_\Delta d^3x \sum_{j=0}^3 \left|\varepsilon^{1/4}\phi\right|^j \le \frac{g}{4}\int_\Delta d^3x \phi^4(x) + O(1)$$

Hence

$$\frac{h^k}{k!}\left\|(D^k\ e^{-\tilde V})(\Delta,\phi)\right\| \le O(1) c^k e^{-\frac{g}{4}\int_\Delta d^3x\phi^4(x)}$$

The sum over k is $O(c)$ if c is small enough. The proof of (5.19)follows easily. The proof of (5.20) follows the same lines but we must take ε sufficiently small depending on L.

Lemma 5.6
Let $p_g(\Delta,\zeta,\phi), p_\mu(\Delta,\zeta,\phi)$ be as given in (4.8). $h = c\varepsilon^{-1/4}$, g,μ as in the inductive hypothesis, and h_* as defined earlier. Then for any $\alpha > 0$, $\kappa = O(1) > 0$, $\xi = O(1) > 0$, $0 \le s < 1$

$$\|p_g(\Delta,\zeta,\phi)\|_h \le C_\alpha \varepsilon^{1/4}(1-s)^{-3/4}\tilde G_{\kappa,\alpha}(\Delta,\zeta) G_\kappa(\Delta,\phi) e^{g(1-s)\xi\int_\Delta d^3x\ \phi^4(x)} \tag{5.21}$$

$$\|p_\mu(\Delta,\zeta,\phi)\|_h \le C_\alpha \epsilon^{7/4-\delta}(1-s)^{-1/2}\tilde G_{\kappa,\alpha}(\Delta,\zeta) G_\kappa(\Delta,\phi) e^{g(1-s)\xi\int_\Delta d^3x\ \phi^4(x)} \tag{5.22}$$

$$\|p_g(\Delta,\zeta,0)\|_{h_*} \le C_{\alpha,L}\varepsilon \tilde G_{\kappa,\alpha}(\Delta,\zeta) \tag{5.23}$$

$$\|p_\mu(\Delta,\zeta,0)\|_{h_*} \le C_{\alpha,L}\varepsilon^{2-\delta} \tilde G_{\kappa,\alpha}(\Delta,\zeta) \tag{5.24}$$

Proof

Undoing the Wick ordering we have

$$p_g(\Delta,\zeta,\phi) = g\int_\Delta d^3x \sum_{j=1}^{3} a_j \zeta^{4-j}(x)\phi^j(x)$$

where the constants a_j are $O(1)$.

$$h^k\left|D^k p_g(\Delta,\zeta,\phi;f^{\times k})\right| \le \varepsilon^{-k/4} g\int_\Delta d^3x\ \sum_{j=1}^{3} a_j|\zeta^{4-j}(x)||\phi^{j-k}(x)|\|f\|^{\times k}_{C^2(\Delta)}$$

Note that since p_g is a third degree polynomial in ϕ, derivatives with $k>3$ vanish. Moreover on the right hand side $j\ge k$. Now use Lemma 5.1, 5.2 to get

$$h^k\left\|D^k p_g(\Delta,\zeta,\phi)\right\| \le C_\alpha g\varepsilon^{-k/4} g^{-\frac{3-k}{4}}(1-s)^{-3/4}\tilde G_{\kappa,\alpha}(\Delta,\zeta)G_\kappa(\Delta,\phi)e^{g(1-s)\xi\int_\Delta d^3x\ \phi^4(x)}$$

Use $g=O(\varepsilon)$, multiply by $\frac{1}{k!}$ and take sum over k to obtain (5.21). The remaining parts are obtained along the same lines.

Define $p(s)=p(s,\Delta,\phi,\zeta)$ by

$$p(s) = sp_g + s^2 p_\mu \tag{5.25}$$

Then $r_1=r_1(\Delta,\phi,\zeta)$ defined by (4.9) is given by

$$r_1 = \frac{1}{2}\int_0^1 ds(1-s)^2 e^{-p(s)-\tilde V}\big(-p'(s)^3+6p'(s)p_\mu\big) \tag{5.26}$$

with $p'(s)=\frac{d}{ds}p(s)=p_g+2sp_\mu$ and $p''(s)=2p_\mu$.

Lemma 5.7

$$\|r_1(\Delta)\|_{h,\hat G_{\kappa,\alpha}} \le C_\alpha\varepsilon^{3/4} \tag{5.27}$$

$$\|r_1(\Delta)\|_{h_*,\tilde G_{\kappa,\alpha}} \le C_{\alpha,L}\varepsilon^{3-\delta} \tag{5.28}$$

Proof
The hypotheses for g,μ also hold for $sg, s^2\mu$ and for $[1-s]g,[1-s^2]\mu$. Since $V+p(s)=V_1(s)+V_2(s)$ with

$$V_1(s)=V(\Delta,\phi+\zeta,C,sg,s^2\mu),\qquad V_2(s)=V(\Delta,\phi,C_{L^{-1}},[1-s]g,[1-s^2]\mu)$$

$$\|r_1(\Delta,\zeta,\phi)\|_h \le \frac{1}{2}\int_0^1 ds(1-s)^2\|e^{-V_1(s)}\|_h\|e^{-V_2(s)}\|_h\Big(\|p'(s)\|_h^3+6\|p'(s)\|_h\|p_\mu\|_h\Big)$$

By Lemma 5.5, the exponential terms are bounded by $4\exp(-(g[1-s]/4)\int\phi^4)$. Use Lemma 5.6 choosing for the regulators the constants $\frac{\kappa}{4}$ and $\frac{\alpha}{3}$ to obtain

$$\|r_1(\Delta,\zeta,\phi)\|_h \le C_\alpha\varepsilon^{3/4}\hat G_{\kappa,\alpha}\int_0^1 ds(1-s)^2(1-s)^{-\frac{9}{4}}e^{-\frac{g}{4}(1-s)\int_\Delta d^3x\ \phi^4(x)}e^{g(1-s)3\xi\int_\Delta d^3x\ \phi^4(x)}$$

Choose $0 < \xi < \frac{1}{12}$ to get (5.27). Equation (5.28) is proved similarly.

Lemma 5.8
Consider $P(\lambda)$ given in (4.9). Then for C independent of ε, but dependent on L,

$$\|P\|_{h,\hat{G}_{\kappa,\alpha},\mathcal{A}} \le C_L|\lambda\varepsilon|^{1/4} \qquad \text{for } |\lambda\varepsilon^{1/4}| \le 1 \tag{5.29}$$

$$\|P\|_{h_*,\tilde{G}_{\kappa,\alpha},\mathcal{A}} \le C_L|\lambda\varepsilon^{1-\delta/2}| \qquad \text{for } |\lambda\varepsilon^{1-\delta/2}| \le 1 \tag{5.30}$$

Proof
(5.29) and (5.30) are immediate from Lemma 5.7, noting that $\lambda\varepsilon^{1-\delta/2}$ in (5.30) is the largest of the several combinations of λ, ε that arise. It comes from the term involving $\mu\lambda^2$.

Estimates for Qe^{-V}
We now turn to the estimate of Qe^{-V}. From (4.4)

$$Q(C, \mathbf{w}, g) = g^2 \sum_{m=1}^{3} a_m Q^{(m,m)}(C, w^{(4-m)}) \tag{5.31}$$

where the a_m are numerical coefficients and the $Q^{(m,m)}$ are given in (4.6). Under an iteration, see Proposition 4.1, we have

$$w^{(p)} \to w^{(p)'} = v^{(p)} + w_L^{(p)}$$

where $p = 1, 2, 3$ and the $v^{(p)}$ are given in Proposition 4.1. Starting with $w_0^{(p)} = 0$ we get after n iterations

$$w_n^{(p)} = \sum_{j=0}^{n-1} v_{L^j}^{(p)} \tag{5.32}$$

We need to first estimate $w_n^{(p)}$ and the limit $\lim_{n\to\infty} w_n^{(p)}$ under appropriate norms.

We consider Banach spaces $\mathcal{W}_p$ of measurable functions with norms $\|\cdot\|_p$, $p = 1, 2, 3$, defined as follows

$$\|f\|_p = ess.\sup_x \left(|x|^{\frac{6p+1}{4}}|f(x)|\right) \tag{5.33}$$

We define the Banach space $\mathcal{W}_1 \times \mathcal{W}_2 \times \mathcal{W}_3$ consisting of vectors $\mathbf{w}$ with the norm

$$\|\mathbf{w}\| = \sup_p \|w_p\|_p$$

Then we have

Lemma 5.9
For L sufficiently large and $\varepsilon > 0$ sufficiently small there exists a constant $c = O(1)$ such that

$$\|\mathbf{w}_n\| \le c/4 \quad \forall n$$

and

$$\|\mathbf{w}_{n+1} - \mathbf{w}_n\| \le c/8 \quad L^{-n/4}$$

so that $\mathbf{w}_n \to \mathbf{w}_*$ *in the norm* $\|\cdot\|$, *and*

$$\|\mathbf{w}_*\| \leq c/4$$

Proof

Let us note first some weak uniform (in L) bounds. Recall that $[\phi] = \frac{3-\varepsilon}{4}$.

$$|\Gamma_L(x)| \leq O(1)|x|^{-2[\phi]} \tag{5.34}$$

$$|C(x)| \leq O(1) \tag{5.35}$$

To see this observe that from the definition of Γ (see Section 1)

$$|\Gamma_L(x)| \leq \int_0^\infty \frac{dl}{l} l^{-2[\phi]} \left| u\left(\frac{x}{l}\right)\right|$$

Let $x \neq 0$. Then, using support properties of u,

$$|\Gamma_L(x)| \leq \int_{|x|}^\infty \frac{dl}{l} l^{-2[\phi]} \; \|u\|_\infty = \frac{2}{3-\varepsilon} \; \frac{1}{|x|^{2[\phi]}} \; \|u\|_\infty$$

which proves (5.34). To prove (5.35) recall

$$|C(x)| \leq \int_1^\infty \frac{dl}{l} l^{-2[\phi]} \left| u\left(\frac{x}{l}\right)\right| \leq O(1)\|u\|_\infty$$

which proves (5.35).

For $p = 1, 2, 3$,

$$\begin{aligned}
|v^{(p)}(x)| &= |C_L^p(x) - C^p(x)| \\
&\leq p \sup_{1\leq q\leq p} |\Gamma_L(x)|^q |C(x)|^{p-q} \leq O(1) \begin{cases} |x|^{-p2[\phi]} & \\ 0 & \text{if } |x| \geq 1 \end{cases}
\end{aligned}$$

where we exploited $C_L = \Gamma_L + C$ and $\Gamma_L(x) = 0$ if $|x| \geq 1$ and (5.34) and (5.35). Hence

$$\|v^{(p)}\|_p \leq O(1)$$

and by scaling $x = L^j x'$

$$\|v_{L^j}^{(p)}\|_p \leq L^{jp2[\phi]} L^{-j\frac{(6p+1)}{4}} \|v^{(p)}\|_p \leq \; c_p/8 \;\; L^{-j/4} \tag{5.36}$$

Define the constant $c = \sup_p c_p$. Then the above estimates lead immediately to the proof of Lemma 5.9, because of (5.32).

Lemma 5.10

$Q(X, \phi)e^{-V(X,\phi)}$ *satisfies the bounds*

$$\left\|Qe^{-V}\right\|_{h,G_\kappa,\mathcal{A}_p} \leq C_p \varepsilon^{1/2} \tag{5.37}$$

$$\left|Qe^{-V}\right|_{h_*,\mathcal{A}_p} \leq C_p \varepsilon^2 \tag{5.38}$$

Proof

$$Qe^{-V} = g^2 \sum_{m=1}^{3} a_m Q^{(m,m)}(C, w^{(4-m)}) e^{-V}$$

$$\left\| Q(X,\phi)e^{-V(X,\phi)} \right\|_h \le g^2 \sum_{m=1}^{3} |a_m| \left\| Q^{(m,m)}(C, w^{(4-m)}, X, \phi) \right\|_h \left\| e^{-V(X,\phi)} \right\|_h \tag{5.39}$$

Here X is a small set, because of the support property of Q. The last factor in the sum will be estimated by Lemma 5.5. From (4.6)

$$Q^{(3,3)}(\tilde{X}, \phi; C, w^{(1)}) = \frac{1}{2} \int_{\tilde{X}} d^3x d^3y : \phi^3(x)\phi^3(y) :_C w^{(1)}(x-y)$$

Undo the Wick ordering, which produces lower order terms with finite coefficients.
Apply $h^k D^k$ with $h = c\varepsilon^{-1/4}$ and use Lemmas 5.1, 5.9 and $g = O(\varepsilon)$. We get

$$\frac{h^k}{k!} \left\| D^k Q^{(3,3)}(\tilde{X}, \phi; C, w^{(1)}) \right\| \le O(1) g^{-3/2} \|w^{(1)}\|_1 \int_{\tilde{X}} d^3x d^3y \frac{1}{|x-y|^{7/4}} G_{\kappa/4}(X,\phi) e^{g/4 \int_X d^3x \phi^4(x)} \tag{5.40}$$

Next turn to $Q^{(m,m)}$, $m = 1, 2$, again in (4.6)

$$Q^{(m,m)}(\tilde{X}, \phi; C, w^{(4-m)}) = -\frac{m^2}{2} \sum_{\mu,\nu=1}^{3} \int_0^1 ds_1 ds_2 \int_{\tilde{X}} d^3x d^3y$$

$$(x-y)_\mu (x-y)_\nu w^{(4-m)}(x-y) : (\phi^{m-1}\nabla_\mu \phi)(y + s_1(x-y))(\phi^{m-1}\nabla_\nu \phi)(y + s_2(x-y)) :_C$$

We consider in turn the cases $m = 2, 1$. We apply $h^k D^k$ with $h = a\varepsilon^{-1/4}$ and use Lemmas 5.1-5.9 and the Sobolev inequality to dominate the $\nabla\phi$ pointwise by the large field regulators.

$$\frac{h^k}{k!} \left\| D^k Q^{(2,2)}(\tilde{X}, \phi; C, w^{(2)}) \right\| \le$$

$$\le O(1) g^{-1/2} \|w^{(2)}\|_2 \sum_{\mu,\nu=1}^{3} \int_{\tilde{X}} d^3x d^3y \frac{|(x-y)_\mu||(x-y)_\nu|}{|x-y|^{13/4}} G_{\kappa/4}(X,\phi) e^{g/4 \int_X d^3x \phi^4(x)}$$

$$\le c g^{-1/2} G_{\kappa/4}(X,\phi) e^{g/4 \int_X d^3x \phi^4(x)} \tag{5.41}$$

We can estimate in the s*$\| \le c/4$

Proof
Let us note first some weak uniform (in L) bounds. Recall that $[\phi] = \frac{3-\varepsilon}{4}$.

$$|\Gamma_L(x)| \le O(1)|x|^{-2[\phi]} \tag{5.42}$$

$$|C(x)| \le O(1) \tag{5.43}$$

To see this observe that from the definition of Γ (see Section 1)

$$|\Gamma_L(x)| \le \int_0^\infty \frac{dl}{l} l^{-2[\phi]} \left| u\left(\frac{x}{l}\right) \right|$$

Let $x \neq 0$. Then, using support properties of u,

$$|\Gamma_L(x)| \leq \int_{|x|}^{\infty} \frac{dl}{l} l^{-2[\phi]} \; \|u\|_\infty = \frac{2}{3-\varepsilon} \; \frac{1}{|x|^{2[\phi]}} \; \|u\|_\infty$$

which proves (5.34). To prove (5.35) recall

$$|C(x)| \leq \int_1^{\infty} \frac{dl}{l} l^{-2[\phi]} \left| u\left(\frac{x}{l}\right)\right| \leq O(1)\|u\|_\infty$$

which proves (5.35).

For $p = 1, 2, 3$,

$$\begin{aligned} |v^{(p)}(x)| &= |C_L^p(x) - C^p(x)| \\ &\leq p \sup_{1\leq q\leq p} |\Gamma_L(x)|^q |C(x)|^{p-q} \leq O(1) \begin{cases} |x|^{-p2[\phi]} & \\ 0 & \text{if } |x| \geq 1 \end{cases} \end{aligned}$$

where we exploited $C_L = \Gamma_L + C$ and $\Gamma_L(x) = 0$ if $|x| \geq 1$ and (5.34) and (5.35). Hence

$$\|v^{(p)}\|_p \leq O(1)$$

and by scaling $x = L^j x'$

$$\|v_{L^j}^{(p)}\|_p \leq L^{jp2[\phi]} L^{-j\frac{(6p+1)}{4}} \|v^{(p)}\|_p \leq \; c_p/8 \;\; L^{-j/4} \tag{5.44}$$

Define the constant $c = \sup_p c_p$. Then the above estimates lead immediately to the proof of Lemma 5.9, because of (5.32).

Lemma 5.10
$Q(X, \phi)e^{-V(X,\phi)}$ *satisfies the bounds*

$$\left\|Qe^{-V}\right\|_{h,G_\kappa,\mathcal{A}_p} \leq C_p \varepsilon^{1/2} \tag{5.45}$$

$$\left|Qe^{-V}\right|_{h_*,\mathcal{A}_p} \leq C_p \varepsilon^2 \tag{5.46}$$

Proof

$$Qe^{-V} = g^2 \sum_{m=1}^{3} a_m Q^{(m,m)}(C, w^{(4-m)}) e^{-V}$$

$$\left\|Q(X,\phi)e^{-V(X,\phi)}\right\|_h \leq g^2 \sum_{m=1}^{3} |a_m| \left\|Q^{(m,m)}(C, w^{(4-m)}, X, \phi)\right\|_h \left\|e^{-V(X,\phi)}\right\|_h \tag{5.47}$$

Here X is a small set, because of the support property of Q. The last factor in the sum will be estimated by Lemma 5.5. From (4.6)

$$Q^{(3,3)}(\tilde{X}, \phi; C, w^{(1)}) = \frac{1}{2} \int_{\tilde{X}} d^3x d^3y : \phi^3(x)\phi^3(y) :_C w^{(1)}(x-y)$$

Undo the Wick ordering, which produces lower order terms with finite coefficients.
Apply $h^k D^k$ with $h = c\varepsilon^{-1/4}$ and use Lemmas 5.1, 5.9 and $g = O(\varepsilon)$. We get

$$\frac{h^k}{k!} \left\|D^k Q^{(3,3)}(\tilde{X}, \phi; C, w^{(1)})\right\| \leq O(1) g^{-3/2} \|w^{(1)}\|_1 \int_{\tilde{X}} d^3x d^3y \frac{1}{|x-y|^{7/4}} G_{\kappa/4}(X,\phi) e^{g/4 \int_X d^3x \phi^4(x)} \tag{5.48}$$

Next turn to $Q^{(m,m)}$, $m=1,2$, again in (4.6)

$$Q^{(m,m)}(\tilde{X},\phi;C,-3\to=3/2$$

and therefore $I^{(p)}=O(L^{3/2})$, etc. ♣5.35 is proved.

Lemma 5.13

$$\left\|Q(e^{-V}-e^{-\tilde{V}})\right\|_{h,\hat{G}_{\kappa,\alpha},\mathcal{A}_p}\leq C_p\varepsilon^{3/4} \tag{5.49}$$

$$\left\|Q(e^{-V}-e^{-\tilde{V}})\right\|_{h_*,\tilde{G}_{\kappa,\alpha},\mathcal{A}_p}\leq C_p\varepsilon^{3} \tag{5.50}$$

Proof

$Q(X)$ is supported on connected polymers with size $|X|\leq 2$. Without loss of generality we do the estimates for $|X|=1$. We can write:

$$Q(\Delta,\zeta+\phi)(e^{-V(\Delta,\zeta+\phi)}-e^{-\tilde{V}(\Delta,\phi)})=Q(\Delta,\zeta+\phi)e^{-\frac{1}{2}V(\Delta,\zeta+\phi)}\int_0^{1/2}ds\,p(\Delta,\zeta,\phi)e^{-(\frac{1}{2}-s)V(\Delta,\zeta+\phi)-s\tilde{V}(\Delta,\phi)}+$$

$$+\int_{1/2}^{1}ds\,Q(\Delta,\zeta+\phi)p(\Delta,\zeta,\phi)e^{-(1-s)V(\Delta,\zeta+\phi)-s\tilde{V}(\Delta,\phi)}$$

whence

$$\left\|Q(\Delta,\zeta+\phi)(e^{-V(\Delta,\zeta+\phi)}-e^{-\tilde{V}(\Delta,\phi)})\right\|_h\leq O(1)(A+B) \tag{5.51}$$

where

$$A=\left\|Q(\Delta,\zeta+\phi)e^{-\frac{1}{2}V(\Delta,\zeta+\phi)}\right\|_h\int_0^{1/2}ds\,\|p(\Delta,\zeta,\phi)\|_h\|e^{-s\tilde{V}(\Delta,\phi)}\|_h$$

and

$$B=\int_{1/2}^{1}ds\,\|Q(\Delta,\zeta+\phi)\|_h\|p(\Delta,\zeta,\phi)\|_h\|e^{-s\tilde{V}(\Delta,\phi)}\|_h$$

To estimate A, use Lemma 5.11, still true for V replaced by $\frac{1}{2}V$, Lemma 5.6, (5.21) with $1-s$ replaced by s, $\delta=\frac{1}{8}$, and lemma 5.5 with g replaced by sg. Observe that $s^{-3/4}$ is integrable. We get

$$A\leq C_\alpha\varepsilon^{3/4}\hat{G}_{\kappa,\alpha}(\Delta,\zeta,\phi) \tag{5.52}$$

To estimate B we use again Lemma 5.5 with g replaced by sg. We estimate $\|p\|_h$ using Lemma 5.6, (5.21) with $1-s$ replaced by s, $\delta=\frac{1}{16}$. We estimate $\|Q(\Delta,\zeta+\phi)\|_h$ as in the proof of Lemma 5.10 with the following change. Expand out polynomials in $\zeta+\phi$, and dominate the ζ using Lemma 5.3. We then get a modified estimate ♣5.30 replacing g with $\frac{s}{4}g$, and $G_\kappa(X,\phi)$ with $\hat{G}_{\kappa,\alpha}(X,\zeta,\phi)$ together with an overall multiplicative factor $C_\alpha s^{-3/2}$ which is integrable in the range under consideration. Then we use lemma 5.5 with g replaced by sg. Putting all this together we get

$$B\leq C_\alpha\varepsilon^{3/4}\hat{G}_{\kappa,\alpha}(\Delta,\zeta,\phi) \tag{5.53}$$

Using (5.52) and (5.53) we get for (5.51)

$$\left\|Q(\Delta,\zeta+\phi)(e^{-V(\Delta,\zeta+\phi)}-e^{-\tilde{V}(\Delta,\phi)})\right\|_h\leq C_\alpha\varepsilon^{3/4}\hat{G}_{\kappa,\alpha}(\Delta,\zeta,\phi) \tag{5.54}$$

It is easy to show that the same estimate holds if $|X|=2$, connected. Hence

$$\left\| Q(e^{-V}-e^{-\tilde V})\right\|_{h,\hat G_{\kappa,\alpha},\mathcal{A}} \le C_{\alpha,L}\varepsilon^{3/4}$$

This proves (5.49). The proof of (5.50) is similar except that we have only fluctuation fields ζ to dominate using Lemma 5.3. This proves Lemma 5.13.

Lemma 5.14
$K(\lambda)$ given by (4.10), satisfies the bounds

$$\|K(\lambda)\|_{h,\hat G_{\kappa,\alpha},\mathcal{A}} \le C_\alpha |\lambda\varepsilon^{1/4-\eta/3}|^2 \qquad \text{for } |\lambda\varepsilon^{1/4-\eta/3}|\le 1 \tag{5.55}$$

$$\|K(\lambda)\|_{h_*,\tilde G_{\kappa,\alpha},\mathcal{A}} \le C_{\alpha,L} |\lambda\varepsilon^{11/12-\eta/3}|^2 \qquad \text{for } |\lambda\varepsilon^{11/12-\eta/3}|\le 1 \tag{5.56}$$

Proof
This follows from lemmas 5.11 and 5.13 and the hypotheses (5.2) and (5.3) on R. The $\lambda\varepsilon^{1/4-\eta/3}$ and $\lambda\varepsilon^{11/12-\eta/3}$ originate from $\lambda^3 R$ contributions.

Lemma 5.15
For any polymer activity K:

$$\|K(X,\zeta)\|_{h_*} \le O(1)\tilde G_{\alpha,\kappa}(X,\zeta)\left[|K(X)|_{h_*} + h^{-n_0}h_*^{n_0}\|K(X)\|_{h,G_\kappa}\right] \tag{5.57}$$

$$\|K(Y,\phi)\|_h \le O(1) e^{\gamma g \int_{Z\setminus Y} d^3y|\phi(y)|^4} G_\kappa(Z,\phi)\left[|K(Y)|_h + L^{-n_0[\phi]}\|K(Y)\|_{L^{[\phi]}h,G_\kappa}\right] \tag{5.58}$$

$$|K^\sharp(X)|_{h_*} \le O(1)2^{|X|}\left[|K(X)|_{h_*} + h^{-n_0}h_*^{n_0}\|K(X)\|_{h,G_\kappa}\right] \tag{5.59}$$

where $\tilde G_{\alpha,\kappa}$ is as defined in (5.12), and n_0 is the maximum number of derivatives appearing in the definition of Kernel and h norms. In (5.58), Y, Z, γ are as described in Lemma 5.1.

Recall that $n_0 = 9$. The superscript $\sharp$ stands for $d\mu_\Gamma(\zeta)$ integration. α is chosen as in Lemma 5.3. Note that we have then

$$\alpha = \alpha(L) = \frac{\kappa'}{h_*^2} \tag{5.60}$$

Proof
First observe:

$$\frac{1}{n_0!}\|(D^{n_0}K)(X,\zeta)\| \le h^{-n_0}\tilde G_{\alpha,\kappa}(X,\zeta)\|K(X)\|_{h,G_\kappa} \tag{5.61}$$

since $G_\kappa(X,\zeta)\le \tilde G_{\alpha,\kappa}(X,\zeta)$.
Now let $n<n_0$. We expand in Taylor series with remainder

$$(D^nK)(X,\zeta;f^{\times n}) = \sum_{m=0}^{n_0-n-1}\frac{1}{m!}(D^{n+m}K)(X,0;f^{\times n},\zeta^{\times m}) +$$

$$+\frac{1}{(n_0-n-1)!}\int_0^1 ds(1-s)^{n_0-n-1}(D^{n_0}K)(X,s\zeta;f^{\times n},\zeta^{\times n_0-n})$$

whence

$$\|(D^nK)(X,\zeta)\| \le \sum_{m=0}^{n_0-n-1} \frac{1}{m!}\|\zeta\|^m_{C^2(X)}\|(D^{n+m}K)(X,0)\|+$$

$$+\frac{1}{(n_0-n-1)!}\int_0^1 ds(1-s)^{n_0-n-1}\|\zeta\|^{n_0-n}_{C^2(X)}G_\kappa(X,s\zeta)\|(D^{n_0}K)(X)\|_{G_\kappa}$$

By Lemma 5.2, with ζ replaced by $\sqrt{1-s^2}\zeta$, and (5.60),

$$h_*^{-p}\|\zeta\|^p_{C^2(X)} \le O(1)\tilde{G}_{\alpha,\kappa}(X,\sqrt{1-s^2}\,\zeta)\frac{1}{(1-s^2)^{p/2}}$$

where O(1) depends on κ' and p and $0 \le s < 1$. With $s=0$ this bound is applied to the terms in the sum over m. For the Taylor remainder term we note that $(1-s)^{(n_0-n)/2-1}$ is integrable since $n_0 > n$. Hence:

$$\frac{h_*^n}{n!}\|(D^nK)(X,\zeta)\| \le O(1)\tilde{G}_{\alpha,\kappa}(X,\zeta)\left[\left(\sum_{m=0}^{n_0-n-1}\frac{(n+m)!}{m!n!}\right)|K(X)|_{h_*}+\right.$$

$$\left.+\frac{n_0!h^{-n_0}h_*^{n_0}}{n!(n_0-n-1)!}\|K(X)\|_{h,G_\kappa}\right] \le$$

$$\le O(1)\tilde{G}_{\alpha,\kappa}(X,\zeta)\left[|K(X)|_{h_*} + h^{-n_0}h_*^{n_0}\|K(X)\|_{h,G_\kappa}\right] \tag{5.62}$$

Summing (5.62) over $0 \le n \le n_0-1$ and adding (5.61) after multiplication by $h_*^{n_0}$ proves (5.57). Equation (5.59) follows from (5.57) using Lemma 5.3. Equation (5.58) is proved in the same way as (5.57) with Lemma 5.1 in the place of Lemma 5.2, h_*, h replaced by h, Lh. This proves Lemma 5.15.

The next lemma gives bounds on $\mathcal{S}(\lambda,K) = \mathcal{R}\circ\mathcal{B}(\lambda,K)$ given in (4.11).

Lemma 5.16
For any $q>0$, there exists c_L such that, for L large,

$$\|\mathcal{S}(\lambda,K)^\natural\|_{h,G_\kappa,\mathcal{A}_p} \le q \qquad \text{when } |\lambda\varepsilon^{1/4-\eta/3}| \le c_L \tag{5.63}$$

$$|\mathcal{S}(\lambda,K)^\natural|_{h_*,\mathcal{A}_p} \le q \qquad \text{when } |\lambda\varepsilon^{11/12-\eta/3}| \le c_L \tag{5.64}$$

When $R=0$ we may set $\eta=0$ in (5.63) and replace $\lambda\varepsilon^{11/12-\eta/3}$ by $\lambda\varepsilon^{1-\delta/2}$ in (5.64).

Proof
From the definition of reblocking (see (4.11)) and subsequent rescaling

$$\begin{aligned}\|\mathcal{S}(\lambda,K)(Z,\phi,\zeta)\|_h \le &\sum_{N+M\ge1}\frac{1}{N!M!}\sum\nolimits_{(X_j),(\Delta_i)\to LZ}\|e^{-\tilde{V}(X_0,\phi_{L^{-1}})}\|_h \\ &\prod\nolimits_{j=1}^N\|K(\lambda,X_j,\phi_{L^{-1}},\zeta_{L^{-1}})\|_h\prod\nolimits_{i=1}^M\|P(\lambda,\Delta_i,\phi_{L^{-1}},\zeta_{L^{-1}})\|_h\end{aligned} \tag{5.65}$$

We rewrite $\tilde{V}(X_0,\phi_{L^{-1}}) = \tilde{V}_L(L^{-1}X_0,\phi)$ and apply Lemma 5.5 (the rewriting gives a better bound by saving factors of 2),

$$\begin{aligned}\|\mathcal{S}(\lambda,K)(Z,\phi,\zeta)\|_h \le &2^{|Z|}\hat{G}_{\kappa,\alpha}(LZ,\zeta_{L^{-1}},\phi_{L^{-1}})\sum_{N+M\ge1}\frac{1}{N!M!}\sum\nolimits_{(X_j),(\Delta_i)\to LZ} \\ &\prod\nolimits_{j=1}^N\|K(\lambda,X_j)\|_{h_L,\hat{G}_{\kappa,\alpha}}\prod\nolimits_{i=1}^M\|\,P(\lambda,\Delta_i)\|_{h_L,\hat{G}_{\kappa,\alpha}}\end{aligned} \tag{5.66}$$

where

$$h_L = L^{-(3-\varepsilon)/4} h$$

Using Lemma 5.4 and $G_{3\kappa}(LZ, \phi_{L^{-1}}) \le G_\kappa(Z, \phi)$ for L large,

$$\begin{aligned}\|\mathcal{S}(\lambda, K)^\natural(Z)\|_{h,G_\kappa} \le & 2^{2|Z|} \sum_{N+M\ge 1} \frac{1}{N!M!} \sum\nolimits_{(X_j),(\Delta_i)\to LZ} \\ & \prod\nolimits_{j=1}^N \|K(\lambda, X_j)\|_{h_L,\hat{G}_{\kappa,\alpha}} \prod\nolimits_{i=1}^M \| P(\lambda, \Delta_i)\|_{h_L,\hat{G}_{\kappa,\alpha}}\end{aligned}$$

$$\|\mathcal{S}(\lambda, K)^\natural\|_{h,G_\kappa,\mathcal{A}_p} \le \sup_\Delta \sum_{X: L^{-1}\bar{X}^L \cap \Delta \neq \emptyset} \mathcal{A}_{2+p}(L^{-1}\bar{X}^L) \|\bar{K}_k(X)\|_{h_L,\hat{G}_{\kappa,\alpha}}$$

Multiply by $\mathcal{A}_{2+p}(Z)$ on the left and, using

$$\mathcal{A}_{2+p}(L^{-1}\bar{X}^L) \le O(1)\mathcal{A}_{-3}(X)$$

$(\mathrm{X}_j), (\Delta_i) \to LZ \prod_{j=1}^N \|K(\lambda, X_j)\|_{h_L,\hat{G}_{\kappa,\alpha}} \prod_{i=1}^M \| P(\lambda, \Delta_i)\|_{h_L,\hat{G}_{\kappa,\alpha}}$

$$\|\mathcal{S}(\lambda, K)^\natural\|_{h,G_\kappa,\mathcal{A}_p} \le \sup_\Delta \sum_{X: L^{-1}\bar{X}^L \cap \Delta \neq \emptyset} \mathcal{A}_{2+p}(L^{-1}\bar{X}^L) \|\bar{K}_k(X)\|_{h_L,\hat{G}_{\kappa,\alpha}}$$

Multiply by $\mathcal{A}_{2+p}(Z)$ on the left and, using

$$\mathcal{A}_{2+p}(L^{-1}\bar{X}^L) \le O(1)\mathcal{A}_{-3}(X)$$

(Lemma 1 in [BDH-est]), by

$$O(1) \prod\nolimits_{j=1}^N \mathcal{A}_{-3}(X_j) \prod\nolimits_{i=1}^M \mathcal{A}_{-3}(\Delta_i)$$

on the right. Fix any Δ and sum both sides over $Z \ni \Delta$. The spanning tree argument of Lemma 7.1 of [BY] controls the sums over $N, M, Z, (X_j), (\Delta_i) \to LZ$ with the result

$$\|\mathcal{S}(\lambda, K)^\natural\|_{h,G_\kappa,\mathcal{A}_p} \le O(1) \sum_{N\ge 1} O(1)^N L^{3N} \left(\|K(\lambda)\|_{h,\hat{G}_{\kappa,\alpha},\mathcal{A}} + \|P(\lambda)\|_{h,\hat{G}_{\kappa,\alpha},\mathcal{A}} \right)^N$$

The proof of (5.63) is completed by Lemmas 5.8 and 5.14. When $R = 0$ we can replace Lemma 5.14 by Lemmas 5.13 and 5.8 which gives the result with $\eta = 0$.

For (5.64) we use (5.65) with $\phi = 0$ and replace h by h_*. We estimate the ζ dependence by the regulator $\tilde{G}_{\kappa,\alpha}$ introduced in (5.12), to obtain, in the place of (5.66),

$$\begin{aligned}\|\mathcal{S}(\lambda, K)(Z, 0, \zeta)\|_{h_*} \le & 2^{|Z|} \tilde{G}_{\kappa,\alpha}(LZ, \zeta_{L^{-1}}, \zeta_{L^{-1}}) \sum_{N+M\ge 1} \frac{1}{N!M!} \sum\nolimits_{(X_j),(\Delta_i)\to LZ} \\ & \prod\nolimits_{j=1}^N \|K(\lambda, X_j)\|_{h_*,\tilde{G}_{\kappa,\alpha}} \prod\nolimits_{i=1}^M \| P(\lambda, \Delta_i)\|_{h_*,\tilde{G}_{\kappa,\alpha}}\end{aligned} \tag{5.67}$$

Then Lemma 5.15 is used to estimate the $\natural$ in $|\mathcal{S}(\lambda,K)^\natural(Z)|_{h_*}$, and the rest is as before. End of proof of Lemma 5.16.

Estimates on relevant parts and flow coefficients from the remainder

Let $(\tilde\alpha_P)$ be the coefficients $(\tilde\alpha_0,\tilde\alpha_{2,0},\tilde\alpha_{2,1},\tilde\alpha_4)$ defined in (4.43) and (4.56). The flow coefficients ξ_R, ρ_R and β_0 are given in (4.54).

Lemma 5.17

$$\|R^\sharp\|_{h,G_{3\kappa},\mathcal{A}_{-2}} \le \varepsilon^{3/4-\eta} \tag{5.68}$$

$$|R^\sharp|_{h_*,\mathcal{A}_{-3}} \le O(1)\varepsilon^{11/4-\eta} \tag{5.69}$$

$$|\tilde\alpha_P|_{\mathcal{A}} \le O(1)\varepsilon^{11/4-\eta} \tag{5.70}$$

$$|\beta_0| \le C\varepsilon^{11/4-\eta} \tag{5.71}$$

$$|\xi_R| \le C\varepsilon^{11/4-\eta} \tag{5.72}$$

$$|\rho_R| \le C\varepsilon^{11/4-\eta} \tag{5.73}$$

Proof
Recall that $\tilde\alpha_P(X)$, are supported on small sets. (5.68)follows from the hypothesis (5.2) and the stability of the large field regulator G_κ. (5.69) follow from the hypotheses (5.3) and lemma 5.16 with $n_0 = 9$ and ε sufficiently small depending on L. (5.70) follows from (5.69), and (4.56). In fact the dominant contribution comes by setting $\tilde V = 0$ because the difference gives additional powers of ε. Then we have

$$\|\tilde\alpha_P\|_{\mathcal{A}} \le O(1)n(P)!h_*^{-n(P)}|1_S R^\sharp|_{h_*,\mathcal{A}}$$

where $n(P)$ is the number of fields in the monomial P and 1_S is the indicator function on small sets. Now use (5.69) to get(5.70). (5.71), (5.72), (5.73) follow from (5.70),and the definitions (4.54), (4.48) and Wick coefficients are $O(1)$. Lemma 5.17 has been proved.

Corollary 5.18
For ε sufficiently small

$$|g'-\bar g| \le \varepsilon^{3/2}, \quad |\mu'| \le C\varepsilon^{2-\delta} \tag{5.74}$$

$$|g'-g| \le \varepsilon^2 \tag{5.75}$$

Proof
It is easy to check from the first of the flow equations (4.17) and the definition of $\bar g$ that

$$g'-\bar g = (g-\bar g)(1-L^{2\varepsilon}ag)+\xi_R$$

and

$$g'-g = (g-\bar g)(-L^{2\varepsilon}ag)+\xi_R$$

$a = O(\log L) > 0$ and the initial Hypothesis implies $g = O(\varepsilon)$ so that for ε sufficiently small $0 < 1-L^{2\varepsilon}ag < 1-ag$. The domain of g in the Hypothesis and the bound (5.95) of Lemma 5.17 imply, for ε sufficiently small, that ξ_R is smaller than the other terms, which gives the two bounds concerning g'. The bound on μ' follows from the second of the flow equations (4.17), the hypothesis on μ, and the bound (5.96) on ρ_R.

The corollary has been proved.

The following lemma proves the stability of V with respect to perturbations by relevant parts F in our model. We state it in the form enunciated as the stability hypothesis in Section 4.2, (103), [BDH-est]. This lemma will be very useful for the control of the extraction formula, as explained in the reference above.
Recall from (4.13) that $F(\lambda) = \lambda^2 F_Q + \lambda^3 F_R$ and from (3.9) that (each part of) F decomposes: $F(X) = \sum_{\Delta\subset X} F(X,\Delta)$.

Lemma 5.19
For any $R > 0$ and $\xi := R\max(|\lambda^2|\varepsilon, |\lambda^3|\varepsilon^{7/4-\eta})$ sufficiently small,

$$\|e^{-\tilde{V}_L(\Delta)-\sum_{X\supset\Delta} z(X)F(\lambda,X,\Delta)}\|_{h,G_\kappa} \le 2^2 \tag{5.76}$$

where $z(X)$ are complex parameters with $|z(X)| \le R$.

Proof
First note that Lemma 5.5 still holds if we replace $\tilde{V}$ by $\tilde{V}_L$ provided ε is sufficiently small. We then have

$$\|e^{-\tilde{V}_L(\Delta)-\sum_{X\supset\Delta} z(X)F(X,\Delta)}\|_h \le 2\ e^{-g_L/4\int_\Delta d^3x\phi^4(x)+\sum_{X\supset\Delta} R\|F(\lambda,X,\Delta)\|_h} \tag{5.77}$$

Recall that the $F(X,\Delta)$ are supported on small sets X. The proof now follows easily from the following

Claim: For ε sufficiently small

$$\|F(\lambda,X,\Delta)\|_h \le C_L\xi\left(\int_\Delta d^3x\varepsilon\phi^4(x) + \varepsilon^{1/2}\|\phi\|^2_{\Delta,1,\sigma} + 1\right) \tag{5.78}$$

where $\|\phi\|^2_{\Delta,1,\sigma}$ is the norm defined in (2.2).
Proof of the Claim: We have

$$\|F(\lambda,X,\Delta)\|_h \le |\lambda|^2\|F_Q(X,\Delta)\|_h + |\lambda|^3\|F_R(X,\Delta)\|_h \tag{5.79}$$

From (4.26)-(4.31)

$$\|F_Q(X,\Delta)\|_h \le O(1)\varepsilon^2\left(\sum_{m=2,4}\|\int_\Delta d^3x :\phi^m:_C(x)\|_h \sup_{x\in\Delta}|f^{(m)}_{1,\tilde{Q}}(x,X,\Delta)| + |\tilde{Q}^{(0,0)}(X,C,v^{(4)})|\right)$$

Undoing the Wick ordering produces lower order terms with $O(1)$ coefficients. Now it is easy to see that for $m = 2,4$

$$\varepsilon\|\int_\Delta d^3x :\phi^m:_C(x)\|_h \le O(1)\varepsilon\int_\Delta d^3x\phi^4(x) + O(1)$$

By Lemma 5.12,

$$\sup_{x\in\Delta}|f^{(m)}_{1,\tilde{Q}}(x,X,\Delta)| \le C_L, \qquad |\tilde{Q}^{(0,0)}(X,C,v^{(4)})| \le C_L$$

Therefore

$$|\lambda^2|R\ \|F_Q(X,\Delta)\|_h \le C_L R|\lambda|^2\varepsilon\left(\varepsilon\int_\Delta d^3x\phi^4(x) + 1\right) \tag{5.80}$$

Next consider F_R, supported on small sets, defined in (4.43), (4.45). Recall (4.49),

$$F_R(X,\Delta,\phi)=\sum_P\int_\Delta d^3x\ \alpha_P(X,\Delta,x)P(\phi(x),\partial\phi(x))$$

By Lemma 5.17 and (4.48)

$$|\alpha_P(X,\Delta,x)|\le C_L\varepsilon^{11/4-\eta}$$

so that

$$\begin{aligned}|\lambda|^3|z(X)|\|F_R(X,\Delta)\|_h \le& C_LR|\lambda|^3\varepsilon^{11/4-\eta}\sum_P\int_\Delta d^3x\ \|P(\phi(x),\partial\phi(x))\|_h\\ \le& C_LR|\lambda|^3\varepsilon^{7/4-\eta}\left(\varepsilon\int_\Delta d^3x\,\phi^4(x)+\varepsilon^{1/2}\|\phi\|^2_{\Delta,1,\sigma}+1\right)\end{aligned}$$

The claim follows by combining this with (5.80).
Note that in the above inequality the Sobolev norm arises only when estiex parameters with $|z(X)|\le R$.

Proof
This is the same as the last proof except that we can use the estimate

$$|F(\lambda,X,\Delta)|_{h_*}\le C_L\xi$$

in place of (5.78). (No need to ensure $R|F(\lambda,X,\Delta)|_{h_*}$ is smaller than $\varepsilon\phi^4$ because stability away from $\phi=0$ is not an issue with the kernel norm). End of proof of Lemma.

Recall (4.36)

$$R_{\text{main}}=\frac{1}{2\pi i}\oint_\gamma\frac{d\lambda}{\lambda^4}\mathcal{E}\left(\mathcal{S}(\lambda,Qe^{-V})^\natural,F_Q(\lambda)\right)\tag{5.81}$$

Lemma 5.21

$$\|R_{\text{main}}\|_{h,G_\kappa,\mathcal{A}}\le C_L\varepsilon^{3/4}\tag{5.82}$$

$$|R_{\text{main}}|_{h_*,\mathcal{A}}\le C_L\varepsilon^{3-3\delta/2}\tag{5.83}$$

Proof
Let

$$J(\lambda)=\mathcal{S}(\lambda,Qe^{-V})^\natural$$

Suppose that $F(\lambda)$ splits, $F(\lambda)=F_0(\lambda)+F_1(\lambda)$, into a field independent part F_0 and F_1 satisfies stability as in (5.76). According to Theorem 6 on page 780 of [BDH-est],

$$\|\mathcal{E}(J(\lambda),F_0,F_1)\|_{h,G_\kappa,\mathcal{A}}\le O(1)\left(\|J(\lambda)\|_{h,G_\kappa,\mathcal{A}_2}+\|f\|_{\mathcal{A}_4}\right)\tag{5.84}$$

provided the norms on the right hand side are less than a small constant $R^{-1}=O(1)$. In the above $|f(X)|\le 2|z(X)|^{-1}$ where the $z(X)$ are the complex parameters introduced in Lemma 5.19. The $f(X)$ are supported on small sets. We choose $|\lambda|=c_L\varepsilon^{-1/4}$. By Lemma 5.19 we have stability (5.76) if ε is small. Therefore (5.84) holds and by combining it with Lemma 5.16,

$$\|\mathcal{E}(J(\lambda), F_0, F_1)\|_{h,G_\kappa,\mathcal{A}} \leq q + O(R^{-1}) \leq O(1)$$

(5.82) follows by choosing the contour γ in (5.81) to be a circle of radius $c_L\varepsilon^{-1/4}$.

The proof of (5.83) follows the same steps but with contour γ being a circle of radius $c_L\varepsilon^{-1+\delta/2}$ chosen so that ξ in Lemma 5.20 is small and the hypothesis of Lemma 5.16 is satisfied. End of proof.

Recall from (4.38) that

$$R_3 = \frac{1}{2\pi i}\oint_\gamma \frac{d\lambda}{\lambda^4(\lambda-1)}\mathcal{E}(\mathcal{S}(\lambda,K)^\natural, F(\lambda)) \tag{5.85}$$

Lemma 5.22

$$\|R_3\|_{h,G_\kappa,\mathcal{A}} \leq C_L\varepsilon^{1-4\eta/3} \tag{5.86}$$

$$|R_3|_{h_*,\mathcal{A}} \leq C_L\varepsilon^{11/3-4\eta/3} \tag{5.87}$$

Proof
This proof follows the same steps as the proof of Lemma 5.21 with contours $|\lambda| = c_L\varepsilon^{1/4-\eta/3}$ and $|\lambda| = c_L\varepsilon^{11/12-\eta/3}$. End of proof.

Lemma 5.23
R_4 as defined in (4.40) satisfies

$$\|R_4\|_{h,G_\kappa,\mathcal{A}} \leq C\varepsilon^{3/2}$$
$$|R_4|_{h_*,\mathcal{A}} \leq C\varepsilon^3$$

Proof
From (4.40)

$$R_4 = \left(e^{-V'} - e^{-\tilde{V}_L}\right)Q(C,\mathbf{w}',g') + e^{-\tilde{V}_L}\left(Q(C,\mathbf{w}',g') - Q(C,\mathbf{w}',g_L)\right)$$

First we observe from (4.39), Lemma 5.17 and Lemma 5.9 the following bounds

$$|g' - g_L| \leq C\varepsilon^2 \tag{5.88}$$

$$|\mu' - \mu_L| \leq C\varepsilon^2 \tag{5.89}$$

$$\|\mathbf{w}'\| \leq c/4 \tag{5.90}$$

We estimate in turn the two terms in the expression for R_4 above. Because of Q each term is supported on small sets.

We write the first term as

$$\left(e^{-V'} - e^{-\tilde{V}_L}\right)Q(C,\mathbf{w}',g') = Q(C,\mathbf{w}',g')e^{-\frac{1}{2}V'}\int_0^1 ds(\tilde{V}_L - V')e^{-(\frac{1}{2}-s)V'-sV_L}$$

Then we bound

$$\|\left(e^{-V'}-e^{-\tilde V_L}\right)Q(C,\mathbf{w}',g')\|_{h,G_\kappa}\le \|Q(C,\mathbf{w}',g')e^{-\frac12 V'}\|_{h,G_\kappa}\int_0^1 ds\|\tilde V_L - V'\|_h\|e^{-(\frac12-s)V'}\|_h\|e^{-sV_L}\|_h \tag{5.91}$$

Using (5.88) and (5.89) we can bound

$$\|\tilde V_L(X)-V'(X)\|_h\le C\frac{\varepsilon}{\gamma}e^{O(1)\gamma\varepsilon\int_X d^3x\phi^4(x)} \tag{5.92}$$

for any $\gamma = O(1) > 0$.
By Lemma 5.5 we can bound

$$\|e^{-(\frac12-s)V'(X)}\|_h\le 2^{|X|}e^{-(\frac12-s)\frac{g'}{4}\int_X d^3x\phi^4(x)} \tag{5.93}$$

$$\|e^{-s)V_L(X)}\|_h\le 2^{|X|}e^{-s\frac{g'}{4}\int_X d^3x\phi^4(x)} \tag{5.94}$$

We now plug into (5.91) the bounds (5.93) and (5.94). We then write the s-integration in (5.91) as the union of the intervals $[0,\frac14$ and $\frac14,1]$. In the first interval we insert the bound (5.92) with γ replaced by $(\frac12-s)\gamma$. In the second interval we insert the same bound with γ replaced by $s\gamma$ and in both cases take $\gamma=O(1)$ sufficiently small. Then the s-integral factor in (5.91) is bounded by $C\varepsilon$. On the other hand the first factor in (5.91) is bounded by $O(1)\varepsilon^{\frac12}$ by virtue of Lemma 5.10. (The factor of $\frac12$ in the exponent does not make a difference). Putting these bounds together and recalling that the Q are supported on small sets we obtain

$$\|\left(e^{-V'}-e^{-\tilde V_L}\right)Q(C,\mathbf{w}',g')\|_{h,G_\kappa,\mathcal{A}}\le C\varepsilon^{\frac32} \tag{5.95}$$

We can now easily bound the second term in the expression for R_4 by noting that

$$|g'^2-g_L^2|\le C\varepsilon^3$$

and then using Lemma 5.10. We again get the bound

$$\|e^{-\tilde V_L}\left(Q(C,\mathbf{w}',g')-Q(C,\mathbf{w}',g_L)\right)\|_{h,G_\kappa,\mathcal{A}}\le C\varepsilon^{\frac32} \tag{5.96}$$

Adding together (5.95) and (5.96) finishes the proof of the first bound in the Lemma. The second bound is easy to prove since all derivatives in the h_* norm are at $\phi=0$. End of proof of Lemma 5.23.

Lemma 5.24
Let X be a small set and let J be normalized as in (4.42). Then we have

$$|D^2J(X,0;f_{L^{-1}}^{\times 2})|\le O(1)L^{-(7-\varepsilon)/2}\|D^2J(X,0)\|\prod_{j=1}^2\|f_j\|_{C^2(L^{-1}X)} \tag{5.97}$$

$$|D^4J(X,0;f_{L^{-1}}^{\times 4})|\le O(1)L^{-(4-\varepsilon)}\|D^4J(X,0)\|\prod_{j=1}^4\|f_j\|_{C^2(L^{-1}X)} \tag{5.98}$$

Proof: See Lemma 15 [BDH-est].

Corollary 5.25
Let $Y=L^{-1}X$ where X is a small set, $Z=L^{-1}\bar X^L$ and let J be normalized as in (4.42). Then

$$|J_L(Y)|_h \le O(1)L^{-(7-\varepsilon)/2}|J(X)|_h \tag{5.99}$$

$$\|J_L(Y)e^{-\tilde V_L(Z\backslash Y)}\|_{h,G_\kappa} \le O(1)L^{-(7-\varepsilon)/2} parameterswith$$

—z(X)—$\le R$.

Proof
This is the same as the last proof except that we can use the estimate

$$|F(\lambda,X,\Delta)|_{h_*} \le C_L\xi$$

in place of (5.78). (No need to ensure $R|F(\lambda,X,\Delta)|_{h_*}$ is smaller than $\varepsilon\phi^4$ because stability away from $\phi=0$ is not an issue with the kernel norm). End of proof of Lemma.

Recall (4.36)

$$R_{\mathrm{main}} = \frac{1}{2\pi i}\oint_\gamma \frac{d\lambda}{\lambda^4}\mathcal{E}\left(\mathcal{S}(\lambda,Qe^{-V})^\natural, F_Q(\lambda)\right) \tag{5.100}$$

Lemma 5.21

$$\|R_{\mathrm{main}}\|_{h,G_\kappa,\mathcal{A}} \le C_L\varepsilon^{3/4} \tag{5.101}$$

$$|R_{\mathrm{main}}|_{h_*,\mathcal{A}} \le C_L\varepsilon^{3-3\delta/2} \tag{5.102}$$

Proof
Let

$$J(\lambda) = \mathcal{S}(\lambda,Qe^{-V})^\natural$$

Suppose that $F(\lambda)$ splits, $F(\lambda)=F_0(\lambda)+F_1(\lambda)$, into a field independent part F_0 and F_1 satisfies stability as in (5.76). According to Theorem 6 on page 780 of [BDH-est],

$$\|\mathcal{E}(J(\lambda),F_0,F_1)\|_{h,G_\kappa,\mathcal{A}} \le O(1)\Big(\|J(\lambda)\|_{h,G_\kappa,\mathcal{A}_2} + \|f\|_{\mathcal{A}_4}\Big) \tag{5.103}$$

provided the norms on the right hand side are less than a small constant $R^{-1}=O(1)$. In of $\tilde F_R$ given in (4.43) and (4.44). $\tilde F_R$ is supported on small sets. We have

$$\|\tilde F_R(X,\phi)\|_h \le \sum_P |\tilde\alpha_P(X)| \int_X d^3x\ \|P(\phi(x),\partial\phi(x))\|_h$$

$$\le O(1)\sum_P |\tilde\alpha_P(X)|\varepsilon^{-1}\left(\varepsilon\int_X d^3x\,\phi^4(x) + \varepsilon^{1/2}\|\phi\|^2_{X,1,\sigma}+1\right)$$

$$\le O(1)\sum_P |\tilde\alpha_P(X)|\varepsilon^{-1}G_\kappa(X,\phi)e^{\gamma g\int_X d^3y\phi^4(y)}$$

for any $\gamma = O(1) > 0$. Hence, using Lemma 5.5

$$\|\tilde F_R(X\phi)e^{-\tilde V(X\phi)}\|_h \le \|\tilde F_R(X\phi)\|_h\|e^{-\tilde V(X\phi)}\|_h \le O(1)\sum_P 2^{|X|}|\tilde\alpha_P(X)|\varepsilon^{-1}G_\kappa(X,\phi)$$

We thus obtain (remembering that $\tilde\alpha_P$ are supported on small sets) on using (5.70)

$$\|\tilde F_R(X\phi)e^{-\tilde V(X\phi)}\|_{h,G_\kappa,\mathcal{A}} \le O(1)\varepsilon^{-1}\sum_P \|\alpha_P\|_{\mathcal{A}} \le O(1)\varepsilon^{7/4-\eta}$$

This proves ♣5.71.
Now we turn to the proof of ♣5.72.
As observed in the proof of Lemma 5.17, for ε sufficiently small (depending on L),

$$|\tilde\alpha_P|_{\mathcal{A}} \le n(P)!h_*^{-n(P)}|1_S R^\sharp|_{h_*,\mathcal{A}} \le O(1)h_*^{-n(P)}\varepsilon^{11/4-\eta}$$

We have from the definition of$\tilde F_R$ given in (4.43)

$$|\tilde F_R(X)|_{h_*} \le O(1)\sum_P |\tilde\alpha_P(X)|h_*^{n_P}$$

whence

$$|\tilde F_R|_{h_*,\mathcal{A}} \le \sum_P |\tilde\alpha_P|_{\mathcal{A}} h_*^{n_P} \le O(1)\varepsilon^{11/4-\eta}$$

which proves ♣5.72.

To get these bounds for $J = R^\sharp - \tilde F_R e^{-\tilde V}$ we apply ♣5.71 and ♣5.72 to the $\tilde F_R e^{-\tilde V}$ part. We bound $R^\sharp$ by Lemma 5.17. End of proof.

Corollary 5.27

$$\|R_{\text{linear}}\|_{h,G_\kappa,\mathcal{A}} \le O(1)L^{-(1-\varepsilon)/2}\varepsilon^{3/4-\eta} \tag{5.104}$$

$$|R_{\text{linear}}|_{h_*,\mathcal{A}} \le O(1)L^{-(1-\varepsilon)/2}\varepsilon^{11/4-\eta} \tag{5.105}$$

Proof

We recall from (4.44) that

$$J = R^\sharp - \tilde F_R e^{-\tilde V}$$

is normalized. Let $1_{\text{s.s}}(X)$ be the indicator function of the event that X is small. Referring to (4.44), the first term in $R_{\text{linear}}(Z)$ is

$$R_{\text{linear,s.s}}(Z) := \sum_{X:L^{-1}\bar X^L = Z} e^{-\tilde V_L(Z\setminus Y)}1_{\text{s.s}}(X)J_L(Y) \tag{5.106}$$

where $Y = L^{-1}X$. By Corollary 5.25 this is bounded in h, G_κ norm by

$$O(1)L^{(7-\varepsilon)/2}\sum_{X:L^{-1}\bar X^L = Z} 1_{\text{s.s}}(X)\Big[|J(X)|_h + \|J(X)\|_{h,G_{3\kappa}}\Big] \tag{5.107}$$

Multiply both sides by $\mathcal{A}(Z)$, using $\mathcal{A}(Z)\le\mathcal{A}(X)$ on the right hand side. Then sum over Z to get the $\mathcal{A}$ norm and use the bounds on J in Lemma 5.17 and Lemma 5.26 to obtain

$$\|R_{\text{linear,s.s}}\|_{h,G_\kappa,\mathcal{A}} \le O(1)L^{-(1-\varepsilon)/2}\varepsilon^{3/4-\eta}$$

where we used an argument on page 790 of [BDH-est] to control the $\sum_{X:L^{-1}\bar{X}^L=Z}$ by $L^{D=3}$ times the sum over X in the definition of $\mathcal{A}$ norm. Similarly the bound on the kernel norm by $O(1)L^{-(1-\varepsilon)/2}\varepsilon^{11/4-\eta}$ comes from the kernel norm bound in Lemma 5.17, Lemma 5.26 and Lemma 5.15.

Let $1_{\mathrm{l.s}}(X)$ be the large set indicator function. The second term in $R_{\mathrm{linear}}(Z)$ is

$$R_{\mathrm{linear,s.s}}(Z) := \sum_{X:L^{-1}\bar{X}^L=Z} e^{-\tilde{V}_L(Z\setminus L^{-1}X)} 1_{\mathrm{l.s}}(X) R_L^{\natural}(L^{-1}X) \tag{5.108}$$

where we have used $J(X) = R^\sharp(X)$ because the subtraction is supported on small sets. This is bounded in the same way as above using Lemma 5.17 except that the necessary $L^{-(1-\varepsilon)/2}$ is obtained for a different reason: For large sets, by (2.7), $\mathcal{A}(Z) \le c_p L^{-4}\mathcal{A}_{-p}(X)$, where $c_p = O(1)$. The Corollary is proved.

Proof of Theorem 1 Concluded

From (4.35)-(4.41), R is the sum of R_i where $i = 1, 2, 3, 4$. By Lemmas 5.21, 5.22, 5.23 and 5.27 with L large and ε small depending on L, the sum satisfies bounds (5.9) and (5.10).

6. STABLE MANIFOLD AND CONVERGENCE TO NON-GAUSSIAN FIXED POINT

6.1

Let $\bar{g}$ be the approximate fixed point of the g flow given by (5.4). Let us define

$$\tilde{g} = g - \bar{g} \tag{6.1}$$

and

$$u = (\tilde{g}, \mu, R, \mathbf{w}) \tag{6.2}$$

Then the RG iteration given by (4.39), (4.41), (4.18) can be written as

$$u' = f(u) \tag{6.3}$$

with components

$$\tilde{g}' = f_g(u) = (2 - L^{\varepsilon})\tilde{g} + \tilde{\xi}(u) \tag{6.4}$$

$$\mu' = f_\mu(u) = L^{\frac{3+\varepsilon}{2}}\mu + \tilde{\rho}(u) \tag{6.5}$$

$$R' = f_R(u) = U(u) \tag{6.6}$$

$$\mathbf{w}' = f_{\mathbf{w}}(u) = \mathbf{v} + \mathbf{w}_L \tag{6.7}$$

with initial

$$u_0 = (\tilde{g}_0, \mu_0, 0, 0) \tag{6.8}$$

Here

$$\tilde{\xi}(u) = -L^{2\varepsilon} a\tilde{g}^2 + \xi(u) \tag{6.9}$$

$$\tilde{\rho}(u) = -L^{2\varepsilon} b(g + \tilde{g})^2 + \rho(u) \tag{6.10}$$

Note that the $\mathbf{w}$ flow is autonomous and solved by (5.32). In Lemma 5.9 it was proved that the $\mathbf{w}$ tends to the fixed point $\mathbf{w}_*$ in an appropriate norm.
Let E be the Banach space consisting of elements u with the (box) norm

$$\|u\| = \sup(\varepsilon^{-3/2}|\tilde{g}|, \varepsilon^{-(2-\delta)}|\mu|, \varepsilon^{-(11/4-\eta)}|||R|||, c^{-1}\|\mathbf{w}\|) \tag{6.11}$$

Here

$$|||R||| = \sup(\varepsilon^2\|R\|_{h,G,\mathcal{A}}, |R|_{h_*,\mathcal{A}}) \tag{6.12}$$

and

$$\|\mathbf{w}\| = \sup_p \|w^{(p)}\|_p$$

The $\|\cdot\|_p$ and $\|\mathbf{w}\|$ norms were defined in (5.33) and the constant c is that of Lemma 5.9.
Let $B(r) \subset E$ be the closed ball of radius r, centered at the origin:

$$B(r) = \{u \in E : \|u\| \leq r\} \tag{6.13}$$

Let $\mathcal{D}$ be the domain of (g, μ, R) specified in (5.1)-(5.3) of the hypothesis stated at the beginning of Section 5. Then we have

$$u \in B(1) \;\Rightarrow\; (g, \mu, R) \in \mathcal{D} \tag{6.14}$$

and then Theorem 1 of Section 5 holds.
Theorem 1, together with the autonomous Lemma 5.9, shows that the ball $B(1)$ would be stable under the RG flow f, except for the unstable direction μ (as evident from (6.5)). The initial unstable parameter μ_0 will have to be fine tuned to a critical function $\mu_0 = \mu_c(g_0)$ which would determine the critical or stable submanifold in which we expect to get a contraction to a fixed point. As observed in [BDH-eps] this is part of the stable manifold theorem in the theory of hyperbolic dynamical systems [S]. In the following we put ourselves in this framework (see appendix 2, Chapter 5 of [S] and Section 5 of [BDH-eps]).
Suppose $u \in B(1)$. Then from theorem 1 of Section 5 we have for $\varepsilon > 0$ sufficiently small (depending on L), which implies in particular $L^\varepsilon = O(1)$,

$$\begin{aligned} |\xi(u)| &\leq O(1)\varepsilon^{11/4-\eta} \\ |\rho(u)| &\leq O(1)L^{3/2}\varepsilon^{11/4-\eta} \\ |||U(u)||| &\leq L^{-1/4}\varepsilon^{11/4-\eta} \end{aligned} \tag{6.15}$$

From (6.9), (6.10) we have by virtue of (6.15) and the estimates $a = O(\log L)$, $b = O(L^{3/2})$, see Lemma 5.12, for $u \in B(1)$:

$$\begin{aligned} |\tilde{\xi}(u)| &\leq O(1)\varepsilon^{11/4-\eta} \\ |\tilde{\rho}(u)| &\leq \varepsilon^{2-\delta} \end{aligned} \tag{6.16}$$

$\tilde{\xi}, \tilde{\rho}, U, \mathbf{w}$ satisfy the following Lipshitz bounds in $B(1/4) \subset B(1)$

Lemma 6.1
Let $u, u' \in B(1/4)$. Then we have the Lipshitz bounds:

$$\begin{aligned} |\tilde{\xi}(u) - \tilde{\xi}(u')| &\leq O(1)\varepsilon^{11/4-\eta}\|u - u'\| \\ |\tilde{\rho}(u) - \tilde{\rho}(u')| &\leq O(L^{3/2})\varepsilon^{5/2}\|u - u'\| \\ |||U(u) - U(u')||| &\leq O(1)L^{-1/4}\varepsilon^{11/4-\eta}\|u - u'\| \\ \|f_{\mathbf{w}}(u) - f_{\mathbf{w}}(u')\| &\leq cL^{-1/4}\|u - u'\| \end{aligned} \tag{6.17}$$

Proof
We shall use the fact that ξ, ρ, U are analytic in $B(1)$. This follows from the algebraic operations in Section 4 together with the analyticity of the extraction map.
Let $\Delta u = u - u'$. Then

$$U(u) - U(u') = \int_0^1 dt \frac{\partial}{\partial t} U(u + t\Delta u)$$

By the Cauchy integral formula

$$\frac{\partial}{\partial t} U(u + t\Delta u) = \frac{1}{2\pi i} \oint_\gamma dz \frac{U(u + z\Delta u)}{(z - t)^2}$$

here $0 \le t \le 1$ and γ is the closed contour

$$\gamma: \; z - t = re^{i\vartheta}, \; r = 1/4\|\Delta u\|^{-1}$$

Note that

$$u + z\Delta u = u + t\Delta u + \frac{1}{4}\frac{\Delta u}{\|\Delta u\|} e^{i\vartheta}$$

Hence

$$\|u + z\Delta u\| \le \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1$$

So that for $z \in \gamma$, $u + z\Delta u \in B(1)$, and hence from (6.15)

$$|||U(u + z\Delta u)||| \le L^{-1/4}\varepsilon^{11/4-\eta}$$

so that

$$\sup_{0\le t\le 1} |||\frac{\partial}{\partial t} U(u + t\Delta u)||| \le O(1)L^{-1/4}\varepsilon^{11/4-\eta}\|\Delta u\|$$

and thus

$$|||U(u) - U(u')||| \le O(1)L^{-1/4}\varepsilon^{11/4-\eta}\|u - u'\|$$

This proves the last inequality of (6.17).
On using (6.16) we have in the same way

$$|\tilde{\xi}(u) - \tilde{\xi}(u')| \le O(1)\varepsilon^{11/4-\eta}\|u - u'\|$$

To get the Lipshitz bound for $\tilde{\rho}$ we first use

$$|\rho(u) - \rho(u')| \le O(1)L^{3/2}\varepsilon^{11/4-\eta}\|u - u'\|$$

Then from (6.10)

$$\begin{aligned} |\tilde{\rho}(u) - \tilde{\rho}(u')| \le& O(L^{3/2})|\tilde{g}' + \tilde{g} + 2\bar{g}||\tilde{g}' - \tilde{g}| + |\rho(u) - \rho(u')| \\ &\le \left(O(L^{3/2})O(\varepsilon)e^{3/2} + O(1)L^{3/2}\varepsilon^{11/4-\eta}\right)\|u - u'\| \\ &\le O(L^{3/2})\varepsilon^{5/2}\|u - u'\| \end{aligned}$$

Finally from (6.7) and the definition of the norms in (5.33) we have

$$\|f_{\mathbf{w}}(u) - f_{\mathbf{w}}(u')\| \le \sup_{1\le p\le 3} \operatorname{ess\,sup}_x \left(|x|^{\frac{6p+1}{4}}|w_L^{(p)}(x) - w_L^{(p)\prime}(x)|\right)$$

Now $w_L^{(p)}(x) = L^2[\phi]w^{(p)}(Lx)$. We then get easily

$$\|f_{\mathbf{w}}(u) - f_{\mathbf{w}}(u')\| \le L^{-1/4}\|\mathbf{w} - \mathbf{w}'\| \le cL^{-1/4}\|u - u'\|$$

and we are done. Lemma 6.1 has been proved.

=u+t$\Delta u + \frac{1}{4}\frac{\Delta u}{\|\Delta u\|}e^{i\vartheta} Hence \|u + z\Delta u\| \le \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1 Sothatforz$∈ γ, $u + z\Delta u \in B(1)$, and hence from (6.15)

$$|||U(u + z\Delta u)||| \le L^{-1/4}\varepsilon^{11/4-\eta}$$

so that

$$\sup_{0\le t\le 1}|||\frac{\partial}{\partial t}U(u+t\Delta u)|||\le O(1)L^{-1/4}\varepsilon^{11/4-\eta}\|\Delta u\|$$

and thus

$$|||U(u)-U(u')|||\le O(1)L^{-1/4}\varepsilon^{11/4-\eta}\|u-u'\|$$

This proves the last inequality of (6.17).
On using (6.16) we have in the same way

$$|\tilde\xi(u)-\tilde\xi(u')|\le O(1)\varepsilon^{11/4-\eta}\|u-u'\|$$

To get the Lipshitz bound for $\tilde\rho$ we first use

$$|\rho(u)-\rho(u')|\le O(1)L^{3/2}\varepsilon^{11/4-\eta}\|u-u'\|$$

Then from (6.10)

$$\begin{aligned}|\tilde\rho(u)-\tilde\rho(u')|\le& O(L^{3/2})|\tilde g'+\tilde g+2\bar g||\tilde g'-\tilde g|+|\rho(u)-\rho(u')|\\ &\le\left(O(L^{3/2})O(\varepsilon)e^{3/2}+O(1)L^{3/2}\varepsilon^{11/4-\eta}\right)\|u-u'\|\\ &\le O(L^{3/2})\varepsilon^{5/2}\|u-u'\|\end{aligned}$$

Finally from (6.7) and the definition of the norms in (5.33) we have

$$\|f_{\mathbf w}(u)-f_{\mathbf w}(u')\|\le\sup_{1\le p\le 3}\ \text{ess}\sup_x\left(|x|^{\frac{6p+1}{4}}|w_L^{(p)}(x)-w_L^{(p)\prime}(x)|\right)$$

Now $w_L^{(p)}(x)=L^2[\phi]w^{(p)}(Lx)$. We then get easily

$$\|f_{\mathbf w}(u)-f_{\mathbf w}(u')\|\le L^{-1/4}\|\mathbf w-\mathbf w'\|\le cL^{-1/4}\|u-u'\|$$

and we are done. Lemma 6.1 has been proved.

Consider now the RG flow (6.3):

$$u_k=f(u_{k-1})$$

with initial condition

$$u_0=(\tilde g_0,\mu_0,0,0)\quad \tilde g_0=g_0-\bar g$$

Theorem 6.2
There exists μ_0 such that for $u_0\in B(1/32)$, $u_k=f(u_{k-1})\in B(1/4)$ for all $k\ge 1$.

Remark
The following proof of existence of global solutions is a textbook argument in the theory of dynamical systems adapted to the present context.

Proof
From the flows (6.4), (6.5) we easily derive after n steps of the RG

$$\tilde g_k=(2-L^\varepsilon)^k\tilde g_0+\sum_{j=0}^{k-1}(2-L^\varepsilon)^{k-1-j}\tilde\xi(u_j),\quad 1\le k\le n$$

$$\mu_k = L^{-\frac{3+\varepsilon}{2}(n-k)}\mu_n - \sum_{j=k}^{n-1} L^{-\frac{3+\varepsilon}{2}(j+1-k)}\tilde{\rho}(u_j), \quad 0 \le k \le n-1$$

Let us fix $\mu_n = \mu_f$ and take $n \to \infty$. We have

$$\tilde{g}_k = (2-L^\varepsilon)^k \tilde{g}_0 + \sum_{j=0}^{k-1}(2-L^\varepsilon)^{k-1-j}\tilde{\xi}(u_j), \quad k \ge 1 \tag{6.18}$$

$$\mu_k = -\sum_{j=k}^{\infty} L^{-\frac{3+\varepsilon}{2}(j+1-k)}\tilde{\rho}(u_j), \quad k \ge 0 \tag{6.19}$$

together with

$$R_k = U(u_{k-1}), \quad k \ge 1 \tag{6.20}$$

We can take the autonomous $\mathbf{w}$ flow, given by (6.7), as solved by (5.32) and need no longer consider it as a flow variable.
Note that for ε sufficiently small (depending on L)

$$0 < 2 - L^\varepsilon < 1 \tag{6.21}$$

Then for $u_j \in B(1)$ the infinite sum of (6.19) converges by (6.21) and (6.16). So μ_0 has now been determined provided (6.18)-(6.20) has a solution.
It is easy to verify that any solution of (6.18)- (6.19), together with the autonomous $\mathbf{w}$ flow, is a solution of the RG flow

$$u_k = f(u_{k-1})$$

Now write (6.18)-(6.20) in the form

$$u_k = F_k(\mathbf{u}) \tag{6.22}$$

where $\mathbf{u} = (u_0, u_1, u_2, ...)$ and F_k has components $(F_k^{(g)}, F_k^{(\mu)}, F_k^{(R)})$ given by the r.h.s. of (6.18), (6.19) and (6.20) respectively.
If we write

$$\mathbf{F}(\mathbf{u}) = (F_0(\mathbf{u}), F_1(\mathbf{u}), ...)$$

then (6.22) can be written as a fixed point equation

$$\mathbf{u} = \mathbf{F}(\mathbf{u}) \tag{6.23}$$

Consider the Banach space $\mathbf{E}$ of sequences $\mathbf{u} = (u_0, u_1, u_2, ...)$ with norm

$$\|\mathbf{u}\| = \sup_{k\ge 0}\|u_k\| \tag{6.24}$$

and the closed ball $\mathbf{B}(r) \subset \mathbf{E}$

$$\mathbf{B}(r) = \{\mathbf{u} : \|\mathbf{u}\| \le r\} \tag{6.25}$$

We shall seek a solution of (6.23) in the closed ball $\mathbf{B}(1/4)$ with initial data $u_0 = (\tilde{g}_0, \mu_0, 0, 0)$ in $B(1/32)$ and $\tilde{g}_0$ held fixed.

We shall need

Lemma 6.3

$$\mathbf{u} \in \mathbf{B}(1/32) \Rightarrow \mathbf{F}(\mathbf{u}) \in \mathbf{B}(1/16) \tag{6.26}$$

Moreover, for $\mathbf{u}, \mathbf{u}' \in \mathbf{B}(1/4)$

$$\|\mathbf{F}(\mathbf{u}) - \mathbf{F}(\mathbf{u}')\| \leq \frac{1}{2}\|\mathbf{u} - \mathbf{u}'\| \tag{6.27}$$

We postpone the proof of this lemma. Given the above lemma 6.3 the proof of theorem 6.2 now follows easily. To this end consider a sequence $\mathbf{u}^{(n)}, n = 1, 2, ...,$ defined by

$$\mathbf{u}^{(n)} = \mathbf{F}(\mathbf{u}^{(n-1)}) \tag{6.28}$$

$$\mathbf{u}^{(0)} \in \mathbf{B}(1/32) \tag{6.29}$$

Claim: $\mathbf{u}^{(n)} \in \mathbf{B}(1/4)$ for all n.
To prove the claim first observe that (6.26) and (6.29) imply

$$\|\mathbf{F}(\mathbf{u}^{(0)}) - \mathbf{u}^{(0)}\| \leq \frac{3}{32}$$

Make the inductive hypothesis $\mathbf{u}^{(j)} \in \mathbf{B}(1/4), \quad j = 0, 1, ..., n-1$ and this is clearly satisfied for $j = 0$. Now using the Lipshitz bound (6.27) we get

$$\|\mathbf{u}^{(n)} - \mathbf{u}^{(n-1)}\| = \|\mathbf{F}(\mathbf{u}^{(n-1)}) - \mathbf{F}(\mathbf{u}^{(n-2)})\| \leq \frac{1}{2^{n-1}}\|\mathbf{u}^{(1)} - \mathbf{u}^{(0)}\| \leq$$

$$\leq \|\mathbf{F}(\mathbf{u}^{(0)}) - \mathbf{u}^{(0)}\| \leq \frac{1}{2^{n-1}}\frac{3}{32}$$

Write

$$\mathbf{u}^{(n)} = \mathbf{u}^{(0)} + \sum_{j=1}^{n}(\mathbf{u}^{(j)} - \mathbf{u}^{(j-1)})$$

Then

$$\|\mathbf{u}^{(n)}\| \leq \frac{1}{32}\left(1 + 3\sum_{j=1}^{n}\frac{1}{2^{j-1}}\right) \leq \frac{7}{32} < \frac{1}{4}$$

and the claim has been proved.
By virtue of the claim and $\|\mathbf{u}^{(n)} - \mathbf{u}^{(n-1)}\| \to 0$ as $n \to \infty$ we have $\mathbf{u}^{(n)} \to \mathbf{u}$ in $\mathbf{B}(1/4)$ and this $\mathbf{u}$ satisfies

$$\mathbf{u} = \mathbf{F}(\mathbf{u})$$

We are done. Thus it only remains to prove lemma 6.3 to complete the proof of theorem 6.2.

Proof of lemma 6.3
First we prove (6.26), and thus take $\mathbf{u} \in \mathbf{B}(1/32)$. From (6.18) and the estimates in (6.16) we have

$$\varepsilon^{-3/2}|F_k^{(g)}(\mathbf{u})| \leq (2 - L^{\varepsilon})\frac{1}{32} + O(1)\varepsilon^{5/4-\eta}\sum_{j=0}^{k-1}(2 - L^{\varepsilon})^{k-1-j} \leq$$

$$\leq \frac{1}{32} + O(1)\frac{\varepsilon^{5/4-\eta}}{L^{\varepsilon} - 1} \leq \frac{1}{32} + O(1)\frac{\varepsilon^{1/4-\eta}}{\log L} \leq \frac{1}{16}$$

since $\eta < \frac{1}{4}$ and ε is sufficiently small.
Similarly from (6.19) and (6.16) we have

$$\varepsilon^{-(2-\delta)}|F_k^{(\mu)}(\mathbf{u})| \leq \sum_{j=k}^{\infty} L^{-\frac{3+\varepsilon}{2}(j+1-k)} \leq L^{-\frac{3+\varepsilon}{2}}(1 - L^{-\frac{3+\varepsilon}{2}}) \leq \frac{1}{16}$$

for L sufficiently large.
Finally from (6.20) and (6.15)

$$\varepsilon^{-(11/4-\eta)}|F_k^{(R)}(\mathbf{u})| \le L^{-1/4} \le \frac{1}{16}$$

This proves (6.26). To prove (6.27), take $\mathbf{u}, \mathbf{u}' \in \mathbf{B}(1/4)$. We can then use the Lipshitz estimates of lemma 6.1. Note that the initial coupling g_0 is held fixed. Then we have

$$\varepsilon^{-3/2}|F_k^{(g)}(\mathbf{u}) - F_k^{(g)}(\mathbf{u}')| \le \sum_{j=0}^{k-1}(2-L^{\varepsilon})^{k-1-j}'s proof of the stable manifold theorem as presented in appendix 2, chapter 5 of [S]. Part of Irwin's proof is replaced by theorem 6.2.$$

Lemma 6.4
Let $u, u' \in B(1/4)$. Then we have

$$\|f_1(u) - f_1(u')\| \le (1-\varepsilon)\|u-u'\| \tag{6.30}$$

and, if $\|u_2 - u_2'\| \ge \|u_1 - u_1'\|$ then

$$\|f_2(u) - f_2(u')\| \ge (1+\varepsilon)\|u-u'\| \tag{6.31}$$

Proof of lemma 6.4
Because $u, u' \in B(1/4)$ we can use throughout lemma 6.1. As always L is sufficiently large and then ε sufficiently small. First we prove (6.30). f_1 has components $(f_g, f_R, f_{\mathbf{w}})$. From (6.4)

$$f_g(u) = (2 - L^{\varepsilon})\tilde{g} + \tilde{\xi}(u)$$

Thus using lemma 6.1

$$\begin{aligned}
\varepsilon^{-3/2}|f_g(u) - f_g(u')| &\le (2-L^{\varepsilon})\|u-u'\| + \varepsilon^{-3/2}|\tilde{\xi}(u) - \tilde{\xi}(u')| \\
&\le (2 - L^{\varepsilon} + O(1)\varepsilon^{5/4-\eta})\|u-u'\| \\
&\le (1-\varepsilon)\|u-u'\|
\end{aligned}$$

for ε sufficiently small.
Since $f_R(u) = U(u)$, we have from lemma 6.1

$$\varepsilon^{-(11/4-\eta)}|||f_R(u) - f_R(u')||| \le (1-\varepsilon)\|u-u'\|$$

for L sufficiently large. Finally from the same lemma

$$c^{-1}\|f_{\mathbf{w}}(u) - f_{\mathbf{w}}(u')\| \le (1-\varepsilon)\|u-u'\|$$

These three inequalities prove (6.30).
Next we turn to (6.31). In this case by assumption $\|u_2 - u_2'\| \ge \|u_1 - u_1'\|$ and hence, since our norms are box norms, we have

$$\|u-u'\| = \|u_2 - u_2'\| = \varepsilon^{-(2-\delta)}|\mu - \mu'|$$

From (6.5)

$$f_\mu(u) = L^{\frac{3+\varepsilon}{2}}\mu + \tilde{\rho}(u)$$

Then, using lemma 6.1,

$$\begin{aligned}
\varepsilon^{-(2-\delta)}|f_\mu(u) - f_\mu(u')| &\ge L^{\frac{3+\varepsilon}{2}}\|u-u'\| - \varepsilon^{-(2-\delta)}|\tilde{\rho}(u) - \tilde{\rho}(u')| \\
&\ge (L^{\frac{3+\varepsilon}{2}} - O(\log L)\varepsilon^{5/2})\|u-u'\| \\
&\ge (1+\varepsilon)\|u-u'\|
\end{aligned}$$

which proves (6.31) and thus completes the proof of lemma 6.4.

Let f^k be the k-fold composition of the RG map f.

Definition 6.5
The stable manifold of f is defined by

$$W^s(f) = \{u \in B(1/32) : f^k(u) \in B(1/4)\ \forall k \geq 0\} \tag{6.32}$$

Write the initial points u as $u = (u_1, u_2)$ with $u_1 = (\tilde{g}_0, 0, 0)$ and $u_2 = \mu_0$. Observe that theorem 6.2 says that there exists for $u \in B(1/32)$ a u_2 such that $f^k(u) \in B(1/4)\ \forall k \geq 0$. We now have

Theorem 6.6
$W^s(f)$ is the graph $\{(u_1, h(u_1)\}$ of a function $u_2 = h(u_1)$ with h Lipshitz continuous with Lip$h \leq 1$. Moreover $f|W^s(f)$ contracts distances and hence has a unique fixed point which attracts all points of $W^s(f)$.

Proof
To prove the first statement it is enough to prove that if in $W^s(f)$ we take two points $u = (u_1, u_2)$ and $u' = (u_1', u_2')$ then

$$\|u_2 - u_2'\| \leq \|u_1 - u_1'\| \tag{6.33}$$

because then for a given u_1 we would have at most one u_2, and by theorem 6.2 there exists such a u_2. This means that $W^s(f)$ is the graph of a function h, $u_2 = h(u_1)$, and moreover

$$\|h(u_1) - h(u_1')\| \leq \|u_1 - u_1'\|$$

Suppose (6.33) is not true. Then

$$\|u_2 - u_2'\| > \|u_1 - u_1'\| \tag{6.34}$$

Then by (6.31) followed by (6.30) gives

$$\|f_2(u) - f_2(u')\| \geq (1+\varepsilon)\|u - u'\| > (1-\varepsilon)\|u - u'\| \geq \|f_1(u) - f_1(u')\|$$

and hence

$$\|f(u) - f(u')\| \geq (1+\varepsilon)\|u - u'\|$$

Now

$$\|f^2(u) - f^2(u')\| = \|f(f(u)) - f(f(u'))\|$$

and by the above and the second part of Lemma 6.4

$$\|f^2(u) - f^2(u')\| \geq (1+\varepsilon)\|f(u) - f(u')\| \geq (1+\varepsilon)^2\|u - u'\|$$

By induction we can prove for all $k \geq 0$

$$\|f^k(u) - f^k(u')\| \geq (1+\varepsilon)^k\|u - u'\|$$

Since $u, u' \in W^s(f)$ the l.h.s. is bounded above by $\frac{1}{2}$ and hence for $k \to \infty$ we have a contradiction because $u \neq u'$ under (6.34).
Hence (6.33) is true and the first statement of theorem 6.6 has been proved.
Now we prove that $f|W^s(f)$ is a contraction. Note that if $u, u' \in W^s(f)$, then

$$\|f_2(u) - f_2(u')\| \leq \|f_1(u) - f_1(u')\| \tag{6.35}$$

We can prove this just as we proved (6.33). Namely assume the contrary and then show in the same way

$$\|f^k(u) - f^k(u')\| \geq (1+\varepsilon)^{k-1} \|f(u) - f(u')\|$$

The l.h.s. is bounded by $\frac{1}{2}$ and so as $k \to \infty$ we get a contradiction because $f(u) \neq f(u')$ under the negation of (6.35). This proves (6.35) which now implies's proof of the stable manifold theorem as presented in appendix 2, chapter 5 of [S]. Part of Irwin's proof is replaced by theorem 6.2.

Lemma 6.4
Let $u, u' \in B(1/4)$. Then we have

$$\|f_1(u) - f_1(u')\| \leq (1-\varepsilon)\|u - u'\| \tag{6.36}$$

and, if $\|u_2 - u_2'\| \geq \|u_1 - u_1'\|$ then

$$\|f_2(u) - f_2(u')\| \geq (1+\varepsilon)\|u - u'\| \tag{6.37}$$

Proof of lemma 6.4
Because $u, u' \in B(1/4)$ we can use throughout lemma 6.1. As always L is sufficiently large and then ε sufficiently small. First we prove (6.30). f_1 has components $(f_g, f_R, f_{\mathbf{w}})$. From (6.4)

$$f_g(u) = (2 - L^{\varepsilon})\tilde{g} + \tilde{\xi}(u)$$

Thus using lemma 6.1

$$\begin{aligned}
\varepsilon^{-3/2}|f_g(u) - f_g(u')| &\leq (2 - L^{\varepsilon})\|u - u'\| + \varepsilon^{-3/2}|\tilde{\xi}(u) - \tilde{\xi}(u')| \\
&\leq (2 - L^{\varepsilon} + O(1)\varepsilon^{5/4-\eta})\|u - u'\| \\
&\leq (1-\varepsilon)\|u - u'\|
\end{aligned}$$

for ε sufficiently small.

Since $f_R(u) = U(u)$, we have from lemma 6.1

$$\varepsilon^{-(11/4-\eta)}|||f_R(u) - f_R(u')||| \leq (1-\varepsilon)\|u - u'\|$$

for L sufficiently large. Finally from the same lemma

$$c^{-1}\|f_{\mathbf{w}}(u) - f_{\mathbf{w}}(u')\| \leq (1-\varepsilon)\|u - u'\|$$

These three inequalities prove (6.30).
Next we turn to (6.31). In this case by assumption $\|u_2 - u_2'\| \geq \|u_1 - u_1'\|$ and hence, since our norms are box norms, we have

$$\|u - u'\| = \|u_2 - u_2'\| = \varepsilon^{-(2-\delta)}|\mu - \mu'|$$

From (6.5)

$$f_\mu(u) = L^{\frac{3+\varepsilon}{2}}\mu + \tilde{\rho}(u)$$

Then, using lemma 6.1,

$$\begin{aligned}
\varepsilon^{-(2-\delta)}|f_\mu(u) - f_\mu(u')| &\geq L^{\frac{3+\varepsilon}{2}}\|u - u'\| - \varepsilon^{-(2-\delta)}|\tilde{\rho}(u) - \tilde{\rho}(u')| \\
&\geq (L^{\frac{3+\varepsilon}{2}} - O(\log L)\varepsilon^{5/2})\|u - u'\| \\
&\geq (1+\varepsilon)\|u - u'\|
\end{aligned}$$

which proves (6.31) and thus completes the proof of lemma 6.4.

Let f^k be the k-fold composition of the RG map f.

Definition 6.5
The stable manifold of f is defined by

$$W^s(f) = \{u \in B(1/32) : f^k(u) \in B(1/4)\ \forall k \geq 0\} \tag{6.38}$$

Write the initial points u as $u = (u_1, u_2)$ with $u_1 = (\tilde{g}_0, 0, 0)$ and $u_2 = \mu_0$. Observe that theorem 6.2 says that there exists for $u \in B(1/32)$ a u_2 such that $f^k(u) \in B(1/4)\ \forall k \geq 0$. We now have

Theorem 6.6
$W^s(f)$ is the graph $\{(u_1, h(u_1))\}$ of a function $u_2 = h(u_1)$ with h Lipshitz continuous with Lip$h \leq 1$. Moreover $f|W^s(f)$ contracts distances and hence has a unique fixed point which attracts all points of $W^s(f)$.

Proof
To prove the first statement it is enough to prove that if in $W^s(f)$ we take two points $u = (u_1, u_2)$ and $u' = (u_1', u_2')$ then

$$\|u_2 - u_2'\| \leq \|u_1 - u_1'\| \tag{6.39}$$

because then for a given u_1 we w