Time fluctuations of the random average process with parabolic initial conditions

L.R.G. Fontes*
Universidade de São Paulo

D.P. Medeiros†
Universidade Federal da Bahia

M. Vachkovskaia‡
Universidade de São Paulo

Abstract

The random average process is a randomly evolving d-dimensional surface whose heights are updated by random convex combinations of neighboring heights. The fluctuations of this process in case of linear initial conditions have been studied before. In this paper, we analyze the case of polynomial initial conditions of degree 2 and higher. Specifically, we prove that the time fluctuations of a initial parabolic surface are of order $n^{1-d/4}$ for $d = 1, 2, 3$; $\sqrt{\log n}$ in $d = 4$; and are bounded in $d \geq 5$. We establish a central limit theorem in $d = 1$. In the bounded case of $d \geq 5$, we exhibit an invariant measure for the process as seen from the average height at the origin and describe its asymptotic space fluctuations. We consider briefly the case of initial polynomial surfaces of higher degree to show that their time fluctuations are not bounded in high dimensions, in contrast with the linear and parabolic cases.

Keywords: random average process, random surfaces, harness process, linear process, smoothing process, voter model, surface fluctuations, central limit theorem

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1 Introduction

The random average process (RAP) was introduced in [3] as a model of a randomly evolving d dimensional surface in $d + 1$ space. The evolution consists of the heights of the surface getting updated, at either discrete or continuous time, by random convex combinations of neighboring heights (see (??) below). In this way, starting out with a given surface, which can be deterministic or itself random, we get at all times evolved surfaces which are random (as functions of the random convex weights and possibly random initial condition).

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Closely related processes are the *harness process* introduced by Hammersley [5] (see also [10]) and the *smoothing process* [1] [6] [7], where height updates consist of *deterministic* convex combinations of neighboring heights plus an additive (for the former process) or multiplicative (for the latter one) random noise. The RAP (as well as the smoothing process) is a special case of Liggett's linear processes ([7], Chapter IX).

A much studied special case of the RAP (one which we discuss only briefly in this paper) is the *voter model* [7] [2]. This corresponds to having the random convex combination almost surely assign total mass to a neighbor chosen at random. As discussed in [3], the behavior of the voter model is rather different from the more general case treated in that paper and also here.

In this paper, we study at length the discrete time RAP with a parabolic initial condition. One of our main results are upper and lower bounds of the same leading order to the time fluctuations of the evolving surface. Under suitable assumptions on the distribution of the convex weights, we obtain $n^{2-d/2}$ as the leading order of the variance (as a function of time n) of the height of the surface at a given site in dimensions $d = 1, 2, 3$; $\log n$ in $d = 4$; and constant for $d \geq 5$ (see Theorem 2.1 below). This compares firstly to the case of linear initial conditions [3], where the analogous variance is of order $\sqrt{n}$ in $d = 1$; $\log n$ in $d = 2$; and constant for $d \geq 3$. The approach and techniques here are also comparable with those of [3]. Here, as there, we have a dual process with the same single time distribution as the RAP (see (6-7) below) which, when centered, is a martingale. It is enough then to study this process (as far as single time distributions are concerned). Variances are then also shown to be related to moments of a space-inhomogeneous Markov chain, here in $2d$-space (see (11,23,25,14) below), rather than in d-space. In view of the extra complication, we keep the analysis simple by making extra assumptions on the distribution of the convex weights vis a vis [3] (see first paragraph of next section). We also take a convenient concrete form for the initial condition (see (8) below). The problem is then further reduced to one involving the same d-dimensional Markov chain that enters the analysis of the linear case in [3], but via a different, if related, quantity (see (31) below). The analysis proceeds indirectly (as in the linear case) by taking generating functions. The argument for the parabolic case will actually involve also the derivative of the generating functions entering the analysis of the linear case. As in [3], we compare with the analogous quantity for a d-dimensional space homogeneous Markov chain (a random walk), and then get the upper bounds (see (33,39,41,42,43) below). The argument for the lower bounds is similar, but simpler. It involves a d-dimensional random walk directly (see (46,51) below).

We also prove, in one dimension, a central limit theorem for the time fluctuations of the surface (see Theorem 2.3 below). As in [3], we verify the hypotheses of a martingale CLT in [4].

The boundedness of the height variance of the RAP in high dimensions seems to be a distinguishing feature of linear and parabolic initial conditions, among polynomial ones. We show in Theorem 5.1 below that, starting out with a cubic, the heights have variance of order at least n^2 in all high enough dimensions. This divergence in time of the fluctuations can be argued for initial polynomials of higher degree as well.

One difference between the initial linear and parabolic cases is the following. Due to the martingale property of the dual processes, mentioned above, and the L_2-boundedness in high dimensions, the RAP's as seen from the average height at the origin with initial linear and parabolic surfaces converge weakly to invariant measures for the dynamics as seen from the average height at the origin, in those dimensions. The spatial fluctuations of these measures can be then studied and they are found to be bounded for linear initial conditions. This is *not* the case for the initial condition here. We show in Theorem 4.1 below that the non trivially scaled space fluctuations of the invariant measures converge weakly to a non trivial limit. The variances also converge to the variance of the limit.

This paper grew out of the PhD research of the second author, which consisted of the RAP with a parabolic initial condition of a different form from the one treated here. [8] contains essentially the same results we present here (except for the CLT and the discussion and result of Section 5), obtained with the same approach and techniques, and more: sharp bounds were obtained in one dimension (see Theorem 2.2 below); fluctuations of the surface as seen from the height at the origin were shown to be bounded for $d \geq 3$, the scaled spatial fluctuations of the limiting invariant measures for the process as seen from the height at the origin arising in this context were proved to converge to non trivial weak limits; and continuous time analogs of the discrete time results were established. The assumptions on the convex weights distribution made in [8] are less restrictive than the ones here. Some of the extra results will be the object of a future paper.

We close this introduction with a comparison to the harness process mentioned above. The underlying space is $\mathbb{Z}^d$. For the case of a deterministic initial surface (as here), translation invariant convex weights (in here, that is the case in the distributional sense) and i.i.d. L_2 mean zero noise, the surface (height vector) at time n can be written as

2 Definitions and main results

We consider a d-dimensional discrete time random average process. At time $t = 0$ we have an initial configuration X_0 in the state space $\mathbb{R}^{\mathbb{Z}^d}$, which is a hypersurface of dimension d. So, $X_0(i)$ is the height of this surface at site $i \in \mathbb{Z}^d$. The evolution is defined as follows. At time n the height of each site will be a random convex combination of the heights of its neighbors at time $n-1$. Let $X_n(i)$ denote the height of site i at time n. We have

Let $\mathcal{F}$ be the sigma-algebra generated by U and let $\mathcal{F}_n$ be the sigma algebra generated by $\{u_m(\cdot,\cdot), 1 \leq m \leq n\}$. Let $\mathbb{P}$ and $\mathbb{E}$ denote the underlying probability and expectation and let ν be the (marginal) distribution of $u_k(\cdot,\cdot)$, which does not depend on $k \geq 1$. We will write $\nu(X)$ for the expectation of a random variable X with respect to ν.

Note that for $x \in \mathbb{Z}^d$

$$\begin{aligned} X_n(x) &= \sum_{j_1,\dots j_n \in \mathbb{Z}^d} \prod_{i=1}^{n} u_{n-i+1}(j_{i-1}, j_i) X_0(j_n) \\ &= \sum_{j_n \in \mathbb{Z}^d} \mathbb{P}(Y_n^{x,n} = j_n \mid \mathcal{F}) X_0(j_n) = \mathbb{E}[X_0(Y_n^{x,n}) \mid \mathcal{F}], \end{aligned} \tag{1}$$

where $(Y_k^{x,n})_{k=0}^n$ is a random walk which starts in x with transition probabilities

$$\mathbb{P}(Y_k^{x,n} = j \mid Y_{k-1}^{x,n} = i, \mathcal{F}) = u_{n-k+1}(i,j). \tag{2}$$

For the process (X_n) to be well defined, it is sufficient that (see [3], Lemma 2.1)

$$\mathbb{E}|X_0(Y_n^{0,n})| < \infty \text{ for all } n. \tag{3}$$

Consider also a random walk $(Y_k^x)_{k\geq 0}$ with transition probabilities

$$\mathbb{P}(Y_k^x = j \mid Y_{k-1}^x = i, \mathcal{F}) = u_k(i,j). \tag{4}$$

Then for all $n \geq 0$

$$\{\mathbb{P}(Y_k^x = \cdot \mid \mathcal{F});\ x \in \mathbb{Z}^d,\ 0 \leq k \leq n\} \stackrel{d}{=} \{\mathbb{P}(Y_k^{x,n} = \cdot \mid \mathcal{F});\ x \in \mathbb{Z}^d,\ 0 \leq k \leq n\}, \tag{5}$$

since $u_k \stackrel{d}{=} u_{n-k+1}$ for all n, k. The unconditional (on $\mathcal{F}$) distributions of Y_i^x and $Y_i^{x,n}$, which are equal in their common time span, make them random walks with transition probabilities

$$\pi(x,y) = \nu[u_1(x,y)] = \nu[u_1(0,y-x)] = \pi(0,y-x)$$

for all $x, y \in \mathbb{Z}^d$. Notice that, by the nearest neighbor assumption on the u's, these random walks are simple and symmetric.

For $n \geq 0$ and any $x \in \mathbb{Z}^d$ define

$$L_n(x) = \mathbb{E}[X_0(Y_n^x) \mid \mathcal{F}]. \tag{6}$$

From (5) we have that for any fixed n

$$X_n := \{X_n(x);\ x \in \mathbb{Z}^d\} \stackrel{d}{=} \{L_n(x);\ x \in \mathbb{Z}^d\} =: L_n. \tag{7}$$

Thus, to establish results about single time distributions of X_n it is sufficient to give an argument for L_n.

In this paper we consider a parabolic initial condition of the form

$$X_0(x) = (\mathbf{1}x)^2 = \left(\sum_{k=1}^d x_k\right)^2, \tag{8}$$

for any $x \in \mathbb{Z}^d$, where $\mathbf{1} \in \mathbb{R}^d$, $\mathbf{1} = (1, \dots, 1)$. This clearly satisfies (3).

Remark 2.1 *Other parabolic forms, like $(\lambda x)^2 = (\sum_{k=1}^d \lambda_k x_k)^2$, where $\lambda \in \mathbb{R}^d$ is a fixed non null vector, can be handled, with essentially the same results and techniques. The form $||x||^2 = \sum_{k=1}^d x_k^2$ was analyzed in [8].*

With the choice (8), we have

$$L_n(x) = \mathbb{E}[(\mathbf{1}Y_n^x)^2 \mid \mathcal{F}]. \tag{9}$$

Since Y_n^x is a random walk starting from x and the u's are symmetric, we have $\mathbb{E}(L_n(x)) = \mathbb{E}((\mathbf{1}Y_n^x)^2) = \mathbb{E}((\mathbf{1}(Y_n^0 + x))^2) = (\mathbf{1}x)^2 + \mathbb{E}((\mathbf{1}Y_n^0)^2)$. Writing $Y_n^0 = \sum_{i=1}^n \Delta_i$, where $\Delta_1, \dots, \Delta_n$ are i.i.d. random vectors such that $\mathbb{P}(\Delta_1 = \pm e_i) = 1/2d$, for $i = 1, \dots, d$, we have $\mathbb{E}((\mathbf{1}Y_n^0)^2) =$

$\sum_{i=1}^{n} \mathbb{E}((\mathbf{1}\Delta_i)^2) = n$, since $\mathbf{1}\Delta_1 = \pm 1$ almost surely. Thus $\mathbb{E}(L_n(x)) = (\mathbf{1}x)^2 + n$. Let $\bar{Y}_n^x = Y_n^x - \mathbb{E}(Y_n^x) = Y_n^x - x$. Then, $(\mathbf{1}Y_n^x)^2 = (\mathbf{1}\bar{Y}_n^x)^2 + 2(\mathbf{1}x)(\mathbf{1}\bar{Y}_n^x) + (\mathbf{1}x)^2$ and

where

$\bar{L}_n(x) = \mathbb{E}((\mathbf{1}\bar{Y}_n^x)^2 \mid \mathcal{F}_n)$ and $\bar{Z}_n(x) = \mathbb{E}(\mathbf{1}\bar{Y}_n^x \mid \mathcal{F}_n)$. Note that, by translation invariance, the distributions of $\bar{L}_n(x)$ and $\bar{Z}_n(x)$ do not depend on x. Let $\bar{L}_n := L_n(0) = \bar{L}_n(0)$ and $\bar{Z}_n := Z_n(0) = \bar{Z}_n(0)$. So,

$L_n(x) = \bar{L}_n(x) + 2(\mathbf{1}x)\bar{Z}_n(x) + (\mathbf{1}x)^2,$

where $\bar{L}_n(x) = \mathbb{E}((\mathbf{1}\bar{Y}_n^x)^2 \mid \mathcal{F}_n)$ and $\bar{Z}_n(x) = \mathbb{E}(\mathbf{1}\bar{Y}_n^x \mid \mathcal{F}_n)$. Note that, by translation invariance, the distributions of $\bar{L}_n(x)$ and $\bar{Z}_n(x)$ do not depend on x. Let $\bar{L}_n := L_n(0) = \bar{L}_n(0)$ and $\bar{Z}_n := Z_n(0) = \bar{Z}_n(0)$. So,

$$\mathbb{V}(L_n(x)) = \mathbb{V}(\bar{L}_n) + 4(\mathbf{1}x)^2\mathbb{V}(\bar{Z}_n) + 4(\mathbf{1}x)\mathbb{C}\mathrm{ov}(\bar{L}_n, \bar{Z}_n). \tag{11}$$

Below, $c_1, c_2, \ldots$ will always denote positive real numbers which may depend only on d. One of our main results is the following.

Theorem 2.1 *For all $x \in \mathbb{Z}^d$, there exist c_1, c_2 such that*

$$c_1\mathfrak{O}(n,d) \leq \mathbb{V}(L_n(x)) \leq c_2\mathfrak{O}(n,d)$$

for all n, where

$$\mathfrak{O}(n,d) = \begin{cases} n^{2-\frac{d}{2}}, & \text{if } d = 1,2,3; \\ \log n, & \text{if } d = 4; \\ \text{constant}, & \text{if } d \geq 5. \end{cases} \tag{12}$$

The proof of Theorem 2.1 will be presented in Section 4.

In dimension 1, it is possible to get a stronger result, for which we do not present a proof here, but rather refer to [8] (Theorem 3.1.3):

Theorem 2.2 *If $d = 1$, then there exists c_3 such that for all $x \in \mathbb{Z}^d$*

$$\frac{\mathbb{V}(L_n(x))}{n^{3/2}} \to c_3, \quad \textit{when } n \to \infty.$$

We establish also a Central Limit Theorem for $\bar{L}_n = L_n(0)$ in dimension 1 (the proof, presented in Section 6, does not use Theorem 2.2). Let $\mathcal{V}_n := \mathbb{V}(\bar{L}_n)$.

Theorem 2.3 *In $d = 1$, the distribution of $\mathcal{V}_n^{-1/2}\bar{L}_n$ converges to a standard Gaussian, as $n \to \infty$.*

Our analysis yields that in dimensions 5 or more there exists an invariant measure for the dynamics of the surface as seen from the average height at the origin. This is related to the almost sure existence of the limits of $\bar{L}_n - n$ and $\bar{Z}_n$ as $n \to \infty$. In Subsection 4.2, we discuss this and prove a result about the asymptotic shape and magnitude of the space fluctuations of this measure.

In Section 3, we state and prove auxiliary results for the arguments of the proofs of our main results. In Section 5, we discuss the case of higher order polynomial initial conditions and prove a result that indicates a substantial difference with the linear and parabolic cases, namely the unboundedness of the time fluctuations at high dimensions.

3 Preliminaries

To prove Theorems 2.1 and 2.3 we will need some lemmas.

Lemma 3.1 *The process $\bar{L}_n - n$ is a martingale with respect to $\{\mathcal{F}_n, n \geq 0\}$.*

Proof. Let $Y_n = Y_n^0$. So, $\mathbb{E}(\bar{L}_n) = \mathbb{E}((\mathbf{1}Y_n)^2) = n$. We have

$$\begin{aligned}
\bar{L}_n &= \mathbb{E}[(\mathbf{1}Y_n)^2 \mid \mathcal{F}_n] \\
&= \sum_{k\in\mathbb{Z}^d}\sum_{j\in\mathbb{Z}^d}(\mathbf{1}j)^2\mathbb{P}(Y_n = j \mid Y_{n-1} = k, \mathcal{F}_n)\mathbb{P}(Y_{n-1} = k \mid \mathcal{F}_n) \\
&= \sum_{k\in\mathbb{Z}^d}\sum_{j\in\mathbb{Z}^d}\{\mathbf{1}[k + (j-k)]\}^2 u_n(k,j)\mathbb{P}(Y_{n-1} = k \mid \mathcal{F}_{n-1}) \\
&= \sum_{k\in\mathbb{Z}^d}(\mathbf{1}k)^2\mathbb{P}(Y_{n-1} = k \mid \mathcal{F}_{n-1}) \\
&\quad +2\sum_{k\in\mathbb{Z}^d}(\mathbf{1}k)\Big(\sum_{j\in\mathbb{Z}^d}[\mathbf{1}(j-k)]u_n(k,j)\Big)\mathbb{P}(Y_{n-1} = k \mid \mathcal{F}_{n-1}) \\
&\quad +\sum_{k\in\mathbb{Z}^d}\Big(\sum_{j\in\mathbb{Z}^d}[\mathbf{1}(j-k)]^2u_n(k,j)\Big)\mathbb{P}(Y_{n-1} = k \mid \mathcal{F}_{n-1}) \\
&= \mathbb{E}[(\mathbf{1}Y_{n-1})^2 \mid \mathcal{F}_{n-1}] + 2\sum_{k\in\mathbb{Z}^d}(\mathbf{1}k)(\mathbf{1}\theta_n(k))\mathbb{P}(Y_{n-1} = k \mid \mathcal{F}_{n-1}) \\
&\quad +\sum_{k\in\mathbb{Z}^d}\mathbb{P}(Y_{n-1} = k \mid \mathcal{F}_{n-1}),
\end{aligned}$$

where $\theta_n(k) = \sum_{j\in\mathbb{Z}^d}(j-k)u_n(k,j)$, since $[\mathbf{1}(j-k)]^2 = 1$ every time $u_n(k,j) \neq 0$, due to the nearest neighbor character of the u's.

Let $W_n = \mathbb{E}[(\mathbf{1}Y_{n-1})(\mathbf{1}\theta_n(Y_{n-1})) \mid \mathcal{F}_n]$, so $\bar{L}_n = \bar{L}_{n-1} + 2W_n + 1$. Note that the distribution of $\theta_n(k)$ does not depend on n or k and $\mathbb{E}(\theta_1(0)) = \mathbb{E}\Big(\sum_{j\in\mathbb{Z}^d} j u_1(0,j)\Big) = 0$.

We have that $\bar{L}_0 = 0$ and, for $n \geq 1$, $\bar{L}_n = \sum_{i=1}^n(2W_i + 1)$. Thus

Lemma 3.2 *Let $(\hat{Y}_n)_{n\geq 0}$ be an independent copy of $(Y_n)_{n\geq 0}$ given $\mathcal{F}$. Then $(Y_n, \hat{Y}_n)$ is a Markov chain in $\mathbb{Z}^d \times \mathbb{Z}^d$ with the following transition probabilities*

Proof.

$$
\begin{aligned}
&\mathbb{P}(Y_n = k_n, \hat{Y}_n = l_n \mid Y_{n-1} = k_{n-1}, \hat{Y}_{n-1} = l_{n-1}, \dots, Y_1 = k_1, \hat{Y}_1 = l_1) \\
&\quad = \mathbb{E}[\mathbb{P}(Y_n = k_n, \hat{Y}_n = l_n \mid Y_{n-1} = k_{n-1}, \hat{Y}_{n-1} = l_{n-1}, \dots, Y_1 = k_1, \hat{Y}_1 = l_1, \mathcal{F}_n)] \\
&\quad = \mathbb{E}[\mathbb{P}(Y_n = k_n \mid Y_{n-1} = k_{n-1}, \mathcal{F}_n)\mathbb{P}(\hat{Y}_n = k_n \mid \hat{Y}_{n-1} = k_{n-1}, \mathcal{F}_n)]. \qquad (13)
\end{aligned}
$$

The second equality is justified by the independence of the random walks Y_n and $\hat{Y}_n$ given $\mathcal{F}_n$ and the Markov property of Y_n and $\hat{Y}_n$.

The result follows from the fact that the right hand side of (13) equals both $\mathbb{E}[\mathbb{P}(Y_n = k_n, \hat{Y}_n = l_n \mid Y_{n-1} = k_{n-1}, \hat{Y}_{n-1} = l_{n-1}, \mathcal{F}_n)] = \mathbb{P}(Y_n = k_n, \hat{Y}_n = l_n \mid Y_{n-1} = k_{n-1}, \hat{Y}_{n-1} = l_{n-1})$ and $\mathbb{E}[u_n(k_{n-1}, k_n)u_n(l_{n-1}, l_n)] = \nu[u_1(k_{n-1}, k_n)u_1(l_{n-1}, l_n)]$. ■

Corollary 3.1 *Let $D_n = Y_n - \hat{Y}_n$ and $S_n = Y_n + \hat{Y}_n$. Then $(D_n, S_n)_{n\geq 0}$ is a Markov chain in $\mathbb{Z}^d \times \mathbb{Z}^d$ with transition probabilities*

$$
\begin{aligned}
&\mathbb{P}(D_n = d_n, S_n = s_n \mid D_{n-1} = d_{n-1}, S_{n-1} = s_{n-1}) \\
&\quad = \nu\Big[u_1\Big(\tfrac{s_{n-1}+d_{n-1}}{2}, \tfrac{s_n+d_n}{2}\Big)u_1\Big(\tfrac{s_{n-1}-d_{n-1}}{2}, \tfrac{s_n-d_n}{2}\Big)\Big]. \qquad (14)
\end{aligned}
$$

Proof. *Straightforward from Lemma 3.2.*

Remark 3.1 *From Corollary 3.1, we have that, if $d_{n-1} = 0$,*

$$
\begin{aligned}
&\mathbb{P}(D_n = d, S_n = s_{n-1} + s \mid D_{n-1} = 0, S_{n-1} = s_{n-1}) \\
&\quad = \mathbb{E}\Big[u_1\Big(\frac{s_{n-1}}{2}, \frac{s_{n-1}}{2} + \frac{s+d}{2}\Big)u_1\Big(\frac{s_{n-1}}{2}, \frac{s_{n-1}}{2} + \frac{s-d}{2}\Big)\Big] \\
&\quad = \mathbb{E}\Big[u_1\Big(0, \frac{s+d}{2}\Big)u_1\Big(0, \frac{s-d}{2}\Big)\Big] = \nu\Big[u_1\Big(0, \frac{s+d}{2}\Big)u_1\Big(0, \frac{s-d}{2}\Big)\Big],
\end{aligned}
$$

where the second equality follows by translation invariant of the u's. On the other hand, if $d_{n-1} \neq 0$, then,

$$
\begin{aligned}
&\mathbb{P}(D_n = d_{n-1} + d, S_n = s_{n-1} + s \mid D_{n-1} = d_{n-1}, S_{n-1} = s_{n-1}) \\
&\quad = \mathbb{E}\Big[u_1\Big(\frac{s_{n-1}+d_{n-1}}{2}, \frac{s_{n-1}+d_{n-1}}{2} + \frac{s+d}{2}\Big) \\
&\qquad\quad \times u_1\Big(\frac{s_{n-1}-d_{n-1}}{2}, \frac{s_{n-1}-d_{n-1}}{2} + \frac{s-d}{2}\Big)\Big] \\
&\quad = \mathbb{E}\Big[u_1\Big(\frac{s_{n-1}+d_{n-1}}{2}, \frac{s_{n-1}+d_{n-1}}{2} + \frac{s+d}{2}\Big)\Big] \\
&\qquad\quad \times \mathbb{E}\Big[u_1\Big(\frac{s_{n-1}-d_{n-1}}{2}, \frac{s_{n-1}-d_{n-1}}{2} + \frac{s-d}{2}\Big)\Big] \\
&\quad = \mathbb{E}\Big[u_1\Big(0, \frac{s+d}{2}\Big)\Big]\mathbb{E}\Big[u_1\Big(0, \frac{s-d}{2}\Big)\Big] = \nu\Big[u_1\Big(0, \frac{s+d}{2}\Big)\Big]\nu\Big[u_1\Big(0, \frac{s-d}{2}\Big)\Big],
\end{aligned}
$$

where we have used the independence and translation invariance of the u's. So,

$$\begin{aligned}\mathbb{P}(D_n = d_{n-1} + d, S_n = s_{n-1} + s \mid D_{n-1} = d_{n-1}, S_{n-1} = s_{n-1}) \\ = \begin{cases} \nu[u_1(0, \frac{s+d}{2}) u_1(0, \frac{s-d}{2})], & \text{if } d_{n-1} = 0 \\ \nu[u_1(0, \frac{s+d}{2})]\nu[u_1(0, \frac{s-d}{2})], & \text{if } d_{n-1} \neq 0, \end{cases}\end{aligned} \tag{15}$$

and thus (D_n, S_n) is space homogeneous in $\{0\} \times \mathbb{Z}^d$ and $(\mathbb{Z}^d \setminus \{0\}) \times \mathbb{Z}^d$ separately, but not in $\mathbb{Z}^d \times \mathbb{Z}^d$.

Remark 3.2 *It follows from (15) that D_n, $n \geq 0$, is a Markov chain with transition probabilities (see also [3], Lemma 2.5)*

$$\gamma(l, k) = \begin{cases} \sum_{j \in \mathbb{Z}^d} \nu[u_1(0, m) u_1(0, j + k)], & \text{if } l = 0, \\ \sum_{j \in \mathbb{Z}^d} \nu[u_1(0, j)]\nu[u_1(l, j + k)], & \text{if } l \neq 0. \end{cases} \tag{16}$$

Our assumptions on the u's make the jumps of D_n have length only either 0 or 2, with $\gamma(0,0) < 1$. The jumps of length 2 can be (only) in any of the coordinate positive and negative directions. All of these possibilities have equal probabilities (which does not depend on the starting point, provided it is in $\mathbb{Z}^d \setminus \{0\}$), that is

$$\gamma(l, l \pm 2e_j) = (1 - \gamma(l, l))/(2d) \quad \text{for all} \quad l \in \mathbb{Z}^d \quad \text{and} \quad j = 1, \ldots, d;$$

$$\gamma(l, l) = \gamma(l', l') \quad \text{if} \quad l, l' \neq 0.$$

Remark 3.3 *Remark 3.1 allows us to construct (D_n, S_n) in the following way. Let $(\tilde{\delta}, \tilde{\xi})$ be distributed as the increments of (D_n, S_n) in $\{0\} \times \mathbb{Z}^d$ and (δ, ξ) as those in $(\mathbb{Z}^d \setminus \{0\}) \times \mathbb{Z}^d$, that is,*

$$\mathbb{P}(\tilde{\delta} = d, \tilde{\xi} = s) = \nu\Big[u_1\Big(0, \frac{s+d}{2}\Big) u_1\Big(0, \frac{s-d}{2}\Big)\Big] \tag{17}$$

and

$$\mathbb{P}(\delta = d, \xi = s) = \nu\Big[u_1\Big(0, \frac{s+d}{2}\Big)\Big]\nu\Big[u_1\Big(0, \frac{s-d}{2}\Big)\Big]. \tag{18}$$

Let $\{(\delta_n(i), \xi_n(i)); i \in \mathbb{Z}^d \setminus \{0\}, n \geq 1\}$ and $\{(\delta_n(0), \xi_n(0)); n \geq 1\}$ be two independent families of i.i.d. random vectors such that $(\delta_1(0), \xi_1(0)) \stackrel{d}{=} (\tilde{\delta}, \tilde{\xi})$ and $(\delta_1(i)), \xi_1(i)) \stackrel{d}{=} (\delta, \xi)$ for all $i \in (\mathbb{Z}^d \setminus \{0\})$. Then

$$(D_n, S_n) = \sum_{i=1}^{n} (\delta_i(D_{i-1}), \xi_i(D_{i-1})). \tag{19}$$

Lemma 3.3 *For all n, given D_{n-1} and $\delta_n(D_{n-1})$, $\xi_n(D_{n-1}) \stackrel{d}{=} -\xi_n(D_{n-1})$.*

Proof. *For all d, given that $D_{n-1} = d_{n-1}$, if $d_{n-1} = 0$, then, by Remark 3.1,*

$$\begin{aligned}
&\mathbb{P}\left(\delta_n(D_{n-1}) = d, \xi_n(D_{n-1}) = s \mid D_{n-1} = 0\right) = \mathbb{P}(\tilde{\delta} = d, \tilde{\xi} = s) \\
= \;& \nu\Big[u_1\Big(0, \frac{s+d}{2}\Big)u_1\Big(0, \frac{s-d}{2}\Big)\Big] \overset{I}{=} \nu\Big[u_1\Big(0, \frac{-s-d}{2}\Big)u_1\Big(0, \frac{-s+d}{2}\Big)\Big] \\
= \;& \mathbb{P}(\tilde{\delta} = d, \tilde{\xi} = -s) = \mathbb{P}\left(\delta_n(D_{n-1}) = d, \xi_n(D_{n-1}) = -s \mid D_{n-1} = 0\right),
\end{aligned}$$

where equality I follows from the symmetry of the u's.

The case $d_{n-1} \neq 0$ is similar. ■

Let H_n be a space homogeneous Markov chain (random walk) with transition probabilities

$$\gamma_H(l,k) = \sum_{j\in\mathbb{Z}^d} \nu(u_1(0,j))\nu(u_1(l,j+k)) \quad \text{for all } l,k \in \mathbb{Z}^d. \tag{20}$$

Remark 3.4 *The transition probabilities of H_n and D_n differ only at the origin. We have also that $0 < \gamma_H(0,0) < \gamma(0,0) < 1$.*

For $0 < s < 1$, let $f(s)$ and $g(s)$ be the power series of $\mathbb{P}(D_n = 0)$ and $\mathbb{P}(H_n = 0)$, respectively; that is, for $0 \le s < 1$

$$f(s) = \sum_{n\ge0} \mathbb{P}(D_n = 0)s^n \text{ and } g(s) = \sum_{n\ge0} \mathbb{P}(H_n = 0)s^n.$$

In [3] (3.7-9), it was shown (with a different notation) that $f(s) = h(s,\gamma)$ and $g(s) = \bar{h}(s,\bar{\gamma})$, where $\gamma := \gamma(0,0)$, $\bar{\gamma} := \gamma_H(0,0)$ and, for $0 \le p < 1$, $h(s,p) := \{1-\phi(s,p)\}^{-1}$, $\bar{h}(s,p) := \{1-\bar{\phi}(s,p)\}^{-1}$, with $\phi(s,p) := ps + (1-p)\psi(s)$, $\bar{\phi}(s,p) := ps + (1-p)\bar{\psi}(s)$ and $\psi(s) := \sum_{n\ge0}\mathbb{P}(T = n)s^{n+1}$, $\bar{\psi}(s) := \sum_{n\ge0}\mathbb{P}(T' = n)s^{n+1}$, where T is the time of the first return to the origin of D_n after first leaving it, and T' is the analogous return time for H_n. Note that, from the nearest neighbor and exchangeability assumptions we made, T has the same distribution as the return time starting from $2e_1$, say, since, after leaving the origin, D_n necessarily jumps to a site of the form $\pm 2e_i$ for some $i = 1,\ldots,d$, and the distribution of this latter return time does not depend on which of these sites D_n starts from, by the coordinate exchangeability assumption on the u's. By the same considerations and Remark 3.4, we conclude that $T \overset{d}{=} T'$ and, thence, $\bar{h}(s,p) \equiv h(s,p)$.

Lemma 3.4 *Let $f^{(k)}$ and $g^{(k)}$ denote the k-th derivatives of f and g, respectively. Then, for $k = 0,1,2,3,4$*

Proof. *For $0 \le s < 1$*

$$\frac{f^{(0)}(s)}{g^{(0)}(s)} = \frac{f(s)}{g(s)} = \frac{1-[\bar{\gamma}s + (1-\bar{\gamma})\psi(s)]}{1-[\gamma s + (1-\gamma)\psi(s)]}. \tag{21}$$

We rewrite (21) as $(1+(1-\bar{\gamma})sC(s))/(1+(1-\gamma)sC(s))$, with $C(s) = [1-\tilde{\psi}(s)]/(1-s) = [\tilde{\psi}(1)-\tilde{\psi}(s)]/(1-s)$, where $\tilde{\psi}(s) := \psi(s)/s = \sum_{n\ge0}\mathbb{P}(T = n)s^n$. Then

$$C(s) = \sum_{n\ge1}\mathbb{P}(T = n)\frac{1-s^n}{1-s} = \sum_{n\ge1}\mathbb{P}(T = n)\sum_{i=1}^{n-1} s^i = \sum_{i\ge0}\mathbb{P}(T > i)s^i.$$

This in turn tends to $\sum_{i\geq 0}\mathbb{P}(T>i)=\mathbb{E}(T)$ *as* $s\uparrow 1$*. It is a well known result for the (nearest neighbor) symmetric random walk in* $\mathbb{Z}^d$ *that* $E(T)=\infty$ *([9], Proposition 18.1). Thus,*

$$\lim_{s\uparrow 1}\frac{f^{(0)}(s)}{g^{(0)}(s)}=\frac{1-\bar\gamma}{1-\gamma}. \tag{22}$$

and the result for $k=0$ *follows, since* $\gamma<1$ *(see Remark 3.2).*

For the next case, notice first that

$$f^{(1)}(s)=[f(s)]^2 d\phi(s,\gamma)/ds=[f(s)]^2[\gamma+(1-\gamma)\psi^{(1)}(s)]$$

and, analogously,

$$g^{(1)}(s)=[g(s)]^2 d\phi(s,\bar\gamma)/ds=[f(s)]^2[\bar\gamma+(1-\bar\gamma)\psi^{(1)}(s)].$$

From (22), there exists a constant $M>1$ *such that* $f(s)\leq Mg(s)$ *for all* $0\leq s<1$*. By Remark 3.4, we can have* M *also satisfy* $1-\gamma\leq M(1-\bar\gamma)$*,* $\gamma\leq M\bar\gamma$*. It follows that* $\lim_{s\uparrow 1}f^{(1)}(s)/g^{(1)}(s)\leq M^3$*.*

For the next case, notice that

$$\begin{aligned} f^{(2)}(s) &= [f(s)]^2\frac{d^2\phi(s,\gamma)}{ds^2}+2[f(s)]^3\left(\frac{d\phi(s,\gamma)}{ds}\right)^2 \\ &= [f(s)]^2(1-\gamma)\psi^{(2)}(s)+2[f(s)]^3[\gamma+(1-\gamma)\psi^{(1)}(s)]^2 \end{aligned}$$

and a similar expression holds for $g^{(2)}(s)$*, with* $\bar\gamma$ *replacing* γ*. From the above considerations, it follows that* $\lim_{s\uparrow 1}f^{(2)}(s)/g^{(2)}(s)\leq 3M^5$*.*

Similarly, we find that

$$\lim_{s\uparrow 1}f^{(3)}(s)/g^{(3)}(s)\leq 13M^7 \quad \textit{and} \quad \lim_{s\uparrow 1}f^{(4)}(s)/g^{(4)}(s)\leq 75M^9. \quad \blacksquare$$

4 Fluctuations of L_n

4.1 Proof of Theorem 2.1

*By Lemma 3.1 and (**??**),*

$$\mathbb{V}(\bar L_n)=4\sum_{i=1}^{n}\mathbb{V}(W_i). \tag{23}$$

Now,

$$\begin{aligned} \mathbb{V}(W_i)=\mathbb{E}(W_i)^2 &= \mathbb{E}\Big[\Big(\sum_{k\in\mathbb{Z}^d}(\mathbf{1}k)(\mathbf{1}\theta_i(k))\mathbb{P}(Y_{i-1}=k\mid\mathcal{F}_{i-1})\Big)^2\Big] \\ &= \mathbb{E}\sum_{k,r\in\mathbb{Z}^d}(\mathbf{1}k)(\mathbf{1}r)(\mathbf{1}\theta_i(k))(\mathbf{1}\theta_i(r))\mathbb{P}(Y_{i-1}=k\mid\mathcal{F}_{i-1})\mathbb{P}(Y_{i-1}=r\mid\mathcal{F}_{i-1}) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k\in\mathbb{Z}^d}(\mathbf{1}k)^2\mathbb{E}\{(\mathbf{1}\theta_i(k))^2\mathbb{P}^2(Y_{i-1}=k\mid\mathcal{F}_{i-1})\}\\
&= \sum_{k\in\mathbb{Z}^d}(\mathbf{1}k)^2\nu[(\mathbf{1}\theta_i(k))^2]\mathbb{E}[\mathbb{P}^2(Y_{i-1}=k\mid\mathcal{F}_{i-1})]\\
&= \sigma^2\sum_{k\in\mathbb{Z}^d}(\mathbf{1}k)^2\mathbb{E}[\mathbb{P}^2(Y_{i-1}=k\mid\mathcal{F}_{i-1})], \qquad (24)
\end{aligned}
$$

by the independence between $\theta_i(\cdot)$ and $\mathcal{F}_{i-1}$, between $\theta_i(k)$ and $\theta_i(r)$ when $k\neq r$, and the zero mean of the latter, where $\sigma^2=\nu[(\mathbf{1}\theta_i(k))^2]=\nu[(\mathbf{1}\theta_1(0))^2]$ is positive by the non degeneracy assumption on the u's.

We have that

$$
\begin{aligned}
&\mathbb{E}[\mathbb{P}^2(Y_{i-1}=k\mid\mathcal{F}_{i-1})]=\mathbb{E}[\mathbb{P}(Y_{i-1}=k\mid\mathcal{F}_{i-1})\mathbb{P}(\hat{Y}_{i-1}=k\mid\mathcal{F}_{i-1})]\\
&= \mathbb{E}[\mathbb{P}(Y_{i-1}=k,\hat{Y}_{i-1}=k\mid\mathcal{F}_{i-1})]\\
&= \mathbb{P}(Y_{i-1}=\hat{Y}_{i-1}=k)=\mathbb{P}\Big(\frac{S_{i-1}}{2}=k,D_{i-1}=0\Big),
\end{aligned}
$$

by the definition of S_n and D_n (see Lemma 3.2 and Corollary 3.1 above). Thus,

$$
\mathbb{V}(W_i)=\sigma^2\sum_{k\in\mathbb{Z}^d}(\mathbf{1}k)^2\mathbb{P}\Big(\frac{S_{i-1}}{2}=k,D_{i-1}=0\Big)=\frac{\sigma^2}{4}\mathbb{E}\left[(\mathbf{1}S_{i-1})^2;D_{i-1}=0\right]. \qquad (25)
$$

Thus from (25) and (23), we get

$$
\mathbb{V}(\bar{L}_n)=\sigma^2\sum_{i=0}^{n-1}\mathbb{E}\left[(\mathbf{1}S_i)^2;D_i=0\right]. \qquad (26)
$$

4.1.1 Upper bound for $\mathbb{V}(\bar{L}_n)$

Writing S_i as a sum of its increments, we get

$$
\begin{aligned}
&\mathbb{E}\left[(\mathbf{1}S_i)^2;D_i=0\right]=\mathbb{E}\Big[\Big(\sum_{j=1}^{i}\mathbf{1}\xi_j(D_{j-1})\Big)^2;\ D_i=0\Big]\\
&= \mathbb{E}\Big[\sum_{j=1}^{i}[\mathbf{1}\xi_j(D_{j-1})]^2+2\sum_{1=k<j}^{i}[\mathbf{1}\xi_k(D_{k-1})][\mathbf{1}\xi_j(D_{j-1})];D_i=0\Big]\\
&=\sum_{j=1}^{i}\mathbb{E}\{[\mathbf{1}\xi_j(D_{j-1}]^2;D_i=0\}) \qquad (27)\\
&\quad+2\sum_{1=k<j}^{i}\mathbb{E}\{[\mathbf{1}\xi_k(D_{k-1})][\mathbf{1}\xi_j(D_{j-1})];D_i=0\}.
\end{aligned}
$$

Since $\xi_j(D_{j-1})=(Y_j-Y_{j-1})+(\hat{Y}_j-\hat{Y}_{j-1})$, we have $|\mathbf{1}\xi_j(D_{j-1})|\le|\mathbf{1}(Y_j-Y_{j-1})|+|\mathbf{1}(\hat{Y}_j-\hat{Y}_{j-1})|\le 2$. So, $(\mathbf{1}\xi_j(D_{j-1}))^2\le 4$ and $\mathbb{E}[(\mathbf{1}\xi_j(D_{j-1}))^2;D_i=0]\le 4\mathbb{P}(D_i=0)$. Thus, for the first term in *the right hand side of (27), we obtain*

$$\sum_{j=1}^{i} \mathbb{E}[(\mathbf{1}\xi_j(D_{j-1}))^2; D_i = 0] \le \sum_{j=1}^{i} 4\mathbb{P}(D_i = 0) = 4i\,\mathbb{P}(D_i = 0). \tag{28}$$

The expectation in the second term in the right hand side of (27) can be written as

$$\begin{aligned}
&\mathbb{E}[(\mathbf{1}\xi_k(D_{k-1}))(\mathbf{1}\xi_j(D_{j-1})); D_i = 0] \\
&\quad= \sum_{l\in\mathbb{Z}^d}\sum_{r\in\mathbb{Z}^d} \mathbb{E}[(\mathbf{1}\xi_k(r))(\mathbf{1}\xi_j(l)); D_{k-1} = r, D_{j-1} = l, D_i = 0] \\
&\quad= \sum_{l,r\in\mathbb{Z}^d}\ \sum_{w_1,w_2\in\mathbb{Z}^d} \mathbb{E}[(\mathbf{1}\xi_k(r))(\mathbf{1}\xi_j(l)); D_{k-1} = r, D_k = r + w_2, \\
&\qquad\qquad D_{j-1} = l, D_j = l + w_1, D_i = 0] \\
&\quad= \sum_{l,r\in\mathbb{Z}^d}\ \sum_{w_1,w_2\in\mathbb{Z}^d}\ \sum_{z_1,z_2\in\mathbb{Z}^d} (\mathbf{1}z_1)(\mathbf{1}z_2) \\
&\qquad \times\mathbb{P}(\xi_k(r) = z_2, \xi_j(l) = z_1, D_{k-1} = r, D_{j-1} = l, D_k = r + w_2, \\
&\qquad\qquad D_j = l + w_1, D_i = 0).
\end{aligned} \tag{29}$$

By the Markov property of D_n and the identities $\{D_j = l + w_1, D_{j-1} = l\} = \{D_{j-1} = l, \delta_j(l) = w_1\}$, $\{D_k = r + w_2, D_{k-1} = r\} = \{D_{k-1} = r, \delta_k(r) = w_2\}$, we get that, for all $k < j$, the latter probability equals $\mathbb{P}(D_i = 0 \mid D_j = l + w_1)$ times

$$\begin{aligned}
&\mathbb{P}(\xi_j(l) = z_1, \delta_j(l) = w_1, \xi_k(r) = z_2, \delta_k(r) = w_2, D_{j-1} = l, D_{k-1} = r) \\
&= \mathbb{P}(\xi_j(l) = z_1, \delta_j(l) = w_1)\mathbb{P}(\xi_k(r) = z_2, \delta_k(r) = w_2, D_{j-1} = l, D_{k-1} = r)
\end{aligned}$$

and thus the right hand side of of (29) can be written as

$$\begin{aligned}
&\sum_{l,r\in\mathbb{Z}^d}\ \sum_{w_1,w_2\in\mathbb{Z}^d}\ \sum_{z_1,z_2\in\mathbb{Z}^d} (\mathbf{1}z_1)(\mathbf{1}z_2)\mathbb{P}(\xi_j(l) = z_1, \delta_j(l) = w_1) \\
&\quad \times\mathbb{P}(\xi_k(r) = z_2, \delta_k(r) = w_2, D_{j-1} = l, D_{k-1} = r)\mathbb{P}(D_i = 0 \mid D_j = l + w_1) \\
&= \sum_{l,r\in\mathbb{Z}^d}\ \sum_{w_1,w_2\in\mathbb{Z}^d}\ \sum_{z_2\in\mathbb{Z}^d} \mathbf{1}\Big\{\sum_{z_1\in\mathbb{Z}^d} z_1\mathbb{P}(\xi_j(l) = z_1, \delta_j(l) = w_1)\Big\}(\mathbf{1}z_2) \\
&\quad \times\mathbb{P}(\xi_k(r) = z_2, \delta_k(r) = w_2, D_{j-1} = l, D_{k-1} = r)\mathbb{P}(D_i = 0 \mid D_j = l + w_1).
\end{aligned}$$

Now, Lemma 3.3 says that

$$\mathbb{P}(\xi_j(l) = z_1, \delta_j(l) = w_1) = \mathbb{P}(\xi_j(l) = -z_1, \delta_j(l) = w_1)$$

and thus the expression within braces above vanishes and consequently so does the second term in the right hand side of (27). Using this and (28) in (27), we get that

$$\mathbb{V}(W_i) \le \sigma^2(i-1)\,\mathbb{P}(D_{i-1} = 0). \tag{30}$$

Now, from (23),

$$\mathbb{V}(\bar{L}_n) \le 4\sigma^2 \sum_{i=1}^{n-1} i\,\mathbb{P}(D_i = 0). \tag{31}$$

(a) *This is already enough to obtain an upper bound of the form (12) in $d = 1$, if we use the following bound. For some c_4*

$$\sum_{i=0}^{n-1} \mathbb{P}(D_i = 0) \leq c_4 n^{1/2}. \tag{32}$$

This was established in [3], Lemma 3.3. From this and (31),

$$\mathbb{V}(\bar{L}_n) \leq 4\sigma^2 \sum_{i=1}^{n-1} i\, \mathbb{P}(D_i = 0) \leq 4\sigma^2 n \sum_{i=1}^{n-1} \mathbb{P}(D_i = 0) \leq 4c_4\sigma^2 n^{3/2}. \tag{33}$$

In $d \geq 2$, this argument does not give the correct order. We also do not know the asymptotic behavior of $\mathbb{P}(D_i = 0)$ as $i \to \infty$ (which we do in $d = 1$ (see Lemma 6.1 below); this would give another argument for (a) above). We thus make a more circuitous argument, via Lemma 3.4. By the latter result, there exists c_5 such that for all $0 \leq s < 1$

$$\sum_{i\geq 1} i\mathbb{P}(D_i = 0)s^i \leq c_5 \sum_{i\geq 1} i\mathbb{P}(H_i = 0)s^i. \tag{34}$$

We also use the well known result that for every $d \geq 1$, there exists c_6 such that

$$\mathbb{P}(H_i = 0) \sim c_6 i^{-d/2}, \tag{35}$$

where, as usually, $a_n \sim b_n$ means that $\lim_{n\to\infty} a_n/b_n = 1$ (see, for example, **P7.9** *in [9]). So, we get that for some c_7*

$$\sum_{i\geq 1} i\mathbb{P}(D_i = 0)s^i \leq c_5 c_7 \sum_{i\geq 1} i^{1-d/2} s^i =: c_5 c_7 \varphi_d(s). \tag{36}$$

Now notice that

$$\sum_{i=1}^{n} i\mathbb{P}(D_i = 0) \leq 4 \sum_{i\geq 1} i\mathbb{P}(D_i = 0)(1 - 1/n)^i \tag{37}$$

for all $n \geq 2$. From this, (36) and (31) , we get

$$\begin{aligned} \mathbb{V}(\bar{L}_n) &\leq 4\sigma^2 \sum_{i=1}^{n} i\mathbb{P}(D_i = 0) \\ &\leq 16c_5c_7\sigma^2 \sum_{i\geq 1} i^{1-d/2}(1 - 1/n)^i = 16c_5c_7\sigma^2\varphi_d(1 - 1/n). \end{aligned} \tag{38}$$

(b) *$\varphi_2(s) = \sum_{i\geq 1} s^i = s/(1 - s)$. Thus, $\varphi_2(1 - 1/n) = n - 1$ and from this and (38) we have*

$$\mathbb{V}(\bar{L}_n) \leq 16c_5c_7\sigma^2 n, \text{ if } d = 2. \tag{39}$$

(c)

$$
\begin{aligned}
(\varphi_3(s))^2 &= \left(\sum_{n\geq 1}\frac{1}{\sqrt{n}}s^n\right)^2 = \sum_{n\geq 1}\left(\sum_{i=1}^{n-1}\frac{1}{\sqrt{i}\sqrt{n-i}}\right)s^n \\
&= \sum_{n\geq 1}\left(\sum_{i=1}^{n-1}\frac{1}{\sqrt{i/n}\sqrt{1-i/n}}\frac{1}{n}\right)s^n
\end{aligned} \tag{40}
$$

Notice that the expression within parentheses in the right hand side of (40) is a Riemann sum for a definite integral which equals π*. Thus, there exists* c_8 *such that*

$$
(\varphi_3(s))^2 \leq \sum_{n\geq 1} c_8 s^n = c_8\, s/(1-s).
$$

Thus $\varphi_3(1-1/n) \leq c_8^{1/2} n^{1/2}$ *and from this and (38) we have*

$$
\mathbb{V}(\bar{L}_n) \leq 16 c_5 c_7 c_8^{1/2}\sigma^2 n^{1/2}, \ \textit{if } d=3. \tag{41}
$$

(d) $\varphi_4(s) = \sum_{i\geq 1} s^i/i = \log[1/(1-s)]$*. Thus* $\varphi_4(1-1/n) = \log n$ *and from this and (38) we have*

$$
\mathbb{V}(\bar{L}_n) \leq 16 c_5 c_7 \sigma^2 \log n, \ \textit{if } d=4. \tag{42}
$$

(e) *Finally, if* $d\geq 5$ *and* $0\leq s<1$*, then*

$$
\varphi_d(s) = \sum_{i\geq 1} i^{-(d/2-1)} s^i \leq \sum_{i\geq 1} i^{-3/2} < \infty.
$$

We get from this and (38) that

$$
\mathbb{V}(\bar{L}_n) \textit{ is bounded in } n \textit{ if } d\geq 5. \tag{43}
$$

Gathering (33), (39), (41), (42) and (43), we have that there exists c_2 *such that*

$$
\mathbb{V}(\bar{L}_n) \leq c_2 \mathfrak{D}(n,d). \tag{44}
$$

4.1.2 Lower bound for $\mathbb{V}(\bar{L}_n)$

We only need to consider $d\leq 4$*. By Jensen's inequality, the expectation in (24) is bounded from below by*

$$
\mathbb{E}^2[\mathbb{P}(Y_{i-1}=k \mid \mathcal{F}_{i-1})] = \mathbb{P}^2(Y_{i-1}=k) = \mathbb{P}(Y_{i-1}=Y'_{i-1}=k), \tag{45}
$$

where Y' *is an independent copy of* Y*. It follows that*

$$
\mathbb{V}(\bar{L}_n) \geq 4\sigma^2 \sum_{i=1}^{n-1}\sum_{k\in\mathbb{Z}^d}(\mathbf{1}k)^2\mathbb{P}^2(Y_i=k). \tag{46}
$$

Thus, if $\varepsilon > 0$,

$$\begin{aligned}\mathbb{V}(\bar{L}_n) &\geq 4\sigma^2\varepsilon\sum_{i=1}^{n-1} i \sum_{k\in\mathbb{Z}^d:\,(\mathbf{1}k)^2>\varepsilon i} \mathbb{P}^2(Y_i = k) \\ &= 4\sigma^2\varepsilon\sum_{i=1}^{n-1} i\left[\sum_{k\in\mathbb{Z}^d}\mathbb{P}^2(Y_i = k) - \sum_{k\in\mathbb{Z}^d:\,(\mathbf{1}k)^2\leq\varepsilon i}\mathbb{P}^2(Y_i = k)\right]. \end{aligned} \tag{47}$$

Notice that $Y_n - Y'_n$ *is a random walk distributed as* H_n *from the previous subsection. Thus,*

$$\sum_{k\in\mathbb{Z}^d}\mathbb{P}^2(Y_i = k) = \sum_{k\in\mathbb{Z}^d}\mathbb{P}(Y_i = Y'_i = k) = \mathbb{P}(Y_i = Y'_i) = \mathbb{P}(H_i = 0). \tag{48}$$

Using (48) in (47), we obtain

$$\mathbb{V}(\bar{L}_n) \geq 4\sigma^2\varepsilon\sum_{i=1}^{n-1} i\left\{\mathbb{P}(H_i = 0) - \sup_k \mathbb{P}(Y_i = k)\mathbb{P}\left((\mathbf{1}Y_i)^2 \leq \varepsilon i\right)\right\}. \tag{49}$$

Using (35) again, there exists c_9 *such that* $\mathbb{P}(H_i = 0) \geq c_9 i^{-d/2}$ *for all* i*. It is also well known that there exists* c_{10} *such that* $\sup_k \mathbb{P}(Y_i = k) \leq c_{10} i^{-d/2}$*. Thus,*

$$\mathbb{V}(\bar{L}_n) \geq 4\sigma^2\varepsilon\sum_{i=1}^{n-1} i^{1-d/2}\left[c_9 - c_{10}\,\mathbb{P}\left(\left(\mathbf{1}\frac{Y_i}{\sqrt{i}}\right)^2 \leq \varepsilon\right)\right]. \tag{50}$$

By the Central Limit Theorem, $Y_i/\sqrt{i}$ *converges weakly to a vector of i.i.d. continuous (Gaussian) random variables, say* V*. Since the function* $(\mathbf{1}\,\cdot)^2$ *is continuous in* $\mathbb{R}^d$*, we have that the probability at the right hand side of (50) converges to* $\mathbb{P}((\mathbf{1}V)^2 \leq \varepsilon)$ *as* $i \to \infty$*. Now* $\mathbf{1}V = \sum_{j=1}^d V_j$ *is a non degenerate Gaussian. Thus,* $\mathbb{P}(\mathbf{1}V = 0) = \mathbb{P}((\mathbf{1}V)^2 = 0) = 0$*. We conclude that there exists* $\varepsilon > 0$ *for which the expression in brackets at the right hand side of (50) is bounded below by a positive constant* c_{11} *for all large enough* i*. Thus, for some* c_{12}*,* c_1*, all* n *and* $d = 1, 2, 3, 4$

$$\mathbb{V}(\bar{L}_n) \geq 4c_{12}\varepsilon\sum_{i=1}^{n-1} i^{1-d/2} \geq c_1\mathfrak{D}(n,d). \tag{51}$$

By Proposition 2.3 of [3],

$$\mathbb{V}(\bar{Z}_n) = \sigma^2\sum_{j=0}^{n-1}\mathbb{P}(D_j = 0). \tag{52}$$

Corollary 3.4 of the same paper states that

$$\mathbb{V}(\bar{Z}_n) \textit{ is of order } \begin{cases} n^{1/2}, & \textit{if } d = 1, \\ \log n, & \textit{if } d = 2 \\ \textit{constant}, & \textit{if } d \geq 3. \end{cases} \tag{53}$$

Thus, from (51), for all $d \geq 1$, $\mathbb{V}(\bar{L}_n)$ dominates $\mathbb{V}(\bar{Z}_n)$. Since the order of $\mathbb{C}ov(\bar{L}_n, \bar{Z}_n)$ is intermediate to the orders of $\mathbb{V}(\bar{Z}_n)$ and $\mathbb{V}(\bar{L}_n)$, it is also dominated by the latter. Thus, by (11), the order of $\mathbb{V}(L_n(x))$ is the same as the order of $\mathbb{V}(\bar{L}_n)$ and, from (51) and (44), the proof of Theorem 2.1 is complete. ■

Note that by (31) and (52), if the non degeneracy assumption on the u's does not hold, then $\mathbb{V}(\bar{L}_n(x))$ and $\mathbb{V}(\bar{Z}_n)$ vanish identically and thus so does $\mathbb{V}(L_n(x))$.

Remark 4.1 *In the case of the voter model, $\gamma = \gamma(0,0) = 1$ and thus $D \equiv 0$. From (26), we get that $\mathbb{V}(\bar{L}_n) = \sigma^2 \sum_{i=0}^{n-1} \mathbb{E}[(\mathbf{1}S_i)^2]$, which one then easily estimates to be of the order of n^2 in all dimensions. From (52), we have that $\mathbb{V}(\bar{Z}_n)$ is of the order of n in all dimensions. Thus, by (11), $\mathbb{V}(L_n(x))$ is of the order of n^2 in all dimensions for all x.*

4.2 Invariant measure in high dimensions and its fluctuations

*Since in 5 and higher dimensions $\bar{L}_n - n$ is an L_2-bounded martingale, it converges almost surely as $n \to \infty$. The same holds for $\bar{Z}_n$ (see [3], Lemma 2.2, and (53) above). We thus have from (2) that $L_n - n$ converges almost surely as $n \to \infty$, say to $\tilde{L}$. We thus have that the distribution of $\tilde{L}$ is invariant for the surface dynamics corresponding to applying the random averaging (**??**) and subtracting 1 at each site. A similar argument as that for Proposition 5.2 in [3] can be made for that. We omit it here. From these facts and (2), we are able to argue the following.*

Theorem 4.1 *The distribution of $(\tilde{L}(x) - (\mathbf{1}x)^2)/|x|$ converges weakly to the distribution of $2\mu\tilde{Z}$ as $|x| \to \infty$, where $\tilde{Z}$ is a non trivial random variable and μ is a real number, provided $\mathbf{1}x/|x| \to \mu$ as $|x| \to \infty$. The variance of $(\tilde{L}(x) - (\mathbf{1}x)^2)/|x|$ converges to the variance of the weak limit.*

Proof. *Let $L' = \lim_{n\to\infty} \bar{L}_n - n$ and $Z' = \lim_{n\to\infty} \bar{Z}_n$. Then, from (2), $\tilde{L}(x) = L'(x) + 2(\mathbf{1}x)Z'(x) + (\mathbf{1}x)^2$ and thus $(\tilde{L}(x) - (\mathbf{1}x)^2)/|x| = (L'(x)/|x|) + 2[(\mathbf{1}x)/|x|]Z'(x)$.*

The distributions of both $L'(x)$ and $Z'(x)$ do not depend on x, since they are the limits of distributions that do not depend on x, and $L'(x)$ and $Z'(x)$ have finite second moments, by the L_2-Martingale Convergence Theorem. We conclude that $L'(x)/|x| \to 0$ as $|x| \to \infty$ in L_2 and in probability. We take $\tilde{Z} = Z'(0)$. The non triviality of $\tilde{Z}$ follows from the positivity of the variance of $Z'(0)$ (which equals $\sigma^2 \sum_{i=0}^{\infty} \mathbb{P}(D_i = 0)$; see (52) above). The result follows. ■

5 Remark on higher degrees

The boundedness of the fluctuations in high dimensions, characteristic of the cases of parabolic (studied above) and linear initial conditions [3], does not necessarily occur for a polynomial initial condition of higher degree. We illustrate with the case of a cubic.

Proposition 5.1 *Let $\hat{L}_n(x) = \mathbb{E}[(\mathbf{1}Y_n^x)^3 \mid \mathcal{F}_n]$, $x \in \mathbb{Z}^d$. For all high enough dimensions, there exists c_{13} such that for all $n \geq 1$*

$$\mathbb{V}(\hat{L}_n(0)) \geq c_{13} n^2. \tag{54}$$

Proof.

$$
\begin{aligned}
\hat{L}_n(0) &= \mathbb{E}[(\mathbf{1}Y_n)^3 \mid \mathcal{F}_n] \\
&= \sum_{k\in\mathbb{Z}^d}\sum_{j=1}^{d}\sum_{\alpha=\pm1}[\mathbf{1}(k+\alpha e_j)]^3 u_n(k,k+\alpha e_j)\mathbb{P}(Y_{n-1}=k \mid \mathcal{F}_{n-1}) \\
&= \sum_{k\in\mathbb{Z}^d}\sum_{j=1}^{d}\sum_{\alpha=\pm1}[(\mathbf{1}k)+\alpha]^3 u_n(k,k+\alpha e_j)\mathbb{P}(Y_{n-1}=k \mid \mathcal{F}_{n-1}) \\
&= \sum_{k\in\mathbb{Z}^d}\sum_{j=1}^{d}\sum_{\alpha=\pm1}[(\mathbf{1}k)^3+3\alpha(\mathbf{1}k)^2+3(\mathbf{1}k)+\alpha] \\
&\quad\times\quad u_n(k,k+\alpha e_j)\mathbb{P}(Y_{n-1}=k \mid \mathcal{F}_{n-1}) \\
&= \hat{L}_{n-1}+3\bar{Z}_{n-1}+3\mathbb{E}[(\mathbf{1}Y_{n-1})^2(\mathbf{1}\theta_n(Y_{n-1})) \mid \mathcal{F}_n]+\mathbb{E}[\mathbf{1}\theta_n(Y_{n-1}) \mid \mathcal{F}_n],
\end{aligned}
$$

where $\bar{Z}_n$ is as in the previous sections. Since the distribution of the u_n's are symmetric and independent of $\mathcal{F}_{n-1}$, $\mathbb{E}(\hat{L}_n \mid \mathcal{F}_{n-1}) = \hat{L}_{n-1}+3\bar{Z}_{n-1}$. From this and the fact that $\bar{Z}_n$ is a martingale we get that $\hat{L}_n - 3n\bar{Z}_n$ is a martingale. So $\hat{L}_n - 3n\bar{Z}_n = 3\sum_{i=1}^n \mathbb{E}[(\mathbf{1}Y_{i-1})^2(\mathbf{1}\theta_i(Y_{i-1})) \mid \mathcal{F}_i] + \sum_{i=1}^n \mathbb{E}[\mathbf{1}\theta_i(Y_{i-1}) \mid \mathcal{F}_i]$. The latter sum equals $\bar{Z}_n$ (see [3], (2.24)). We thus get that $\hat{L}_n - 3(n+1)\bar{Z}_n$ is also a martingale with $\hat{L}_n - 3(n+1)\bar{Z}_n = 3\sum_{i=1}^n \hat{W}_i$, where $\hat{W}_i := \mathbb{E}[(\mathbf{1}Y_{i-1})^2(\mathbf{1}\theta_i(Y_{i-1})) \mid \mathcal{F}_i]$.

The variance of $\hat{W}_i$ can be (roughly) estimated as follows

$$
\begin{aligned}
\mathbb{V}(\hat{W}_i) &= \sigma^2\mathbb{E}\sum_{k\in\mathbb{Z}^d}(\mathbf{1}k)^4\mathbb{P}^2(Y_{i-1}=k \mid \mathcal{F}_{i-1}) \\
&= \sigma^2\mathbb{E}[(\mathbf{1}S_{i-1})^4; D_{i-1}=0] \le 16\sigma^2\,(i-1)^4\,\mathbb{P}(D_{i-1}=0),
\end{aligned}
$$

where in the above inequality we used the fact that S has jumps of length at most 2. Thus $\mathbb{V}(\hat{L}_n)$ is bounded above by constant times $\sum_{i=1}^\infty i^4\mathbb{P}(D_i=0)$. From Lemma 3.4, we can obtain an upper bound for the latter sum by replacing $\mathbb{P}(D_i=0)$ in it by constant times $\mathbb{P}(H_i=0)$. We conclude that the sum is bounded by constant times $\sum_{i=1}^\infty i^{4-d/2}$, which is finite if $d \ge 11$. On the other hand (52,53) above tells us that the fluctuations of $\bar{Z}_n$ are positive and bounded for $d \ge 3$ and thus $\mathbb{V}(\hat{L}_n)$ is of order n^2, if $d \ge 11$. ■

Remark 5.1 *It is possible, with a similar estimation as the one done in Sub-subsection 4.1.1, to get a sharper estimate of the variance of $\sum_{i=1}^n \hat{W}_i$ and then obtain an upper bound of constant times $\sum_{i=1}^n i^{2-d/2}$ for $\mathbb{V}(\hat{L}_n)$. This and (53) above would imply that the term $3(n+1)\bar{Z}_n$ gives the dominating contribution for the variance of $\hat{L}_n$ in $d \ge 2$ and thus the conclusion of Proposition 5.1 would hold for these dimensions. (The case of $d=1$ would demand further analysis, since the contributions of $3(n+1)\bar{Z}_n$ and $\sum_{i=1}^n \hat{W}_i$ would be of the same order and one would have to rule out cancelations.) We chose not to make an exhaustive analysis here, but rather to indicate the phenomenon of unboundedness of the fluctuations in all high dimensions as simply as possible.*

Remark 5.2 *Unboundedness of the fluctuations in high dimensions should also occur for higher degree initial polynomials such as $X_0(x) = (\mathbf{1}x)^k$, $k \ge 4$. Results similar to Proposition 5.1 should hold for those cases, with similar arguments.*

6 Central limit theorem

In $d = 1$, the following holds.

Lemma 6.1 *Let D be as in the previous sections. Then*

$$\mathbb{P}(D_n = 0 \mid D_0 = 0) \simeq n^{-1/2}. \tag{55}$$

From here on, $a_n \simeq b_n$ means that $\lim_{n\to\infty} a_n/b_n$ exists and is positive. **Proof.** *This is loosely argued in Remark 3.3 of [3]. We repeat that argument. The result follows from* **P20.2** *of [9], Lemma 3.4 and the fact that (1) of* **P20.2** *of [9] holds for H. We leave more details to the reader.* ■

We will also use the fact that $\mathbb{P}(D_n = 0 \mid D_0 = m) \leq \mathbb{P}(D_n = 0 \mid D_0 = 0)$ for all m. (See proof of Lemma 3.1 in [3] for an argument.)

Proof of Theorem 2.3. *To prove Theorem 2.3, it is enough that we verify the two conditions of the Corollary to Theorem 3.2 of [4]. In the notation of that reference, $X_{ni} = \mathcal{V}_n^{-1/2}(2W_i+1)$ and $\mathcal{F}_{ni} = \mathcal{F}_i$ for all i, n. We recall that $W_i = \mathbb{E}[Y_{i-1}\theta_i(Y_{i-1}) \mid \mathcal{F}_i] = \sum_k k\theta_i(k)\mathbb{P}(Y_{i-1} = k \mid \mathcal{F}_{i-1})$.*

6.1 First condition

Since $\mathcal{V}_n$ is of the order of $n^{3/2}$, it is enough to show that

$$\frac{1}{n^{3/2}} \sum_{i=1}^{n} \mathbb{E}(W_i^2; |W_i| > \varepsilon n^{3/4} \mid \mathcal{F}_{i-1}) \to 0 \tag{56}$$

in probability as $n \to \infty$ for any $\varepsilon > 0$. For that, it suffices to prove that

$$\frac{1}{n^{3/2}} \sum_{i=1}^{n} \mathbb{E}(W_i^2; |W_i| > \varepsilon n^{3/4}) \to 0 \tag{57}$$

as $n \to \infty$. By Cauchy-Schwarz, the following is an upper bound for the above expectation.

$$\sqrt{\mathbb{E}(W_i^4)\,\mathbb{P}(|W_i| > \varepsilon n^{3/4})} \tag{58}$$

Now,

$$\begin{aligned}
\mathbb{E}(W_i^4) &= \mathbb{E}\Big\{\sum_k k\theta_i(k)\mathbb{P}(Y_{i-1} = k \mid \mathcal{F}_{i-1})\Big\}^4 \\
&= c_{14}\sum_k k^4 \mathbb{E}(\mathbb{P}^4(Y_{i-1} = k \mid \mathcal{F}_{i-1})) \\
&\quad + c_{14}\mathbb{E}\sum_{k<l} k^2 l^2 \mathbb{P}^2(Y_{i-1} = k \mid \mathcal{F}_{i-1})\mathbb{P}^2(Y_{i-1} = l \mid \mathcal{F}_{i-1}) \\
&\leq c_{14}\sum_k k^4 \mathbb{E}(\mathbb{P}^2(Y_{i-1} = k \mid \mathcal{F}_{i-1})) + c_{14}\Big\{\sum_k k^2 \mathbb{E}(\mathbb{P}^2(Y_{i-1} = k \mid \mathcal{F}_{i-1}))\Big\}^2 \\
&\leq c_{15}\mathbb{E}(S_{i-1}^4; D_{i-1} = 0) + c_{15}\mathbb{E}^2(S_{i-1}^2; D_{i-1} = 0)
\end{aligned}$$

$$\leq \ c_{15}\sqrt{\mathbb{E}(S_{i-1}^8)}\sqrt{\mathbb{P}(D_{i-1}=0)}+c_{15}\mathbb{E}^2\left(S_{i-1}^2;D_{i-1}=0\right)\leq c_{16}\,i^{7/4}$$

for some c_{14}, c_{15}, c_{16}. *For the last inequality, we used (25,30), Lemma 6.1 and* $\mathbb{E}(S_{i-1}^8) \leq c_{17}\mathbb{E}(Y_{i-1}^8) \simeq i^4$ *for some* c_{17}. *Now,*

$$\begin{aligned}\mathbb{P}(|W_i|>\varepsilon n^{3/4}) \ &\leq \ \varepsilon^{-2}\,n^{-3/2}\mathbb{E}(W_i^2)=\varepsilon^{-2}\,n^{-3/2}\mathbb{V}(W_i)\\ &\leq \ \varepsilon^{-2}\,n^{-3/2}\sigma^2 i\,\mathbb{P}(D_{i-1}=0)\leq c_{18}n^{-3/2}i^{-1/2},\end{aligned}$$

where the second inequality is (30), and the last one follows from Lemma 6.1. Thus

$$\begin{aligned}n^{-3/2}\textstyle\sum_{i=1}^n \mathbb{E}[W_i^2;|W_i|>\varepsilon n^{3/4})] &\leq c_{16}c_{18}n^{-3/2}\textstyle\sum_{i=1}^n\sqrt{i^{7/4}}\sqrt{n^{-3/2}i^{1/2}}\\ &\simeq n^{-9/4}\textstyle\sum_{i=1}^n i^{9/8}\simeq n^{-1/8}\to 0\end{aligned}$$

as $n\to\infty$. *We have proved (57) and thus (56).*

6.2 Second condition

In the notation of [4], it can be written as

$$V_n^2 := \mathcal{V}_n^{-1}\sum_{i=1}^n\Big(\mathbb{E}[(2W_i+1)^2 \mid \mathcal{F}_{i-1}]-\mathbb{E}[(2W_i+1)^2]\Big)\to 0 \tag{59}$$

in probability as $n\to\infty$. *Since* $\mathcal{V}_n$ *is of the order of* $n^{3/2}$, *and also noting that* $\mathbb{E}(W_i \mid \mathcal{F}_{i-1})=0$ *for all* $i\geq 1$, *it is enough to argue that*

$$n^{-3/2}\sum_{i=1}^n[\mathbb{E}(W_i^2 \mid \mathcal{F}_{i-1})-\mathbb{E}(W_i^2)]\to 0 \tag{60}$$

in probability as $n\to\infty$. *We write the above expression as*

$$\begin{aligned}&n^{-3/2}\sum_{i=1}^n\Big(\mathbb{E}\{\mathbb{E}^2[Y_{i-1}\theta_i(Y_{i-1}) \mid \mathcal{F}_i] \mid \mathcal{F}_{i-1}\}-\mathbb{E}[\mathbb{E}^2\left(Y_{i-1}\theta_i(Y_{i-1}) \mid \mathcal{F}_i\right)]\Big)\\ = \ &\sigma^2 n^{-3/2}\sum_{i=1}^n\sum_{k\in\mathbb{Z}}k^2\{\mathbb{P}^2(Y_{i-1}=k \mid \mathcal{F}_{i-1})-\mathbb{E}[\mathbb{P}^2(Y_{i-1}=k \mid \mathcal{F}_{i-1})]\}\\ = \ &\sigma^2 n^{-3/2}\sum_{i=0}^{n-1}\Big(\mathbb{E}(S_i^2;D_i=0 \mid \mathcal{F}_i)-\mathbb{E}(S_i^2;D_i=0)\Big).\end{aligned} \tag{61}$$

We will argue that the variance of the latter expression tends to 0 *as* $n\to\infty$. *We write the variance of the sum as*

$$\begin{aligned}&\sum_{j=1}^{n-1}\mathbb{E}\Big(\sum_{i=0}^{n-1}[\mathbb{E}(S_i^2;D_i=0 \mid \mathcal{F}_{i\wedge j})-\mathbb{E}(S_i^2;D_i=0 \mid \mathcal{F}_{i\wedge(j-1)})]\Big)^2\\ = \ &\sum_{j=1}^{n-1}\mathbb{E}\Big(\sum_{i=j}^{n-1}[\mathbb{E}(S_i^2;D_i=0 \mid \mathcal{F}_j)-\mathbb{E}(S_i^2;D_i=0 \mid \mathcal{F}_{j-1})]\Big)^2.\end{aligned} \tag{62}$$

Now the inner sums in the right hand side of (62) can be written as

$$\begin{aligned}
&\sum_{i=j}^{n-1}\sum_{k,m} E(S_i^2; D_i=0 \mid S_j=k, D_j=m)(\mathbb{P}(S_j=k, D_j=m \mid \mathcal{F}_j)\\
&\qquad -\mathbb{P}(S_j=k, D_j=m \mid \mathcal{F}_{j-1}))\\
&=\sum_{i=j}^{n-1}\sum_{k,m,l,l'} \mathbb{E}(S_i^2; D_i=0 \mid S_j=k, D_j=m)\,\mathbb{P}(Y_{j-1}=l, \hat{Y}_{j-1}=l' \mid \mathcal{F}_{j-1})\\
&\quad \times[\mathbb{P}(S_j=k, D_j=m \mid Y_{j-1}=l, \hat{Y}_{j-1}=l', \mathcal{F}_j)\\
&\qquad -\mathbb{P}(S_j=k, D_j=m \mid Y_{j-1}=l, \hat{Y}_{j-1}=l')].
\end{aligned} \tag{63}$$

Let $u_{n,k} := u_n(k, k+1)$. *Note that the all possible values for the pair* (k,m) *are:* $(l+l',\ l-l'-2)$, $(l+l'-2,\ l-l')$, $(l+l'+2,\ l-l')$, $(l+l',\ l-l'+2)$, *and*

$$\begin{aligned}
\mathbb{P}(S_j=l+l', D_j=l-l'-2 \mid Y_{j-1}=l, \hat{Y}_{j-1}=l', \mathcal{F}_j) &= (1-u_{j,l})u_{j,l'}\\
\mathbb{P}(S_j=l+l'-2, D_j=l-l' \mid Y_{j-1}=l, \hat{Y}_{j-1}=l', \mathcal{F}_j) &= (1-u_{j,l})(1-u_{j,l'})\\
\mathbb{P}(S_j=l+l'+2, D_j=l-l' \mid Y_{j-1}=l, \hat{Y}_{j-1}=l', \mathcal{F}_j) &= u_{j,l}u_{j,l'}\\
\mathbb{P}(S_j=l+l', D_j=l-l'+2 \mid Y_{j-1}=l, \hat{Y}_{j-1}=l', \mathcal{F}_j) &= (1-u_{j,l'})u_{j,l}.
\end{aligned}$$

Let $A^n_{j,l,l'}$ *denote*

$$\begin{aligned}
&\sum_{i=j}^{n-1}\Big(\mathbb{E}(S_i^2; D_i=0 \mid S_j=l+l', D_j=l-l'-2)\\
&\qquad -\mathbb{E}(S_i^2; D_i=0 \mid S_j=l+l'-2, D_j=l-l')\Big).
\end{aligned} \tag{64}$$

The right hand side of (63) can be rewritten as

$$wherebarmeanscentering(thatis,$$

$\bar{X} = X - \mathbb{E}(X)$, *if* X *is an integrable random variable). Substituting (6.2) into (62), we get that the expectation inside the first sum on the right hand side there can be upper bounded above by constant times the sum of three terms, one of which has the following form*

$$\sum_{l,l'}[A^n_{j,l,l'}\overline{u_{j,l'}(1-u_{j,l})} - A^n_{j,l,l'-2}\overline{u_{j,l'}} + A^n_{j,l+2,l'}\overline{u_{j,l'}u_{j,l}}]\mathbb{P}(Y_{j-1}=l, \hat{Y}_{j-1}=l' \mid \mathcal{F}_{j-1}),$$

where bar means centering (that is, $\bar{X} = X - \mathbb{E}(X)$, *if* X *is an integrable random variable). Substituting (6.2) into (62), we get that the expectation inside the first sum on the right hand side there can be upper bounded above by constant times the sum of three terms, one of which has the following form*

$$V^2_{n,j} := \mathbb{E}\Big(\sum_{l,l'} A^n_{j,l,l'}\overline{u_{j,l'}(1-u_{j,l})}\mathbb{P}(Y_{j-1}=l, \hat{Y}_{j-1}=l' \mid \mathcal{F}_{j-1})\Big)^2, \tag{66}$$

and the other two are similar, with $A^n_{j,l,l'}\overline{u_{j,l'}(1-u_{j,l})}$ replaced by $A^n_{j,l,l'-2}\overline{u_{j,l'}}$ and $A^n_{j,l+2,l'}\overline{u_{j,l'}u_{j,l}}$, respectively. We will analyze (66) only. The other two terms are similar and can be treated in an analogous way.

We go back to $A^n_{j,l,l'}$. Observe that, from Lemma 3.3, we can represent S_n in one dimension as $\sum_{i=1}^n \hat\xi_i\eta_i(D_{i-1})$, where $\hat\xi_1,\hat\xi_2,\dots$ are i.i.d. with $\mathbb{P}(\hat\xi_1=2)=\mathbb{P}(\hat\xi_1=-2)=1/2$ and $\eta_i(l)\stackrel{d}{=}1\!\!1(\xi_i(l)\neq 0)$ for all i,l, where $1\!\!1(\cdot)$ is the indicator function. Clearly, $\eta_i=1\!\!1(S_i\neq S_{i-1})$. Now, from the nearest neighbor character of the jumps of $Y,\hat Y$, we have that $1\!\!1(S_i\neq S_{i-1})=1\!\!1(D_i=D_{i-1})$. We can thus rewrite the summands of (64) as

$$
\begin{aligned}
&\mathbb{E}[(l+l'+S_i-S_j)^2; D_i=0\mid D_j=l-l'-2]\\
&\qquad -\mathbb{E}[(l+l'-2+S_i-S_j)^2; D_i=0\mid D_j=l-l']\\
=\ &\mathbb{E}\Big[\Big(l+l'+\sum_{k=j+1}^{i}\hat\xi_k\eta_k(D_{k-1})\Big)^2; D_i=0\mid D_j=l-l'-2\Big]\\
&-\mathbb{E}\Big[\Big(l+l'-2+\sum_{k=j+1}^{i}\hat\xi_k\eta_k(D_{k-1})\Big)^2; D_i=0\mid D_j=l-l'\Big]\\
=\ &\Big[(l+l')^2\mathbb{P}(D_i=0\mid D_j=l-l'-2)-(l+l'-2)^2\mathbb{P}(D_i=0\mid D_j=l-l')\\
&+4\sum_{k=j+1}^{i}\Big\{\mathbb{E}[\eta_k(D_{k-1}); D_i=0\mid D_j=l-l'-2]\\
&\qquad -\mathbb{E}[\eta_k(D_{k-1}); D_i=0\mid D_j=l-l']\Big\}\Big]\\
=\ &(l+l')^2[\mathbb{P}(D_i=0\mid D_j=l-l'-2)-\mathbb{P}(D_i=0\mid D_j=l-l')]\\
&+4(l+l'-1)\mathbb{P}(D_i=0\mid D_j=l-l')\\
&+4\sum_{k=j+1}^{i}\Big[\mathbb{P}(D_i=0,D_k=D_{k-1}\mid D_j=l-2)\\
&\qquad -\mathbb{P}(D_i=0,D_k=D_{k-1}\mid D_j=l)\Big].
\end{aligned}\tag{67}
$$

We write the expression within brackets in the sum above as

$$
\begin{aligned}
&\sum_{h\in\mathbb{Z}}\mathbb{P}(D_i=0\mid D_k=h)\gamma(h,h)\big[\mathbb{P}(D_{k-1}=h\mid D_j=l-l'-2)\\
&\qquad -\mathbb{P}(D_{k-1}=h\mid D_j=l-l')\big]\\
=\ &\bar\gamma\sum_{h\in\mathbb{Z}}\mathbb{P}(D_{i-1}=0\mid D_{k-1}=h)\big[\mathbb{P}(D_{k-1}=h\mid D_j=l-l'-2)\\
&\qquad -\mathbb{P}(D_{k-1}=h\mid D_j=l-l')\big]\\
+\ &(\gamma-\bar\gamma)\mathbb{P}(D_i=0\mid D_k=0)\big[\mathbb{P}(D_{k-1}=0\mid D_j=l-l'-2)\\
&\qquad -\mathbb{P}(D_{k-1}=0\mid D_j=l-l')\big],
\end{aligned}\tag{68}
$$

where, as before, $\gamma=\gamma(0,0)$ and $\bar\gamma$ is the common value of $\gamma(h,h)$ for $h\neq 0$. We have also used time homogeneity in the equality above. The first term of the right hand side of (68) thus *becomes*

$$\bar{\gamma}(\mathbb{P}(D_{i-1}=0 \mid D_j=l-l'-2)-\mathbb{P}(D_{i-1}=0 \mid D_j=l-l')). \tag{69}$$

Substituting (69) into (68) and this into (67), we get that $A^n_{j,l,l'}$ *is the sum in* $i=j,\dots,n$ *of*

$$\begin{aligned}
&(l+l')^2[\mathbb{P}(D_i=0 \mid D_j=l-l'-2)-\mathbb{P}(D_i=0 \mid D_j=l-l')]\\
&\quad+4(l+l'-1)\mathbb{P}(D_i=0 \mid D_j=l-l')\\
&\quad+4\bar{\gamma}(i-j)[\mathbb{P}(D_{i-1}=0 \mid D_j=l-l'-2)-\mathbb{P}(D_{i-1}=0 \mid D_j=l-l')]\\
&\quad+4(\gamma-\bar{\gamma})\sum_{k=j+1}^{i}\mathbb{P}(D_i=0 \mid D_k=0)[\mathbb{P}(D_{k-1}=0 \mid D_j=l-l'-2)\\
&\qquad\qquad-\mathbb{P}(D_{k-1}=0 \mid D_j=l-l')]\\
&\quad=(l+l')^2a_{i,j,l,l'}+4(l+l'-1)\mathbb{P}(D_i=0 \mid D_j=l-l')+4\bar{\gamma}(i-j)a_{i-1,j,l,l'}\\
&\quad+4(\gamma-\bar{\gamma})\sum_{k=j+1}^{i}\mathbb{P}(D_i=0 \mid D_k=0)a_{k-1,j,l,l'},
\end{aligned} \tag{70}$$

where $a_{i,j,l,l'}:=\mathbb{P}(D_i=0 \mid D_j=l-l'-2)-\mathbb{P}(D_i=0 \mid D_j=l-l')$. *We have that* $\sum_{i=j}^n a_{i,j,l,l'}$ *is uniformly bounded in* l,l', j *and* n *(see, e.g., [3], Lemma 4.3). So is* $\sum_{i=j}^n |a_{i-1,j,l,l'}|$, *by the nearest neighbor character of* D *(in* $2\mathbb{Z}$*), which makes* $a_{i,j,l,l'}\geq 0$ *for all* $i\geq j$ *or* $a_{i,j,l,l'}\leq 0$ *for all* $i\geq j$. *We have also that* $\sum_{i=j}^n \mathbb{P}(D_i=0 \mid D_j=l-l')\leq\sum_{i=j}^n \mathbb{P}(D_i=0 \mid D_j=0)\leq c_{19}\sqrt{n}$ *for all* j,n, *by (53) and the remark after the proof of Lemma 6.1 above. From these bounds, we get* $\sum_{i=j}^n\sum_{k=j+1}^i \mathbb{P}(D_i=0 \mid D_k=0)|a_{k-1,j,l,l'}|\leq c_{20}\sqrt{n}$. *We conclude from these facts and (70) that* $|A^n_{j,l,l'}|$ *is bounded above by constant times*

$$(l+l')^2+|l+l'-1|\sqrt{n}+n. \tag{71}$$

Thus (66) can be bounded above by constant times the sum of three terms, the first of which can be written as

$$\begin{aligned}
\sum_{l,l'}\sum_{k,k'}&(l+l')^2(k+k')^2|\mathbb{E}(\overline{u_{j,l'}(1-u_{j,l})u_{j,k'}(1-u_{j,k})})|\\
&\times\mathbb{P}(Y_{j-1}=l,Y'_{j-1}=l',\hat{Y}_{j-1}=k,\tilde{Y}_{j-1}=k'),
\end{aligned} \tag{72}$$

where $Y,Y',\hat{Y}$ *and* $\tilde{Y}$ *are independent given* $\mathcal{F}$. *The inner expectation above does not vanish only if* $\{l,l'\}\cap\{k,k'\}\neq\emptyset$. *Thus (72) can be bounded above by constant times*

$$\begin{aligned}
&\sum_{l,l',k}(l+l')^2(k+l')^2\mathbb{P}(Y_{j-1}=l,Y'_{j-1}=\tilde{Y}_{j-1}=l',\hat{Y}_{j-1}=k)\\
=\;&\mathbb{E}((Y_{j-1}+Y'_{j-1})^2(\hat{Y}_{j-1}+\tilde{Y}_{j-1})^2;\hat{D}_{j-1}=0),
\end{aligned}$$

where $\hat{D}=Y'-\tilde{Y}$. *The latter term above can be upper bounded, using Cauchy-Schwarz (and the fact that* $\hat{D}\overset{d}{=}D$*), by*

Second term. *The second term of the bound to (66) is*

$$n\sum_{l,l'}\sum_{k,k'}|(l+l'-1)(k+k'-1)||\mathbb{E}(\overline{u_{j,l'}(1-u_{j,l})}\,\overline{u_{j,k'}(1-u_{j,k})})|$$
$$\times\mathbb{P}(Y_{j-1}=l,Y'_{j-1}=l',\hat{Y}_{j-1}=k,\tilde{Y}_{j-1}=k'). \tag{73}$$

Similarly as in the reasoning for the first term, there exists c_{21} such that (73) is bounded above by constant times

$$\begin{aligned}
& n\sum_{l,l',k}|(l+l'-1)(k+l'-1)|\mathbb{P}(Y_{j-1}=l,Y'_{j-1}=\tilde{Y}_{j-1}=l',\hat{Y}_{j-1}=k)\\
\leq\; & n\mathbb{E}(|(Y_{j-1}+Y'_{j-1}-1)(\hat{Y}_{j-1}+\tilde{Y}_{j-1}-1)|;\hat{D}_{j-1}=0)\\
\leq\; & n\sqrt{\mathbb{E}((Y_{j-1}+Y'_{j-1}-1)^2(\hat{Y}_{j-1}+\tilde{Y}_{j-1}-1)^2)}\sqrt{\mathbb{P}(D_{j-1}=0)}\\
\leq\; & n\sqrt{\mathbb{E}((Y_{j-1}+Y'_{j-1}-1)^4)}\sqrt{\mathbb{P}(D_{j-1}=0)}\leq c_{21}nj^{3/4}.
\end{aligned} \tag{74}$$

Third term. *The third term of the bound to (66) is*

$$n^2\sum_{l,l'}\sum_{k,k'}|\mathbb{E}(\overline{u_{j,l'}(1-u_{j,l})}\,\overline{u_{j,k'}(1-u_{j,k})})|$$
$$\times\mathbb{P}(Y_{j-1}=l,Y'_{j-1}=l',\hat{Y}_{j-1}=k,\tilde{Y}_{j-1}=k'). \tag{75}$$

Similarly as in the reasoning for the first and second terms, there exists c_{22} such that (75) is bounded above by constant times

*From (**??**), (74) and (**??**), we conclude that there exists c_{23} such that*

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*Instituto de Matemática e Estatística — Universidade de São Paulo
R. do Matão 1010, Cidade Universitária — 05508-900 São Paulo SP — Brasil
lrenato@ime.usp.br, marina@ime.usp.br*

*Instituto de Matemática — Universidade Federal da Bahia
Av. Ademar de Barros s/n, Campus de Ondina — 40170-110 Salvador Ba — Brasil
medeiros@ufba.br*