Quasimodes with Exponentially Small Errors Associated with Elliptic Periodic Rays

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Abstract

The aim of this paper is to construct compactly supported Gevrey quasimodes with exponentially small discrepancy for the Laplace operator with Dirichlet boundary conditions in a domain X with analytic boundary. The quasimodes are associated with a non-degenerate elliptic closed broken geodesic γ in X. We find a Cantor family Λ of invariant tori of the corresponding Poincaré map which is Gevrey smooth with respect to the transversal variables (the frequencies). Quantizing the Gevrey family Λ, we construct quasimodes with exponentially small discrepancy. As a consequence, we obtain a large amount of resonances exponentially close to the real axis for suitable compact obstacles.

This is a new version of mparc 02-181, where the beginning of the proof of Proposition 6.1 is revised.

1 Introduction

The motivation for this paper is the modified Lax-Phillips conjecture about the existence of resonances near the real axis for trapping obstacles. It is known by a result of Stefanov [13] that if there exists a compactly supported C^∞-quasimode for the corresponding exterior problem, then the counting function $N(r)$ of resonances of modulus less or equal r lying sufficiently "close" to the real axis can be estimated from below by the counting function of the quasimode. A sharp lower bound $N(r) \geq C(1 + r^n)$ has been obtained in this way in [13] for the so called Helmholtz resonator in $\mathbf{R}^n$, $n \geq 2$, with a smooth compact boundary, where $N(r)$ counts with multiplicities all the resonances λ of modulus $\leq r$ such that $0 < \operatorname{Im}\lambda \leq C_N|\lambda|^{-N}$ for each $N \in \mathbf{N}$ and some $C_N > 0$. The aim of this paper is to construct Gevrey quasimodes, that is quasimodes with exponentially small

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discrepancy, for the Helmholtz resonator with analytic boundary. As an consequence we localize resonances exponentially close to the real axis and we obtain a sharp lower bound for their counting function.

Consider the Laplace operator Δ_D with Dirichlet boundary conditions in a domain $X \subset \mathbf{R}^n$, $n \geq 2$, with a smooth compact boundary Γ. We are going to construct quasimodes for Δ_D with exponentially small discrepancy. They will be associated to appropriate periodic broken geodesics γ in X. We impose the following two conditions:

(H$_1$) γ is a broken closed elliptic geodesic and the corresponding Poincaré map is N-elementary, $N \geq 4$, and non-degenerate.

(H$_2$) Γ is analytic in a neighborhood of the vertices of γ.

The precise meaning of (H$_1$) will be explained in Sect. 3. We note that (H$_1$) is stable under small C^∞ perturbations of the boundary.

We define a Gevrey quasimode for the Laplace operator Δ_D in X with Dirichlet boundary conditions as follows: Given $\varrho > 1$, we say that $\mathcal{Q} = \{(u_\nu, \lambda_\nu), \nu \in \mathcal{M}\}$, $\mathcal{M} \subset \mathbf{Z}^n$ being an unbounded index set, is a *Gevrey G^ϱ-quasimode* of Δ_D if $u_\nu \in C_0^\infty(\overline{X})$, $\lambda_\nu > 0$, $\lim_{\nu\to\infty} \lambda_\nu \;\to\; +\;\infty$ and

$$\begin{cases} ||(\Delta + \lambda_\nu^2) u_\nu||_{L^2(X)} &= O(e^{-c\lambda_\nu^{1/\varrho}}) \quad \text{in} \quad X \\ u_\nu|_\Gamma &= 0 \\ \langle u_\nu, u_{\nu'} \rangle_{L^2(X)} &= \delta_{\nu,\nu'}, \;\; \nu, \nu' \in \mathcal{M} \end{cases} \tag{1.1}$$

for some constant $c > 0$, where $\delta_{\nu,\nu'} = 0$ if $\nu \neq \nu'$ and $\delta_{\nu,\nu} = 1$. We define the support of $\mathcal{Q}$ by $\operatorname{supp}\mathcal{Q} = \cap_{p=1}^\infty \overline{\cup_{|\nu|\geq p} \operatorname{supp} u_\nu}$. Fix $\varrho > 2n+1$ if $n \geq 3$ and $\varrho > 11/2$ if $n = 2$. Our main result is:

Theorem 1 *Let (H$_1$) and (H$_2$) be fulfilled. Then for each compact neighborhood K of γ in $\overline{X}$ there exists a Gevrey G^ϱ-quasimode $\mathcal{Q} = \{(u_\nu, \lambda_\nu), \nu \in \mathcal{M}\}$ for Δ_D with support in K. Moreover,*

$$\#\{\nu \in \mathcal{M} : \lambda_\nu \leq \lambda\} \;=\; V\lambda^n \,+\, o(\lambda^n)\,, \quad \lambda \to +\infty, \tag{1.2}$$

where V is a positive constant.

The quasimodes we are going to construct are associated with a Cantor family Λ of Kolmogorov-Arnold-Moser (KAM) invariant tori of the Poincaré map P of γ. We are going to find λ_ν by means of the quantization relations (5.2) and (5.12) involving "Birkhoff invariants" on Λ of the "quantum Poincaré map" associated with γ. The constant V in (1.2) is given by the volume of the "flow-out" of Λ by the billiard flow in $S^*X := \{(x,\xi) \in T^*X : |\xi| = 1\}$ (see (5.9)). Quasimodes satisfying (1.2) and having discrepancy of the form $O_N(\lambda_\nu^{-N})$ for each $N > 0$ have been constructed by Lazutkin (see [6] and the references there) and Colin de Verdière [4], see also [3] and [9].

As a consequence of Theorem 1 (see [6] and [10]), we can localize eigenvalues of Δ_D in each interval $[\lambda_\nu^2 - \exp(-\frac{c}{2}|\lambda_\nu|^{1/\varrho}), \lambda_\nu^2 + \exp(-\frac{c}{2}|\lambda_\nu|^{1/\varrho})$, $|\nu| \gg 1$, provided that X is bounded. Suppose now that $X = \mathbf{R}^n \setminus \mathcal{O}$, where $\mathcal{O}$ is a bounded open set with analytic boundary. Combining Theorem 1 with a result of Stefanov [13], we obtain a sharp lower bound of the counting function of resonances (counted with multiplicities) of Δ_D in the upper half plane which are exponentially close to the real axis. Recall that the multiplicity of each resonance $0 \neq \lambda \in \mathrm{Res}(\Delta_D)$ is given by $m(\lambda) = \mathrm{rank} \int_{|z-\lambda|=\varepsilon} zR_\chi(z)dz$, $0 < \varepsilon \ll 1$, where $R_\chi(z)$ is the meromorphic continuation of $\chi(\Delta_D - z^2)^{-1}\chi$, from $\mathrm{Im}\,\lambda < 0$ to the whole complex plane $\mathbf{C}$ if n is odd, and to the logarithmic Riemann surface if n is even, $\chi \in C_0^\infty(\mathbf{R}^n)$, $\chi = 1$ in a neighborhood of $\mathcal{O}$, and $m(\lambda)$ does not depend on χ. By a result of Burq [2], there exist positive constants c_0 and C_1 such that all the resonances λ of Δ_D satisfy $\mathrm{Im}\,\lambda \geq C_1 e^{-c_0|\lambda|}$. Consider the sequence of resonances $\mathbf{N} \ni j \to z_j \in \mathrm{Res}(\Delta_D)$ of Δ_D counted with multiplicities (each $z \in \mathrm{Res}(\Delta_D)$ appears exactly $m(z)$ times successively in the sequences $\{z_j\}_{j\in\mathbf{N}}$). Fix $0 < c_1 < c/2$, where $c > 0$ is introduced in (1.1). Using Corollary 1 and Remark 5 in [13], we localize resonances near λ_ν and obtain a sharp lower bound of their counting function.

Corollary 1 *Let $X = \mathbf{R}^n\backslash\mathcal{O}$, where $\mathcal{O}$ is a compact obstacle with analytic boundary. Suppose that (H_1) holds. Then there exists an one-to-one (injective) maping $\mathcal{M} \ni \nu \to j_\nu \in \mathbf{N}$ such that*

$$C_1'\, e^{-c_0\lambda_\nu} < \mathrm{Im}\, z_{j_\nu} \leq C_2\, \lambda_\nu^{n+2} e^{-c\lambda_\nu^{1/\varrho}} \ , \quad |\mathrm{Re}\, z_{j_\nu} - \lambda_\nu| \leq C_2\, e^{-c_1\lambda_\nu^{1/\varrho}} \ ,$$

for each $\nu \in \mathcal{M}$, where $C_1' > 0$ and $C_2 \gg 1$. Moreover, the counting function of these resonances satisfies $\#\{\nu \in \mathcal{M} : 0 < \mathrm{Re}\, z_{j_\nu} \leq r\} = V r^n + o(r^n)$ as $r \to +\infty$, where V is the constant in Theorem 1.

We point out that the statement of Corollary 1 is stable under small analytic perturbations of the boundary. Moreover, Theorem 1 and Corollary 1 hold also if we replace the Euclidean metric with a smooth Riemannian metric which is analytic in a neighborhood of the closed geodesic γ. We stress also that we give in Sect. 5.1 a procedure of finding λ_ν from the quantization relations (5.2) and (5.12).

Resonances exponentially close to the real axis have been found for the reduced wave equation in $\mathbf{R}^n$ in the case of spherically symmetric media by Ralston [11] and recently by Vodev [15]. Such resonances were obtained by Vodev [14] for the Neumann problem in linear elasticity for obstacles with analytic boundaries. The reason for the existence of resonances exponentially close to the real axis in that case is that Rayleigh waves are supported in the elliptic region of the boundary. Semi-classical resonances for Schrödinger type operators exponentially close to the real axis and associated with Gevrey families of KAM tori have been found in [10]. Here we extend some of the techniques developed in [10] to the case of discrete Hamiltonian systems such as the Poincaré and the quantum Poincaré maps associated to γ. We would like to note that in general KAM results are easier to state for discrete Hamiltonian systems than for Hamiltonians but more difficult to prove.

The article is organized as follows: In Sect. 2 we reduce the problem to a spectral problem at the boundary Γ following a standard procedure in geometric optics. Consider the periodic broken geodesic $\gamma : [0,T] \to X$ with vertices $\gamma_j = \gamma(t_j) \in \Gamma$, $0 = t_1 < t_2 < \cdots < t_m < t_{m+1} = T$. Recall that $\gamma(t)$ moves on a straight line with unit velocity in each interval (t_j, t_{j+1}), $1 \leq j \leq m$, and it reflects at the boundary Γ of X by the usual law of geometric optics. Denote by $\widetilde{\gamma}(t) = (\gamma(t), \xi(t))$, $t \neq t_j$, the corresponding broken bicharacteristic in $S^*X = \{(x,\xi) \in T^*X : |\xi| = 1\}$, and by $\rho_j = (\gamma_j, \eta_j)$, $\rho_{m+1} = \rho_1$, its "vertices" in $B^*\Gamma = \{(y,\eta) \in T^*\Gamma : |\eta| \le 1\}$, where $\eta_j = \xi(t_j - 0)|_{T_{\gamma_j}\Gamma} = \xi(t_j + 0)|_{T_{\gamma_j}\Gamma}$. We fix $\sigma > 1$ such that $\sigma - 1 \ll 1$, and consider an outgoing parametrix of the reduced wave equation $(\Delta + \lambda^2)U_1(\lambda) = O(e^{-c|\lambda|^{1/\sigma}})$ in X with initial data in a small neighborhood Γ_1 of γ_1 such that the restriction $U_1(\lambda)|_{\Gamma_1}$ is a pseudodifferential operator $\Psi_1(\lambda)$ with a large parameter λ (λ-PDO) with a Gevrey symbol of class S^σ (see Appendix), the G^σ-frequency set of which, $\mathrm{WF}^\sigma(\Psi_1(\lambda))$, is contained in a small neighborhood of $(\gamma_1, \eta_1) \in B^*\Gamma$. Moreover, we suppose that $\Psi_1(\lambda)$ coincides with the identity mapping microlocally near (γ_1, η_1). The operator $U_1(\lambda)$ is a Fourier integral operator with a large parameter λ (λ-FIO) which can be represented by oscillatory integrals with analytic phase functions and amplitudes in S^σ. When reaching the boundary again in a small neighborhood Γ_2 of γ_2, we take an outgoing parametrix $U_2(\lambda)$ in X such that $(U_1(\lambda) + U_2(\lambda))|_{\Gamma_2} = 0$. Repeating this procedure m times we come back to Γ_1. Then $U(\lambda) := U_1(\lambda) + \cdots + U_m(\lambda)$ satisfies $(\Delta + \lambda^2)U(\lambda) = O(e^{-c|\lambda|^{1/\sigma}})$ in X and by construction $U(\lambda)_\Gamma = \Psi_1(\lambda) - \bar{M}(\lambda) + O(e^{-c|\lambda|^{1/\varrho}})$, where $\bar{M}(\lambda)$ is a λ-FIO. The corresponding canonical relation is just the graph of the Poincaré map associated to $\tilde{\gamma}$ and it can be represented as an oscillatory integral with an analytic phase function and amplitude in S^σ. The operator $\bar{M}(\lambda)$ is often referred to as a quantum Poincaré map associated to γ. To obtain (u_ν, λ_ν) in (1.1) it will be sufficient to solve the spectral problem $\bar{M}(\lambda_\nu)u_\nu = u_\nu$ modulo $O(e^{-c|\lambda|^{1/\varrho}})u_\nu$, where the G^ϱ-frequency set of $\{(u_\nu, \lambda_\nu)\}$ is contained in a small neighborhood of ρ_1. To do this we shall obtain a "normal" form of $\bar{M}(\lambda)$, in other words, we shall separate microlocally the variables near a large invariant set of the Poincaré map.

In Sect. 3 we obtain a large Cantor family of *analytic* KAM tori of the Poincaré map P which is *Gevrey smooth* with respect to the "transversal variables" (the frequencies) $\omega \in \Omega_\kappa$ in the sense of Whitney. The Cantor set $\Omega_\kappa \subset \mathbf{R}^{n-1}$ is given by the Diophantine condition (3.5), where the exponent $\tau > n-1$ and the small constant $\kappa > 0$ are fixed, and it has a positive Lebesgue measure. Moreover, for each $\tau' > \max(\tau, 5/2)$, we obtain a symplectic normal form of P near the family of KAM tori in the Gevrey class $G^{\tau'+2}$. In other words, we find polar symplectic coordinates $(\varphi, I) \in \mathbf{T}^{n-1} \times D$, $\mathbf{T} := \mathbf{R}/2\pi\mathbf{Z}$, $D \subset \mathbf{R}^{n-1}_+$, an open set, and a Gevrey function $K \in G^{\tau'+2}(D)$ such that

$$P(\varphi, I) \ = \ (\varphi + \nabla K(I), I) + R_0(\varphi, I),$$

where $R_0 \in G^{1,\tau'+2}(\mathbf{T}^{n-1} \times D)$ (analytic with respect to φ and $G^{\tau'+2}$ with respect to I) and R_0 is *flat* at $\mathbf{T}^{n-1} \times E_\kappa$, where E_κ is a Cantor set of positive measure in D such that the map $\omega : E_\kappa \to \Omega_\kappa$, $\omega(I) = \nabla K(I)$, is bijective. Then each $\mathbf{T}^{n-1} \times \{I\}$, $I \in E_\kappa$, is an invariant torus of P with a vector of rotation $\omega/2\pi = \nabla K(I)/2\pi$. We denote that torus by Λ_ω and by Λ the family of all such tori. The set $\Lambda \subset B^*\Gamma$ has a positive Lebesgue measure. Since R_0 is Gevrey smooth and flat at $\mathbf{T}^{n-1} \times E_\kappa$, it is also *exponentially small* near $\mathbf{T}^{n-1} \times E_\kappa$ (see (3.6)). As in [9], Corollary I.2, this leads to "effective stability" of the quasiperiodic motion of the billiard flow near the flow-out $\mathcal{T} \subset S^*X$ of the family Λ of invariant tori. In other words, we have stability in finite but exponentially large time intervals. To obtain the normal form, we use a similar result for analytic Hamiltonians [9] and a result of Kuksin and Pöschel [5]. We give the following geometric interpretation of $K(I)$, $I \in E_\kappa$. Denote by $\mathcal{T}_\omega$ the flow-out of Λ_ω, $\omega = \nabla K(I) \in \Omega_\kappa$, by the broken bicharacteristic flow. Then $\mathcal{T}_\omega$ is homeomorphic to $\mathbf{T}^n$ and $I = (I_1, \ldots, I_{n-1})$ is just the action along suitable cycles in Λ_ω. It turns out that $K(I)$, $I \in E_\kappa$, is uniquely determined modulo a constant, which can be fixed so that $-K(I)$ is the action along a suitable cycle of $\mathcal{T}_\omega$ (see Sect. 3).

Starting from the symplectic normal form of P, we obtain in Sect. 4 a "Birkhoff normal form" of the quantum Poincaré map $\overline{M}(\lambda)$ near Λ. We have $\overline{M}(\lambda) = e^{i\pi(m+\vartheta/4)}M(\lambda)$, where ϑ is the Maslov's index corresponding to γ and the principal symbol of $M(\lambda)$ equals one. Conjugating $M(\lambda)$ with suitable λ-FIOs with Gevrey symbols we obtain a λ-FIO, $M_0(\lambda)$, the distribution kernel $M_0(\lambda, x, y)$ of which is of the form

$$\left(\frac{\lambda}{2\pi}\right)^{n-1} \int_{\mathbf{R}^{n-1}} \exp(i\lambda(\langle x-y, I\rangle - K(I) + R(x,I)))\,(p^0(I,\lambda) + r(x,I,\lambda))\,dI \qquad (1.3)$$

modulo $O(e^{-c|\lambda|^{1/\varrho}})$. Here $R \in G^{1,\mu}(\mathbf{T}^{n-1}\times D)$, $\mu > \tau'+2$, p^0 and r are Gevrey symbols in $S_\ell(\mathbf{T}^{n-1}, D)$, $\ell = (\sigma, \nu, \varrho)$ (see Appendix), compactly supported with respect to I in D, and the "perturbations" R and r are *flat* at $\mathbf{T}^{n-1}\times E_\kappa$. In other words, we have a separation of the variables in (1.3) at $\mathbf{T}^{n-1}\times E_\kappa$. The operator $M_0(\lambda)$ acts in the space $L^2(\mathbf{T}^{n-1}, \mathbf{L})$, where $\mathbf{L}$ is a flat Hermitian line bundle over $\mathbf{T}^{n-1}$. An orthonormal basis e_p, $p \in \mathbf{Z}^{n-1}$, of $L^2(\mathbf{T}^{n-1}, \mathbf{L})$ is given by $e_p(x) = \exp(i\langle p + \vartheta_0/4, x\rangle)$, where $\vartheta_0 = (2,\ldots,2)$. We call (1.3) "quantum Birkhoff normal form" of $M(\lambda)$ and $p(I,\lambda), I \in E_\kappa$, quantum Birkhoff invariants. The construction of the normal form (1.3) is the main technical part in the paper. We look for formal Gevrey symbols

$$a(\varphi, I, \lambda) \;=\; \sum_{j=0}^{\infty} a_j(\varphi, I)\lambda^{-j}\;, \quad p^0(I,\lambda) \;=\; \sum_{j=0}^{\infty} p_j^0(I)\lambda^{-j}$$

in $S_\ell(\mathbf{T}^{n-1}, D)$. We have $a_0 = p_0^0 = 1$, while the functions a_j, $j \geq 1$, are solutions of the "homological equation"

$$\frac{1}{i}\left[a_j(\varphi - \nabla K(I), I) - a_j(\varphi, I)\right] \;=\; f_j(\varphi, I) \qquad (1.4)$$

in $\mathbf{T}^{n-1}\times E_\kappa$. The functions $p_j^0(I)$, $j \geq 1$, are chosen so that the average of $f_j(\varphi, I)$ on $\mathbf{T}^{n-1}$ is 0. Here f_j does not depend on a_j, p_j^0, but it does depend on a_k and p_k^0 for $1 \leq k \leq j-1$. The main difficulty is to find Gevrey estimates for a_j, p_j^0, and f_j which are uniform with respect to j (see (4.9)). Note that the Gevrey index ϱ in (1.1) and (1.3) is related to the loss of derivatives in the the "homological equation" (see Proposition 6.1), which can not be avoided because of small divisors. In this section we use some of the techniques developed in [10]. The main difference comes from the fact that here we look for a normal form of a λ-FIO, while in [10] we have found a normal form of a λ-PDO which is much easier. A natural idea would be to write $M_0(\lambda)$ as an exponent of a λ-PDO modulo an operator the full symbol of which vanishes to infinite order at $\mathbf{T}^{n-1}\times E_\kappa$ and then to use the results in [10]. This could be done but it involves ones more the Diophantine conditions and we lose Gevrey regularity.

In Sect. 5 we obtain the quasimode $\mathcal{Q}$. We are looking for eigenvalues $Z_\nu(\lambda)$ of $M_0(\lambda)$ with eigenfunctions e_p, where $|\lambda - \lambda_\nu^0| \leq C$, $C = \text{const.} \gg 1$, and λ_ν^0, $\nu = (p,q) \in \mathbf{N}^{n-1}\times\mathbf{N}$, is given by the quantization condition (5.2). This condition makes the contribution of R and r in $M_0(\lambda)e_p$ exponentially small and for such λ we obtain $M_0(\lambda)e_p = Z_\nu(\lambda)e_p$ modulo $O(e^{-c|\lambda|^{1/\varrho}})$. To find λ_ν we solve the equation $e^{i\pi(m+\vartheta/4)}Z_\nu(\lambda) = 1$ modulo $O(e^{-c|\lambda|^{1/\varrho}})$

writing it in the form (5.12) and then using the implicit function theorem. We prove also the orthogonality relations.

In Sect. 6 we solve the homological equation in Gevrey classes. In the Appendix we collect some facts about Gevrey symbols.

2 Reduction at the boundary.

To prove Theorem 1 we use an outgoing parametrix for the Helmholtz equation near Γ. Fix $\sigma > 1$. Let $\Gamma_j \subset \Gamma$ be small compact neighborhoods of the vertices γ_j of γ, $j = 1, \dots, m+1$, where $\rho_j = (\gamma_j, \eta_j)$ and $\rho_{m+1} = \rho_1$. Denote by Ψ_j a λ-pseudodifferential operator with a Gevrey symbol in S^σ (see Appendix), supported in a small compact neighborhood of ρ_j, $j = 1, \dots, m+1$, and with a large parameter $\lambda \in \mathcal{D} := \{z \in \mathbf{C} : |\mathrm{Im}\, z| \le D_0, \mathrm{Re}\, z \ge 1\}$, $D_0 > 0$ being fixed. Consider the parametrix $H_j : C_0^\infty(\Gamma) \to C^\infty(X)$, of the problem

$$\begin{cases} (\Delta + \lambda^2) H_j(\lambda) u &= O_s(e^{-C|\lambda|^{1/\sigma}}) u, \quad C > 0, \\ H_j(\lambda) u|_{\Gamma_j} &= \Psi_j u \,. \end{cases}$$

This is a Fourier integral operator with a large parameter $\lambda \in \mathcal{D}$, the distribution kernel of which can be written microlocally as an oscillatory integral with an analytic non-degenerate phase function and Gevrey amplitude in S^σ. Hereafter

$$O_s(e^{-C|\lambda|^{1/\sigma}}) \; : \; L_2(\Gamma) \longrightarrow H^s(X)$$

stands for a family of bounded operators depending on λ with norms $\le C_1 e^{-C|\lambda|^{1/\sigma}}$, $C_1 > 0$ and $H^s(X)$, $s \ge 0$ is the Sobolev space. Denote by $\imath^*$ the map of restriction on Γ, $\imath^* u = u|_\Gamma$. The canonical relation of the λ-FIO $\imath^* H_j(\lambda)$ is just the graph of the billiard ball map B defined in a neighborhood of (γ_j, η_j). We may suppose that the Gevrey G^σ-frequency set WF^σ of $\imath^* H_j(\lambda)$ (see Appendix) is contained in $T^*\Gamma_j \times T^*\Gamma_{j+1}$ for any $j = 1, \dots, m$ and that $(\Psi_{j+1} \imath^* H_j(\lambda) - \imath^* H_j(\lambda)) u|_{\Gamma_{j+1}} = O(e^{-C|\lambda|^{1/\sigma}}) u$. Consider the operator

$$G(\lambda) u = H_1(\lambda) u - H_2(\lambda) \imath^* H_1(\lambda) u + \cdots + (-1)^{m-1} H_m(\lambda) \imath^* H_{m-1}(\lambda) \cdots \imath^* H_1(\lambda) u$$

for $u \in C_0^\infty(\Gamma_1)$. Obviously

$$\begin{cases} (\Delta + \lambda^2) G(\lambda) u &= O_s(e^{-C|\lambda|^{1/\sigma}}) u \,, \\ \imath^* G(\lambda) u &= \Psi_1(\lambda) u - \bar{M}(\lambda) u + O_s(e^{-C|\lambda|^{1/\sigma}}) u \,, \end{cases}$$

for any $u \in C_0^\infty(\Gamma)$, where $\bar{M}(\lambda) = (-1)^m \Psi_{m+1} \imath^* H_m(\lambda) \imath^* H_{m-1}(\lambda) \cdots \imath^* H_1(\lambda)$, and the symbol of Ψ_{m+1} is 1 in a neighborhood of the support of the symbol of Ψ_1 and equals 0 outside Γ_{m+1}. We are going to find $\lambda_\nu > 0$ and $v_\nu \in L^2(\Gamma)$ such that $G(\lambda_\nu) v_\nu|_\Gamma = O_s(e^{-c\lambda_\nu^{1/\varrho}}) v_\nu$, $c > 0$. To do so we have to solve the equation

$$(\bar{M}(\lambda) - \Psi_1) u = O_s(e^{-c\lambda^{1/\varrho}}) u \,. \tag{2.1}$$

Observe that $\bar{M}(\lambda)$ is a λ-FIO of order 0 with a large parameter $\lambda \in \mathcal{D}$. The canonical relation of $\bar{M}(\lambda)$ is given by the graph of the Poincaré map $P = B^m : W_0 \to W$, where

W_0 and W are neighborhoods of ρ_1 in $B^*\Gamma$. Moreover, the principal symbol of $\bar{M}(\lambda)$ can be identified with the function

$$\sigma(\bar{M})(y,\eta,\lambda) \;=\; \exp(i[\lambda L(y,\eta) + (m+\vartheta/4)\pi])\,,\;\; (y,\eta)\in W_0\,,$$

where ϑ is the Maslov index corresponding to the broken closed bicharacteristic $\widetilde{\gamma}$. Here L is the "return time function", i.e. $\Phi^{L(y,\eta)}(y,\eta) = P(y,\eta)$, $(y,\eta)\in W_0$, $t\to\Phi^t(y,\eta)$ being the broken bicharacteristic issuing from (y,η). We set $\bar{M}(\lambda) = e^{i(m+\vartheta/4)\pi}M(\lambda)$. Then the principal symbol of $M(\lambda)$ equals one, modulo the Liouville factor $\exp(i\lambda L(y,\eta))$.

3 Birkhoff normal form of P in Gevrey classes

We are going to explain (H_1) in more details. Consider a periodic broken geodesics $\gamma : [0,T]\to X$ with vertices $\gamma_j = \gamma(t_j)\in\Gamma$, $0 = t_1 < t_2 < \cdots < t_m < t_{m+1} = T$, $1\le j\le m$. Let $\widetilde{\gamma}(t) = (\gamma(t),\xi(t))$, $t\ne t_j$, be the corresponding broken bicharacteristic of S^*X with "vertices" $\rho_j = (\gamma_j,\eta_j)$, $\rho_{m+1} = \rho_1$, in $B^*\Gamma$. The billiard ball map B is *analytic* and symplectic in a neighborhood of ρ_j in $B^*\Gamma$ because of (H_2). Recall that for $(y,\eta)\in B^*\Gamma$ close to ρ_j, $B(y,\eta)$ is obtained as follows: Consider the bicharacteristic $(x(t),\xi(t))$, $t\ge 0$, in S^*X issuing from (y,η), i.e. $x(0) = y$ and $\xi|_{T_x\Gamma} = \eta$, $\langle\xi, n(x)\rangle > 0$, $n(x)$ being the inward unit normal to Γ and denote by $t_1(y,\eta) > 0$ the first time of intersection with the boundary. The function t_1 is obtained by the implicit function theorem and it is analytic near ρ_j. We set $B(y,\eta) = (x(t_1(y,\eta)),\xi(t_1(y,\eta))|_{T_x\Gamma})$. Obviously, $\{\rho_1,\rho_2,\ldots,\rho_m\}$ is a periodic orbit of B, $B\rho_j = \rho_{j+1}$, for $j<m$, $B\rho_m = \rho_1$, and ρ_1 is a fixed point of the (local) Poincaré map $P = B^m : W_0\to W$, where W_0 and W are neighborhoods of ρ_1 in $B^*\Gamma$. We assume that γ is elliptic, which means that the eigenvalues of the linear Poincaré map DP_{ρ_1} are of the form $e^{\pm i\alpha_j^0}$, $j = 1,\ldots,n-1$, $0<\alpha_j^0\le\pi$. We suppose also that P is N-elementary for some $N\ge 4$, which means that

$$\sum_{j=1}^{n-1}\alpha_j^0\,k_j\notin 2\pi\mathbf{Z},\;\forall\, k = (k_1,\ldots,k_{n-1})\in\mathbf{Z}^{n-1},\;\; 0<|k_1|+\cdots+|k_{n-1}|\le N. \tag{3.1}$$

Then P admits an analytic Birkhoff normal form near $\rho_1 = (0,0)$. In other words, there exist analytic symplectic local coordinates $(q,p) : W_0\to T^*\mathbf{R}^{n-1}$, $(q(\rho_1),p(\rho_1)) = (0,0)$, such that in the corresponding polar symplectic coordinates $(\theta,r)\in\mathbf{T}^{n-1}\times V$, V being the intersection of a ball of radius $r_0\ll 1$ with a proper open cone in $\mathbf{R}_+^{n-1}$ having vertex at 0, the map P becomes

$$P_0(\theta,r) = (\theta + \nabla S(r) + O(|r|^{(N-1)/2}), r + O(|r|^{(N+1)/2})),\quad r\to 0, \tag{3.2}$$

where S is a polynomial of degree $\le N/2$ and $\nabla S(0) = (\alpha_1^0,\ldots,\alpha_{n-1}^0)$. Denote by B_1, the matrix $(\partial_i\partial_j S(0))_{i,j=1}^{n-1}$ and make the non-degeneracy assumption

$$det\; B_1\ne 0. \tag{3.3}$$

We consider Gevrey functions of the following type: Given a domain $D\subset\mathbf{R}^{n-1}$ and reals $\sigma,\mu\ge 1$, we say that $f\in G^{\sigma,\mu}(\mathbf{T}^{n-1}\times D)$ if for every compact subset Y of D there exists $C = C(Y) > 0$ such that

$$\sup_{(\varphi,I)\in\mathbf{T}^n\times Y}|\partial_\varphi^\beta\partial_I^\alpha f(\varphi,I)|\;\le\; C^{|\alpha|+|\beta|+1}\,\beta!^{\,\sigma}\,\alpha!^{\,\mu}\;,\;\forall\,\alpha,\,\beta\in\mathbf{N}^{n-1}\,, \tag{3.4}$$

where $\mathbf{N}$ stands for the set of all non-negative integers. Hereafter, given a multi-index $\alpha \in \mathbf{N}^d$, $d \geq 1$, we write $|\alpha| = \sum_{j=1}^d |\alpha_j|$. The solutions u_ν, λ_ν of (2.1) will be associated with a Gevrey smooth family of invariant tori of the Poincaré map P in a neighborhood of ρ_1. Fix $C_1 > 1$ and for each $0 < a \leq a_0 \ll 1$ set

$$D \;=\; D_a \;=\; \{I \in \Gamma : \, C_1^{-1} a \leq I_j \leq C_1 a, \; j = 1, \ldots, n-1\},$$

and denote by $\Omega = \Omega_a \;= \nabla S(D_a)$. Choosing $a_0 > 0$ sufficiently small, the map ∇S : $D_a \to \Omega_a$ becomes a diffeomorphism. Fix $\tau > n-1$, and $0 < \varepsilon \ll 1$ and set $\kappa = \kappa_a = \varepsilon a$. Consider the set of all $\omega \in \mathbf{R}^{n-1}$ defined by the Diophantine condition:

$$|\langle \omega, k \rangle - 2\pi k_n| \;\geq\; \frac{\kappa}{(\sum_{j=1}^n |k_j|)^\tau}, \quad \forall\, (k, k_n) \in \mathbf{Z}^n \backslash \{0\}. \tag{3.5}$$

Let Ξ_κ be the set of all $\omega \in \Omega$ satisfying (3.5) and having distance $\geq \kappa$ to the boundary of Ω. We define Ω_κ to be the set of points of positive Lebesgue density in Ξ_κ. In other words, $\omega \in \Omega_\kappa$ if for any neighborhood U of ω in Ω the Lebesgue measure of $U \cap \Xi_\kappa$ is positive. Obviously, Ω_κ and Ξ_κ have the same Lebesgue measure, which is positive if $\varepsilon > 0$ is sufficiently small. Fix $\tau' > \max(\tau, 5/2)$. Recall that a symplectic diffeomorphism $\chi : \mathbf{T}^{n-1} \times D \to \mathbf{T}^{n-1} \times D$ is called exact symplectic if χ admits a generating function $\Phi \in C^\infty(\mathbf{T}^{n-1} \times D)$, which is determined modulo a constant by

$$\chi(\varphi + \nabla_I \Phi(\varphi, I), I) = (\varphi, I + \nabla_\varphi \Phi(\varphi, I)) \;,\; (\varphi, I) \in \mathbf{T}^{n-1} \times D\,.$$

Theorem 3.1 *Suppose that (3.3) holds. Then for each $0 < a \leq a_0$, a_0 sufficiently small, there exists an exact symplectic diffeomorphism χ_0 of Gevrey class $G^{1,\tau'+2}$ mapping $\mathbf{T}^{n-1} \times D$ into itself, a $G^{\tau'+2}$-diffeomorphism $\omega : D \to \Omega$, and $K \in G^{\tau'+2}(D)$ such that the exact symplectic map*

$$P^0 = \chi_0^{-1} \circ P_0 \circ \chi_0 \in G^{1,\tau'+2}(\mathbf{T}^{n-1} \times D; \mathbf{T}^{n-1} \times D)$$

has a generating function $Q(\varphi, I) \;=\; -K(I) \;+\; R(\varphi, I)$ in $G^{1,\tau'+2}$, where

$$D_I^\alpha(\omega(I) - \nabla K(I)) \;=\; 0 \quad \text{and} \quad D_I^\alpha R(\varphi, I) \;=\; 0\;, \quad \forall\, \alpha \in \mathbf{N}^{n-1}\,,$$

on $E_\kappa := \omega^{-1}(\Omega_\kappa)$ and $\mathbf{T}^{n-1} \times E_\kappa$ respectively. The symplectic map χ_0 is exact symplectic with generating function $\Phi \in G^{1,\tau'+2}(\mathbf{T}^{n-1} \times D)$ and for each $|\alpha| \leq (N-1)/2$ and $0 < q < 1$, we have

$$\begin{aligned} &\left| D_\varphi^\beta D_I^\alpha \Phi(\varphi, I) \right| \;+\; |D_I^\alpha(\omega(I) - \nabla S(I))| \\ &\leq\; C^{|\alpha|+|\beta|+1} \, a^{q(N-1)/2-|\alpha|} \;\beta!\,,\; \forall \beta \in \mathbf{N}^{n-1}\,, \end{aligned}$$

for any $(\varphi, I) \in \mathbf{T}^{n-1} \times D$, where $C = C_q$ does not depend on a.

To prove the theorem we use Theorem 2, [5]. We write the real analytic map P as a symplectic map in a neighborhood of the origin in $\mathbf{R}^{2n-2}$ of the form $z \to \Lambda z + \hat{F}(z)$, where $\Lambda = DP(0)$ and $\hat{F}(z) = O(z^2)$. Moreover, $\Lambda = \exp(JA)$, where A is a real symmetric matrix and

$$J \;=\; \begin{pmatrix} 0 & \mathrm{Id} \\ \text{-Id} & 0 \end{pmatrix}.$$

Applying Theorem 2, [5], we find a 2π-periodic Hamiltonian $\widetilde{H}(t,z) = \langle Az, z\rangle + \hat{H}(t,z)$ which is jointly real analytic in some neighborhood of the origin in $\mathbf{R}^{2n-2}$ in z, and in t, and such that P is just the flow $\exp(tX_{\widetilde{H}})_{|t=2\pi}$ at time 2π. Because of (3.1) we can find a Birkhoff normal form up to order N of the 2π-periodic Hamiltonian $\tilde{H}$ [1], which in fact coincides with S. This gives a normal form of the non-degenerated Hamiltonian $H = \tau + \widetilde{H}(t,z) + (\tau + \widetilde{H}(t,z))^2$, $(t,\tau,z) \in \mathbf{T}\times\mathbf{R}\times\mathbf{R}^{2n-2}$ in a neighborhood of $\{(t,0,0) : t\in\mathbf{T}\}$, and we deduce Theorem 3.1 from Theorem I.1, [9], using the same argument as in the proof of Corollary I.3, [9].

Set $\chi = \chi_1\circ\chi_0$, where χ_0 is given by Theorem 3.1 and χ_1 is the symplectic map which puts P in the Birkhoff normal form (3.2). Then χ is an exact symplectic diffeomorphism. Let $\Omega \ni \omega \to I(\omega) \in D$ be the inverse map to the diffeomorphism $I \to \omega(I)$. Then $E_\kappa = I(\Omega_\kappa)$ and each

$$\Lambda_\omega \;=\; \chi(\mathbf{T}^{n-1}\times\{I(\omega)\})\,,\;\; \omega\in\Omega_\kappa\,,$$

is an invariant torus of P with frequency ω. Moreover, we can find two positive constants C_1 and c depending only on the constant C in (3.4) of R, such that for every $\alpha,\beta\in\mathbf{N}^{n-1}$ the following estimate holds

$$|\partial_\varphi^\beta\partial_I^\alpha R(\varphi,I)| \;\le\; C_1^{|\alpha|+|\beta|+1}\,\beta!\,\alpha!^{\,\tau'+2}\,\exp\left(-c\,|E-I|^{-\frac{1}{\tau'+1}}\right)\,, \tag{3.6}$$

for each $(\varphi,I)\in\mathbf{T}^{n-1}\times Y$, $I\notin E$, where $|E-I| = \inf_{I'\in E}|I'-I|$ is the distance to the compact set E (see [9], (I,3)). Using (3.6) we can obtain as in [9], Corollary I.2, effective stability of the quasiperiodic motion of the discrete dynamical system P^k, $k\in\mathbf{Z}$, near the union Λ of Λ_ω, $\omega\in\Omega_\kappa$.

We are going to provide a geometric interpretation of $K(I)$, $I=(I_1,\dots,I_{n-1})\in E_\kappa$. Obviously $I_j = (2\pi)^{-1}\int_{\gamma_j}\xi dx$, where γ_j, $j=1,\dots,n-1$, form a basis of cycles in Λ_ω, $\omega = \nabla K(I)$. Consider the "flow-out" $\mathcal{T}_\omega$ of Λ_ω by the bicharacteristic flow. Let $\rho = \chi(\varphi,I)\in\Lambda_\omega$. We denote by γ_{n1} the broken bicharacteristic arc in $\mathcal{T}_\omega$ issuing from ρ and having endpoint at $P(\rho)$, and by γ_{n2} an arc in Λ_ω connecting $P(\rho)$ and ρ such that $\gamma_{n2}^0 := \chi^{-1}(\gamma_{n2})$ is given by $s\to(\varphi+(1-s)\nabla K(I), I)$, $s\in[0,1]$. Let γ_n be the union of the two arcs. We are going to show that $K(I)$, $I\in E_\kappa$, is uniquely determined modulo a constant and we have

$$K(I) \;=\; -\int_{\gamma_n}\xi dx\,,\;\; I\in E_\kappa \tag{3.7}$$

choosing the constant appropriately. Note that the right hand side of (3.7) depends only on the homotopy class of the loop γ_n in the "Lagrangian" manifold $\mathcal{T}_\omega$. To prove (3.7) we use the Poincaré identity

$$P^*(\xi dx) = \xi dx + dL,$$

where ξdx is the fundamental one form at $T^*\Gamma$ and $L(\rho)$ stands for the action along the broken bicharacteristic γ_{n1}. Since χ is exact symplectic we have $\chi^*(\xi dx) = Id\varphi + d\Psi$ with a suitable smooth function Ψ. Combining the two equalities we obtain

$$(P^0)^*(Id\varphi) - Id\varphi = d((L\circ\chi) + \Psi - \Psi\circ P^0).$$

This implies

$$I\nabla K(I) - K(I) = L(\chi(\varphi,I)) + \Psi(\varphi,I) - \Psi(P^0(\varphi,I)) + C\,,\;\; (\varphi,I)\in\mathbf{T}^{n-1}\times E_\kappa\,, \tag{3.8}$$

and we choose K so that $C=0$. On the other hand,

$$\int_{\gamma_{n2}} \xi dx \;=\; \int_{\gamma^0_{n2}} (Id\varphi + d\Psi) \;=\; -I\nabla K(I) + \Psi(\varphi, I) - \Psi(P^0(\varphi, I))\,,$$

and we obtain (3.7).

4 Quantum Birkhoff normal form of $M(\lambda)$ in Gevrey classes

Consider the map $\chi = \chi_1 \circ \chi_0$. Recall that χ_1^{-1} is a composition of an analytic symplectic map $W_0 \ni (y,\eta) \to (q,p) \in T^*\mathbf{R}^{n-1}$ and the map $(q,p) \to (\theta, r)$, which puts (q,p) in polar symplectic coordinates, $q_j = \sqrt{2r_j}\cos\theta_j$, $p_j = \sqrt{2r_j}\sin\theta_j$, $j = 1,\ldots,n-1$. Using the restriction of χ to $\mathbf{T}^{n-1} \times \{I(\omega)\}$, $\omega \in \Omega_\kappa$, we identify the first cohomology groups $H^1(\Lambda_\omega, \mathbf{Z}) = H^1(T^{n-1}, \mathbf{Z}) = \mathbf{Z}^{n-1}$, and we denote by $\vartheta_0 \in \mathbf{Z}^{n-1}$ the Maslov class of the invariant torus $\Lambda(\omega)$. It is easy to see that $\vartheta_0 = (2,\ldots,2)$, which comes in fact from the map $(\theta, r) \to (q,p)$. We set up the parameters of the Gevrey classes. In (3.5) we fix $\tau > n-1$ when $n \geq 3$ and $\tau > 3/2$ when $n = 2$. Denote $\nu = \tau + n + 1$ and fix τ' such that $\tau + n - 1 \; > \tau' \; > \; \max(\tau, 5/2)$. Then fix μ such that $\nu \; > \; \mu \; > \; \tau' + 2$, choose $\sigma > 1$ sufficiently close to 1 such that

$$\nu \; > \; \mu \; > \; \sigma(\tau' + 1) + 1\,, \tag{4.1}$$

and set

$$\varrho \;=\; \sigma(\nu + 1)\,.$$

Then ϱ could be any number bigger than $\nu + 1$ and sufficiently close to $\nu + 1$, hence, ϱ could be any number $> 2n+1$ when $n \geq 3$ and $> 11/2$ when $n = 2$.

As in [4] we consider the flat Hermitian line bundle $\mathbf{L}$ over $\mathbf{T}^{n-1}$ which is associated to the class ϑ_0. The sections f in $\mathbf{L}$ can be identified canonically with functions $\widetilde{f} : \mathbf{R}^{n-1} \to \mathbf{C}$ so that

$$\widetilde{f}(x + 2\pi p) \;=\; e^{i\frac{\pi}{2}\langle \vartheta_0, p\rangle}\widetilde{f}(x) \tag{4.2}$$

for each $x \in \mathbf{R}^{n-1}$ and $p \in \mathbf{Z}^{n-1}$. An orthonormal basis of $L^2(\mathbf{T}^{n-1}, \mathbf{L})$ is given by e_m, $m \in \mathbf{Z}^{n-1}$, where

$$\widetilde{e}_m(x) \;=\; \exp\left(i\langle m + \vartheta_0/4, x\rangle\right).$$

As in [10], Sect. II, using Theorem 3.1, we conjugate $M(\lambda)$ with a λ-FIO

$$T : C^\infty(\mathbf{T}^{n-1}, \mathbf{L}) \to C^\infty(\Gamma_1)\,.$$

We look for $T = T_1 \circ T_0$, where T_1 is associated to χ_1 and T_0 to χ_0. As in [10], we write microlocally the distribution kernel of

$$T_1 : C^\infty(\mathbf{T}^{n-1}, \mathbf{L}) \to C^\infty(\Gamma_1)$$

as an oscillatory integral with analytic phase function and full symbol in the class S^σ. The operator

$$T_0 : C^\infty(\mathbf{T}^{n-1}, \mathbf{L}) \to C^\infty(\mathbf{T}^{n-1}, \mathbf{L})$$

is given by

$$\widetilde{T_0(\lambda)}u(x) = \left(\frac{\lambda}{2\pi}\right)^{n-1} \int_{\mathbf{R}^{2n-2}} e^{i\lambda(\langle x-y,I\rangle+\Phi(x,I))}\, b(x,I,\lambda)\, \widetilde{u}(y)\, dIdy\,,$$

$u \in C^\infty(\mathbf{T}^{n-1}, \mathbf{L})$, where $\Phi \in G^{1,\tau'+2}$ is given by Theorem 3.1, b is a Gevrey symbol in the class $S_{\widetilde{\ell}}(\mathbf{T}^{n-1} \times D)$, $\widetilde{\ell} = (\sigma, \mu, \sigma+\mu-1)$, and b is uniformly compactly supported with respect to I in D. As in [10], Sect. II, we obtain an operator $M_1(\lambda) = T(\lambda)^{-1}M(\lambda)T(\lambda)$ of the form

$$\widetilde{M_1(\lambda)}u(x) = \left(\frac{\lambda}{2\pi}\right)^{n-1} \int_{\mathbf{R}^{2n-2}} e^{i\lambda(\langle x-y,I\rangle+Q(x,I))}\, p(x,I,\lambda)\, \widetilde{u}(y)\, dIdy, \tag{4.3}$$

$u \in C^\infty(\mathbf{T}^{n-1}, \mathbf{L})$, where p is a Gevrey symbol in the class $S_{\widetilde{\ell}}(\mathbf{T}^{n-1} \times D)$,

$$p(\varphi, I; \lambda) \sim \sum_{j=0}^{\infty} p_j(\varphi, I)\lambda^{-j}\,,$$

and $p_0(\varphi, I) = 1$. The phase function $Q(x,I) = -K(I) + R(x,I)$ is given by Theorem 3.1, and it is uniquely defined modulo a constant. Comparing the Liouville factors in the principal symbols of $M_1(\lambda)$ and $M(\lambda)$ and using (3.8), we obtain as in [3], Sect. 3.3, that K is given by (3.7). For each $a \in C^\infty(\mathbf{T}^{n-1} \times D)$ we set

$$\mathcal{L}a(\varphi, I) = a(\varphi - \omega(I), I) - a(\varphi, I)\,,$$

with $\omega(I)$ introduced by Theorem 3.1, $\omega(I) = \nabla K(I)$ on E_κ. Denote

$$K_0(I,\xi) = \int_0^1 \nabla_I K(I + \tau\xi)d\tau$$

and set $D_\xi = (1/i)\partial_\xi$.

Lemma 4.1 *Let $M_0(\lambda)$ be a λ-FIO of the form (4.3) with amplitude $p^0(I,\lambda)$, independent of the angle variable, and let $A(\lambda)$ be a λ-PDO with full symbol $a(\varphi, I, \lambda)$ acting on $C^\infty(\mathbf{T}^{n-1}, \mathbf{L})$ such that*

$$a \sim \sum_{j=0}^{\infty} a_j(\varphi, I)\lambda^{-j}, \quad p^0 \sim \sum_{j=0}^{\infty} p_j^0(I)\lambda^{-j}\,, \tag{4.4}$$

are uniformly compactly supported with respect to I in D and $a_0 = p_0^0 = 1$ in a neighborhood of E_κ. Suppose that a and p^0 belong to $S_\ell(\mathbf{T}^{n-1} \times D)$, $\ell = (\sigma, \mu, \varrho)$. Then

$$M_1(\lambda)A(\lambda) - A(\lambda)M_0(\lambda) = R^0(\lambda) + R^1(\lambda), \tag{4.5}$$

where R^0 and R^1 are λ-FIOs of the form (4.3) with amplitudes $r^0, r^1 \in S_\ell(\mathbf{T}^{n-1} \times D)$. Moreover, r^1 vanishes to infinite order for $I \in E_\kappa$ and

$$r^0 \sim \sum_{j=0}^{\infty} r_j(\varphi, I)\lambda^{-j},$$

where

$$r_j(\varphi, I) \;=\; \frac{1}{i}(\mathcal{L}\, a_j)(\varphi, I) \;+\; p_j(\varphi, I) \;-\; p_j^0(I) \;+\; F_j(\varphi, I)\,,$$

$$F_1(\varphi, I) = 0,\;\; F_j(\varphi, I) \;=\; F_{j1}(\varphi, I) \;-\; F_{j2}(\varphi, I)\,,\; j \geq 2\,,$$

and

$$F_{j1}(\varphi, I) \;=\; \sum_{\substack{r+s+|\gamma|=j \\ 1\leq s<j}} \frac{1}{\gamma!}\left[D_\xi^\gamma \left(p_r(\varphi, I+\xi)\, \partial_\varphi^\gamma\, a_s(\varphi - K_0(I,\xi), I)\right)\right]_{|\xi=0}\,,$$

$$F_{j2}(\varphi, I) \;=\; \sum_{s=1}^{j-1} a_s(\varphi, I)\, p_{j-s}^0(I)\,.$$

Proof. We write the operator $M_1(\lambda)A(\lambda)$ in the form (4.3) with amplitude given by

$$g(x, I, \lambda) = \left(\frac{\lambda}{2\pi}\right)^{n-1} \int_{\mathbf{R}^{2n-2}} e^{i\lambda(\langle x-z, \xi-I\rangle + Q(x,\xi) - Q(x,I))}\, p(x,\xi,\lambda) a(z, I, \lambda)\, d\xi dz\;,$$

which belongs to $C^\infty(\mathbf{T}^{n-1}\times D)$ for each λ fixed. Changing the variables we write $g(x, I, \lambda)$ in the form

$$\left(\frac{\lambda}{2\pi}\right)^{n-1} \int_{\mathbf{R}^{2n-2}} e^{-i\lambda\langle v,\eta\rangle}\, p(x, I+\eta, \lambda) a(v + x - K_0(I,\eta) + H_0(x, I, \eta), I, \lambda)\, d\eta dv\;,$$

where $H_0(x, I, \eta) = \int_0^1 \nabla_I R(x, I+\tau\eta) d\tau$ and all its derivatives vanish for $\eta = 0$, $I \in E_\kappa$. Using the Taylor formula for the amplitude at $v = 0$ and integrating by parts we get $g(x, I, \lambda) \sim \sum_{j=1}^\infty g_j(x, I)\lambda^{-j} \in S_\ell(\mathbf{T}^{n-1}\times D)$, where

$$g_j(x, I) \;=\; \sum_{r+s+|\gamma|=j} \frac{1}{\gamma!}\left[D_\eta^\gamma \left(p_r(x, I+\eta)\, \partial_x^\gamma\, a_s(x - K_0(I,\eta) + H_0(x, I, \eta), I)\right)\right]_{|\eta=0}\,.$$

In the same way we write $A(\lambda)M_0(\lambda)$ in the form (4.3) with amplitude $h(x, I, \lambda)$ given by the oscillatory integral

$$\left(\frac{\lambda}{2\pi}\right)^{n-1} p^0(I,\lambda) \int_{\mathbf{R}^{2n-2}} e^{i\lambda(\langle x-z, \xi-I\rangle + Q(z,I) - Q(x,I))}\, a(x, \xi, \lambda) d\xi dz\,.$$

Changing the variables we obtain $h = h^0 + h^1$, where $h^0(x, I, \lambda) = a(x, I, \lambda)p^0(I, \lambda)$ and $h^1(x, I, \lambda)$ is given by

$$\left(\frac{\lambda}{2\pi}\right)^{n-1} p^0(I,\lambda) \int_{\mathbf{R}^{2n-2}} e^{-i\lambda\langle v,\eta\rangle}[a(x, \eta + I + H_1(x, v, I), \lambda) - a(x, \eta + I, \lambda)]\, d\eta dv\;,$$

where $H_1(x, v, I) = \int_0^1 \nabla_x R(x+\tau v, I) d\tau$ and all its derivatives vanish for $I \in E_\kappa$. Then $h^1 \in S_\ell(\mathbf{T}^{n-1}\times D)$ and $\partial_I^\beta h^1 = 0$ for each $I \in E_\kappa$ and any β. This proves the lemma. $\Box$

We are going to find $a, p^0 \in S_\ell(\mathbf{T}^{n-1}\times D)$, p^0 independent of φ such that $r^0 \equiv 0$ on $\mathbf{T}^{n-1}\times E_\kappa$.

Proposition 4.1 *There exist Gevrey symbols $a, p^0 \in S_\ell(\mathbf{T}^{n-1}\times D)$ of the form (4.4) such that $r^0 \equiv 0$ on $\mathbf{T}^{n-1}\times E_\kappa \times \mathcal{D}$.*

Proof. We take $a_0 = p_0^0 = 1$ in $\mathbf{T}^{n-1} \times D$. We are going to find a_j and p_j^0 for $j \geq 1$.

We solve the equations $r_j = 0,\ j \geq 1$, as in [10], Sect. 5. First we set

$$p_j^0(I) \ = \ (2\pi)^{1-n} \int_{\mathbf{T}^{n-1}} (p_j(\varphi, I) \ + \ F_j(\varphi, I))\, d\varphi\,, \tag{4.6}$$

then, using Proposition 6.1, we find a_j such that

$$\left\{ \begin{array}{rcl} \frac{1}{i}\mathcal{L}a_j(\varphi, I) & = & f_j(\varphi, I)\,,\ (\varphi, I) \in \mathbf{T}^{n-1} \times E_\kappa\,, \\ \int_{\mathbf{T}^{n-1}} a_j(\varphi, I) d\varphi & = & 0\,,\ I \in D\,, \end{array} \right. \tag{4.7}$$

where $f_j(\varphi, I) = p_j^0(I) - p_j(\varphi, I) - F_j(\varphi, I)$. For $j = 1$ we obtain

$$p_1^0(I) \ = \ (2\pi)^{1-n} \int_{\mathbf{T}^{n-1}} p_1(\varphi, I)\, d\varphi,$$

and

$$\frac{1}{i}\mathcal{L}a_1(\varphi, I) \ = \ p_1^0(I) \ - \ p_1(\varphi, I)\,, \quad \int_{\mathbf{T}^{n-1}} a_1(\varphi, I)\, d\varphi \ = \ 0.$$

Denote by $\mathbf{N}_+ = \mathbf{N}\backslash\{0\}$ the set of all positive integers. Since $p \in S_{\widetilde{\ell}}(\mathbf{T}^{n-1} \times D)$, the functions p_j, $j \in \mathbf{N}_+$, satisfy the estimates

$$|D_I^\alpha D_\varphi^\beta p_j(\varphi, I)| \leq \widetilde{C}_0^{|\alpha|+|\beta|+j} \alpha!^\mu \beta!^\sigma (j!)^{\sigma+\mu-1}$$

$$\leq C_0^{|\alpha|+|\beta|+j} \alpha!\beta!\Gamma((\mu-1)|\alpha| + (\sigma-1)|\beta| + (\sigma+\mu-1)(j-1)+1), \tag{4.8}$$

for $(\varphi, I) \in \mathbf{T}^{n-1} \times D$ and each $\alpha,\, \beta \in \mathbf{N}^{n-1}$, where $\Gamma(s)$ is the Euler Γ function, and $\widetilde{C}_0$ and C_0 are suitable positive constants. In particular, using Proposition 6.1 we find a_1 satisfying

$$|D_I^\alpha D_\varphi^\beta a_1(\varphi, I)| \ \leq \ 2RC_0\, \alpha!\, C^{|\alpha|+|\beta|}\, \Gamma((\mu-1)|\alpha| + \sigma|\beta| + \varrho)\,,$$

choosing $C > C_0$. Fix $j \geq 2$ and suppose that there exist $p_k^0(I)$ and $a_k(\varphi, I)$, $1 \leq k \leq j-1$, satisfying (4.6) and (4.7), and such that

$$\begin{array}{rcl} |D_I^\alpha p_k^0(I)| & \leq & d^{k-1/2} C^{|\alpha|}\, \alpha!\, \Gamma((\mu-1)|\alpha| + (k-1)\varrho + 2\sigma)\,, \\ |D_I^\alpha D_\varphi^\beta a_k(\varphi, I)| & \leq & d^k C^{|\alpha|+|\beta|}\, \alpha!\, \Gamma((\mu-1)|\alpha| + \sigma|\beta| + k\varrho)\,, \end{array} \tag{4.9}$$

for any $(\varphi, I) \in \mathbf{T}^{n-1} \times D$ and $\alpha, \beta \in \mathbf{N}^{n-1}$, where $d \geq 2RC_0$. , We shall prove that p_j^0 and a_j satisfy the same estimates for $k = j$ with $d \gg 1$ independent of j.

First we estimate the derivatives of

$$b_{k,\gamma}(\varphi, I) = \frac{1}{\gamma!} \left[\partial_\xi^\gamma a_k(\varphi - K_0(I, \xi), I)\right]_{|\xi=0}\,,\ I \in D\,.$$

For that we fix $C_1 > 0$ such that for each $\alpha, \gamma \in \mathbf{N}^{n-1}$, $|\alpha| + |\gamma| \neq 0$, we have

$$|D_I^\alpha D_\xi^\gamma K_0(I, \xi)| \ \leq \ C_1^{|\alpha+\gamma|}\, \alpha!\gamma!\, ((|\alpha+\gamma|-1)!)^{\,\mu-1}\,,\ (I, \xi) \in D \times D\,. \tag{4.10}$$

Lemma 4.2 *Let $C > 4C_1$. Then there is a constant $C_2 > 0$ independent of j and d, such that for each $\alpha, \beta, \gamma \in \mathbf{N}^{n-1}$ and $1 \le k \le j-1$ we have*

$$|D_I^\alpha D_\varphi^\beta b_{k,\gamma}(\varphi, I)| \;\le\; C_2\, d^k\, C^{|\alpha|+|\beta|+|\gamma|}\, \alpha!\, \Gamma((\mu-1)(|\alpha|+|\gamma|) + \sigma|\beta| + k\varrho + \sigma)\,. \qquad (4.11)$$

Moreover, if $\gamma \neq 0$ we have

$$|D_I^\alpha D_\varphi^\beta b_{k,\gamma}(\varphi, I)| \;\le\; C_2\, d^k\, C^{|\alpha|+|\beta|+|\gamma|}\, \alpha!\, \Gamma((\mu-1)(|\alpha|+|\gamma|-1) + \sigma|\beta| + k\varrho + \sigma)\,. \quad (4.12)$$

Proof. We write

$$D_I^\alpha D_\varphi^\beta b_{k,\gamma}(\varphi, I) \;=\; \frac{1}{\gamma!} \sum_{\alpha' \le \alpha} \binom{\alpha}{\alpha'} \left(D_\eta^{\alpha'} \partial_\xi^\gamma D_I^{\alpha-\alpha'} D_\varphi^\beta a_k(\varphi - K_0(\eta,\xi), I) \right)_{|\eta = I,\, \xi = 0}\,. \quad (4.13)$$

We are going to estimate each term $A_{\alpha'}$ of (4.13). First, assuming that $\gamma \neq 0$, we shall prove (4.12). Set $z = (\eta, \xi)$ and $\delta = (\alpha', \gamma)$. Then $\delta \neq 0$ and using Faa de Bruno formula (see [7], Corollary 5.5) we get

$$\begin{aligned} |A_{\alpha'}| \;&\le\; \frac{1}{\gamma!} \sum_{\mathcal{B}(\alpha')} \binom{\alpha}{\alpha'} \frac{\alpha'!\gamma!}{\ell!\,\delta^1! \cdots \delta^p!} \left| \left(D_\varphi^{\ell+\beta} D_I^{\alpha-\alpha'} a_k \right)(\varphi - \nabla K(I), I) \right| \\ &\times\; \left| \left(D_z^{\delta^1} K_0 \right)(I,0) \right| \dots \left| \left(D_z^{\delta^p} K_0 \right)(I,0) \right|\,, \end{aligned}$$

where the index set $\mathcal{B}(\alpha')$ consists of all $\ell \in \mathbf{N}^{n-1}$, $\delta^1, \dots, \delta^p \in \mathbf{N}^{2n-2}$, $p \le |\delta|$, such that

$$1 \le |\ell| = p \le |\delta| = |\alpha'| + |\gamma|\,,\ \delta^1 + \cdots + \delta^p = \delta\,,\ |\delta^1| \ge 1, \dots, |\delta^p| \ge 1\,.$$

Using (4.9) and (4.10), we obtain

$$\begin{aligned} |A_{\alpha'}| \;&\le\; \alpha!\, d^k \sum_{\mathcal{B}(\alpha')} C^{|\alpha|-|\alpha'|+p+|\beta|}\, C_1^{|\alpha'|+|\gamma|} \\ &\times\; \Gamma\left((\mu-1)|\alpha-\alpha'| + \sigma(p+|\beta|) + k\rho\right) \left((|\delta^1|-1)! \cdots (|\delta^p|-1)!\right)^{\mu-1}\,. \end{aligned} \qquad (4.14)$$

Since $|\delta^j| \ge 1$, $j = 1, \dots, p$, and $\mu > 2$, we get by Stirling's formula

$$\left((|\delta^1|-1)! \cdots (|\delta^p|-1)!\right)^{\mu-1} \;\le\; (|\delta|-p)!^{\,\mu-1} \;\le\; A\,\Gamma((\mu-1)(|\delta|-p)+1)\,,$$

where $A > 0$ depends only on μ. Then using the inequality

$$\Gamma(x)\Gamma(y) \le 2\Gamma(x+y-1),\ x, y \ge 1\,, \qquad (4.15)$$

we estimate the second line in (4.14) by

$$\begin{aligned} &2A\,\Gamma\left((\mu-1)|\alpha-\alpha'| + \sigma(p+|\beta|) + (\mu-1)((|\delta|-p) + k\rho\right) \\ &\le\; 4A\, \frac{\Gamma\left((\mu-1)|\alpha| + \sigma|\beta| + (\mu-1)(|\gamma|-1) + k\rho + \sigma\right)}{\Gamma((\mu-\sigma-1)(p-1)+1)}\,, \end{aligned}$$

where $\mu - \sigma - 1 > \sigma\tau' > 0$. Observe that for any $m \in \mathbf{N}$, the number of multi-indices $\alpha = (\alpha_1, \dots, \alpha_p)$ with $|\alpha| = m$ is given by $\binom{m+p-1}{p-1}$ (see [12], Sect. 1.2). Then the number of partitions $(\delta^1, \cdots, \delta^p)$ of $\delta = (\delta_1, \ldots, \delta_{2n-2})$ in the index set $\mathcal{B}(\alpha')$ is dominated by

$$\binom{\delta_1 + p - 1}{p-1} \cdots \binom{\delta_{2n-2} + p - 1}{p-1} \ \leq \ 2^{|\alpha'|+|\gamma|} 2^{2n(p-1)} .$$

Summing up with respect to $p \geq 1$ in (4.14) and using that $C > 4C_1$ we obtain

$$|A_{\alpha'}| \ \leq \ C_3 d^k \, C^{|\alpha|+|\beta|+|\gamma|} \, 2^{-|\alpha'|} \, \alpha! \, \Gamma\left((\mu-1)|\alpha| + \sigma|\beta| + (\mu-1)(|\gamma|-1) + k\rho + \sigma\right) \ ,$$

where $C_3 > 0$ depends only on C, C_1, μ, σ and n. Summing up with respect to α' we prove (4.12).

It remains to prove (4.11). If $\gamma = \alpha' = 0$, we just use (4.9). If $\gamma = 0$ but $\alpha' \neq 0$ we use the argument above. This completes the proof of Lemma 4.2. □

We are able now to estimate the derivatives of F_j.

Lemma 4.3 *Let $d > C > 4C_0 > 1$, where C_0 is the constant in (4.8). Then there is $R_1 > 0$ independent of d and j such that for any $\alpha, \beta \in \mathbf{N}^{n-1}$ we have*

$$|D_I^\alpha D_\varphi^\beta F_{j1}(\varphi, I)| \ \leq \ R_1 d^{j-1} C^{\alpha+|\beta|} \, \alpha! \, \Gamma((\mu-1)\alpha + \sigma|\beta| + (j-1)\varrho + 2\sigma)$$

for $(\varphi, I) \in \mathbf{T}^{n-1} \times D$.

Proof. Consider

$$B_{r,s,\gamma}(\varphi, I) = \frac{1}{\gamma!} \left[D_\xi^\gamma (p_r(\varphi, I+\xi) \, \partial_\varphi^\gamma \, a_s(\varphi - K_0(I,\xi), I)) \right]_{|\xi=0} .$$

where

$$2 \leq r + s + |\gamma| = j \ , \quad 1 \leq s \leq j-1 \, . \tag{4.16}$$

First we suppose that $r = 0$. Since $s \leq j-1$ we have $\gamma \neq 0$. Then using (4.12) we get

$$\begin{aligned} &|D_I^\alpha D_\varphi^\beta B_{0,s,\gamma}(\varphi, I)| \\ \leq \ & C_2 \, d^s \, C^{|\alpha|+|\beta|+|\gamma|} \, \alpha! \, \Gamma((\mu-1)(|\alpha|+|\gamma|-1) + \sigma(|\beta|+|\gamma|) + s\varrho + \sigma) \\ \leq \ & C_2 \, d^{j-1} \, C^{|\alpha|+|\beta|} \, \alpha! \, \Gamma((\mu-1)|\alpha| + \sigma|\beta| + (\mu+\sigma-1)(|\gamma|-1) + s\varrho + 2\sigma) \, . \end{aligned} \tag{4.17}$$

Suppose now that $r \geq 1$. We have

$$\begin{aligned} |D_I^\alpha D_\varphi^\beta B_{r,s,\gamma}(\varphi, I)| \leq & \sum_{\alpha' \leq \alpha} \sum_{\beta' \leq \beta} \sum_{\gamma' \leq \gamma} \frac{1}{\gamma!} \binom{\alpha}{\alpha'} \binom{\beta}{\beta'} \binom{\gamma}{\gamma'} \, (\gamma - \gamma')! \\ & \times |D_I^{\alpha'+\gamma'} D_\varphi^{\beta'} p_r(\varphi, I)| \, |D_I^{\alpha-\alpha'} \partial_\varphi^{\gamma+\beta-\beta'} b_{s,\gamma-\gamma'}(\varphi, I)| \, . \end{aligned} \tag{4.18}$$

Taking into account (4.8), (4.9) and (4.11), we estimate each term Σ of (4.18) by

$$\begin{aligned} |\Sigma| \quad \leq \quad & \frac{d^s}{\gamma!} \binom{\alpha}{\alpha'} \binom{\beta}{\beta'} \binom{\gamma}{\gamma'} \, (\alpha'+\gamma')! \, \beta'! \, (\alpha-\alpha')! \, (\gamma-\gamma')! \\ \times \quad & \Gamma((\mu-1)|\gamma'+\alpha'| + (\sigma-1)|\beta'| + (\sigma+\mu-1)(r-1) + 1) \\ \times \quad & \Gamma((\mu-1)|\alpha-\alpha'| + \sigma|\gamma+\beta-\beta'| + (\mu-1)|\gamma-\gamma'| + s\varrho + \sigma) \\ \times \quad & C_2 \, C_0^r \, C^{|\alpha|+|\beta|+|\gamma|} \left(\tfrac{C_0}{C}\right)^{|\alpha'|+|\gamma'|+|\beta'|} . \end{aligned}$$

We have

$$\frac{1}{\gamma!}\binom{\alpha}{\alpha'}\binom{\beta}{\beta'}\binom{\gamma}{\gamma'}(\gamma'+\alpha')!\,\beta'!\,(\alpha-\alpha')!\,(\gamma-\gamma')!$$
$$\leq\ \alpha!\,2^{|\alpha'|+|\gamma'|}\,|\beta|\cdots(|\beta-\beta'|+1)\,.$$

Multiplying it by

$$\Gamma(|\beta-\beta'|+(\mu-1)|\alpha-\alpha'|+\sigma|\gamma|+(\sigma-1)|\beta-\beta'|+(\mu-1)|\gamma-\gamma'|+s\varrho+\sigma),$$

and using that $x\Gamma(x)=\Gamma(x+1)$, $x>0$, we estimate the product by

$$\alpha!\,2^{|\alpha'|+|\gamma'|}\,\Gamma((\mu-1)|\alpha-\alpha'|+\sigma|\gamma|+\sigma|\beta|-(\sigma-1)|\beta'|+(\mu-1)|\gamma-\gamma'|+s\varrho+\sigma)\,.$$

Then multiplying it by

$$\Gamma((\mu-1)(|\alpha'|+|\gamma'|)+(\sigma-1)|\beta'|+(\sigma+\mu-1)(r-1)+1)$$

and taking into account (4.15), we estimate the product by

$$\alpha!\,2^{|\alpha'|+|\gamma'|}\,\Gamma((\mu-1)|\alpha|+\sigma|\beta|+(\sigma+\mu-1)(|\gamma|+r-1)+s\varrho+\sigma)\,.$$

Since $d>C>4C_0>1$, we estimate $|\Sigma|$ by

$$d^{j-1}\,C_2'\,C^{|\alpha|+|\beta|}\,\alpha!\,\Gamma((\mu-1)|\alpha|+\sigma|\beta|+(\sigma+\mu-1)(|\gamma|+r-1)+s\varrho)\,2^{-|\alpha'+\beta'+\gamma'|}\,.$$

Summing up with respect to $m=(\alpha',\beta',\gamma')$ we estimate

$$|D_I^\alpha\partial_\varphi^\beta B_{r,s,\gamma}(\varphi,I)|$$
$$\leq\ d^{j-1}\,\alpha!\,C_3\,C^{|\alpha|+|\beta|}\,\Gamma((\mu-1)|\alpha|+\sigma|\beta|+(\sigma+\mu-1)(|\gamma|+r-1)+s\varrho+\sigma)\,, \qquad (4.19)$$

where $C_3=C_2'\sum_{m\in\mathbf{N}^{3n-3}}2^{-m}$. On the other hand,

$$1<s\varrho<(\sigma+\mu-1)(|\gamma|+r-1)+s\varrho=(j-1)\varrho-\delta(|\gamma|+r-1),$$

where $\delta=\varrho-\mu-\sigma+1>\mu(\sigma-1)+1>0$, since $\varrho=\sigma(\nu+1)$ and $\nu>\mu$. Using (4.15), we get for $|\gamma|+r\geq 1$,

$$\Gamma((\mu-1)|\alpha|+\sigma|\beta|+(\sigma+\mu-1)(|\gamma|+r-1)+s\varrho+2\sigma)$$
$$\leq\frac{\Gamma((\mu-1)|\alpha|+\sigma|\beta|+(j-1)\varrho+2\sigma)}{\Gamma(\delta(|\gamma|+r-1)+1)}\,.$$

Applying it to (4.17) and (4.19) we obtain

$$|D_I^\alpha\partial_\varphi^\beta B_{r,s,\gamma}(\varphi,I)|\leq d^{j-1}\,C_4\,C^{|\alpha|+|\beta|}\,\alpha!\,\frac{\Gamma((\mu-1)|\alpha|+\sigma|\beta|+(j-1)\nu+2\sigma)}{\Gamma(\delta(|\gamma|+r-1)+1)}\,.$$

where C_4 does not depend on d. Summing up with respect to γ and r we get

$$|D_I^\alpha D_\varphi^\beta F_{j1}(\varphi,I)|\leq R_1\,d^{j-1}\,C^{|\alpha|+|\beta|}\,\alpha!\,\Gamma((\mu-1)|\alpha|+\sigma|\beta|+(j-1)\varrho+2\sigma)\,,$$

where $R_1=C_4\sum_{|\gamma|+r\geq 1}\Gamma(\delta(|\gamma|+r-1)+1)^{-1}$. We have proved the lemma. □

We are ready to estimate $p_j^0(I)$. In view of (4.7) we have

$$p_j^0(I) \;=\; (2\pi)^{1-n}\int_{\mathbf{T}^{n-1}}(p_j(\varphi,I)+F_{j1}(\varphi,I))d\varphi\,.$$

Then (4.8) and Lemma 4.3 imply

$$\begin{aligned}
|D_I^\alpha p_j^0(I)| \;&\leq\; R_1\, d^{j-1}\, C^{|\alpha|}\,\alpha!\,\Gamma((\mu-1)|\alpha|+(j-1)\varrho+2\sigma)\\
&+\; C_0^{|\alpha|+j+1}\,\alpha!\,\Gamma((\mu-1)|\alpha|+(\sigma+\mu-1)(j-1)+1)\\
&\leq\; d^{j-1/2}C^{|\alpha|}\,\alpha!\,\Gamma((\mu-1)|\alpha|+(j-1)\varrho+2\sigma)\,,
\end{aligned}$$

since $\varrho=\sigma\nu+\sigma>\sigma+\mu-1$. Here we choose d sufficiently large as a function of R_1 and C_0. This proves the first part of (4.9) for $k=j$.

It remains to estimate $F_{j2}(\varphi,I)$ and $a_j(\varphi,I)$. As in [10], Lemma V.2, we obtain:

Lemma 4.4 *There is $R_2>0$ independent of d and j such that for any $\alpha,\beta\in\mathbf{Z}_+^{n-1}$ we have*

$$|D_I^\alpha D_\varphi^\beta F_{j2}(\varphi,I)| \;\leq\; R_2 d^{j-\frac{1}{2}}C^{|\alpha|+|\beta|}\,\alpha!\,\Gamma((\mu-1)|\alpha|+\sigma|\beta|+(j-1)\varrho+2\sigma)$$

for $(\varphi,I)\in\mathbf{T}^{n-1}\times D$.

Proof. Using (4.9) for $k\leq j-1$, we obtain

$$\begin{aligned}
|D_I^\alpha D_\varphi^\beta(a_s(\varphi,I)p_{j-s}^0(I))| \leq\; & d^{j-\frac{1}{2}}C^{|\alpha|+|\beta|}\sum_{\gamma\leq\alpha}\binom{\alpha}{\gamma}\,\gamma!\,\Gamma((\mu-1)|\gamma|+\sigma|\beta|+s\varrho)\\
&\times(\alpha-\gamma)!\,\Gamma((\mu-1)|\alpha-\gamma|+(j-s-1)\varrho+2\sigma).
\end{aligned}$$

Recall that $1\leq s\leq j-1$ and $\mu-1>\tau'+1>7/2$. Using Lemma 4.5 below we complete the proof of the lemma as in [10], Lemma V.2. $\square$

As in [10], Lemma A.3 we prove

Lemma 4.5 *Let $\mu>9/2$. Then there exists a positive constant M such that for any $x,y\in\mathbf{N}$ and $p\geq1,\ q\geq1$, we have*

$$\Gamma((\mu-1)x+p)\,\Gamma((\mu-1)y+q) \;\leq\; M\,\Gamma((\mu-1)(x+y)+p+q)\,B(p,q)^{1/3}\binom{x+y}{x}^{-5/6}.$$

Finally, combining Lemma 4.3 and Lemma 4.4 we estimate the right hand side of the first equation in (4.7) as follows:

$$|D_I^\alpha D_\varphi^\beta f_j(\varphi,I)| \;\leq\; M_3\,d^{j-\frac{1}{2}}\,C^{|\alpha|+|\beta|}\,\alpha!\,\Gamma((\mu-1)|\alpha|+\sigma|\beta|+(j-1)\varrho+2\sigma)\,,\ \forall\alpha,\beta\in\mathbf{N}^{n-1}\,,$$

where M_3 does not depend on j and d. Now applying Proposition 6.1 we find a_j satisfying (4.7) and the second inequality in (4.9) on $\mathbf{T}^{n-1}\times D$ for $k=j$, choosing $d\gg1$. This proves Proposition 4.1. $\square$

Using Lemma 4.1 we obtain symbols a and p^0 in $S_\ell(\mathbf{T}^{n-1}\times D)$, $\ell=(\sigma,\mu,\varrho)$, such that

$$M_1(\lambda)A(\lambda)-A(\lambda)M_0(\lambda)=R_1(\lambda)+R'(\lambda)\,, \tag{4.20}$$

where $R_1(\lambda)$ and $R'(\lambda)$ are λ-FIOs, $\lambda\in\mathcal{D}$, of the form (4.3). The amplitude $r\in S_\ell(\mathbf{T}^{n-1}\times D)$ of $R_1(\lambda)$ vanishes to infinite order for $I\in E_\kappa$, and the amplitude r' of $R'(\lambda)$ is in the residue class $S_\ell^{-\infty}(\mathbf{T}^{n-1}\times D)$ (see Appendix) and we have

$$R'(\lambda)=O_{s,s'}\left(e^{-c|\lambda|^{1/\rho}}\right)\;:\;H^s(\mathbf{T}^{n-1};\mathbf{L})\;\longrightarrow\;H^{s'}(\mathbf{T}^{n-1};\mathbf{L})\;,\;s,s'\in\mathbf{R}\,, \tag{4.21}$$

where H^s is the corresponding Sobolev space. Recall that the distribution kernel of $\widetilde{M_0(\lambda)}$ is

$$\widetilde{M_0}(x,y,I,\lambda)=\left(\frac{\lambda}{2\pi}\right)^{n-1}\int_{\mathbf{R}^{2n-2}}e^{i\lambda(\langle x-y,I\rangle+Q(x,I))}\,p^0(I,\lambda)\,dI\,.$$

One can consider $e^{i\pi(m+\vartheta/4)}\widetilde{M_0(\lambda)}$ as a "quantum Birkhoff normal form" of the "quantum Poincaré map" $\overline{M_0}(\lambda)$ and the function $p^0(I,\lambda), I\in E_\kappa$, as its Birkhoff invariants.

5 Construction of quasimodes. Orthogonality relations.

5.1 Construction of quasimodes

Consider the λ-FIO $R_1(\lambda)$ having the form (4.3) with phase function Q and amplitude $r\in S_\ell$, which is uniformly compactly supported with respect to I in D. Given $C_1>0$ we extend Q and r for complex I as in (A.4). Changing the contour of integration and using (A.7) we write the distribution kernel of $R_1(\lambda)$ as follows

$$\left(\frac{1}{2\pi}\right)^{n-1}\int_{\mathbf{R}^{n-1}}e^{i(\langle x-y,I\rangle+\lambda Q(x,I/\lambda))}\,r(x,I/\lambda,\lambda)\,dI\;+\;O\left(e^{-c|\lambda|^{1/(\mu-1)}}\right)\;,\;\lambda\in\mathcal{D}\,.$$

This implies

$$\widetilde{R_1(\lambda)}e_p(x)\;=\;e^{i\lambda Q(x,(p+\vartheta_0/4)/\lambda)}r(x,(p+\vartheta_0/4)/\lambda,\lambda)\widetilde{e}_p(x)+O\left(e^{-c|\lambda|^{1/(\mu-1)}}\right)\widetilde{e}_p(x)\,, \tag{5.1}$$

for $\lambda\in\mathcal{D}$, where for each $s,s'\in\mathbf{R}$ the reminder term is $O(e^{-c|\lambda|^{1/(\mu-1)}}):H^s\to H^{s'}$.

The index set $\mathcal{M}$ and the frequencies I_ν of the quasimode $\mathcal{Q}$ we are going to construct are determined as follows: The pair $\nu=(p,q)\in\mathbf{N}^{n-1}\times\mathbf{N}$ belongs to $\mathcal{M}$ if there exists $I_\nu\in E_\kappa$ and $\lambda_\nu^0>0$ such that the following quantization conditions hold:

$$\lambda_\nu^0 I_\nu\;=\;p+\vartheta_0/4+O(1)\;,\quad -\lambda_\nu^0 K(I_\nu)+\pi(m+\vartheta/4)\;=\;2\pi q+O(1)\,, \tag{5.2}$$

as $|p|+|q|\to\infty$. We set

$$\lambda_\nu^0\;=\;-\frac{2\pi q-\pi(m+\vartheta/4)}{K(I_\nu)}\,.$$

Recall that $K|_{E_\kappa}<0$ is given by (3.7) and m is the number of vertices of the broken geodesic γ. We set $\lambda^0=\lambda_\nu^0$ in (A.4) and suppose that λ satisfies $|\lambda-\lambda_\nu^0|<C_2$, where $C_2\gg 1$ is fixed. Then (5.2) holds if λ_ν^0 is replaced by λ, and using (5.1) and Remark A.1,

we obtain $R_1(\lambda)e_p = O\left(e^{-c|\lambda|^{1/(\mu-1)}}\right) e_p$, where $c > 0$. In the same way, using (A.4) and Remark A.1 for p^0 and $Q(x, I/\lambda) + K(I/\lambda)$, we get

$$\widetilde{M_0(\lambda)} e_p(x) = Z_\nu(\lambda)\, e_p + O\left(e^{-c|\lambda|^{1/(\mu-1)}}\right) \widetilde{e}_p(x)\,,$$

where $Z_\nu(\lambda) = e^{-i\lambda K((p+\vartheta_0/4)/\lambda)} p^0((p+\vartheta_0/4)/\lambda, \lambda, \lambda_\nu^0)$.

More precisely, $p^0(I,\lambda) = 1 + p_1^0(I)\lambda^{-1} + \cdots$ is given by (A.3) with $\lambda^0 = \lambda_\nu^0$, and $K(z, \lambda_\nu^0)$ and $p^0(z, \lambda, \lambda_\nu^0)$, $z = I + iY$, are defined by (A.4). We write

$$Z_\nu(\lambda) = \exp\left(-i\lambda K((p+\vartheta_0/4)/\lambda, \lambda_\nu^0) + \operatorname{Log} p^0((p+\vartheta_0/4)/\lambda, \lambda, \lambda_\nu^0)\right).$$

We are going to solve the equation $e^{i\pi(m+\vartheta/4)} Z_\nu(\lambda) = 1$ modulo $O\left(e^{-c|\lambda_\nu^0|^{1/(\mu-1)}}\right)$. For that we are looking for a small perturbation $\lambda = \lambda_\nu$ of λ_ν^0 such that

$$\begin{aligned} -\lambda K((p+\vartheta_0/4)/\lambda, \lambda_\nu^0) + \pi(m+\vartheta/4) &+ \operatorname{Log} p^0((p+\vartheta_0/4)/\lambda, \lambda, \lambda_\nu^0) \\ &= 2\pi q + O\left(e^{-c|\lambda_\nu|^{1/(\mu-1)}}\right). \end{aligned} \tag{5.3}$$

Suppose for a moment that λ_ν exist. Then for each $\nu = (p,q) \in \mathcal{M}$ we set

$$v_\nu^0 := T(\lambda_\nu)A(\lambda_\nu)e_p \quad \text{and} \quad u_\nu^0 := G(\lambda_\nu)v_\nu^0 = G(\lambda_\nu)T(\lambda_\nu)A(\lambda_\nu)e_p\,. \tag{5.4}$$

In view of (5.3)

$$(\bar{M}(\lambda_\nu) - \mathrm{Id})v_\nu^0 = O\left(e^{-c|\lambda_\nu|^{1/\varrho}}\right) v_\nu^0\,, \tag{5.5}$$

and we get

$$\left|\begin{aligned} (\Delta + \lambda_\nu^2)\, u_\nu^0(x) &= O_s\left(e^{-c|\lambda_\nu|^{1/\varrho}}\right) u_\nu^0\,, \text{ in } X\,, \\ u_\nu^0|_\Gamma &= O_s\left(e^{-c|\lambda_\nu|^{1/\varrho}}\right) u_\nu^0 \end{aligned}\right.$$

Then modifying u_ν^0 by an exponentially small function, if necessary, we obtain

$$\left|\begin{aligned} \|(\Delta + \lambda_\nu^2)\, u_\nu^0(x)\|_{H^s} &= O_s\left(e^{-c|\lambda_\nu|^{1/\varrho}}\right) \|u_\nu^0\|_{L^2(X)}\,, \\ u_\nu^0|_\Gamma &= 0\,. \end{aligned}\right. \tag{5.6}$$

We are going to solve (5.3). Introduce a small parameter $\varepsilon_\nu = \left(\lambda_\nu^0\right)^{-1}$ and put

$$\left\{\begin{aligned} \lambda_\nu &= \lambda_\nu^0(1 + \varepsilon_\nu v_\nu)\,, \\ p + \vartheta_0/4 &= \lambda_\nu(I_\nu + \varepsilon_\nu w_\nu)\,, \end{aligned}\right.$$

where $|v_\nu|$, $\|w_\nu\| = O(1)$. Substituting it in (5.3) and using (5.2) we obtain

$$\left\{\begin{aligned} w_\nu + v_\nu I_\nu + \varepsilon v_\nu w_\nu &= r_\nu^1 \\ K(I_\nu)v_\nu + \langle \nabla K(I_\nu), w_\nu\rangle + \varepsilon F_\nu(v_\nu, w_\nu) &= r_\nu^2 \end{aligned}\right. \tag{5.7}$$

for $\varepsilon = \varepsilon_\nu$, where $r_\nu^j \in \mathbf{C}$, $r_\nu^j = O(1)$ and $\operatorname{Im} r_\nu^j = O(\varepsilon_\nu)$, $j = 1,2$. Here $F_\nu : \mathbf{C}^{n-1} \times \mathbf{C}^{n-1} \to \mathbf{C}$ are uniformly compactly supported with respect to ν,

$$\overline{\partial}_v F_\nu(v,w)\ ,\ \overline{\partial}_w F_\nu(v,w) = O\left(e^{-c|\lambda_\nu|^{1/(\mu-1)}}\right) \text{ for } |\operatorname{Im} v|\,,\ |\operatorname{Im} w| = O(1)\,,$$

uniformly with respect to ν. Moreover, for $|\alpha|+|\beta|\leq 2$ we have

$$|\partial_v^\alpha \partial_w^\beta F_\nu(v,w)| \;\leq\; \widetilde{C}\,, \tag{5.8}$$

where $\widetilde{C}$ does not depend on ν. On the other hand, $I_\nu = O(a)$, $0 < a \leq a_0 \ll 1$, and $-K(I) = L(\varrho_1) + O(a)$ in view of (3.7), where $L(\varrho_1)$ is the length of γ. Fix $0 < a \ll 1$. Applying the implicit function theorem we get $v_\nu(\varepsilon)$ and $w_\nu(\varepsilon)$, $0 < \varepsilon \leq \varepsilon^0$, satisfying (5.7), where ε^0 is independent of ν. Then setting $\varepsilon = \varepsilon_\nu$, we obtain v_ν and w_ν for $|\nu| \gg 1$.

5.2 Asymptotics of the counting function

In view of (5.1), we have $\nu = \lambda_\nu^0(I_\nu, -K(I_\nu)/2\pi) + O(1)$, $\nu\in\mathcal{M}$, $|\nu|\to\infty$. Set $D(r) = \{\tau(I, -K(I)/2\pi)\, :\; I\in E_\kappa, 1\leq\tau\leq r\}$ and denote by $D(r)+C$, $C>0$, the set of all $x\in\mathbf{R}^n$ within a distance $\leq C$ to $D(r)$. Then we have

$$N^*(r) \;:=\; \#\{\nu\in\mathcal{M} :\, \lambda_\nu\leq r\} \;=\mathrm{Vol}\,(D(r)+C) + o(r^n)\,,\; r\to+\infty$$

for some $C>0$. As in [3], Sect. 4.4, we obtain $\mathrm{Vol}\,((D(r)+C)\backslash D(r)) = o(r^n)$, and calculating the volume of $D(r)$ we find

$$N^*(r) \;=\frac{r^n}{2\pi n}\int_{E_\kappa} (\langle I, \nabla K(I)\rangle - K(I))\,dI \;+\; o(r^n)\,.$$

On the other hand, using (3.8) and taking into account that P^0 and χ are symplectic, hence, volume preserving, we evaluate the integral above by

$$\frac{1}{(2\pi)^{n-1}}\int_{\mathbf{T}^{n-1}\times E_\kappa} L(\chi(\varphi, I))\,d\varphi dI \;=\; \frac{1}{(2\pi)^{n-1}}\int_\Lambda L(\rho)\,d\rho\,,$$

L being the "return time function" for the Poincaré map P, i.e. $\Phi^{L(\rho)}(\rho) = P(\rho)$, where $t\to\Phi^t(\rho)$ is the broken bicharacteristic starting from ρ. This implies

$$N^*(r) \;=\; \left(\frac{r}{2\pi}\right)^n \frac{\mathrm{Vol}\,\mathcal{T}}{n} \;+\; o(r^n)\,,\; r\to+\infty\,, \tag{5.9}$$

where $\mathcal{T}$ is the flow-out by the broken bicharacteristic flow of the union Λ of the invariant tori Λ_ω, $\omega\in\Omega_\kappa$, and $\mathrm{Vol}\,\mathcal{T}$ stands for the Liouville measure of $\mathcal{T}$ on S^*X.

5.3 Orthogonality relations

First we show that λ_ν can be chosen real-valued. Taking into account (5.4), we obtain

$$\|u_\nu^0\|_{L^2(X)} \;\leq C\,,\; \forall\nu\in\mathcal{M}\,, \tag{5.10}$$

where $C>0$ is a constant. Then using (5.6) we estimate the normal derivative $\partial_\eta u_\nu^0|_\Gamma$, η being the unit inner normal to Γ, as follows

$$\|(\partial_\eta u_\nu^0)|_\Gamma\|_{L^2(\Gamma)} \;\leq\; O(1)\|u_\nu^0\|_{H^2(X)} \;\leq\; O(|\lambda_\nu|^2)\,.$$

On the other hand, $(\partial_\eta u_\nu^0)|_\Gamma = \imath^*\partial_\eta G(\lambda_\nu)v_\nu^0$, $v_\nu^0 = T(\lambda_\nu)A(\lambda_\nu)e_p$. It is easy to see that

$$\imath^*\partial_\eta G(\lambda) \;=\; B(\lambda)(\bar{M}(\lambda) \,+\, \mathrm{Id}\,) \,+\, O\left(e^{-c|\lambda_\nu|^{1/\varrho}}\right)\,,$$

where $B(\lambda)$ is a λ-PDO on Γ which is elliptic in a neighborhood W_1 of ρ_1. Moreover, the G^ϱ-frequency set of $\{v_\nu^0\}_{\nu\in\mathcal{M}}$ is contained in W_1 choosing $0 < a \le a_0 \ll 1$. Then using (5.5) and (5.10) we get

$$\|u_\nu^0\|_{L^2(X)} \ \ge\ \tilde{C}(1+|\lambda_\nu|)^{-2}\|T(\lambda_\nu)A(\lambda_\nu)e_p\|_{L^2(\Gamma)} - O\left(e^{-c|\lambda_\nu|^{1/\varrho}}\right) \ \ge\ C(1+|\lambda_\nu|)^{-2} \quad (5.11)$$

for $\nu = (p,q) \in \mathcal{M}$, where C and $\tilde{C}$ are positive constants. By Green's formula we get

$$C|\lambda_\nu^2 - \overline{\lambda_\nu}^2| \ \le\ |\langle \lambda_\nu^2 u_\nu^0, u_\nu^0\rangle \ -\ \langle u_\nu^0, \lambda_\nu^2 u_\nu^0\rangle| \ \le\ O\left(e^{-c|\lambda_\nu|^{1/\varrho}}\right),$$

which allows us to take λ_ν in $\mathbf{R}$. From now we suppose that $\lambda_\nu > 0$. In view of (5.3), λ_ν is a small perturbation of λ_ν^0, and it satisfies

$$\begin{aligned} &-\lambda_\nu K((p+\vartheta_0/4)/\lambda_\nu) + \pi(m+\vartheta/4) \\ &+ \operatorname{Log} p^0((p+\vartheta_0/4)/\lambda_\nu, \lambda_\nu) \ =\ 2\pi q + O\left(e^{-c|\lambda_\nu|^{1/(\mu-1)}}\right). \end{aligned} \quad (5.12)$$

To obtain the orthogonality relations, we consider three cases. First we suppose that $\lambda_\nu - \lambda_{\nu'} \ge \lambda_\nu/2$. As in [3], Sect. 4.2, we write $\langle u_\nu^0, u_{\nu'}^0\rangle$ as an oscillatory integral with a large parameter λ_ν and an additional parameter $\lambda_{\nu'}/\lambda_\nu \in (0,1/2]$. The amplitude of that integral belongs to $S_{\mu,\varrho}$ and the phase function is Gevrey G^μ smooth and it has no critical points. Then integrating by parts we get

$$|\langle u_\nu^0, u_{\nu'}^0\rangle_{L^2(X)}| \ \le\ Ce^{-c|\lambda_\nu|^{1/\varrho}},$$

where c and C are positive constants. Suppose now that

$$\exp\left(-\frac{c}{3}|\lambda_\nu|^{1/\varrho}\right) \ \le\ \lambda_\nu \ -\ \lambda_{\nu'} \ \le\ \frac{\lambda_\nu}{2}.$$

Applying Green's formula we get

$$|\langle u_\nu^0, u_{\nu'}^0\rangle_{L^2(X)}| \ \le\ \exp\left(\frac{c}{3}|\lambda_\nu|^{1/\varrho}\right) |(\lambda_\nu^2 - \lambda_{\nu'}{}^2)\langle u_\nu^0, u_{\nu'}^0\rangle| \ \le\ C\exp\left(-\frac{c}{3}|\lambda_\nu|^{1/\varrho}\right).$$

Consider the case

$$|\lambda_\nu \ -\ \lambda_{\nu'}| \ \le\ \min\left[\exp\left(-\frac{c}{3}|\lambda_\nu|^{1/\varrho}\right), \exp\left(-\frac{c}{3}|\lambda_{\nu'}|^{1/\varrho}\right)\right].$$

Given $\nu \in \mathcal{M}$ we denote by $\mathcal{I}_\nu$ the connected component of the union

$$\cup_{\nu'\in\mathcal{M}}\left[\lambda_{\nu'} \ -\ \exp\left(-\frac{c}{3}\lambda_{\nu'}^{1/\varrho}\right),\ \lambda_{\nu'} \ +\ \exp\left(-\frac{c}{3}\lambda_{\nu'}^{1/\varrho}\right)\right]$$

which contains λ_ν and set $\mathcal{M}_\nu = \{\nu' \in \mathcal{M} \ :\ \lambda_{\nu'} \in \mathcal{I}_\nu\}$. Using (5.9) we estimate the length of $\mathcal{I}_\nu$ by

$$|\mathcal{I}_\nu| \le C\lambda_\nu^n \exp\left(-\frac{c}{3}\lambda_\nu^{1/\varrho}\right).$$

We consider now the linear space $\mathcal{F}_\nu = \mathrm{Span}\,\{u^0_{\nu'} : \nu' \in \mathcal{M}_\nu\}$ and set $J_\nu = \#\mathcal{M}_\nu$ (the cardinality of $\mathcal{M}_\nu$). Let

$$v_\delta \;=\; \sum_j \delta_j u^0_{\nu_j} \;\in \mathcal{F}_\nu \;,\;\; \delta = (\delta_1, \ldots, \delta_{J_\nu}) \neq 0 \;. \tag{5.13}$$

Fix $0 < c_1 < c/3$ and note that

$$u^0_{\nu'} \;=\; G(\lambda_\nu)T(\lambda_\nu)A(\lambda_\nu)e_{p'} \;+\; O\left(\exp(-c_1\lambda_\nu^{1/\varrho})\right) e_{p'} \;,$$

uniformly with respect to $\nu \in \mathcal{M}_\nu$ and $\nu' \in \mathcal{M}_\nu$. It follows from (5.3) that $q' = q$ for each $\nu' = (p', q') \in \mathcal{M}_\nu$. Then the dimension of $\mathcal{F}_\nu$ is $J_\nu = \#\mathcal{M}_\nu$ and we obtain $J_\nu = O(\lambda_\nu^n)$ in view of (5.9). Moreover,

$$\|v_\delta\|_{L^2(X)} \leq C_1 \|\delta\| \;, \quad \|\delta\| = (\sum |\delta_j|^2)^{1/2} \;,$$

where C_1 is independent of ν. By (5.6) we get

$$\left| \begin{array}{rcl} \|(\,\Delta \,+\, \lambda_\nu^2\,)\; v_\delta(x)\|_{H^s} & \leq & C_s\, \|\delta\|\, e^{-c_1|\lambda_\nu|^{1/\varrho}} \;, \\ v_\delta|_\Gamma & = & 0\,. \end{array} \right. \tag{5.14}$$

On the other hand, as in the proof of (5.11) we get

$$\begin{array}{rcl} \|v_\delta\|_{L^2(X)} & \geq & C_2(1+\lambda_\nu)^{-2}\, \|(\partial_\eta v_\delta)|_\Gamma\|_{L^2(\Gamma_1)} \\ & \geq & C_3(1+\lambda_\nu)^{-2}\, \|\sum_j \delta_j e_{p_j}\|_{L^2(\mathbf{T}^{n-1},\mathbf{L})} \\ & - & O\left(e^{-c_1|\lambda_\nu|^{1/\varrho}}\right)\, \|v_\delta\|_{L^2(X)} \;\geq\; C_4(1+\lambda_\nu)^{-2}\, \|\delta\| \,, \end{array} \tag{5.15}$$

where the constants are independent of ν. We fix in each $\mathcal{F}_\nu$ an orthonormal basis $v_{\delta(\nu,k)}$, $k = 1, \ldots, J_\nu$, given by (5.13) with some $\delta = \delta(\nu, k)$. Using (5.15) we get

$$C \;\leq\; \|\delta(\nu, k)\| \;\leq\; C^{-1}(1+\lambda_\nu)^2 \tag{5.16}$$

where $C > 0$ does not depend on ν and k. Let $v_{\delta(\nu',k')}$ belongs to the space $\mathcal{F}_{\nu'}$ corresponding to $\nu' \notin \mathcal{M}_\nu$. Then using the estimates in the first two cases as well as (5.16) and the relation $J_\nu = O(\lambda_\nu^n)$, we estimate the scalar product $\langle v_{\delta(\nu,k)}, v_{\delta(\nu',k')}\rangle$ by

$$\min\,(\lambda_\nu^{2n+4} \exp(-\frac{c}{3}\lambda_\nu^{1/\varrho}),\; \lambda_{\nu'}^{2n+4} \exp(-\frac{c}{3}\lambda_{\nu'}^{1/\varrho}))\,.$$

Finally, using Gramm-Schmidt procedure we obtain u_ν in (1.1) as linear combinations of $v_{\delta(\nu,k)}$ taking $c = c_1/2$.

By construction the G^ϱ micro-support $\mathrm{WF}^\varrho(\mathcal{Q})$ of $\mathcal{Q}$ (see Appendix) is contained in the flow-out by the bicharacteristic flow in $S^\star X$ of the union of the invariant tori Λ.

6 On the homological equation in Gevrey classes

Suppose that $f \in G^{\sigma,\mu}(\mathbf{T}^{n-1} \times D)$ satisfies

$$\left|D_I^\alpha D_\varphi^\beta f(\varphi, I)\right| \;\leq\; d_0\, C^{|\alpha|+|\beta|}\; \alpha!\,\Gamma((\mu-1)|\alpha| + \sigma|\beta| + q) \;,\;\; (\varphi, I) \in \mathbf{T}^{n-1} \times D, \tag{6.1}$$

for any $\alpha, \beta \in \mathbf{N}^{n-1}$, and some $C \geq 1$ and $q > 0$. Let

$$\int_{\mathbf{T}^{n-1}} f(\varphi, I) d\varphi \;=\; 0\,, \quad (\varphi, I) \in \mathbf{T}^{n-1} \times E_\kappa. \tag{6.2}$$

We are going to find $u \in G^{\sigma,\mu}(\mathbf{T}^{n-1} \times D)$, which solves the "homological" equation

$$\begin{cases} u(\varphi - \omega(I), I) - u(\varphi, I) \;=\; f(\varphi, I)\,, \; (\varphi, I) \in \mathbf{T}^{n-1} \times E_\kappa, \\ \int_{\mathbf{T}^{n-1}} u(\varphi, I) d\varphi \;=\; 0\,, \; I \in D, \end{cases} \tag{6.3}$$

and provide the corresponding Gevrey estimates for the derivatives of u. Recall that $\omega : D \to \Omega$ is a diffeomorphism of Gevrey class $G^{\tau'+2}(D, \Omega)$ given by Theorem 3.1, and there exists $C_0 > 0$ such that for each $\alpha \in \mathbf{N}_+^{n-1}$ we have:

$$|D_I^\alpha \omega(I)| \;\leq\; C_0^{|\alpha|}\, \alpha!\, ((|\alpha| - 1)!)^{\tau'+1}\,, \quad I \in D\,. \tag{6.4}$$

Proposition 6.1 *Let $f \in G^{\sigma,\mu}(\mathbf{T}^{n-1} \times D)$ satisfy (6.1) and (6.2), where $\sigma > 1$ and $\mu - 1 > \sigma(\tau' + 1)$. Then there exists $u \in G^{\sigma,\mu}(\mathbf{T}^{n-1} \times D)$ satisfying (6.3) and the estimate*

$$\left| D_I^\alpha D_\varphi^\beta u(\varphi, I) \right| \;\leq\; R\, d_0\, C^{\alpha + |\beta| + \nu}\; \alpha!\, \Gamma\left((\mu - 1)|\alpha| + \sigma|\beta| + \sigma(\nu - 1) + q\right)\,, \tag{6.5}$$

for any $(\varphi, I) \in \mathbf{T}^{n-1} \times D$ and $\alpha, \beta \in \mathbf{N}^{n-1}$, where the constant $R > 0$ does not depend on d_0.

Proof. Define the residue $[z]_{2\pi}$ of $z \in \mathbf{R}$ by $z = [z]_{2\pi} + 2\pi k_n\,,\; -\pi < [z]_{2\pi} \leq \pi\,,\; k_n \in \mathbf{Z}$, and for $0 \neq k \in \mathbf{Z}^n$ consider

$$z_k(I) = 1 - e^{-i\langle \omega(I), k\rangle} + \; \kappa |k|^{-\tau}\, \psi\left(|k|^\tau [\langle \omega(I), k\rangle]_{2\pi}\right)\,.$$

Here $\psi \in G^{1+\epsilon}(D)$, satisfies $0 \leq \psi \leq 1$, $\psi(x) = 0$ for $|x| \geq \kappa/2$, and $\psi(x) = 1$ for $|x| \leq \kappa/4$, $0 < \kappa \ll 1$ being the small constant in (3.5), and we fix $0 < \epsilon < \mu - 1 - \sigma(\tau' + 1)$. Choosing $C_0 \gg 1$ sufficiently big we suppose that $|D_x^p \psi(x)| \leq C_0^{p+1} p!^{1+\epsilon}$, $x \in \mathbf{R}$, $p \in \mathbf{Z}_+$. It is easy to see that

$$|z_k(I)| \;\geq\; \frac{\kappa_0}{|k|^\tau}\,, \quad I \in D,\; k \in \mathbf{Z}^{n-1} \backslash \{0\}\,, \tag{6.6}$$

where $\kappa_0 = \kappa/2\pi$. Indeed, writing

$$z_k(I) = 1 - \cos\left([\langle \omega(I), k\rangle]_{2\pi}\right) + \; \kappa|k|^{-\tau}\, \psi\left(|k|^\tau [\langle \omega(I), k\rangle]_{2\pi}\right) + i \sin\left([\langle \omega(I), k\rangle]_{2\pi}\right)\,,$$

and estimating below $\operatorname{Re} z_k(I)$, respectively $|\operatorname{Im} z_k(I)|$, when $|[\langle \omega(I), k\rangle]_{2\pi}|$ belongs to the union $[0, \kappa|k|^{-\tau}/4] \cup [\pi/2, \pi]$, respectively to the interval $[\kappa|k|^{-\tau}/4, \pi/2]$, we obtain (6.6).

Consider the Fourier expansions of f and u

$$f(\varphi, I) \;=\; \sum_{k \in \mathbf{Z}^{n-1}} e^{i\langle k, \varphi\rangle} f_k(I)\;,\; u(\varphi, I) \;=\; \sum_{k \in \mathbf{Z}^{n-1}} e^{i\langle k, \varphi\rangle} u_k(I),$$

where

$$f_k(I) \;=\; (2\pi)^{1-n} \int_{\mathbf{T}^{n-1}} e^{-i\langle k, \varphi\rangle}\, g(\varphi, I)\, d\varphi,$$

and $u_k(I)$ is defined in the same way. We set $u_0 = 0$, and

$$u_k(I) \;=\; -z_k(I)^{-1} f_k(I), \quad I \in D,\; 0 \neq k \in \mathbf{Z}^{n-1}\,.$$

Then u satisfies (6.3) since $z_k(I) = 1 - \exp(-i\langle \omega(I), k\rangle)$ on E_κ. We are going to find suitable Gevrey estimates for $u_k(I)$. To this end we estimate the derivatives of $z_k(I)^{-1}$.

Lemma 6.1 *There is $\widetilde{C}_0 > 0$ depending only on C_0, κ, and n such that*

$$|D_I^\alpha z_k(I)| \le \widetilde{C}_0^{|\alpha|}\, \alpha!^{\,1+\epsilon} \max_{1\le p\le|\alpha|} \left(|k|^{\tau p+p-\tau}((|\alpha|-p)!)^{\tau'+1}\right), \tag{6.7}$$

for any $I\in D$, $0\neq k\in \mathbf{Z}^{n-1}$, $0\neq\alpha\in\mathbf{N}^{n-1}$.

Proof. Applying Faa de Bruno formula for $z_{k1}(I) = 1-e^{-i\langle\omega(I),k\rangle}$, $\alpha\neq 0$, $k\neq 0$, we get

$$|D_I^\alpha z_{k1}(I)| \;\le\; \sum_{p=1}^{|\alpha|} \sum_{\substack{\alpha^1+\cdots+\alpha^p=\alpha\\ |\alpha^1|\ge 1,\dots,|\alpha^p|\ge 1}} \frac{\alpha!}{\alpha^1!\cdots\alpha^p!}\left|\langle D_I^{\alpha^1}\omega(I),k\rangle\right|\dots\left|\langle D_I^{\alpha^p}\omega(I),k\rangle\right|.$$

As in the proof of Lemma 4.2, given $1\le p\le|\alpha|$, we estimate the number of partitions $\alpha^1,\dots,\alpha^p$ of α in the index set of the second sum by $2^{|\alpha|}2^{n(p-1)}$. Then choosing $C_1 = 2^{n+2}C_0$ and using (6.4) we get (6.7) for z_{k1}. Set $\psi_k(x) = \kappa|k|^{-\tau}\psi(x|k|^\tau)$ and $z_{k2}(I) = \kappa|k|^{-\tau}\,\psi\left(|k|^\tau[\langle\omega(I),k\rangle]_{2\pi}\right) = \psi_k([\langle\omega(I),k\rangle]_{2\pi})$. For any $k\neq 0$ fixed and I in the support of $z_{k2}(I)$ we have $[\langle\omega(I),k\rangle]_{2\pi} = \langle\omega(I),k\rangle - 2\pi k_n$, where $k_n\in\mathbf{Z}$ is locally constant. Moreover, $|D_x^p\psi_k(x)| \le C_0^{p+1}p!^{1+\epsilon}|k|^{\tau p-\tau}$, $x\in\mathbf{R}$, $p\in\mathbf{Z}_+$, and using the Faa de Bruno formula for $z_{k2}(I)$, we get

$$|D_I^\alpha z_{k2}(I)| \le \sum_{p=1}^{|\alpha|} \sum_{\substack{\alpha^1+\cdots+\alpha^p=\alpha\\ |\alpha^1|\ge 1,\dots,|\alpha^p|\ge 1}} C_0^{p+1}p!^{\,\epsilon}\,|k|^{\tau p+p-\tau}\,C_0^{|\alpha|}\alpha!\,\left((|\alpha^1|-1)!\cdots(|\alpha^p|-1)!\right)^{\tau'+1},$$

which implies (6.7) for z_{k2} as above. □

Lemma 6.2 *There exists a positive constant C_1 depending only on C_0, κ and n such that*

$$|D_I^\alpha(z_k(I)^{-1})| \;\le\; C_1^{|\alpha|+1}\,\alpha!^{\,1+\epsilon} \max_{0\le j\le|\alpha|}\left(|k|^{\tau j+j+\tau}(|\alpha|-j)!^{\,\tau'+1}\right), \tag{6.8}$$

for any $I\in D$, $k\in\mathbf{Z}^{n-1}\backslash\{0\}$, $\alpha\in\mathbf{N}^{n-1}$.

Proof. Applying Faa de Bruno formula for $\alpha\neq 0$ we get

$$\left|D_I^\alpha\left(z_k(I)^{-1}\right)\right| \;\le\; \sum_{p=1}^{|\alpha|} \sum_{\substack{\alpha^1+\cdots+\alpha^p=\alpha\\ |\alpha^1|\ge 1,\dots,|\alpha^p|\ge 1}} \frac{\alpha!}{\alpha^1!\cdots\alpha^p!}\,|z_k(I)|^{-p-1}\left|D_I^{\alpha^1}z_k(I)\right|\dots\left|D_I^{\alpha^p}z_k(I)\right|.$$

Using (6.6) and Lemma 6.1 we estimate

$$\left|D_I^\alpha\left(z_k(I)^{-1}\right)\right| \;\le\; \left(\frac{\widetilde{C}_0}{\kappa_0}\right)^{|\alpha|+1}\alpha!^{\,1+\epsilon}\,|k|^\tau\sum_{p=1}^{|\alpha|} \sum_{\substack{\alpha^1+\cdots+\alpha^p=\alpha\\ |\alpha^1|\ge 1,\dots,|\alpha^p|\ge 1}} \prod_{s=1}^{p}|k|^{(\tau+1)j_s}(|\alpha^s|-j_s)!^{\,\tau'+1},$$

where $1\le j_s\le|\alpha^s|$. Choosing $C_1 = 2^{n+2}\kappa_0^{-1}\widetilde{C}_0$ we prove (6.8) as above. □

For any $m\in\mathbf{N}$ we set $\langle k\rangle_m = 1+|k_1|^m+\cdots+|k_n|^m$, $0\neq k\in\mathbf{Z}^{n-1}$. Next, for any $j\in\mathbf{N}$ we denote $m(j) = [(\tau'+1)j+\tau]+n$, where $[p]$, $p-1<[p]\le p$, stands for the integer part of $p \in \mathbf{R}$. Set $W(k) = (1+|k|)^{n-1+\varepsilon_0}$ with $0 < \varepsilon_0 < \min\{\tau' - \tau, [\tau] - \tau + 1\}$. Then

$$\tau j + j + \tau + n - 1 + \varepsilon_0 \ < \ m(j) \ \leq \ (\tau' + 1) j + \nu - 1 \, , \ \forall j \in \mathbf{N} \, ,$$

with $\nu = \tau + n + 1$. Taking into account the inequality $|k|^m \ \leq \ n^m \langle k \rangle_m \, , \ m \in \mathbf{N}, \ 0 \neq k \in \mathbf{Z}^{n-1}$, and using Lemma 6.2, we get

$$W(k) \left| D_I^\alpha \left(z_k(I)^{-1} \right) \right| \ \leq \ C_2^{|\alpha|+1} \, \alpha!^{\,1+\epsilon} \max_{0 \leq j \leq |\alpha|} \left((|\alpha| - j)!^{\,\tau'+1} \langle k \rangle_{m(j)} \right) \, , \tag{6.9}$$

for any $I \in D, \ \alpha \in \mathbf{N}^{n-1}$, and $0 \neq k \in \mathbf{Z}^{n-1}$, with a constant $C_2 > 0$ depending only on $n, \ \kappa$, and C_1. Integrating by parts, we obtain

$$k^\beta k_j^m D_I^\alpha f_k(I) \ = \ (2\pi)^{1-n} \int_{\mathbf{T}^{n-1}} e^{-i\langle k, \varphi \rangle} \, D_\varphi^\beta D_{\varphi_j}^m D_I^\alpha f(\varphi, I) \, d\varphi \, .$$

Then using (6.1) we get

$$\left| k^\beta \langle k \rangle_m D_I^\alpha f_k(I) \right| \ \leq \ n d_0 \, C^{|\alpha|+|\beta|+m} \, \alpha! \, \Gamma((\mu - 1)|\alpha| + \sigma|\beta| + \sigma m + q) \, , \tag{6.10}$$

for any $I \in D, \ 0 \neq k \in \mathbf{Z}^{n-1}$, and any $m \in \mathbf{N}$, $\alpha, \beta \in \mathbf{N}^{n-1}$. Consider

$$A_k \ := \ W(k) \left| k^\beta D_I^\alpha u_k(I) \right| \ = \ \sum_{0 \leq \gamma \leq \alpha} \frac{\alpha!}{\gamma!(\alpha - \gamma)!} W(k) \left| D_I^\gamma \left(z_k(I)^{-1} \right) \right| \, \left| k^\beta D_I^{\alpha-\gamma} f_k(I) \right| \, .$$

By Stirling's formula we have $p!^{\,\tau'+1} \ \leq \ C_3^p \, \Gamma((\tau' + 1)p + 1)$, $p \geq 0$, with some constant $C_3 > 0$. Then using (6.9) we obtain

$$W(k) \left| D_I^\gamma \left(z_k(I)^{-1} \right) \right| \, \left| k^\beta D_I^{\alpha-\gamma} f_k(I) \right|$$

$$\leq \ (C_2 C_3)^{|\gamma|+1} \, \gamma!^{\,1+\epsilon} \, \Gamma((\tau' + 1)(|\gamma| - j_\gamma) + 1) \ \left| k^\beta \langle k \rangle_{m(j_\gamma)} D_I^{\alpha-\gamma} f_k(I) \right| \, ,$$

where $0 \leq j_\gamma \leq |\gamma|$. On the other hand, $\gamma!^{\,\epsilon} \ \leq \ \widetilde{C}_3^{|\gamma|+1} \, \Gamma(\epsilon|\gamma| + 1)$, and using (4.15), we estimate $\gamma!^{\,\epsilon} \, \Gamma((\tau' + 1)(|\gamma| - j_\gamma) + 1) \leq 2\widetilde{C}_3^{|\gamma|+1} \Gamma((\tau' + 1)(|\gamma| - j_\gamma) + \epsilon|\gamma| + 1)$. Taking into account (6.10) and using again (4.15) we obtain

$$A_k \ \leq \ 4 n d_0 \, \alpha! \sum_{0 \leq \gamma \leq \alpha} C_4^{|\gamma|+1} \, C^t \, \Gamma(s) \ ,$$

where $C_4 = C_2 C_3 \widetilde{C}_3$ and

$$t = |\alpha - \gamma| + |\beta| + m(j_\gamma) \ ,$$

$$s = (\mu - 1)|\alpha - \gamma| + \sigma|\beta| + q + \sigma m(j_\gamma) + (\tau' + 1)(|\gamma| - j_\gamma) + \epsilon|\gamma|.$$

Set $\delta = \mu - 1 - \sigma(\tau' + 1) - \epsilon > 0$. We have

$$\begin{aligned} s \ &\leq \ (\mu - 1)|\alpha - \gamma| + \sigma|\beta| + \sigma(\nu - 1) + q + \sigma(\tau' + 1) j_\gamma + (\tau' + 1)(|\gamma| - j_\gamma) + \epsilon|\gamma| \\ &= \ (\mu - 1)|\alpha| + \sigma|\beta| + \sigma(\nu - 1) + q - \delta|\gamma| - (\sigma - 1)(\tau' + 1)(|\gamma| - j_\gamma) \, . \end{aligned}$$

Moreover, $t \le |\alpha| + |\beta| + m(|\gamma|) - |\gamma|$. Since $\delta > 0$, $\sigma > 1$, $|\gamma| \ge j_\gamma$ and $C \ge 1$, we obtain

$$A_k \ \le \ d_0\, d_1\, C^{|\alpha|+|\beta|}\, \alpha! \sum_{0\le\gamma\le\alpha} C_4^{|\gamma|+1}\, C^{m(|\gamma|)-|\gamma|}\, \frac{\Gamma((\mu-1)|\alpha| + \sigma|\beta| + \sigma(\nu-1) + q)}{\Gamma(\delta|\gamma|+1)}\,,$$

where $d_1 = 16n \max\limits_{1\le x\le 2} \Gamma(x)^{-1}$. This implies

$$A_k \ \le \ R_1\, d_0\, C^{|\alpha|+|\beta|}\, \alpha!\, \Gamma((\mu-1)|\alpha| + \sigma|\beta| + \sigma(\nu-1) + q)\,,$$

where $R_1 = d_1 \sum_{\gamma\in\mathbf{N}^{n-1}} \frac{C_4^{|\gamma|+1}\, C^{m(|\gamma|)-|\gamma|}}{\Gamma(\delta|\gamma|+1)}$ and $m(|\gamma|) - |\gamma| \le \tau'|\gamma| + \nu - 1$. Finally, summing up $W(k)^{-1}A_k$, we obtain

$$\left| D_I^\alpha D_\varphi^\beta v(\varphi, I)\right| \ \le \ R\, d_0\, C^{|\alpha|+|\beta|}\, \alpha!\, \Gamma((\mu-1)|\alpha| + \sigma|\beta| + \sigma(\nu-1) + q)\,,$$

where $R = R_1 \sum_{k\in\mathbf{Z}^{n-1}} W(k)^{-1}$. The constant $R > 0$ is independent of d_0. The proof of the proposition is complete. □

Appendix

We collect here some facts about Gevrey symbols.

Let M be either $\mathbf{T}^m$, $m \ge 1$, or a bounded domain in $\mathbf{R}^m$. Denote by D a bounded domain in $\mathbf{R}^m$ and $\mathcal{D} := \{z \in \mathbf{C} : |z| \ge 1,\ |\mathrm{Im}\, z| \le \widetilde{C}\}$, where $\widetilde{C} > 0$ is fixed. Fix $\sigma, \mu > 1$, $\varrho \ge \sigma + \mu - 1$, and set $\ell = (\sigma, \mu, \varrho)$. We use a class of formal Gevrey symbols $FS_\ell(M \times D)$ defined as follows. Consider a sequence of smooth functions $g_j \in C_0^\infty(M \times D)$, $j \in \mathbf{Z}_+$ such that $\mathrm{supp}\, g_j$ is contained in a fixed compact subset of $M \times D$. We say that

$$\sum_{j=0}^{\infty} g_j(\varphi, I)\, \lambda^{-j}\ , \quad \lambda \in \mathcal{D}\,, \tag{A.1}$$

is a formal Gevrey symbol in $FS_\ell(M \times D)$ if there exists a positive constant C such that g_j satisfies the estimates

$$\sup_{M\times D} |\partial_\varphi^\beta \partial_I^\alpha g_j(\varphi, I)| \ \le \ C^{j+|\alpha|+|\beta|+1}\, \beta!^{\,\sigma}\, \alpha!^{\,\mu}\, j!^{\,\varrho} \tag{A.2}$$

for any α, β and j.

The function $g(\varphi, I, \lambda)$, $(\varphi, I) \in M \times \mathbf{R}^m$, is called a realization of the formal symbol (A.1) in $M \times D$ if for each $\lambda \in \mathcal{D}$ it is smooth with respect to (φ, I) and has compact support in $M \times D$, and if there exists a positive constant C_1 such that

$$\sup_{\mathbf{Q}} |\partial_\varphi^\beta \partial_I^\alpha (g(\varphi, I, \lambda) - \sum_{j=0}^{N} g_j(\varphi, I)\lambda^{-j})| \ \le \ |\lambda|^{-N-1}\, C_1^{N+|\alpha|+|\beta|+2}\, \beta!^{\,\sigma}\, \alpha!^{\,\mu}\, (N+1)!^{\,\varrho}$$

for any multi-indices α, β and $N \in \mathbf{N}$, where $\mathbf{Q} = M \times D \times \mathcal{D}$. For example, one can take

$$g(\varphi, I, \lambda) \ = \ \sum_{j \le \varepsilon|\lambda|^{1/\varrho}} g_j(\varphi, I)\, \lambda^{-j}\,,$$

where $0 < \varepsilon \le \varepsilon_0 \ll 1$ depends only on the constant C_1 and the dimension of M and D. We denote by $S_\ell(M \times D)$ the corresponding class of symbols. In order to work with symbols holomorphic with respect to λ we fix $0 < \varepsilon < 2\varepsilon_0$, choose $\lambda^0 \ge 1$ such that $|\lambda - \lambda^0| < \lambda^0/2$ and take

$$g^0(\varphi, I, \lambda) \;=\; \sum_{j<\varepsilon(\lambda^0)^{1/\varrho}} g_j(\varphi, I)\lambda^{-j}\,. \tag{A.3}$$

Note that

$$\left|D_I^\alpha(g(\varphi,I,\lambda) - g^0(\varphi,I,\lambda))\right| \;\le\; C^{|\alpha|+1}\,\alpha!^{\,\mu}\,e^{-c|\lambda|^{1/\varrho}}\,,\;\; |\lambda - \lambda^0| < \lambda^0/2\,,$$

where $c, C > 0$ are independent of λ^0. We say that $g \in S_\ell^{-\infty}(M \times D)$ if

$$\sup_{\mathbf{Q}} |\partial_\varphi^\beta \partial_I^\alpha\, g(\varphi, I, \lambda)| \;\le\; |\lambda|^{-N}\, C^{N+|\alpha+\beta|+1}\,\beta!^{\,\sigma}\,\alpha!^{\,\mu}\,N!^{\,\varrho}$$

for $\lambda \in \mathcal{D}$, $\forall N \in \mathbf{N}$, and for any multi-indices $\alpha, \beta \in \mathbf{N}^m$, or equivalently

$$\sup_{\mathbf{Q}} |\partial_\varphi^\beta \partial_I^\alpha\, g(\varphi, I, \lambda)| \;\le\; C_1^{|\alpha+\beta|+1}\,\beta!^{\,\sigma}\,\alpha!^{\,\mu}\,\exp(-c|\lambda|^{1/\varrho})$$

for some $C_1,\ c > 0$, and any $\lambda \in \mathcal{D}$, $\alpha,\ \beta\ \in \mathbf{N}^m$. Moreover, given $f, g \in S_\ell(X \times D)$, we say that f is equivalent to g ($f \sim g$) if $f - g \in S_\ell^{-\infty}(X \times D)$. It is not hard to prove that any two realizations of $\sum_{j=0}^\infty g_j\lambda^{-j}$ in $S_\ell(X \times D)$ are equivalent. When $\sigma = \mu$ and $\varrho = 2\sigma - 1$, we set $S^\sigma = S_\ell$ and $S^{\sigma,-\infty} = S_\ell^{-\infty}$.

To each symbol $g \in S_\ell(M \times D)$, $\dim M = \dim D = m$, we associate an λ-PDO by

$$P(\lambda)u(x) \;=\; \left(\frac{\lambda}{2\pi}\right)^m \int_{\mathbf{R}^{2m}} e^{i\lambda\langle x-y,\xi\rangle}\, g(x,\xi,\lambda)\,u(y)d\xi dy,\;\; u \in C_0^\infty(M).$$

It is well defined modulo $\exp(-c|\lambda|^{1/\varrho})$. Indeed, for any $g \in S_\ell^{-\infty}$ we have

$$||P(\lambda)u||_{L^2} \;\le\; C\exp(-c|\lambda|^{1/\varrho})||u||_{L^2}\,,\;\; u \in C_0^\infty(M),$$

with some positive constants c and C.

Let $u(x,\lambda)$ be a family of smooth functions in M for $\lambda \in \mathcal{D}$. The G^ϱ micro-support $\mathrm{WF}^\varrho(u) \subset T^*(M)$ of u is defined as follows: $(x_0,\xi_0) \notin \mathrm{WF}^\varrho(u)$ if there exists $c > 0$ and compact neighborhoods U of x_0 and V of ξ_0 such that for any G^ϱ function v with compact support in U

$$\int e^{-i\lambda\langle x,\xi\rangle}\,v(x)u(x,h)dx \;=\; O\left(e^{-c|\lambda|^{1/\varrho}}\right)\,,$$

uniformly with respect to $\xi \in V$. Obviously, $(x_0,\xi_0,x_0,-\xi_0)$ does not belong to the G^ϱ microsupport of the distribution kernel of the λ-PDO $P(\lambda)$ above if its amplitude $p \in S_\ell$ belongs to $S_\ell^{-\infty}$ in a neighborhood of (x_0,ξ_0).

We are going to extend symbols in $S_\ell(\mathbf{T}^{n-1} \times D)$ in the complex space with respect to I. Given $g \in S_\ell$ with a compact support with respect to I in D and $\delta > 0$, we set

$$g(\varphi, I+iY, \lambda, \lambda_0) \;=\; \sum_{|\gamma|<(\delta\lambda^0)^{1/(\mu-1)}} \frac{i^{|\gamma|}Y^\gamma}{\gamma!}\,\partial_I^\gamma\, g(\varphi, I, \lambda)\,,\; Y \in \mathbf{R}^{n-1}\,,\; |Y| \le \widetilde{C}|\lambda|^{-1}\,, \tag{A.4}$$

where $\lambda \in \mathcal{D}$, $\lambda^0 \geq 1$ and $\widetilde{C} > 0$. We write also $g(\varphi, z, \lambda)$ instead of $g(\varphi, z, \lambda, \lambda_0)$. Since $g \in S_\ell$, we have

$$|D_I^\alpha D_\varphi^\beta g(\varphi, I, \lambda, \lambda_0)| \;\leq\; C^{|\alpha|+|\beta|+1}\, \alpha!^{\,\mu} \beta!^{\,\sigma}\,,$$

for $(\varphi, I) \in \mathbf{T}^{n-1} \times D$ and $\lambda \in \mathcal{D}$. Choose $\lambda^0 \geq 1$ and $\delta > 0$ so that

$$|\lambda - \lambda^0| < \lambda^0/2 \quad , \quad 0 < \delta < \frac{1}{eC\widetilde{C}2^{\mu+1}}\,. \tag{A.5}$$

Then we have

$$|D_I^\alpha D_\varphi^\beta g(\varphi, z, \lambda, \lambda_0)| \;\leq\; d_0\,(2^\mu C)^{|\alpha|+|\beta|+1}\, \alpha!^{\,\mu} \beta!^{\,\sigma}\ ,\ z = I + iY\,, \tag{A.6}$$

$$|D_I^\alpha D_\varphi^\beta \overline{\partial}_z g(\varphi, z, \lambda, \lambda_0)| \;\leq\; d_0\, e\,(2^\mu C)^{|\alpha|+|\beta|+1}\, \alpha!^{\,\mu} \beta!^{\,\sigma}\, e^{-c|\lambda|^{1/(\mu-1)}}\,, \tag{A.7}$$

for $(\varphi, I) \in \mathbf{T}^{n-1} \times D$, $|Y| \leq \widetilde{C}|\lambda|^{-1}$, and $\lambda \in \mathcal{D}$, $|\lambda| \gg L$, where $c = (2\delta/3)^{1/(\mu-1)}$, the constant $d_0 > 0$ depends only on n, and $L > 0$ is a constant. To prove (A.6), we estimate the $D_I^\alpha D_\varphi^\beta$-derivative of each term in (A.4) by

$$\frac{(\alpha+\gamma)!^{\,\mu}}{\gamma!}\beta!^{\,\sigma} C^{|\alpha|+|\beta|+|\gamma|+1}\left(\frac{\widetilde{C}}{|\lambda|}\right)^{|\gamma|} \;\leq\; e^{-|\gamma|}\, C^{|\alpha|+|\beta|+1}\, 2^{\mu|\alpha|}\, \alpha!^{\,\mu}\beta!^{\,\sigma}\,. \tag{A.8}$$

Indeed, since $\gamma!\, \leq |\gamma|^{|\gamma|}$, we obtain for $|\lambda| \geq L \gg 1$

$$\frac{(\alpha+\gamma)!^{\,\mu}}{\gamma!}\, C^{|\gamma|}\left(\frac{\widetilde{C}}{|\lambda|}\right)^{|\gamma|} \;\leq\; d_1 \alpha!^{\,\mu}\, 2^{\mu|\alpha|}\left(\frac{2^\mu C\widetilde{C}|\gamma|^{\mu-1}}{|\lambda|}\right)^{|\gamma|}\,.$$

Then using (A.5) we obtain

$$\frac{2^\mu C\widetilde{C}|\gamma|^{\mu-1}}{|\lambda|} \;\leq\; \frac{\delta 2^\mu C\widetilde{C}\lambda^0}{|\lambda|} \;\leq\; \frac{1}{e}\,,$$

which proves (A.8).

Notice that the $\overline{\partial}_z$-derivatives of all terms in (A.4) with $|\gamma| < (\delta\lambda^0)^{1/(\mu-1)} - 1$ cancel, and then using (A.8) for the remaining terms we obtain (A.7).

Remark A.1. Suppose that $g(\varphi, I, \lambda)$ vanishes up to infinite order at each $I \in E_\kappa$. Then g satisfies (3.6) uniformly with respect to $\lambda \in \mathcal{D}$. Choose δ as in (A.5) with C replaced by $C_2 = \max(C, C_1)$, where C_1 is the corresponding constant in (3.6) which depends only on C. Then for each $z = I + iY$ with $|E_\kappa - z| \leq C_0|\lambda|^{-1}$ and $C_0 > 0$ fixed, we have

$$|D_I^\alpha D_\varphi^\beta g(\varphi, I+iY, \lambda, \lambda_0)| \;\leq\; d_2\,(2^\mu C_2)^{|\alpha|+|\beta|+1}\, \alpha!^{\,\mu}\beta!^{\,\sigma}\, e^{-c|\lambda|^{1/(\mu-1)}}\,.$$

To prove the remark, we apply (3.6) to each term of (A.4) and then use (A.8).

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