A Pedestrian Introduction to Gamow Vectors

R. de la Madrid
Institute for Scientific Interchange (ISI), Villa Gualino, Viale Settimio Severo 65, I-10133, Torino, Italy
E-mail: `rafa@isiosf.isi.it`

M. Gadella
Departamento de Física Teórica, Facultad de Ciencias, 47011 Valladolid, Spain

Abstract

The Gamow vector description of resonances is compared with the S-matrix and the Green function descriptions using the example of the square barrier potential. By imposing different boundary conditions on the time independent Schrödinger equation, we obtain either eigenvectors corresponding to real eigenvalues and the physical spectrum or eigenvectors corresponding to complex eigenvalues (Gamow vectors) and the resonance spectrum. We show that the poles of the S matrix are the same as the poles of the Green function and are the complex eigenvalues of the Schrödinger equation subject to a purely outgoing boundary condition. The intrinsic time asymmetry of the purely outgoing boundary condition is discussed. Finally, we show that the probability of detecting the decay within a shell around the origin of the decaying state follows an exponential law if the Gamow vector (resonance) contribution to this probability is the only contribution that is taken into account.

I. INTRODUCTION

Most elementary particles are only quasistable states decaying through various interactions and thus have finite lifetimes of various orders of magnitude.[1] Several theoretical descriptions have been proposed for quasistable states. Three of the most widely used descriptions are the S matrix, the Gamow vector, and the Green function. We shall use the example of nonrelativistic potential scattering, in particular the square barrier potential, to demonstrate the connection between these three descriptions.

Experimentally, resonances often appear as peaks in the cross section that resemble the well-known Breit-Wigner distribution. The Breit-Wigner distribution has two characteristic parameters: the energy E_R at which the peak reaches its maximum, and its width Γ_R at half-maximum. The inverse of Γ_R is the lifetime of the decaying state.[2] The peak of the cross section with Breit-Wigner shape is related to a first-order pole of the S matrix in the energy representation $S(E)$ at the complex number $z_R = E_R - i\Gamma_R/2$. The theoretical expression of the cross section in terms of $S(E)$ fits the shape of the experimental cross section in the neighborhood of E_R (see, for example, Ref. 3). This is why the first-order pole of the S matrix is often taken as the theoretical definition of a resonance. The Green function description treats a resonance as a pole of the Green function when analytically continued to the whole complex plane.

Although a resonance has a finite lifetime, it is otherwise assigned all the properties that are also attributed to stable particles, such as angular momentum and charge. For example, a radioactive nucleus has a finite lifetime, but otherwise it possesses all the properties of stable nuclei; in fact, it is included in the periodic table of the elements along with the stable nuclei. Therefore, it seems natural to seek a theoretical description that provides "particle status" to quasistable states. The description of a resonance by Gamow vectors allows us to interpret resonances as autonomous exponentially decaying physical systems.

The energy eigenfunction with complex eigenvalue was originally introduced by Gamow in his paper on α decay of atomic nuclei,[4] and used by a number of authors (see, for example, Refs. 5–11 and references therein). The real part of the complex eigenvalue is associated with the energy of the resonance, and the imaginary part is associated with the inverse of the lifetime. Gamow eigenfunctions have an exponentially decaying time evolution, in accordance with the exponential law observed in α decay of radioactive nuclei.[12–15]

Gamow's treatment was heuristic though, and could not be made mathematically rigorous within the Hilbert space theory, because self-adjoint operators on a Hilbert space can only have real eigenvalues. A rigorous mathematical treatment of Gamow vectors needs an extension of Hilbert space to the Rigged Hilbert Space (RHS).[16–18] The RHS was first introduced in physics in order to justify Dirac's bra-ket formalism.[19–21] In RHS language, Gamow vectors are eigenkets of a (dual) extension of the self-adjoint Hamiltonian. This extension can surely have complex eigenvalues. Using the RHS formalism, one can prove that the time evolution of the Gamow vectors is governed by a semigroup, expressing time asymmetry on the microscopic level. One can also obtain an exact golden rule[3] for the decay of the state described by the Gamow vectors. This golden rule reduces to the Fermi-Dirac golden rule in the Born approximation.

There are several pedagogical papers on Gamow vectors[22,23] and related topics such as α decay,[23,24] barrier penetration,[25,26] and exponential decay[25–28] (this list is not exhaustive). Some of these papers consider alternatives to the Gamow, S matrix, and Green function descriptions. Especially suggestive is the approach of Refs. 23 and 25, where among other methods Feynman's path integral is used. In this paper, we discuss at the graduate student level some of the issues not covered by Refs. 22–28.

In Sec. II A, we calculate the Dirac kets for the square barrier potential. These Dirac kets are monoenergetic eigensolutions of the time independent Schrödinger equation. They are not square normalizable, and therefore they cannot represent a wave packet. Actually, they are members of a continuous basis that expands the wave functions in the Dirac basis vector expansion.

The well-known expression for the S matrix in the energy representation is provided in Sec. II B. The resonances will be defined as the poles of the S matrix. In Sec. II C, we calculate the Gamow vectors as the solutions of the time independent Schrödinger equation with complex eigenvalues subject to purely outgoing boundary conditions.

The Green function and its poles are calculated in Sec. II D. It will be apparent that the poles of the Green function are the same as the poles of the S matrix and are the complex eigenvalues obtained from the purely outgoing boundary condition. The residue of the Green function at the resonance energy is expressed in terms of the Gamow eigenfunction.

In Sec. II E, the basis vector expansion generated by the Gamow vectors is provided. Sections III A and III B discuss the time asymmetry built into the purely outgoing boundary condition. Section III C treats the exponential decay law of the Gamow vectors.

In the paper, there will be some mathematical interludes enclosed by brackets. These interludes are not essential to understand the paper (although they are essential to prove other fundamental results[16,18]) and may be skipped in the first reading. They are recommended to readers who are seriously interested in the mathematical framework that supports the Gamow vector approach to resonance scattering.

II. RESONANCES FOR A SQUARE BARRIER POTENTIAL

A. Time Independent Schrödinger Equation

We consider a three-dimensional square barrier potential of height $V_0 > 0$ and calculate the energy kets using the time independent Schrödinger equation

$$H|E\rangle = E|E\rangle\,. \tag{1}$$

Equation (1) is an eigenvalue equation of the Hamiltonian H. When the eigenvalues E belong to the continuous spectrum of H, then the solutions $|E\rangle$, also called *Dirac kets*, are given by vectors that lie outside the Hilbert space, that is, they are not square integrable. Following von Neumann, these kets are sometimes interpreted as states that are "very near a proper state."[29] Following Dirac,[30] we interpret these kets as members of a continuous basis system that spans the space of wave functions. Therefore, every wave function φ can be expanded in terms of the Dirac kets $|E,l,m\rangle \equiv |E\rangle$ as

$$\varphi = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int_0^{\infty} dE\,|E,l,m\rangle\langle E,l,m|\varphi\rangle\,. \tag{2}$$

[Although Dirac introduced Eq. (2) on heuristic grounds, its mathematical rigor was later established by the Gelfand-Maurin theorem[31] (also called the nuclear spectral theorem) within the Rigged Hilbert Space (see also Refs. 16, 18–21, 32, and 33).]

To calculate the possible set of (real) eigenvalues and their corresponding eigenvectors in our example, we solve Eq. (1) in the position representation,

$$\langle\vec{x}|H|E\rangle = \left(\frac{-\hbar^2}{2m}\nabla^2 + V(\vec{x})\right)\langle\vec{x}|E\rangle = E\langle\vec{x}|E\rangle\,, \tag{3}$$

where ∇^2 is the three-dimensional Laplacian and

$$V(\vec{x}) = V(r) = \begin{cases} 0 & 0<r<a \\ V_0 & a<r<b \\ 0 & b<r<\infty\,. \end{cases} \tag{4}$$

Because $V(\vec{x})$ is spherically symmetric, we can use spherical coordinates $\vec{x} \equiv (r, \theta, \phi)$ to solve Eq. (3), which in spherical coordinates reads

$$\langle r, \theta, \phi | H | E, l, m \rangle = \Big(\frac{-\hbar^2}{2m} \frac{1}{r} \frac{\partial^2}{\partial r^2} r + \frac{\hbar^2 l(l+1)}{2mr^2} + V(r) \Big) \langle r, \theta, \phi | E, l, m \rangle = E \langle r, \theta, \phi | E, l, m \rangle \, . \tag{5}$$

By separating the radial and angular dependences,

$$\langle r, \theta, \phi | E, l, m \rangle \equiv \langle r | E \rangle_l \, \langle \theta, \phi | l, m \rangle \equiv \frac{1}{r} \chi_l(r; E) Y_{l,m}(\theta, \phi), \tag{6}$$

where $Y_{l,m}(\theta, \phi)$ are the spherical harmonics, we obtain for the radial part

$$\Big(\frac{-\hbar^2}{2m} \frac{d^2}{dr^2} + \frac{\hbar^2 l(l+1)}{2mr^2} + V(r) \Big) \chi_l(r; E) = E \chi_l(r; E) \, . \tag{7}$$

In this section, we shall restrict ourselves to the case of zero orbital angular momentum (the higher-order case is treated in the appendix). We write $\chi_{l=0}(r; E) \equiv \chi(r; E)$ and obtain

$$-\frac{\hbar^2}{2m} \frac{d^2}{dr^2} \chi(r; E) + V(r) \chi(r; E) = E \chi(r; E) \, . \tag{8}$$

The solutions of Eq. (8) for each value of E are the kets (more precisely, the radial part of the energy kets in the position representation $\langle \vec{x} | E \rangle = \langle r, \theta, \phi | E, 0, 0 \rangle = \frac{\chi(r;E)}{r} Y_{0,0}(\theta, \phi)$). In our case,

$$\chi(r; E) = \begin{cases} \alpha_1 e^{ikr} + \beta_1 e^{-ikr} & 0 < r < a \\ \alpha_2 e^{iQr} + \beta_2 e^{-iQr} & a < r < b \\ \mathcal{F}_1 e^{ikr} + \mathcal{F}_2 e^{-ikr} & b < r < \infty \, , \end{cases} \tag{9}$$

where

$$k = \sqrt{\frac{2m}{\hbar^2} E} \tag{10}$$

is the wave number of the particle, and

$$Q = \sqrt{k^2 - \frac{2m}{\hbar^2} V_0} = \sqrt{\frac{2m}{\hbar^2} (E - V_0)} \, . \tag{11}$$

The coefficients α, β, and $\mathcal{F}$ are functions of k (and therefore of the energy E). Their possible values are restricted by the boundary conditions that we need to impose on the solutions (9) of the radial Schrödinger equation. Some of these boundary conditions are due to the fact that at least the second derivative of χ must be well defined, which is why χ must be a continuous function of r with a continuous derivative. Another boundary condition makes the solution vanish at the origin (see Eq. (12a)). This condition can be understood by viewing the analogous one-dimensional problem of Eq. (8). In the one-dimensional case the potential is infinite for $r < 0$. The function $\mathcal{F}_2 e^{-ikr}$ represents an incoming probability wave of amplitude $\mathcal{F}_2$, and $\mathcal{F}_1 e^{ikr}$ represents an outgoing wave of amplitude $\mathcal{F}_1$. Because there cannot be any transmission into the region $r < 0$, χ must vanish at the origin.[34] Mathematically, this condition is related to the self-adjointness of the Hamiltonian.[35,16] Finally, the eigenfunction $\chi(r;E)$ is assumed to be bounded (see Eq. (12f)). We impose this boundedness condition because we want the set of eigenvalues to coincide with the Hilbert space (that is, the physical) spectrum of the Hamiltonian. In our case, the Hilbert space spectrum is $[0,\infty)$ (see Refs. 35 and 16). Then, the boundary conditions that we impose upon Eq. (9) read

$$\chi(0;E) = 0 \tag{12a}$$
$$\chi(a-0;E) = \chi(a+0;E) \tag{12b}$$
$$\chi'(a-0;E) = \chi'(a+0;E) \tag{12c}$$
$$\chi(b-0;E) = \chi(b+0;E) \tag{12d}$$
$$\chi'(b-0;E) = \chi'(b+0;E) \tag{12e}$$
$$|\chi(r;E)| < \infty\,. \tag{12f}$$

If E is a negative real number (or complex), then there are no coefficients of $\chi(r;E)$ in (9) for which the boundary conditions (12) can be satisfied unless they are trivially zero. To be more precise, if E is a complex or a negative real number, the corresponding eigensolution $\chi(r;E)$ of Eq. (8) does not satisfy the boundary condition (12f), even though it satisfies the other boundary conditions (12a)–(12e). If E is a real number in $[0,\infty)$, the corresponding eigenfunction satisfies all the boundary conditions in (12). Therefore, the boundedness of the eigenkets is what forbids the negative (and the complex) energies and singles out the physical (Hilbert space) spectrum.

If we use the notation $\alpha = 2i\alpha_1$, we rewrite the boundary conditions (12) as the following equations for the coefficients:

$$\alpha_2 e^{iQa} + \beta_2 e^{-iQa} = \alpha \sin(ka) \tag{13a}$$
$$iQ(\alpha_2 e^{iQa} - \beta_2 e^{-iQa}) = \alpha k \cos(ka) \tag{13b}$$
$$\mathcal{F}_1 e^{ikb} + \mathcal{F}_2 e^{-ikb} = \alpha_2 e^{iQb} + \beta_2 e^{-iQb} \tag{13c}$$
$$ik(\mathcal{F}_1 e^{ikb} - \mathcal{F}_2 e^{-ikb}) = iQ(\alpha_2 e^{iQb} - \beta_2 e^{-iQb})\,. \tag{13d}$$

After straightforward but tedious calculations, we find that

$$\alpha_2(k) = \frac{1}{2} e^{-iQa}\left[\sin(ka) + \frac{k}{iQ}\cos(ka)\right]\alpha(k) \tag{14a}$$
$$\beta_2(k) = \frac{1}{2} e^{iQa}\left[\sin(ka) - \frac{k}{iQ}\cos(ka)\right]\alpha(k) \tag{14b}$$
$$\begin{aligned}\mathcal{F}_1(k) = \frac{e^{-ikb}}{4}\Big[&(1+\frac{Q}{k})e^{iQ(b-a)}(\sin(ka) + \frac{k}{iQ}\cos(ka)) \\ &+ (1-\frac{Q}{k})e^{-iQ(b-a)}(\sin(ka) - \frac{k}{iQ}\cos(ka))\Big]\alpha(k)\end{aligned} \tag{14c}$$
$$\begin{aligned}\mathcal{F}_2(k) = \frac{e^{ikb}}{4}\Big[&(1-\frac{Q}{k})e^{iQ(b-a)}(\sin(ka) + \frac{k}{iQ}\cos(ka)) \\ &+ (1+\frac{Q}{k})e^{-iQ(b-a)}(\sin(ka) - \frac{k}{iQ}\cos(ka))\Big]\alpha(k)\,.\end{aligned} \tag{14d}$$

Thus, for each E in $[0, \infty)$, there exists a solution of the eigenvalue equation in the position representation

$$\langle r, \theta, \phi | E \rangle = \frac{\chi(r; E)}{r} Y_{0,0}(\theta, \phi) = \frac{\chi(r; E)}{r} \sqrt{\frac{1}{4\pi}}\,, \qquad 0 \leq E < \infty \tag{15}$$

with $\chi(r; E)$ given by

$$r\langle r|E\rangle_{l=0} = \chi(r; E) = \begin{cases} \alpha(k)\sin(kr) & 0 < r < a \\ \alpha_2(k)e^{iQr} + \beta_2(k)e^{-iQr} & a < r < b \\ \mathcal{F}_1(k)e^{ikr} + \mathcal{F}_2(k)e^{-ikr} & b < r < \infty\,. \end{cases} \tag{16}$$

The coefficients $\alpha_2(k)$, $\beta_2(k)$, and $\mathcal{F}_{1,2}(k)$ in Eq. (16) are given by Eq. (14) in terms of $\alpha(k)$. Usually, $\alpha(k)$ is chosen such that the kets in Eq. (16) are δ-normalized, although in the following sections we shall assume that $\alpha(k) = 1$.

Equation (15) means that for each energy eigenvalue E in the spectrum $[0, \infty)$ of the Hamiltonian, there exists an eigensolution of the Hamiltonian satisfying the boundary conditions (12). We can expand any wave function φ in terms of these eigenfunctions (for $l = 0$) as

$$\varphi(r, \theta, \phi) = \int_0^\infty dE \frac{\chi(r; E)}{r} Y_{0,0}(\theta, \phi)\, \varphi(E)\,, \tag{17}$$

or in bra-ket notation

$$\langle r, \theta, \phi | \varphi \rangle = \int_0^\infty dE\, \langle r, \theta, \phi | E \rangle \langle E | \varphi \rangle\,. \tag{18}$$

We can interpret (18) by saying that any wave function φ is a continuous linear superposition of the eigenfunctions $\langle r, \theta, \phi | E \rangle$ in Eq. (15). Each eigenfunction is weighted by $\langle E | \varphi \rangle$, which represents the wave function in the energy representation.

[Mathematically, the eigenkets $\langle r, \theta, \phi | E \rangle$ allow us to go from the energy representation $\varphi(E) = \langle E | \varphi \rangle$ to the position representation $\varphi(\vec{x}) \equiv \langle r, \theta, \phi | \varphi \rangle$ and vice versa, that is, they are continuous transition matrix elements. Because the monoenergetic $\langle r, \theta, \phi | E \rangle$ are not square integrable, they are not in the Hilbert space. Thus Hilbert space methods are not sufficient to handle them, and an extension of those methods is needed. As shown in Refs. 16, 18, 32, and 33, the extension that seems to be the most convenient is the Rigged Hilbert Space.]

B. S-Matrix Approach

We consider now the scattering process of a particle beam off the square barrier potential. The S matrix for this process relates the incoming (or prepared) and the outgoing (or detected) beams. We write the S matrix in the energy and angular momentum representation,

$$\langle {}^-E, l, m | E', l', m'^+ \rangle = S_l(E)\, \delta(E - E')\, \delta_{l,l'}\, \delta_{m,m'}\,, \tag{19}$$

where $|E,l,m^{\pm}\rangle$ are the kets that solve the Lippmann-Schwinger equation, and $\langle^{\pm}E,l,m|$ are their corresponding bras. In the position representation, these Lippmann-Schwinger eigenfunctions are given by (see Refs. 36–38)

$$\langle r|E^{+}\rangle = \frac{-1}{2i}\frac{\chi(r;E)}{\mathcal{F}_2} \tag{20a}$$

$$\langle r|E^{-}\rangle = \frac{1}{2i}\frac{\chi(r;E)}{\mathcal{F}_1}. \tag{20b}$$

The probability amplitude of detecting an out-state ψ^- in an in-state φ^+ is (see Refs. 3 and 16)

$$(\psi^-,\varphi^+) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int_0^{\infty} dE\langle\psi^-|E,l,m^-\rangle S_l(E)\langle^+E,l,m|\varphi^+\rangle\,. \tag{21}$$

In this section, we restrict ourselves to the case of zero angular momentum, $l=0$, that is, to the first term of Eq. (21). The partial S matrix for the case of zero angular momentum will be denoted by $S(E)\equiv S_{l=0}(E)$.

Because the S matrix relates incoming and outgoing waves far outside the interaction region, it suffices to focus on the region $r>b$. In this region, we have an incoming spherical wave e^{-ikr}/r with amplitude $\mathcal{F}_2(k)$ and an outgoing spherical wave e^{ikr}/r with amplitude $\mathcal{F}_1(k)$. Therefore, the S matrix in the energy representation[36,39] is given by the ratio

$$S(E)\equiv S(k) = -\frac{\mathcal{F}_1(k)}{\mathcal{F}_2(k)}\,, \tag{22}$$

where k is given by Eq. (10). It is easy to verify that the S matrix is unity when there is no potential,

$$\lim_{V_0\to 0} S(k) = 1\,, \tag{23}$$

and that $S^*(k)S(k)=1$, that is, $|S(k)|=1$ for every k.

The S matrix $S(k)$ in Eq. (22) is well defined for each real value of k, because its denominator is never zero when k is real. The $l=0$ resonances are associated with the poles of the analytic continuation of $S(k)$ into the entire complex plane. The relation $k^2=2mE/\hbar^2$ provides a Riemann surface for this extension in a natural way. The analytic continuation of the numerator and the denominator of $S(k)$ yield two analytic functions $\mathcal{F}_{1,2}(k)$. Therefore, the continuation of $S(k)$ is analytic except at its poles. These are precisely the zeros of the denominator of $S(k)$ (see Ref. 40),

$$\mathcal{F}_2(k) = 0\,, \tag{24}$$

which leads to

$$(1-\frac{Q}{k})e^{iQ(b-a)}[\sin(ka)+\frac{k}{iQ}\cos(ka)] + (1+\frac{Q}{k})e^{-iQ(b-a)}[\sin(ka)-\frac{k}{iQ}\cos(ka)] = 0\,. \tag{25}$$

The solutions of Eq. (25) are the (S-matrix) resonances of the square barrier potential. Equation (25) has a denumerable infinite number of complex resonance energy solutions.

These solutions come in pairs $E_R \pm i\Gamma_R/2$ (see Fig. 1). The pole $E_R - i\Gamma_R/2$ is associated with the decaying part of the resonance, and is located on the lower half-plane of the second sheet of the two-sheeted Riemann surface corresponding to the square root mapping (see Fig. 1a). The pole $E_R + i\Gamma_R/2$ is associated with the growing part of the resonance, and is located on the upper half-plane of the second sheet of the Riemann surface (see Fig. 1b). In the wave number plane, this pair of energy poles corresponds to a pair of poles $\pm\mathrm{Re}(k) - i\mathrm{Im}(k)$ in the lower half-plane that are mirror images of one another with respect to the imaginary axis (see Fig. 2).

The width of the resonance increases as the energy increases, and therefore the lifetime $\tau = 1/\Gamma_R$ decreases. The resonances whose energies are below the threshold $E = V_0$ are close to the real axis. As E increases, the resonances move away from the real axis toward infinity.

C. Gamow Vector Approach

In this section, we determine the state vectors of the resonances of the square barrier potential (4) as solutions of the time independent Schrödinger equation with complex eigenvalues. The state vectors will be eigenvectors as in Eq. (1). Because in Sec. II A we found all the eigenvectors of H with the boundary conditions (12) and determined that they all have positive real eigenvalues, we will have to change these boundary conditions to purely outgoing boundary conditions.[5] We will see that there is a one-to-one correspondence between the complex poles of the analytically continued S matrix and the complex eigenvalues obtained from purely outgoing boundary conditions. These eigenvectors of the Hamiltonian with complex eigenvalues, which are associated with the poles of the S matrix, are called Gamow vectors or Gamow kets.

The time independent Schrödinger equation for the Gamow vectors is given by

$$H|z_R\rangle = z_R|z_R\rangle\,, \tag{26}$$

where z_R and $|z_R\rangle$ represent the complex resonance eigenvalue and the Gamow vector, respectively. In the position representation, Eq. (26) becomes

$$\langle\vec{x}|H|z_R\rangle = z_R\langle\vec{x}|z_R\rangle, \tag{27}$$

or

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{x})\right)\langle\vec{x}|z_R\rangle = z_R\langle\vec{x}|z_R\rangle\,. \tag{28}$$

After writing Eq. (28) in spherical coordinates and considering only the case of zero angular momentum, we obtain

$$-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}\chi(r;z_R) + V(r)\chi(r;z_R) = z_R\chi(r;z_R)\,. \tag{29}$$

The solution of Eq. (29) for each value of z_R is the s-Gamow vector (more precisely, the radial part of the s-Gamow vector in the position representation $\langle\vec{x}|z_R\rangle = \langle r,\theta,\phi|z_R,0,0\rangle = 1/r\,\chi(r;z_R)Y_{0,0}(\theta,\phi)$). The solution of Eq. (29) has the same form as Eq. (9)

$$\chi(r;z_R)=\begin{cases}\alpha_1 e^{ikr}+\beta_1 e^{-ikr} & 0<r<a\\ \alpha_2 e^{iQr}+\beta_2 e^{-iQr} & a<r<b\\ \mathcal{F}_1 e^{ikr}+\mathcal{F}_2 e^{-ikr} & b<r<\infty\,,\end{cases} \tag{30}$$

but now the wave number

$$k=\sqrt{\frac{2m}{\hbar^2}z_R} \tag{31}$$

of the resonance is *complex*, and

$$Q=\sqrt{k^2-\frac{2m}{\hbar^2}V_0}=\sqrt{\frac{2m}{\hbar^2}(z_R-V_0)}\,. \tag{32}$$

Because the second derivative of $\chi(r;z_R)$ must be well defined, the solution (30) must be continuous with continuous derivatives as in (12b)–(12e). For the Gamow vector we shall also choose the condition that the radial part $\chi(r;z_R)$ vanishes at the origin as in Eq. (12a). However, at infinity we choose the purely outgoing boundary condition, which means that far from the potential region, the solution reduces to an exponential of the type e^{ikr}, but not of the type e^{-ikr}. If k is complex, this purely outgoing boundary condition does not mean that there are only outgoing waves. In fact, we have outgoing waves only when $\mathrm{Re}(k)$ is positive, and incoming waves when $\mathrm{Re}(k)$ is negative. Thus the boundary conditions that we impose on the Gamow vectors are

$$\chi(0;z_R)=0 \tag{33a}$$
$$\chi(a-0;z_R)=\chi(a+0;z_R) \tag{33b}$$
$$\chi'(a-0;z_R)=\chi'(a+0;z_R) \tag{33c}$$
$$\chi(b-0;z_R)=\chi(b+0;z_R) \tag{33d}$$
$$\chi'(b-0;z_R)=\chi'(b+0;z_R) \tag{33e}$$
$$\chi(r;z_R)\sim e^{ikr}\,,\qquad r\to\infty\,. \tag{33f}$$

The purely outgoing boundary condition (33f) is often written as

$$\lim_{r\to\infty}\frac{d\chi(r;z_R)}{dr}-ik\chi(r;z_R)=0\,. \tag{34}$$

One can easily check that Eq. (34) is equivalent to Eq. (33f).

The boundary condition (33f) leads by Eq. (30) to $\mathcal{F}_2=0$. Because this condition is the same as the condition (25) for the complex poles of the S matrix (22), the set of complex eigenvalues z_R must include the set of complex solutions of Eq. (25) (which are the S-matrix resonance poles). We shall show that these two sets of solutions are the same.

If we define $\alpha=2i\alpha_1$, the boundary conditions (33) can be written in terms of the coefficients as

$$\alpha_2 e^{iQa}+\beta_2 e^{-iQa}=\alpha\sin(ka) \tag{35a}$$
$$iQ(\alpha_2 e^{iQa}-\beta_2 e^{-iQa})=\alpha k\cos(ka) \tag{35b}$$
$$\mathcal{F}_1 e^{ikb}=\alpha_2 e^{iQb}+\beta_2 e^{-iQb} \tag{35c}$$
$$ik\mathcal{F}_1 e^{ikb}=iQ(\alpha_2 e^{iQb}-\beta_2 e^{-iQb})\,. \tag{35d}$$

If we write this set of linear equations as a matrix equation, we obtain

$$\begin{pmatrix} \sin(ka) & 0 & -e^{iQa} & -e^{-iQa} \\ k\cos(ka) & 0 & -iQe^{iQa} & iQe^{-iQa} \\ 0 & e^{ikb} & -e^{iQb} & -e^{-iQb} \\ 0 & ike^{ikb} & -iQe^{iQb} & iQe^{-iQb} \end{pmatrix} \begin{pmatrix} \alpha \\ \mathcal{F}_1 \\ \alpha_2 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} . \tag{36}$$

This is a homogeneous system of four equations with four unknowns. The system has a non-trivial solution if, and only if, the determinant of the coefficients is equal to zero,

$$\begin{vmatrix} \sin(ka) & 0 & -e^{iQa} & -e^{-iQa} \\ k\cos(ka) & 0 & -iQe^{iQa} & iQe^{-iQa} \\ 0 & e^{ikb} & -e^{iQb} & -e^{-iQb} \\ 0 & ike^{ikb} & -iQe^{iQb} & iQe^{-iQb} \end{vmatrix} = 0\,. \tag{37}$$

Straightforward calculations lead to

$$(1-\frac{Q}{k})e^{iQ(b-a)}[\sin(ka)+\frac{k}{iQ}\cos(ka)]+(1+\frac{Q}{k})e^{-iQ(b-a)}[\sin(ka)-\frac{k}{iQ}\cos(ka)]=0\,. \tag{38}$$

Thus we obtain exactly the same condition as the condition (25) for the poles of the S matrix. If we compare the boundary conditions (12) imposed upon the Dirac kets with the boundary conditions (33) imposed upon the Gamow vectors, we can see that the boundary condition that singles out the resonance spectrum is the purely outgoing boundary condition (33f).

As we mentioned earlier, the solutions of Eq. (38) (or Eq. (25)) always come in pairs. The eigenvector corresponding to the complex eigenvalue $E_R - i\Gamma_R/2$ is the decaying Gamow vector in the position representation, whose radial part, up to a normalization factor, is

$$\chi^{\rm decaying}(r;z_R) = \left\{ \begin{array}{ll} \sin(k_d r) & 0<r<a \\ \alpha_2(k_d)e^{iQ_d r}+\beta_2(k_d)e^{-iQ_d r} & a<r<b \\ \mathcal{F}_1(k_d)e^{ik_d r} & b<r<\infty\,, \end{array} \right. \tag{39}$$

where $k_d = \sqrt{2m/\hbar^2\,(E_R - i\Gamma_R/2)}$ and $Q_d^2 = k_d^2 - 2m/\hbar^2\, V_0$.

For $E_R + i\Gamma_R/2$, we obtain the growing Gamow vector in the position representation. Its radial part, up to a normalization factor, is

$$\chi^{\rm growing}(r;z_R^*) = \left\{ \begin{array}{ll} \sin(k_g r) & 0<r<a \\ \alpha_2(k_g)e^{iQ_g r}+\beta_2(k_g)e^{-iQ_g r} & a<r<b \\ \mathcal{F}_1(k_g)e^{ik_g r} & b<r<\infty\,, \end{array} \right. \tag{40}$$

where $k_g = \sqrt{2m/\hbar^2\,(E_R + i\Gamma_R/2)}$ and $Q_g^2 = k_g^2 - 2m/\hbar^2\, V_0$.

[Gamow eigenfunctions are not square integrable, that is, they do not belong to the Hilbert space. Thus, as Dirac kets, they must be handled by methods that extend those available in the Hilbert space framework. The Rigged Hilbert Space provides those methods (see for example, Refs. 16 and 18).]

D. Green Function Approach

In this section, we perform the calculations for the resonance energies using the Green function method. The radial Green function satisfies (see for instance, Refs. 36 and 41)

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2}+V(r)-E\right)G(r,r';E)=-\delta(r-r'), \tag{41}$$

subject to various boundary conditions. Equation (41) says that for $r\neq r'$, $G(r,r';E)$ obeys the radial Schrödinger equation (8). At $r=r'$ it is continuous, but its derivative has a discontinuity due to the delta function,

$$\frac{\partial}{\partial r}G(r'+0,r';E)-\frac{\partial}{\partial r}G(r'-0,r';E)=\frac{2m}{\hbar^2}\,. \tag{42}$$

We want G to satisfy the same kind of boundary conditions at zero and at infinity that we imposed on the other methods. We shall also require G to be regular at zero and a purely outgoing boundary condition at infinity. The expression for this Green function is[36,41]

$$G(r,r';E)=\frac{2m}{\hbar^2}\frac{\chi(r_<;E)\psi(r_>;E)}{W(\chi,\psi)}\,, \tag{43}$$

where $r_<$ refers to the minimum and $r_>$ to the maximum of r and r'. The function χ is the solution of Eq. (8) that vanishes at the origin, ψ is the solution of Eq. (8) that satisfies a purely outgoing boundary condition at infinity, and $W(\chi,\psi)$ is their Wronskian. In this case χ is just the solution in Eq. (16). The function ψ satisfies Eq. (8) and the boundary conditions

$$\psi(a-0;E)=\psi(a+0;E) \tag{44a}$$
$$\psi'(a-0;E)=\psi'(a+0;E) \tag{44b}$$
$$\psi(b-0;E)=\psi(b+0;E) \tag{44c}$$
$$\psi'(b-0;E)=\psi'(b+0;E) \tag{44d}$$
$$\psi(r;E)\sim e^{ikr}\,, \qquad r\to\infty\,, \tag{44e}$$

The solution ψ is then given by

$$\psi(r;k)=\begin{cases} a_1(k)e^{ikr}+b_1(k)e^{-ikr} & 0<r<a\\ a_2(k)e^{iQr}+b_2(k)e^{-iQr} & a<r<b\\ e^{ikr} & b<r<\infty\,,\end{cases} \tag{45}$$

where the coefficients $a(k)$ and $b(k)$ are functions of k that make ψ satisfy the boundary conditions of Eq. (44). The Wronskian of χ and ψ is

$$W(\chi,\psi)(k)=\chi(r)\psi'(r)-\chi'(r)\psi(r)=2ik\mathcal{F}_2(k)\,. \tag{46}$$

From Eqs. (43) and (46), we obtain the Green function,

$$G(r,r';k)=\frac{2m}{\hbar^2}\frac{\chi(r_<;k)\psi(r_>;k)}{2ik\mathcal{F}_2(k)}\,. \tag{47}$$

The poles of this Green function are simply the zeros of its denominator.[40] Those poles are the same as the S-matrix poles (25) and as the complex eigenvalues z_R in Eq. (38).

It is worth noting that the Green function (47) is an outgoing Green function only when $\mathrm{Re}(k)$ is positive. If $\mathrm{Re}(k)$ is negative, the Green function (47) has a purely incoming character (cf. Sec. III). Also, as k approaches the real positive k axis or as E approaches the right-hand cut from above, the Green function G becomes the *outgoing* Green function G^+; as k approaches the negative real axis (from above) or E the right-hand cut from below, G becomes the *incoming* Green function G^-.[36]

The residue of the Green function (47) at a decaying resonance wave number is given by

$$\mathrm{res}\,[G(r,r';k)]_{k=k_d} = \frac{2m}{\hbar^2}\frac{1}{2ik_d\mathcal{F}_1(k_d)\mathcal{F}_2'(k_d)}\chi^{\mathrm{decaying}}(r_<;k_d)\chi^{\mathrm{decaying}}(r_>;k_d)\,. \tag{48}$$

A similar relation between the residue of $G(r,r';k)$ at the growing wave number pole k_g and the growing Gamow eigenfunction (40) holds.

E. Complex Basis Vector Expansion

The Dirac kets are basis vectors that were used to expand a wave function φ as in Eq. (18). This expansion is known as the Dirac basis vector expansion.[30] The Gamow vectors are also basis vectors. The expansion generated by the Gamow vectors is called the complex basis vector expansion. However, the Gamow vectors do not form a complete basis. The complex basis vector expansion needs an additional set of Dirac kets corresponding to the energies that lie in the negative real axis of the second sheet of the Riemann surface.[3,16] This has been realized by other authors[7,10,11] who used the Green function approach. In this section, we shall expand a wave function in terms of the Gamow vectors (which contain the resonance contribution) and a continuous set of Dirac kets (which is interpreted as a background contribution).

[Rigorously speaking, the complex basis vector expansion is not valid for every normalizable wave function, that is, for every element of the Hilbert space, but only for those square integrable functions that are also Hardy functions.[16,18,42]]

Experimentally, we can only measure the probability for an event to take place. In a scattering experiment this is the transition probability from an in-state φ^+ to an out-state ψ^-. The amplitude of this probability is calculated from the scalar product between φ^+ and ψ^-. In terms of the quantities that we already have calculated, this amplitude is given by[3,16]

$$\left(\psi^-,\varphi^+\right) = \int_0^\infty \langle\psi^-|E^-\rangle S(E)\langle E^+|\varphi^+\rangle\, dE\,. \tag{49}$$

We now extract the resonance contribution from Eq. (49). This resonance contribution is carried by the Gamow vectors. In order to do so, we deform the contour of integration into the lower half-plane of the second sheet of the Riemann surface for the S matrix, where the resonances poles are located (see Fig. 3a). If we use the results in Ref. 42, we can write Eq. (49) as

$$(\psi^-,\varphi^+) = \int_0^{-\infty} \langle\psi^-|E^+\rangle\langle E^+|\varphi^+\rangle dE - 2\pi i \sum_{n=0}^{\infty} r_n \langle\psi^-|z_{d,n}^-\rangle\langle^+ z_{d,n}|\varphi^+\rangle, \qquad (50)$$

where $z_{d,n} = E_n - i\Gamma_n/2$ denotes the nth decaying pole and r_n denotes the residue of $S(E)$ at $z_{d,n}$. In Eq. (50), the integration is done on the negative real semiaxis of the second sheet of the Riemann surface. The series in Eq. (50) can be shown to be convergent.[42] Omitting ψ^- in (50), we obtain the complex basis vector expansion for the in-states,

$$\varphi^+ = \int_0^{-\infty} |E^+\rangle\langle^+ E|\varphi^+\rangle dE - 2\pi i \sum_{n=0}^{\infty} r_n |z_{d,n}^-\rangle\langle^+ z_{d,n}|\varphi^+\rangle\,. \qquad (51)$$

In Eq. (51), the infinite sum contains the resonances contribution, while the integral is interpreted as the background contribution.

Similarly, we obtain the complex basis vector expansion for the out-state ψ^-,[42] but now we deform the contour of integration into the upper half-plane of the second sheet of the Riemann surface, where the growing resonance poles are located (see Fig. 3b). The result is

$$\psi^- = \int_0^{-\infty} |E^-\rangle\langle E^-|\psi^-\rangle dE + 2\pi i \sum_{n=0}^{\infty} r_n^* |z_{g,n}^{*}{}^+\rangle\langle^- z_{g,n}^*|\psi^-\rangle\,, \qquad (52)$$

where $z_{g,n}^* = E_n + i\Gamma_n/2$ is the nth growing pole, and r_n^* is the residue of $S(E)$ at $z_{g,n}^*$. The integration in Eq. (52) is performed on the negative real semiaxis of the second sheet of the Riemann surface. The series in Eq. (52) has been shown to be convergent.[42]

III. TIME ASYMMETRY OF THE PURELY OUTGOING BOUNDARY CONDITION

The semigroup time evolution of the Gamow vectors expresses the time asymmetry built into them.[3,16,18] We will show here that the purely outgoing boundary condition that singles out the resonance energies also has an intrinsic time asymmetry. To be more precise, we will show that the purely outgoing boundary condition should be read as *purely outgoing* only for the decaying part of a resonance and as *purely incoming* for the growing part of the resonance. Obviously, the purely incoming boundary condition is the time reversed of the purely outgoing one. Therefore the growing Gamow vector can be viewed as the time reversed of the decaying Gamow vector.

A. Outgoing Boundary Condition in Phase

First, we study the meaning of the purely outgoing boundary condition when it is imposed on the decaying part of the resonance. The complex energy associated with the decaying part of a resonance is $z_d = E_R - i\Gamma_R/2$ ($E_R, \Gamma_R > 0$), which lies in the lower half-plane of the second sheet of the Riemann surface (see Fig. 1a). Its wave number $k_d = \mathrm{Re}(k) - i\mathrm{Im}(k)$ ($\mathrm{Re}(k)$, $\mathrm{Im}(k) > 0$) lies in the fourth quadrant of the wave number plane (see Fig. 2). The decaying Gamow vector χ^{decaying} in Eq. (39) was obtained after imposing the purely outgoing boundary condition (33f) on Eq. (30). If we had not imposed this condition, we would had obtained a solution of the form (30), and every complex number would had been an eigenvalue of the Hamiltonian. In the region $r > b$, this solution would had been the sum of two linearly independent solutions:

$$\begin{aligned}\chi^{\text{decaying}}_{\text{incoming}}(r,t) &= \mathcal{F}_2 e^{-ik_d r} e^{-iz_d t/\hbar} \\ &= \left(\mathcal{F}_2 e^{-\text{Im}(k) r - \Gamma_R t/(2\hbar)}\right) e^{-i\text{Re}(k) r - iE_R t/\hbar} \,, \qquad r > b \,, \end{aligned} \tag{53}$$

which we call the incoming decaying Gamow vector, and

$$\begin{aligned}\chi^{\text{decaying}}_{\text{outgoing}}(r,t) &= \mathcal{F}_1 e^{ik_d r} e^{-iz_d t/\hbar} \\ &= \left(\mathcal{F}_1 e^{\text{Im}(k) r - \Gamma_R t/(2\hbar)}\right) e^{i\text{Re}(k) r - iE_R t/\hbar} \,, \qquad r > b \,, \end{aligned} \tag{54}$$

which we call the outgoing decaying Gamow vector. These names come from the standard interpretation of plane waves with a complex exponent (see for instance Ref. 43): the exponential with a purely imaginary exponent (the term that carries the phase) is interpreted as the term that governs the propagation of the wave, and the exponential with the real exponent is interpreted as the term that just changes the amplitude of the wave on the surfaces of equal phase.[43] We are going to interpret Eqs. (53) and (54) in the same fashion. The terms between parentheses in Eqs. (53) and (54) determine the amplitude of the waves. The propagation of $\chi^{\text{decaying}}_{\text{outgoing}}$ is governed by $e^{i\text{Re}(k) r} e^{-iE_R t/\hbar}$, and therefore $\chi^{\text{decaying}}_{\text{outgoing}}$ is an outgoing wave (in phase). Analogously, the propagation of $\chi^{\text{decaying}}_{\text{incoming}}$ is governed by $e^{-i\text{Re}(k) r} e^{-iE_R t/\hbar}$, and thus $\chi^{\text{decaying}}_{\text{incoming}}$ is an incoming wave (in phase). Imposing the purely outgoing boundary condition $\mathcal{F}_2 = 0$ is tantamount to forbidding $\chi^{\text{decaying}}_{\text{incoming}}$. Thus for the decaying part of the resonance, the purely outgoing boundary condition allows only purely outgoing waves.

The meaning of the purely outgoing boundary condition applied to the growing part of the resonance is the opposite. The growing energy eigenvalue $z_g = E_R + i\Gamma_R/2$ (E_R, $\Gamma_R > 0$) lies in the upper half-plane of the second sheet of the Riemann surface (see Fig. 1b), and its wave number $k_g = -\text{Re}(k) - i\text{Im}(k)$ ($\text{Re}(k)$, $\text{Im}(k) > 0$) lies in the third quadrant of the wave number plane (see Fig. 2). The growing Gamow vector χ^{growing} in Eq. (40) was obtained after imposing the condition $\mathcal{F}_2 = 0$ on Eq. (30). If we had not imposed this condition, in the region $r > b$, the solution would had been the sum of two linearly independent solutions:

$$\begin{aligned}\chi^{\text{growing}}_{\text{incoming}}(r,t) &= \mathcal{F}_1 e^{ik_g r} e^{-iz_g t/\hbar} \\ &= \left(\mathcal{F}_1 e^{\text{Im}(k) r + \Gamma_R t/(2\hbar)}\right) e^{-i\text{Re}(k) r - iE_R t/\hbar} \,, \qquad r > b \,, \end{aligned} \tag{55}$$

which we call the incoming growing Gamow vector, and

$$\begin{aligned}\chi^{\text{growing}}_{\text{outgoing}}(r,t) &= \mathcal{F}_2 e^{-ik_g r} e^{-iz_g t/\hbar} \\ &= \left(\mathcal{F}_2 e^{-\text{Im}(k) r + \Gamma_R t/(2\hbar)}\right) e^{i\text{Re}(k) r - iE_R t/\hbar} \,, \qquad r > b \,, \end{aligned} \tag{56}$$

which we call the outgoing growing Gamow vector. The names also come from the standard interpretation[43] of plane waves with a complex exponent. Therefore, the purely outgoing boundary condition $\mathcal{F}_2 = 0$, when applied to the growing part of a resonance, bans $\chi^{\text{growing}}_{\text{outgoing}}$ and allows only purely incoming waves.

B. Outgoing Boundary Condition in Probability Density

In Sec. III A, we showed how the time asymmetry built into the purely outgoing boundary condition affected the phase of the Gamow vectors. In this section, we show the same time asymmetry but now consider the probability density of the Gamow vectors.

For the decaying part of the resonance, the probability densities (before imposing the purely outgoing boundary condition) are obtained by taking the square of the absolute value of Eq. (53)

$$\begin{aligned}\rho^{\rm decaying}_{\rm incoming}(r,t) = |\chi^{\rm decaying}_{\rm incoming}(r,t)|^2 &= |\mathcal{F}_2|^2 e^{-2{\rm Im}(k)r - \Gamma_R t/\hbar} \\ &= |\mathcal{F}_2|^2 e^{-\Gamma_R/\hbar (t+r/v)}\, , \qquad r > b\, , \end{aligned} \tag{57}$$

which we call the incoming decaying probability density. Similarly, the square of the absolute value of Eq. (54) yields

$$\begin{aligned}\rho^{\rm decaying}_{\rm outgoing}(r,t) = |\chi^{\rm decaying}_{\rm outgoing}(r,t)|^2 &= |\mathcal{F}_1|^2 e^{2{\rm Im}(k)r - \Gamma_R t/\hbar} \\ &= |\mathcal{F}_1|^2 e^{-\Gamma_R/\hbar (t-r/v)}\, , \qquad r > b\, , \end{aligned} \tag{58}$$

which we call the outgoing decaying probability density ($v = \Gamma_R/(2\hbar {\rm Im}(k))$). By imposing the purely outgoing boundary condition $\mathcal{F}_2 = 0$, we allow only Eq. (58) and forbid Eq. (57), which we interpret by saying that we have a purely outgoing probability density condition for the decaying part of the resonance.

For the growing part of the resonance, the probability densities (before imposing $\mathcal{F}_2 = 0$) are the square of the absolute value of Eq. (55),

$$\begin{aligned}\rho^{\rm growing}_{\rm incoming}(r,t) = |\chi^{\rm growing}_{\rm incoming}(r,t)|^2 &= |\mathcal{F}_1|^2 e^{2{\rm Im}(k)r + \Gamma_R t/\hbar} \\ &= |\mathcal{F}_1|^2 e^{\Gamma_R/\hbar (t+r/v)}\, , \qquad r > b\, , \end{aligned} \tag{59}$$

which we call the incoming growing probability density. Similarly, from Eq. (56), we have

$$\begin{aligned}\rho^{\rm growing}_{\rm outgoing}(r,t) = |\chi^{\rm growing}_{\rm outgoing}(r,t)|^2 &= |\mathcal{F}_2|^2 e^{-2{\rm Im}(k)r + \Gamma_R t/\hbar} \\ &= |\mathcal{F}_2|^2 e^{\Gamma_R/\hbar (t-r/v)}\, , \qquad r > b\, , \end{aligned} \tag{60}$$

which we call the outgoing growing probability density. For this growing part, the condition $\mathcal{F}_2 = 0$ only allows waves with purely incoming probability densities.

In short, the purely outgoing boundary condition (33f) must be read as purely outgoing (in phase or in probability density) only for the decaying part of the resonance and as purely incoming (in phase or in probability density) for the growing part of the resonance.

C. Exponential Decay Law of the Gamow Vectors

We want to determine the probability $\mathcal{P}_{\Delta r_0}(t)$ of detecting the decaying state within a shell of width Δr_0 outside the potential region ($r > b$). This is the probability that is measured by the counting rate of a detector placed, for example, outside a radioactive nucleus from which an α-particle is emitted. We assume that the detector surrounds the nucleus completely and that is at a distance $r_0 > b$ from the center $r = 0$.

Theoretically, the probability $\mathcal{P}_{\Delta r_0}(t)$ to observe an in-state φ^+ at time t within the interval Δr_0 around the surface $r = r_0$ is given by

$$\mathcal{P}_{\Delta r_0}(t) = \int d\Omega \int_{\Delta r_0} r^2\, dr |\langle r, \theta, \phi | \varphi^+(t) \rangle|^2 \, . \tag{61}$$

Experimentally, the probability of finding the decaying state particle around r_0, that is, the counting rate of the detector, is not defined for all times t: a resonance must be first prepared before the system can decay. The time at which the preparation of the resonance is finished and at which the decay starts can be chosen arbitrarily (we choose it to be 0). For example, the α particle emitted by an α-unstable nucleus travels at speed $v = \Gamma_R/(2\hbar \mathrm{Im}(k))$ and reaches the point r_0 at the time $t(r_0) = r_0/v$. For times less than $t(r_0)$, the α particle is not there yet, and therefore the counting rate measured by a detector placed at r_0 is zero for times $t < r_0/v$. Whatever would have been counted by the detector before the instant $t(r_0)$ at r_0 cannot be connected with the decaying state. Thus the theoretical probability to detect a resonance at r_0 should be zero for $t < r_0/v$. This is an instance of the time asymmetry built into a decaying process.

Experimentally as well, the decay of unstable systems usually follows the exponential law (cf. Refs. 12–15).

[Hilbert space cannot accommodate either the time asymmetry of $\mathcal{P}_{\Delta r_0}(t)$[22] or the exponential decay law.[44] To account for these two features, we should use the Rigged Hilbert Space.[3,16,18] In this formulation, the Gamow vectors have an asymmetric time evolution given by a semigroup, which accounts for the time asymmetry of a resonant process. The behavior of the semigroup evolution is in contrast to the time-symmetric Hilbert space time evolution, which is given by a group.]

We are going to show that the exponential decay law holds if we consider only the resonance (Gamow vector) contribution to the probability (61). In Sec. II E, we used the Gamow vectors as basis vectors to expand the in-state φ^+ in terms of the background and the resonance contribution (see Eq. (51)). To calculate the resonance contribution to the probability (61), we approximate φ^+ by the Gamow vector by neglecting the background term in Eq. (51),

$$\varphi^+(r, \theta, \phi) \simeq \psi^D(r, \theta, \phi) = \frac{\chi^{\text{decaying}}(r)}{r} Y_{0,0}(\theta, \phi) \, . \tag{62}$$

Thus the resonance contribution to the probability is

$$\mathcal{P}_{\Delta r_0}(t) \simeq \int d\Omega \int_{\Delta r_0} r^2 dr |\langle r, \theta, \phi | \psi^D(t) \rangle|^2 \, . \tag{63}$$

The time evolution of the Gamow vector[3,16,18] is given by

$$\psi^D(t) = e^{-iHt/\hbar} \psi^D = e^{-i(E_R t - i\Gamma_R/2)t/\hbar} \psi^D \, , \tag{64}$$

and therefore

$$\langle r, \theta, \phi | \psi^D(t) \rangle = e^{-i(E_R - i\Gamma_R/2)t/\hbar} \, \frac{\chi^{\text{decaying}}(r)}{r} Y_{0,0}(\theta, \phi) \, . \tag{65}$$

If we substitute Eq. (65) into Eq. (63), we obtain

$$\begin{aligned}
\mathcal{P}_{\Delta r_0}(t) &\simeq |e^{-\Gamma_R/(2\hbar)t}|^2 \int_{\Delta r_0} dr |\chi^{\rm decaying}(r)|^2 \\
&= e^{-\Gamma_R t/\hbar} \int_{\Delta r_0} dr |\mathcal{F}_1(k)|^2 |e^{i({\rm Re}(k) - i{\rm Im}(k))r}|^2 \\
&= e^{-\Gamma_R t/\hbar} |\mathcal{F}_1(k)|^2 \int_{r_0}^{r_0+\Delta r_0} dr e^{2{\rm Im}(k) r} \\
&= e^{-\Gamma_R t/\hbar} |\mathcal{F}_1(k)|^2 e^{2{\rm Im}(k) r_0} \, \frac{e^{2{\rm Im}(k)\Delta r_0} - 1}{2{\rm Im}(k)} \\
&\simeq e^{-\Gamma_R t/\hbar} |\mathcal{F}_1(k)|^2 e^{2{\rm Im}(k) r_0} \, \Delta r_0 \\
&= |\mathcal{F}_1(k)|^2 \Delta r_0 \, e^{-\Gamma_R/\hbar (t - r_0/v)} \, , \qquad t > r_0/v \, ,
\end{aligned} \tag{66}$$

where we have used the approximation Δr_0 small in the next to the last step. Therefore,

$$\mathcal{P}_{\Delta r_0}(t) \simeq |\mathcal{F}_1(k)|^2 \Delta r_0 e^{-\Gamma_R/\hbar (t - r_0/v)} \, , \qquad t > r_0/v \, . \tag{67}$$

Equation (67) represents the resonance contribution to the counting rate measured by a detector placed at r_0. This resonance contribution reaches its maximum at $t = r_0/v$ and decreases exponentially as time goes on. Therefore, the Gamow vector (resonance) contribution to the probability $\mathcal{P}_{\Delta r_0}(t)$ follows the exponential decay law.

IV. CONCLUSIONS

We have studied the different kinds of boundary conditions that we need to impose on the time independent Schrödinger equation in order to obtain either Dirac kets (scattering states) and the physical spectrum or Gamow vectors (resonance states) and the resonance spectrum. By imposing the boundary condition of boundedness (12f), we obtain the Dirac kets and the spectrum $[0, \infty)$. And by imposing the purely outgoing boundary condition (33f), we obtain the Gamow vectors and the resonance spectrum of Fig. 1. This purely outgoing boundary condition produces the same resonance spectrum as the S matrix and the Green function.

The Gamow vectors have been used as basis vectors in the complex basis vector expansions (51) and (52). However, they do not form a complete basis, and therefore a continuous set of Dirac kets was added to complete them. The expansions (51) and (52) extract the resonance contribution out of the normalized in- and out-states, respectively.

We have uncovered the time asymmetry that arises from the purely outgoing boundary condition. We have seen that the purely outgoing boundary condition should be read as *purely outgoing* only for the decaying part of the resonance, and as *purely incoming* for the growing part of the resonance.

The exponential law has been shown to hold if the background term of the complex basis vector expansion is neglected—only the resonance (Gamow vector) contribution to the probability is taken into consideration. These conclusions are not restricted to the example given here and are in fact applicable to a large class of potentials that includes potentials with compact support.

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APPENDIX A: HIGHER ORDER ANGULAR MOMENTUM

We extend the analysis done in Sec. II to the case in which the angular momentum $l > 0$.

1. Eigenkets

The radial part $\chi_l(r;E)$ of $\langle r,\theta,\phi|E,l,m\rangle$ satisfies Eq. (7). The solution of this equation can be written as

$$\chi_l(r;E) = \begin{cases} \alpha_{l1}\hat{h}_l^+(kr) + \beta_{l1}\hat{h}_l^-(kr) & 0<r<a \\ \alpha_{l2}\hat{h}_l^+(Qr) + \beta_{l2}\hat{h}_l^-(Qr) & a<r<b \\ \mathcal{F}_{l1}\hat{h}_l^+(kr) + \mathcal{F}_{l2}\hat{h}_l^-(kr) & b<r<\infty\,, \end{cases} \tag{A1}$$

where k and Q are given by Eqs. (10) and (11) respectively. The functions $\hat{h}_l^+(kr)$ and $\hat{h}_l^-(kr)$ are the Riccati-Hankel functions.[45] The boundary conditions that $\chi_l(r;E)$ is to satisfy are given in Eq. (12). If we write $\alpha_l = 2i\alpha_{l1}$, these boundary conditions can be written in terms of the coefficients as

$$\alpha_{l2}\hat{h}_l^+(Qa) + \beta_{l2}\hat{h}_l^-(Qa) = \alpha_l\hat{j}_l(ka) \tag{A2a}$$

$$\alpha_{l2}\hat{h}_l^{+\prime}(Qa) + \beta_{l2}\hat{h}_l^{-\prime}(Qa) = \frac{k}{Q}\alpha_l\hat{j}_l'(ka) \tag{A2b}$$

$$\mathcal{F}_{l1}\hat{h}_l^+(kb) + \mathcal{F}_{l2}\hat{h}_l^-(kb) = \alpha_{l2}\hat{h}_l^+(Qb) + \beta_{l2}\hat{h}_l^-(Qb) \tag{A2c}$$

$$\mathcal{F}_{l1}\hat{h}_l^{+\prime}(kb) + \mathcal{F}_{l2}\hat{h}_l^{-\prime}(kb) = \frac{Q}{k}(\alpha_{l2}\hat{h}_l^{+\prime}(Qb) + \beta_{l2}\hat{h}_l^{-\prime}(Qb))\,, \tag{A2d}$$

where $\hat{j}_l = \frac{1}{2i}(\hat{h}_l^+ - \hat{h}_l^-)$, $\hat{h}_l^{+\prime}(kb) = \frac{d\hat{h}_l^+(z)}{dz}|_{z=kb}$, and so on. Then

$$\alpha_{l2}(k) = \alpha_l(k)\frac{\hat{j}_l(ka)\hat{h}_l^{-\prime}(Qa) - \frac{k}{Q}\hat{j}_l'(ka)\hat{h}_l^-(Qa)}{\hat{h}_l^+(Qa)\hat{h}_l^{-\prime}(Qa) - \hat{h}_l^{+\prime}(Qa)\hat{h}_l^-(Qa)} \tag{A3a}$$

$$\beta_{l2}(k) = \alpha_l(k)\frac{\hat{j}_l(ka)\hat{h}_l^{+\prime}(Qa) - \frac{k}{Q}\hat{j}_l'(ka)\hat{h}_l^+(Qa)}{\hat{h}_l^-(Qa)\hat{h}_l^{+\prime}(Qa) - \hat{h}_l^{-\prime}(Qa)\hat{h}_l^+(Qa)} \tag{A3b}$$

$$\mathcal{F}_{l1}(k) = \frac{[\hat{h}_l^+(Qb)\hat{h}_l^{-\prime}(kb) - \frac{Q}{k}\hat{h}_l^{+\prime}(Qb)\hat{h}_l^-(kb)][\hat{j}_l(ka)\hat{h}_l^{-\prime}(Qa) - \frac{k}{Q}\hat{j}_l'(ka)\hat{h}_l^-(Qa)]}{[\hat{h}_l^+(kb)\hat{h}_l^{-\prime}(kb) - \hat{h}_l^{+\prime}(kb)\hat{h}_l^-(kb)][\hat{h}_l^+(Qa)\hat{h}_l^{-\prime}(Qa) - \hat{h}_l^{+\prime}(Qa)\hat{h}_l^-(Qa)]}\alpha_l(k)$$
$$+ \frac{[\hat{h}_l^-(Qb)\hat{h}_l^{-\prime}(kb) - \frac{Q}{k}\hat{h}_l^{-\prime}(Qb)\hat{h}_l^-(kb)][\hat{j}_l(ka)\hat{h}_l^{+\prime}(Qa) - \frac{k}{Q}\hat{j}_l'(ka)\hat{h}_l^+(Qa)]}{[\hat{h}_l^+(kb)\hat{h}_l^{-\prime}(kb) - \hat{h}_l^{+\prime}(kb)\hat{h}_l^-(kb)][\hat{h}_l^-(Qa)\hat{h}_l^{+\prime}(Qa) - \hat{h}_l^{-\prime}(Qa)\hat{h}_l^+(Qa)]}\alpha_l(k) \tag{A3c}$$

$$\mathcal{F}_{l2}(k) = \frac{[\hat{h}_l^+(Qb)\hat{h}_l^{+\prime}(kb) - \frac{Q}{k}\hat{h}_l^{+\prime}(Qb)\hat{h}_l^+(kb)][\hat{j}_l(ka)\hat{h}_l^{-\prime}(Qa) - \frac{k}{Q}\hat{j}_l'(ka)\hat{h}_l^-(Qa)]}{[\hat{h}_l^-(kb)\hat{h}_l^{+\prime}(kb) - \hat{h}_l^{-\prime}(kb)\hat{h}_l^+(kb)][\hat{h}_l^+(Qa)\hat{h}_l^{-\prime}(Qa) - \hat{h}_l^{+\prime}(Qa)\hat{h}_l^-(Qa)]}\alpha_l(k)$$
$$+ \frac{[\hat{h}_l^-(Qb)\hat{h}_l^{+\prime}(kb) - \frac{Q}{k}\hat{h}_l^{-\prime}(Qb)\hat{h}_l^+(kb)][\hat{j}_l(ka)\hat{h}_l^{+\prime}(Qa) - \frac{k}{Q}\hat{j}_l'(ka)\hat{h}_l^+(Qa)]}{[\hat{h}_l^-(kb)\hat{h}_l^{+\prime}(kb) - \hat{h}_l^{-\prime}(kb)\hat{h}_l^+(kb)][\hat{h}_l^-(Qa)\hat{h}_l^{+\prime}(Qa) - \hat{h}_l^{-\prime}(Qa)\hat{h}_l^+(Qa)]}\alpha_l(k)\,. \tag{A3d}$$

The lth radial ket reads up to a normalization factor as

$$\chi_l(r;E) = \begin{cases} \alpha_l(k)\hat{j}_l(kr) & 0<r<a \\ \alpha_{l2}(k)\hat{h}_l^+(Qr) + \beta_{l2}(k)\hat{h}_l^-(Qr) & a<r<b \\ \mathcal{F}_{l1}(k)\hat{h}_l^+(kr) + \mathcal{F}_{l2}(k)\hat{h}_l^-(kr) & b<r<\infty\,. \end{cases} \tag{A4}$$

2. S-Matrix Approach

Now we calculate the lth partial S matrix and its poles. Because

$$\hat{h}_l^\pm \sim e^{\pm i(kr - l\pi/2)}\,, \qquad r\to\infty\,, \tag{A5}$$

the function $\mathcal{F}_{l1}\hat{h}_l^+$ may be interpreted as an outgoing wave with amplitude $\mathcal{F}_{l1}$ and $\mathcal{F}_{l2}\hat{h}_l^-$ may be interpreted as an incoming wave with amplitude $\mathcal{F}_{l2}$. Thus the expression for the lth partial S matrix in the energy representation is

$$S_l(k) = -\frac{\mathcal{F}_{l1}(k)}{\mathcal{F}_{l2}(k)}\,. \tag{A6}$$

The resonances are associated with the poles of the analytic continuation of Eq. (A6). The functions $\mathcal{F}_{l1}$ and $\mathcal{F}_{l2}$ are analytic, and therefore the poles of the S_l matrix (A6) are the zeros of its denominator.[40] Then, the condition $\mathcal{F}_{l2}(k)=0$ provides the lth resonance energies,

$$[\hat{h}_l^+(Qb)\hat{h}_l^{+\prime}(kb) - \frac{Q}{k}\hat{h}_l^{+\prime}(Qb)\hat{h}_l^+(kb)][\hat{j}_l(ka)\hat{h}_l^{-\prime}(Qa) - \frac{k}{Q}\hat{j}_l'(ka)\hat{h}_l^-(Qa)]$$
$$- [\hat{h}_l^-(Qb)\hat{h}_l^{+\prime}(kb) - \frac{Q}{k}\hat{h}_l^{-\prime}(Qb)\hat{h}_l^+(kb)][\hat{j}_l(ka)\hat{h}_l^{+\prime}(Qa) - \frac{k}{Q}\hat{j}_l'(ka)\hat{h}_l^+(Qa)] = 0\,. \tag{A7}$$

3. Gamow Vector Approach

As we did for $l=0$, we shall prove that the S-matrix poles are the same as the complex eigenvalues obtained from the purely outgoing boundary condition.

The Gamow vector in the position representation satisfies

$$\left(\frac{-\hbar^2}{2m}\frac{1}{r}\frac{\partial^2}{\partial r^2}r+\frac{\hbar^2 l(l+1)}{2mr^2}+V(r)\right)\langle r,\theta,\phi|z_R,l,m\rangle = z_R\langle r,\theta,\phi|z_R,l,m\rangle\,. \tag{A8}$$

If we write $\langle r,\theta,\phi|z_R,l,m\rangle := \frac{1}{r}\chi_l(r;z_R)Y_{l,m}(\theta,\phi)$, then the radial part fulfills

$$\left(\frac{-\hbar^2}{2m}\frac{d^2}{dr^2}+\frac{\hbar^2 l(l+1)}{2mr^2}+V(r)\right)\chi_l(r;z_R) = z_R\chi_l(r;z_R)\,. \tag{A9}$$

The general solution of (A9) is

$$\chi_l(r;z_R)=\begin{cases}\alpha_{l1}\hat{h}_l^+(kr)+\beta_{l1}\hat{h}_l^-(kr) & 0<r<a\\ \alpha_{l2}\hat{h}_l^+(Qr)+\beta_{l2}\hat{h}_l^-(Qr) & a<r<b\\ \mathcal{F}_{l1}\hat{h}_l^+(kr)+\mathcal{F}_{l2}\hat{h}_l^-(kr) & b<r<\infty\,,\end{cases} \tag{A10}$$

where k is the complex wave number (31) and Q is given by Eq. (32). The boundary conditions that the Gamow vectors satisfy are equivalent to the case $l=0$. Now $\hat{h}_l^+$ plays the role of outgoing wave and $\hat{h}_l^-$ that of the incoming wave,

$$\chi_l(0;z_R)=0 \tag{A11a}$$
$$\chi_l(a-;z_R)=\chi_l(a+;z_R) \tag{A11b}$$
$$\chi_l'(a-;z_R)=\chi_l'(a+;z_R) \tag{A11c}$$
$$\chi_l(b-;z_R)=\chi_l(b+;z_R) \tag{A11d}$$
$$\chi_l'(b-;z_R)=\chi_l'(b+;z_R) \tag{A11e}$$
$$\chi_l(r;z_R)\sim\hat{h}_l^+(kr)\,,\qquad r\to\infty\,. \tag{A11f}$$

If we define $\alpha_l=2i\alpha_{l1}$, Eq. (A11) can be written in terms of the coefficients of $\chi_l(r;z_R)$ as

$$\alpha_{l2}\hat{h}_l^+(Qa)+\beta_{l2}\hat{h}_l^-(Qa)=\alpha_l\hat{j}_l(ka) \tag{A12a}$$
$$\alpha_{l2}\hat{h}_l^{+\prime}(Qa)+\beta_{l2}\hat{h}_l^{-\prime}(Qa)=\frac{k}{Q}\alpha_l\hat{j}_l'(ka) \tag{A12b}$$
$$\mathcal{F}_{l1}\hat{h}_l^+(kb)=\alpha_{l2}\hat{h}_l^+(Qb)+\beta_{l2}\hat{h}_l^-(Qb) \tag{A12c}$$
$$\mathcal{F}_{l1}\hat{h}_l^{+\prime}(kb)=\frac{Q}{k}(\alpha_{l2}\hat{h}_l^{+\prime}(Qb)+\beta_{l2}\hat{h}_l^{-\prime}(Qb))\,. \tag{A12d}$$

In the matrix representation Eq. (A12) looks like

$$\begin{pmatrix}-\hat{j}_l(ka) & 0 & \hat{h}_l^+(Qa) & \hat{h}_l^-(Qa)\\ \frac{-k}{Q}\hat{j}_l'(ka) & 0 & \hat{h}_l^{+\prime}(Qa) & \hat{h}_l^{-\prime}(Qa)\\ 0 & -\hat{h}_l^+(kb) & \hat{h}_l^+(Qb) & \hat{h}_l^-(Qb)\\ 0 & -\hat{h}_l^{+\prime}(kb) & \frac{Q}{k}\hat{h}_l^{+\prime}(Qb) & \frac{Q}{k}\hat{h}_l^{-\prime}(Qb)\end{pmatrix}\begin{pmatrix}\alpha_l\\ \mathcal{F}_{l1}\\ \alpha_{l2}\\ \beta_{l2}\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\,. \tag{A13}$$

This system has non-trivial solutions iff the determinant of the coefficients is equal to zero,

$$\begin{vmatrix} -\hat{j}_l(ka) & 0 & \hat{h}_l^+(Qa) & \hat{h}_l^-(Qa) \\ \frac{-k}{Q}\hat{j}_l'(ka) & 0 & \hat{h}_l^{+\prime}(Qa) & \hat{h}_l^{-\prime}(Qa) \\ 0 & -\hat{h}_l^+(kb) & \hat{h}_l^+(Qb) & \hat{h}_l^-(Qb) \\ 0 & -\hat{h}_l^{+\prime}(kb) & \frac{Q}{k}\hat{h}_l^{+\prime}(Qb) & \frac{Q}{k}\hat{h}_l^{-\prime}(Qb) \end{vmatrix} = 0\,. \tag{A14}$$

Straightforward calculations then lead to

$$\begin{aligned} &[\hat{h}_l^+(Qb)\hat{h}_l^{+\prime}(kb) - \frac{Q}{k}\hat{h}_l^{+\prime}(Qb)\hat{h}_l^+(kb)][\hat{j}_l(ka)\hat{h}_l^{-\prime}(Qa) - \frac{k}{Q}\hat{j}_l'(ka)\hat{h}_l^-(Qa)] \\ &- [\hat{h}_l^-(Qb)\hat{h}_l^{+\prime}(kb) - \frac{Q}{k}\hat{h}_l^{-\prime}(Qb)\hat{h}_l^+(kb)][\hat{j}_l(ka)\hat{h}_l^{+\prime}(Qa) - \frac{k}{Q}\hat{j}_l'(ka)\hat{h}_l^+(Qa)] = 0\,. \end{aligned} \tag{A15}$$

Equation (A15) is just Eq. (A7), which was found in the previous section using the S matrix approach.

4. Green Function Approach

The lth radial Green function satisfies

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2} + \frac{\hbar^2 l(l+1)}{2mr^2} + V(r) - E\right) G_l(r,r';E) = -\delta(r-r')\,. \tag{A16}$$

It expression is given by

$$G_l(r,r';E) = \frac{2m}{\hbar^2}\,\frac{\chi_l(r_<;E)\psi_l(r_>;E)}{W(\chi_l,\psi_l)}\,. \tag{A17}$$

The function χ_l is the solution of the time independent Schrödinger equation (7) that vanishes at the origin. Therefore, $\chi_l(r;E)$ is given by Eq. (A4). The function ψ_l also satisfies Eq. (7), but with the boundary conditions

$$\psi_l(a-) = \psi_l(a+) \tag{A18a}$$

$$\psi_l'(a-) = \psi_l'(a+) \tag{A18b}$$

$$\psi_l(b-) = \psi_l(b+) \tag{A18c}$$

$$\psi_l'(b-) = \psi_l'(b+) \tag{A18d}$$

$$\psi_l(r) \sim \hat{h}_l^+(kr)\,, \qquad r\to\infty\,. \tag{A18e}$$

Then

$$\psi_l(r;k) = \begin{cases} a_{l1}(k)\hat{h}_l^+(kr) + b_{l1}(k)\hat{h}_l^-(kr) & 0<r<a \\ a_{l2}(k)\hat{h}_l^+(Qr) + b_{l2}(k)\hat{h}_l^+(Qr) & a<r<b \\ \hat{h}_l^+(kr) & b<r<\infty\,, \end{cases} \tag{A19}$$

where the a and b coefficients express the continuity conditions in Eq. (A18). The lth Green function is given by

$$G_l(r, r'; k) = \frac{2m}{\hbar^2} \frac{\chi_l(r_<; k)\, \psi_l(r_>; k)}{2ik\mathcal{F}_{2l}(k)}\,. \tag{A20}$$

The poles of Eq. (A20) are the zeros of its denominator.[40] This is the same condition as was found in the S-matrix approach and leads to the same result as the Gamow vector approach.

REFERENCES

[1] D. E. Groom *et al.*, "Review of particle physics," Eur. Phys. J. C**15**, 1–878 (2000).

[2] V. E. Weisskopf and E. P. Wigner, "Berechnung der natürlichen Linienbreite auf Grund der Diracsichen Licht theory," Z. Phys. **63**, 54–73 (1930); "Uber die natürlichen Linienbreite in der Strahlung des harmonischen Oszillators," Z. Phys. **65**, 18–27 (1930).

[3] A. Bohm, *Quantum Mechanics: Foundations and Applications* (Springer-Verlag, New York, 1986).

[4] G. Gamow, "Zur Quantentheorie de Atomkernes," Z. Phys. **51**, 204–212 (1928).

[5] A. F. J. Siegert, "On the derivation of the dispersion formula for nuclear reactions," Phys. Rev. **56**, 750–752 (1939).

[6] R. E. Peierls, "Complex eigenvalues in scattering theory," Proc. R. Soc. London, Ser. A **253**, 16–36 (1959).

[7] E. Hernández and A. Mondragón, "Resonant states in momentum representation," Phys. Rev. C **29**, 722–738 (1984); A. Mondragón, E. Hernández, and J. M. Velázquez Arcos, "Resonances and Gamow states in non-local potentials," Ann. Phys. (Leipzig) **48**, 503–517 (1991).

[8] M. Baldo, L. S. Ferreira, and L. Streit, "Eigenvalue problem for Gamow vectors and a separable approximation for the N-N interaction," Phys. Rev. C **36**, 1743–1746 (1987). L. S. Ferreira, E. Maglione, R. J. Liotta, "Nucleon resonances in deformed nuclei," Phys. Rev. Lett. **78**, 1640–1643 (1997).

[9] C. G. Bollini, O. Civitarese, A. L. De Paoli, and M. C. Rocca, "Gamow states as continuous linear functionals over analytical test functions," J. Math. Phys. **37**, 4235–4242 (1996).

[10] G. García-Calderón and R. Peierls, "Resonant states and their uses," Nucl. Phys. A **265**, 443–460 (1976).

[11] T. Berggren, "On the use of resonant states in eigenfunction expansions of scattering and reaction amplitudes," Nucl. Phys. A **109**, 265–287 (1968).

[12] A. A. Rollesfon and R. M. Prior, "An advanced undergraduate nuclear lifetime experiment," Am. J. Phys. **46**, 1007–1008 (1978).

[13] F. W. Supon and J. J. Kraushaar, "Radioactive half-life measurements in a freshman or sophomore laboratory," Am. J. Phys. **51**, 761–763 (1983).

[14] D. Desmarais and J. L. Duggan, "The study of alpha-particle decay schemes of heavy nuclei," Am. J. Phys. **58**, 1079–1085 (1990).

[15] K. Ruddick, "Determination of the half-life of ^{212}Po," Am. J. Phys. **63**, 658–660 (1995).

[16] R. de la Madrid, *Quantum mechanics in rigged Hilbert space language*, Ph.D. thesis, Universidad de Valladolid, Valladolid, 2001. Available at `http://www.isi.it/~rafa/`.

[17] J.-P. Antoine, "Quantum mechanics beyond Hilbert space," in *Irreversibility and Causality*, A. Bohm, H.-D. Doebner, and P. Kielanowski, eds. (Springer-Verlag, Berlin, 1998), pp. 3–33.

[18] A. Bohm and M. Gadella, *Dirac kets, Gamow Vectors, and Gelfand Triplets*, Springer Lectures Notes in Physics Vol. 348 (Springer, Berlin, 1989).

[19] J.-P. Antoine, "Dirac formalism and symmetry problems in quantum mechanics. I. General Dirac formalism, II. Dirac formalism and symmetry problems in quantum mechanics," J. Math. Phys. **10**, 53–69, 2276–2290 (1969).

[20] A. Bohm, "The Rigged Hilbert Space in Quantum Mechanics," *Boulder Lectures in Theoretical Physics, 1966*, Vol. 9A (Gordon and Breach, New York, 1967).

[21] J. E. Roberts, "The Dirac bra and ket formalism," J. Math. Phys. **7**, 1097–1104 (1966); ibid., "Rigged Hilbert spaces in quantum mechanics," Commun. Math. Phys. **3**, 98–119 (1966).
[22] A. Bohm, M. Gadella, and G. B. Mainland, "Gamow vectors and decaying states," Am. J. Phys. **57**, 1103–1108 (1989).
[23] B. R. Holstein, "Understanding alpha decay," Am. J. Phys. **64**, 1061–1071 (1996).
[24] M. G. Fuda, "Time-dependent theory of alpha decay," Am. J. Phys. **52**, 838–842 (1984).
[25] B. R. Holstein, "Bound states, virtual states, and nonexponential decay via path integrals," Am. J. Phys. **51**, 897–901 (1983).
[26] H. Massmann, "Illustration of resonances and the law of exponential decay in a simple quantum-mechanical problem," Am. J. Phys. **53**, 679–683 (1985).
[27] H. Jakobovits, Y. Rothschild, and J. Levitan, "The approximation to the exponential decay law," Am. J. Phys. **63**, 439–443 (1995).
[28] D. S. Onley and A. Kumar, "Time dependence in quantum mechanics study of a simple decaying system," Am. J. Phys. **60**, 432–439 (1992).
[29] J. von Neumann, *Mathematische Grundlagen der Quantertheorie* (Springer, Berlin, 1931); English translation *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, 1955).
[30] P. A. M. Dirac, *The Principles of Quantum Mechanics* (Clarendon Press, Oxford, 1958).
[31] I. M. Gelfand and N. Y. Vilenkin, *Generalized Functions*, Vol. IV (Academic Press, New York, 1964); K. Maurin, *Generalized Eigenfunction Expansions and Unitary Representations of Topological Groups* (Polish Scientific Publishers, Warsaw, 1968).
[32] R. de la Madrid, A. Bohm, and M. Gadella, "Rigged Hilbert space treatment of continuous spectrum," Fortsch. Phys. **50**, 185–216 (2002); quant-ph/0109154.
[33] R. de la Madrid, "Rigged Hilbert space approach to the Schrödinger equation," J. Phys. A: Math. Gen. **35**, 319–342 (2002); quant-ph/0110165.
[34] C. Cohen-Tannoudji, B. Diu, and F. Laloë, *Quantum Mechanics* (Wiley, New York, 1977), Vol. II, Chap. VIII.
[35] N. Dunford and J. Schwartz, *Linear Operators* (Interscience Publishers, New York, 1963), Vol. II.
[36] R. G. Newton, *Scattering Theory of Waves and Particles* (McGraw-Hill, New York, 1966); ibid. (Springer-Verlag, New York, 1982), 2nd ed.
[37] J. R. Taylor, *Scattering Theory* (Wiley, New York, 1972).
[38] V. I. Kukulin, V. M. Krasnopol'sky, and J. Horacek, *Theory of Resonances* (Kluwer Academic Publishers, Dordrecht, 1989).
[39] H. M. Nussenzveig, *Causality and Dispersion Relations* (Academic Press, New York, 1972).
[40] J. E. Marsden, *Basic Complex Analysis* (Freeman, San Francisco, 1975).
[41] R. de la Madrid, "Formal and precise derivation of the Green functions for a simple potential," Chaos, Solitons and Fractals **12**, 2689–2695 (2001); quant-ph/0107096.
[42] M. Gadella, "Derivation of Gamow vectors for resonances in cut-off potentials," Lett. Math. Phys. **41**, 279–290 (1997).
[43] M. Born and E. Wolf, *Principles of Optics: Electromagnetic Theory of Propagation, Interference and Diffraction of Light* (Pergamon Press, 1980); J. Jackson, *Classical Electrodynamics* (John Wiley & Sons, New York, 1962), 2nd ed.
[44] L. A. Khalfin, "Contribution to the decay theory of a quasi-stationary state,"

Sov. Phys. JETP **6**, 1053–1063 (1958).
[45] G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, Cambridge, 1958).

FIGURES

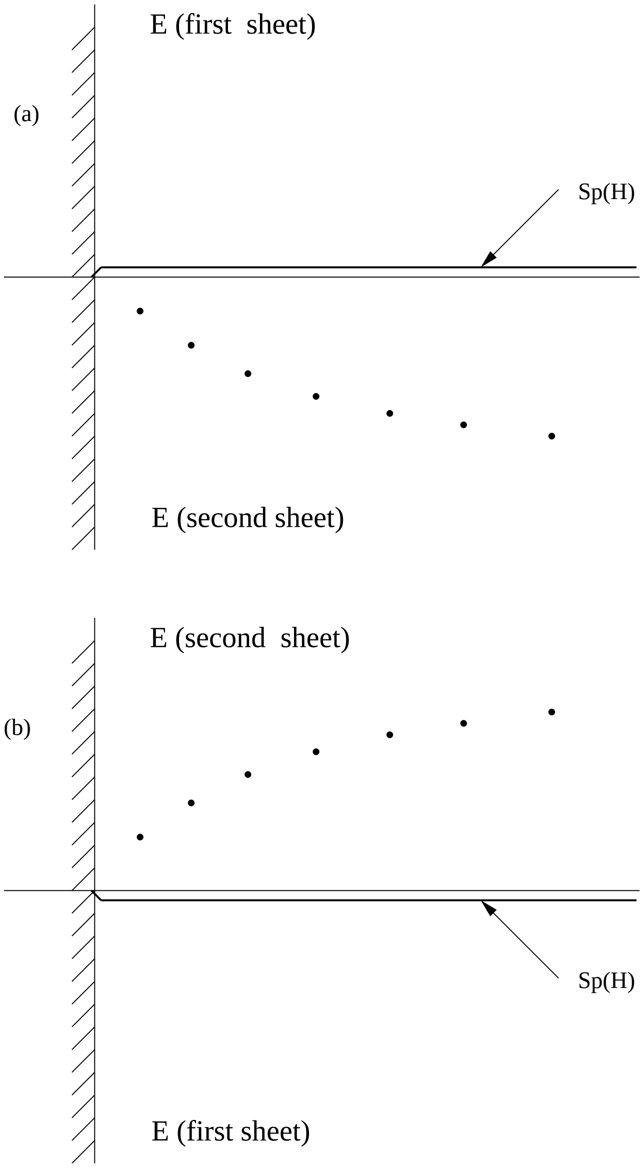

FIG. 1. The resonance energies of the square barrier potential.

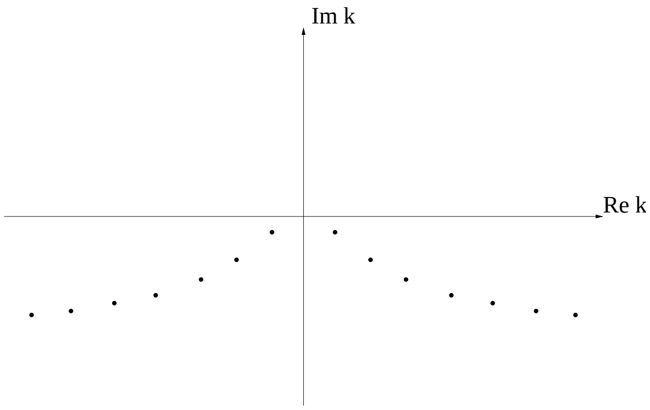

FIG. 2. The resonance wave numbers of the square barrier potential.

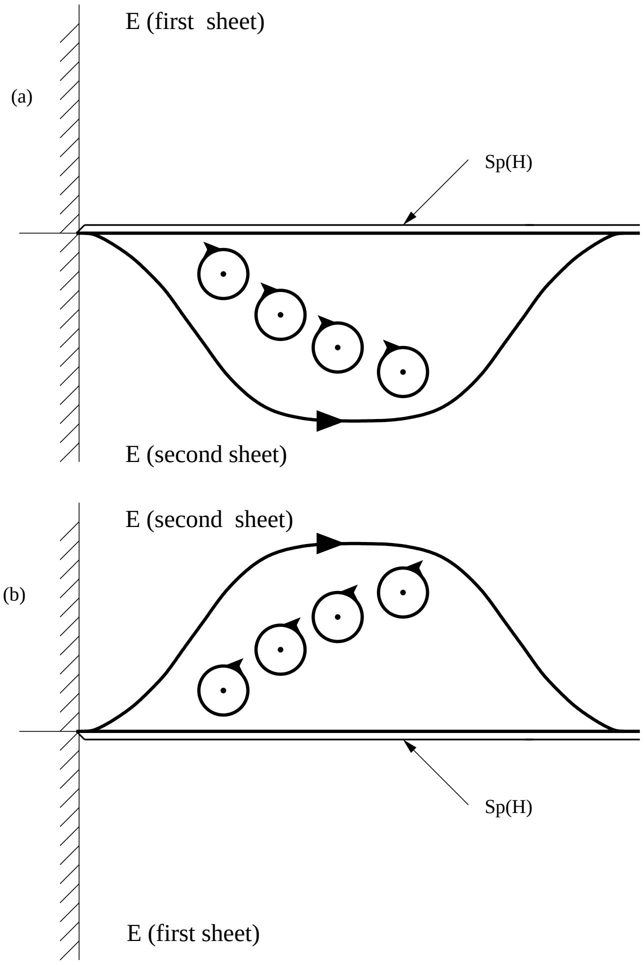

FIG. 3. Deformation of the path of integration into the second sheet of the energy Riemann surface; (a) for the decaying states and (b) for the growing states.