Proof of Bose-Einstein Condensation for Interacting Gases

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Using a specially tuned mean-field Bose gas as a reference system, we establish a positive lower bound on the condensate density for continuous Bose systems with superstable two-body interactions and a finite gap in the one-particle excitations spectrum, *i.e.* we prove for the first time standard homogeneous Bose-Einstein condensation for such interacting systems.

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INTRODUCTION

The long-standing problem of proving the existence of a Bose-Einstein condensation (BEC) in non-ideal real Bose gases with a standard two-body interaction has recently been warmed up by the great success of observing this phenomenon in trapped gases. Notice that much earlier BEC has been observed in liquid He^4, which however remained always under discussion.

In the present note we announce our result together with a sketch of the proof about the existence of conventional BEC in realistic Bose gases with superstable two-body interactions and with gap in the one-particle excitation energy spectrum. To the best of our knowledge, this is a first proof of its sort for homogeneous systems. We are not using any scaling limits, (*e.g.* type van der Waals limits [1–3]) or truncation of particle interactions [4, 5]. We prove that BEC occurs by constructing a positive lower bound for the condensate density which is valid for low enough temperature and appropriate large density of particles.

We consider a gas of interacting Bosons in cubic boxes $\Lambda = L \times L \times L \subset \mathbb{R}^3$ with periodic boundary conditions. We look here in detail at the 3 dimensional case, and comment later on the case of other dimensions. Denote by $V = L^3$ the volume of the box Λ. The Hamiltonian for the volume Λ of this system reads

$$H^{\Delta}_{\Lambda,g} = T^{\Delta}_{\Lambda} - \mu N_{\Lambda} + gU_{\Lambda}, \qquad g > 0, \tag{1}$$

where T^{Δ}_{Λ} is the kinetic energy with gap $\Delta > 0$ in its spectrum,

$$T^{\Delta}_{\Lambda} = \sum_{k\in\Lambda^*} \frac{\hbar^2 k^2}{2m} a^{\dagger}_k a_k - \Delta a^{\dagger}_0 a_0. \tag{2}$$

The sum k runs over the set Λ^*, dual to Λ, *i.e.* $\Lambda^* = \{k \in \mathbb{R}^3; k_\alpha = 2\pi n_\alpha/L; n_\alpha = 0, \pm 1, \dots; \alpha = 1, 2, 3\}$. The operators $a^{\dagger}_k$ and a_k are the Bose creation and annihilation operators for mode k. As usual, the number operators are denoted by $N_k = a^{\dagger}_k a_k$, $N_\Lambda = \sum_{k\in\Lambda^*} N_k$ is the total number operator in the volume Λ.

We assume *a priori* the presence of a gap Δ in the one-particle excitations spectrum, isolating the lowest energy level. In this letter we make no comments about the origin of this spectral gap, however its presence can be realised in several ways, a gap can be created using attractive boundary conditions [6, 7] or part of the inter-particle interaction could have caused this gap and as such being effectively incorporated [5].

Of course μ is the chemical potential and the interaction between the particles is modelled by the two-body interaction term

$$U_\Lambda = \frac{1}{2}\int_{\Lambda^2} \mathrm{d}x\mathrm{d}y \; a^{\dagger}(x)a^{\dagger}(y)v(x-y)a(y)a(x), \tag{3}$$

where $a^{\dagger}(x), a^{\dagger}(y)$ and $a(y), a(x)$ are the creation and annihilation operators for the Bose particles at $x, y \in \mathbb{R}^3$. The interaction potential v is assumed to be spherically symmetric, superstable [8], *i.e.* it satisfies the inequality

$$\sum_{1\le i<j\le n} v(x_i - x_j) \ge \frac{A}{2V}n^2 - Bn \tag{4}$$

for some constants $A > 0$, $B \ge 0$, and all $n \in \mathbb{N}$, $x_i \in \Lambda$. Consequently, the interaction term (3) satisfies

$$U_\Lambda \ge \frac{A}{2V}N^2_\Lambda - BN_\Lambda. \tag{5}$$

This superstability property is, together with the spectral gap (2), the physical foundation of our proof. Intuitively, one might understand that condensation in the groundstate (*i.e.* $k = 0$), which is energetically isolated by a gap Δ can survive the switching-on of a non-catastrophic interaction, and that fluctuations must be of a macroscopical size to overcome this gap and lift particles out of the isolated groundstate. More technical ingredients of the proof are the convexity properties of thermodynamical potentials, such as the pressure, and the use of an optimal choice for the constants A, B in the superstability criterion (4). Indeed, it was proved [8] that continuous L^1-functions of positive type $v : \mathbb{R}^3 \to \mathbb{R}$ are superstable potentials if and only if

$$\hat{v}(0) = \int_{\mathbb{R}^3} \mathrm{d}x \; v(x) > 0. \tag{6}$$

Moreover, Lewis, Pulè, and de Smedt [9] proved the existence of optimal constants $A = \hat{v}(0)(1-\epsilon)$ and

$B = v(0)/2$ in (4) for this type of potentials. Here, $\epsilon > 0$ is an arbitrarily small positive constant. This optimal choice is of determining importance in our proof of Bose-Einstein condensation.

SKETCH OF PROOF

The main idea of the proof is to estimate the Bose condensate of the full model (1) by the condensate of a particularly chosen reference system for which one knows the occurrence of condensation. The clever choice of this reference system is the subtle point of our proof. The reference system is a so-called mean-field Bose gas, an exactly solvable model of Bosons [10–16] (for a review see [5]) defined by Hamiltonians as

$$H^{\Delta}_{\Lambda,g,\lambda} = T^{\Delta}_{\Lambda} - \mu N_{\Lambda} + g\frac{\lambda}{2V}N^2_{\Lambda}. \tag{7}$$

The kinetic energy operator T^{Δ}_{Λ} (2) is as for the general interacting system (1), but the interaction term (3) is replaced by a mean-field interaction term.

The reference system (7), a mean-field Bose gas, emerges as the *van der Waals* limit of the fully interacting system [1–3]. In that case, the constant λ equals $\hat{v}(0)$ (6), which means that the van der Waals limit reduces the full interaction to the $\hat{v}(0)$ contribution. In our proof we tune the constant λ in order to get the best possible lower bound for the condensate density of the full interaction model.

Remark that our reference system (7) does show Bose condensation for large enough densities (*i.e.* for μ large) at any given temperature. Moreover, systems of the type of our reference system have better properties than the ideal Bose gas which is in many ways a *pathological* model, *e.g.* in the sense that there is no equivalence of ensembles [17, 18] and in the sense that the chemical potential has a zero upperbound in order to safeguard thermodynamical stability. These are the reasons for our strategy of using the free of those pathologies reference system (7).

Thermodynamic properties of reference the system (7). It is well known that our reference system (7) is a soluble models. In the case of vanishing gap $\Delta = 0$, the complete solution can be found at several places in the literature [5, 10–16]. In particular there is condensation for all dimensions $D \geq 3$ at any temperature for densities large enough. It is a student exercise to work out now the case with gap $\Delta > 0$. There is one main difference with the gapless case, namely the presence of the gap provokes a shift in the chemical potential and its treshholds, and one gets condensation in all dimensions $D \geq 1$. One derives straightforwardly that for the reference model (7) one gets condensation for all values of the chemical potential satisfying,

$$\mu > g\lambda\rho^P(\beta, -\Delta) - \Delta.$$

where $\rho^P(\beta, -\Delta)$ is the total density of the perfect Bose Gas (PBG) at inverse temperature β and chemical potential equal to $-\Delta$. Moreover, the condensate density

$$\rho^{\Delta}_{0,g,\lambda}(\beta,\mu) = \lim_{V\to\infty} \frac{1}{V}\langle N_0\rangle_{H^{\Delta}_{\Lambda,g,\lambda}}(\beta,\mu),$$

i.e. the density at the zero mode in the thermodynamic limit ($V \to \infty$) of the grand-canonical Gibbs states $\langle\cdot\rangle^{(\beta,\mu)}_{H_\Lambda}$, in volumes Λ for a certain choice of temperature and chemical potential (β,μ) and Hamiltonian H_Λ, is explicitly given by

$$\rho^{\Delta}_{0,g,\lambda} = \frac{\mu + \Delta}{g\lambda} - \rho^P(\beta, -\Delta). \tag{8}$$

One also computes the total density for given (β,μ) to be

$$\rho^{\Delta}_{g,\lambda} = \lim_{V\to\infty} \frac{1}{V}\langle N_\Lambda\rangle_{H^{\Delta}_{\Lambda,g,\lambda}} = \frac{\mu}{g\lambda}. \tag{9}$$

To prove Bose condensation in the full model (1), we subtract from (3) the van der Waals-like behaving part of the interaction proportional to $N^2_\Lambda/2V$, tune this with a factor, taking into account the optimal stability constants, and add it to the kinetic-energy term. The latter serves as our reference system (7), from which we establish a lower bound on the condensate density $\rho^{\Delta}_{0,g}(\beta,\mu) = \lim_{V\to\infty}\langle N_0/V\rangle_{H^{\Delta}_{\Lambda,g}}$ in the fully interacting system (1). In the lemma below, a lower bound on $\rho^{\Delta}_{0,g}(\beta,\mu)$ is given.

Lemma 1 *The condensate density $\rho^{\Delta}_{0,g}(\beta,\mu)$ in the thermodynamic limit of grand-canonical Gibbs states of interacting systems (1) with superstable two-body potentials v (3), has the following lower bound:*

$$\begin{aligned}\rho^{\Delta}_{0,g}(\beta,\mu) \geq{}& \frac{\mu}{g\hat{v}(0)(1-\epsilon)} + \frac{g\hat{v}(0)}{2\Delta}\left(\rho^P(\beta,-\Delta)\right)^2 \\ &- \frac{(\mu+\Delta)^2}{2\Delta g\hat{v}(0)}\frac{\epsilon}{(1-\epsilon)^2} - \frac{\mu+\Delta}{\Delta(1-\epsilon)}\rho^P(\beta,-\Delta) \\ &- \frac{gv(0)}{2\Delta}\rho^{(\Delta=0)}(\beta,\mu) - \rho^P_c(\beta).\end{aligned} \tag{10}$$

Here $\rho^{(\Delta=0)}_g(\beta,\mu)$ is the total density of the interacting gas without gap (1), and $\epsilon > 0$ is an arbitrarily small constant. $\rho^P(\beta,-\Delta)$ refers to the total density of the PBG at inverse temperature β and chemical potential $\mu = -\Delta$, $\rho^P_c(\beta)$ is the critical density of the PBG. The bound is valid for values $\mu > g\hat{v}(0)(1-\epsilon)\rho^P_c(\beta)$, and dimensions $D = 3$.

Idea of the proof. Using the Bogoliubov convexity inequality [5] one gets:

$$\frac{g}{V}\langle W^{\lambda}_{\Lambda}\rangle_{H^{\Delta}_{\Lambda,g}} \leq p_\Lambda[H^{\Delta}_{\Lambda,g,\lambda}] - p_\Lambda[H^{\Delta}_{\Lambda,g}] \leq \frac{g}{V}\langle W^{\lambda}_{\Lambda}\rangle_{H^{\Delta}_{\Lambda,g,\lambda}}. \tag{11}$$

This gives upper and lower bounds on the difference of the thermodynamic pressure $p_\Lambda[H_\Lambda]$ of the mean-field reference Bose gas (7) and the full model (1). The operator W_Λ^λ is the difference between the interactions of the fully interacting and the mean-field Bose gases, *i.e.* $W_\Lambda^\lambda = U_\Lambda - \frac{\lambda}{2V}N_\Lambda^2$. The expectation values in (11) can be estimated using, for the lower bound, the superstability properties of the interaction, and, for the upper bound, the properties of the equilibrium states of the mean-field reference Bose gas. The lower bound in (11) follows from (5), and from the tuning of the interaction parameter λ for the mean-field reference Bose gas (7) to the constant A in (5),

$$\frac{g}{V}\langle W_\Lambda^A\rangle_{H^\Delta_{\Lambda,g}} \geq -\frac{gB}{V}\langle N_\Lambda\rangle_{H^\Delta_{\Lambda,g}}. \qquad (12)$$

Using the mode by mode gauge invariance of the Gibbs states of the mean-field Bose gas we arrive at the following upper bound for the pressure difference (11),

$$\frac{g}{V}\langle W_\Lambda^A\rangle_{H^\Delta_{\Lambda,g,A}} \leq \frac{g}{V^2}\langle CN_\Lambda^2 - \frac{\hat{v}(0)}{2}N_0^2\rangle_{H^\Delta_{\Lambda,g,A}}, \qquad (13)$$

where $C = \hat{v}(0) - A/2$. It follows from the properties of the mean-field Bose gas that the expectation values in the rhs of (13) in the limit ($V \to \infty$) are given by $gC(\rho_A^\Delta(\beta,\mu))^2 - g\hat{v}(0)(\rho_{0,A}^\Delta(\beta,\mu))^2/2$.

The pressure $p_\Lambda[H_\Lambda^\Delta]$ is an increasing convex function of $\Delta \geq 0$. Since the condensate density $\rho_{0,g}^\Delta(\beta,\mu)$ is the derivative of the pressure with respect to Δ, by convexity we find a lower bound for the condensate density $\rho_{0,g}^\Delta(\beta,\mu)$, given by

$$\frac{1}{V}\langle N_0\rangle_{H^\Delta_{\Lambda,g}} \geq \frac{p_\Lambda[H^\Delta_{\Lambda,g}] - p_\Lambda[H^{(\Delta=0)}_{\Lambda,g}]}{\Delta}. \qquad (14)$$

Analogously, by virtue of the same convexity property, the difference of the pressures between the mean-field Bose gas with gap and without gap, is bounded from below by the condensate density for the mean-field gas without gap, *i.e.*

$$\frac{p_\Lambda[H^\Delta_{\Lambda,g,A}] - p_\Lambda[H^{(\Delta=0)}_{\Lambda,g,A}]}{\Delta} \geq \frac{1}{V}\langle N_0\rangle_{H^{(\Delta=0)}_{\Lambda,g,A}}. \qquad (15)$$

Adding up these two inequalities and using the Bogoliubov convexity inequality (11) once at $\Delta > 0$ and once at $\Delta = 0$, together with the bounds (12) and (13), yields the following lower bound for the condensate density $\rho_{0,g}^\Delta(\beta,\mu)$ in the thermodynamic limit ($V \to \infty$), *i.e.*

$$\rho_{0,g}^\Delta(\beta,\mu) \geq \rho_{0,g,A}^{(\Delta=0)}(\beta,\mu) + g\frac{\hat{v}(0)}{2\Delta}(\rho_{0,g,A}^\Delta(\beta,\mu))^2 - \frac{g}{\Delta}\left(B\rho_g^{(\Delta=0)}(\beta,\mu) + C(\rho_{g,A}^\Delta(\beta,\mu))^2\right). \qquad (16)$$

The lower bound (10) is now found using the explicit expressions for total density and the condensate density of the mean-field Bose gas with gap, (9) and (8), and the well-known expression for the condensate density in the gapless mean-field model, *i.e.* if $\mu > g\lambda\rho_c^P(\beta)$,

$$\rho_{0,g,\lambda}^{(\Delta=0)} = \lim_{V\to\infty}\frac{1}{V}\langle N_0\rangle_{H^{(\Delta=0)}_{\Lambda,g,\lambda}} = \frac{\mu}{g\lambda}\rho_c^P(\beta),$$

where $\rho_c^P(\beta)$ is the critical density for the perfect Bose Gas at inverse temperature β. As a last step, we need the optimal superstability constants for continuous L^1 potentials of positive type [9], *i.e.* we take $A = (1-\epsilon)\hat{v}(0)$, and $B = v(0)/2$, and get the expression for the lower bound (10) in the lemma. *QED*

Now we get our main result:

Theorem 2 *Consider a three dimensional system of interacting Bose particles (1), with a superstable two-body interaction. Take any $\eta > 0$, fix a temperature and a chemical potential (β,μ) such that $\mu > g\hat{v}(0)(\rho_c^P(\beta)+2\eta)$, then there exists a minimal gap Δ_{min} such that for all $\Delta \geq \Delta_{min}$,*

$$\left|\frac{g\hat{v}(0)}{2\Delta}\left(\rho^P(\beta,-\Delta)\right)^2 - \frac{gv(0)}{2\Delta}\rho_g^{(\Delta=0)}(\beta,\mu) - 2\frac{\mu+\Delta}{\Delta}\rho^P(\beta,-\Delta)\right| < \eta$$

For these values of the gap Δ we have $\rho_{0,g}^\Delta(\beta,\mu) > 0$, and hence proved condensation.

Proof. It is a consequence of Lemma 1 and the fact that $\epsilon > 0$ can be chosen arbitrarily small. *QED*

DISCUSSION

First of all let us remark that our proofs hold without any change in all dimensions $D \geq 3$. For $D = 1$ or 2 a similar lower bound on the condensate density can be derived, on the basis of modified convexity arguments (14)–(15), *i.e.* one has to consider pressure differences of the form $p_\Lambda[H_\Lambda^\Delta] - p_\Lambda[H_\Lambda^{\Delta_0}]$, with $0 < \Delta_0 < \Delta$, instead of with $\Delta_0 = 0$ in (14)–(15). This yields the substitution of $\rho_{0,g,A}^{\Delta_0}(\beta,\mu)$ and $\rho_g^{\Delta_0}(\beta,\mu)$ for $\rho_{0,g,A}^{(\Delta=0)}(\beta,\mu)$ and $\rho_g^{(\Delta=0)}(\beta,\mu)$ in (16). Hence, also in one and two dimensional interacting Bose gases with gap (1), Bose condensation is proved, in contrast to the Bogoliubov–Hohenberg theorem [19] which yields the absence of BEC for $1D$ or $2D$ translation invariant continuous Bose systems without gap.

Remark that our bound (10) contains a term $\rho_g^{(\Delta=0)}(\beta,\mu)$, the total density of the fully interacting model (1) without gap ($\Delta = 0$), which is not explicitly known. This term can be eliminated using an alternative reference system, yielding a new bound. This alternative reference system is an adaptation of the original reference systems (7) with an extra interaction term, linear in the total number operator N_Λ (cf. (5)), *i.e.* yielding a shift in the chemical potential. The derivation of this lower bound on the Bose condensate density, is then as above. It is explicitly given by

$$\begin{aligned}\rho^{\Delta}_{0,g}(\beta,\mu) \geq \frac{2\mu + gv(0)}{g\hat{v}(0)(1-\epsilon)} + \frac{g\hat{v}(0)}{2\Delta}\left(\rho^{P}(\beta,-\Delta)\right)^2 \\ - \frac{(2\mu+2\Delta+gv(0))^2}{2\Delta g\hat{v}(0)}\frac{\epsilon}{(1-\epsilon)^2} - \rho^{P}_{c}(\beta) \\ - \frac{2\mu+2\Delta+gv(0)}{\Delta\hat{v}(0)(1-\epsilon)}\left(\frac{v(0)}{2} + \rho^{P}(\beta,-\Delta)\right), \quad (17)\end{aligned}$$

for $\mu \geq g\hat{v}(0)1-\epsilon)\rho^{P}_{c}(\beta)$. The properties of this bound are completely analogous to the properties of Lemma 1. However, as all quantities in this bound are explicitly known, it is a useful one to compute numerical estimations on the condensate density.

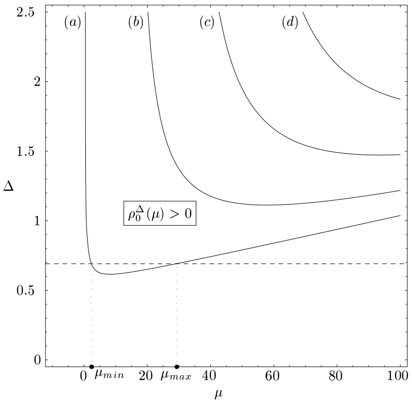

FIG. 1: (μ, Δ)-graph of the lower bound.

An example of such a computation is represented in Fig. 1. In this (μ, Δ) graph the values of the lower bound are given, for a choice of the parameters $\beta = g = v(0) = \hat{v}(0) = \frac{\hbar^2}{2m} = 1$ and $\epsilon = 0.01$. The lowest line (a) is the line for which the lower bound (17) is zero, *i.e.* above this curve we have BEC, the higher lines (b), (c), and (d), are the contours where the numerical values of the lower bound are $15, 30$ resp. 45. Clearly, the density of the condensate increases with increasing μ (*i.e.* increasing total density) and increasing gap Δ. For a given chemical potential μ, there is a minimal value for the gap Δ (cf. Theorem 2) below which we cannot prove condensation. Notice that, if one fixes Δ larger than this minimal value, our lower bound yields BEC between a μ_{min} and a μ_{max} of the chemical potential. The μ_{max} increases with decreasing ϵ. Considering the high μ regime and taking ϵ of order μ^{-2}, one estimates the absolute lowest bound on Δ in order to have a positive lower bound on de condensate density (17), *i.e.* condensation occurs when

$$\Delta > g\left(\frac{v(0)}{2} + \hat{v}(0)\rho^{P}(\beta,-\Delta)\right),$$

meaning that for non-zero interaction (3) ($g > 0$), there is a non-zero lower bound on the gap-width. This minorant is proportional to the coupling constant $g \geq 0$.

Finally we want to stress that our results are for homogeneous systems. On the other hand we have to mention here the interesting recent result about the condensation for trapped Bose gases [20], *i.e.* for inhomogeneous systems where also a rigorous proof is given of Bose condensation in a realistic setting.

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