Correlation at Low Temperature: I. Exponential Decay

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Abstract

The present paper generalizes the analysis [18, 2] of the correlations for a lattice system of real-valued spins at low temperature. The Gibbs measure is assumed to be generated by a fairly general pair potential (Hamiltonian function). The novelty, as compared to [18, 2], is that the single-site (self-) energies of the spins are not required to have only a single local minimum and no other extrema. Our derivation of exponential decay of correlations goes through the spectral analysis of a deformed Laplacian closely related to the Witten Laplacian studied in [18, 2]. We prove that this Laplacian has a spectral gap above zero and argue that this implies exponential decay of the correlations.

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I Introduction

The present paper is devoted to the study of probability measures on $\mathbb{R}^\Lambda$ given in form of a Boltzmann weight,

$$d\mu(x) \; := \; \mathcal{Z}_\beta^{-1} \, e^{-2\beta H(x)} \, d^\Lambda x \, , \tag{I.1}$$

where Λ is a large, but finite set, $H \in C^2(\mathbb{R}^\Lambda; \mathbb{R})$ is twice differentiable, and $\mathcal{Z}_\beta^{-1}$ is a normalization factor.

In the context of (classical) statistical mechanics, measures of the form (I.1) are called *lattice gases* or *lattice spin systems*. In this part of theoretical physics, $\mathbb{R}^\Lambda = \{x : \Lambda \to \mathbb{R}\}$ is intepreted as the space of configurations of continuous spins on a lattice Λ, and $H \in C^2(\mathbb{R}^\Lambda; \mathbb{R})$ is a Hamiltonian function whose values $H(x)$ represent the energy of a given spin configuration x.

The properties of (an appropriate sequence labeled by Λ of) measures of the form (I.1) in the limit $|\Lambda| \to \infty$ is called *the thermodynamic limit* and is of central importance to statistical mechanics, as statistical mechanics deals with macroscopic properties of these physical systems for large lattices. In contrast, statements about (I.1), whose validity requires $|\Lambda|$ to be bounded above by a constant, are useless for most applications in statistical mechanics. (Exceptions are, e.g., spin glass systems.) For this reason, we insured that all statements in this paper are meant to hold *uniformly in* $|\Lambda|$, unless explicitly stated otherwise.

Moreover, $\beta > 0$ corresponds to inverse temperature, $\beta = (k_B T)^{-1}$, and our interest is focused on the *low-temperature* regime, $\beta \gg 1$. (Here k_B is Boltzmann's constant.)

The measures (I.1) are studied in this paper by means of a *twisted Witten Laplacian* $\Delta_{H,Q}$ which is a selfadjoint operator acting on (a dense domain in) the space $\mathcal{H} = \bigwedge(\mathbb{R}^\Lambda)$ of square-integrable, antisymmetric forms on $\mathbb{R}^\Lambda$. Denoting by d the usual exterior differentiation on $\mathcal{H}$, we introduce the *twisted exterior complex* as the conjugation of d by certain exponential weights,

$$\begin{aligned} d_{H,Q} \;\; &:= \;\; \frac{1}{\beta} \, e^{-\beta[H - d\Gamma(Q)]} \, d \, e^{\beta[H - d\Gamma(Q)]} \\ &= \;\; \frac{1}{\beta} \, e^{\beta \, d\Gamma(Q)} \, e^{-\beta H} \, d \, e^{\beta H} \, e^{-\beta \, d\Gamma(Q)} \, . \end{aligned} \tag{I.2}$$

Here, Q is a suitably chosen matrix-valued function on $\mathbb{R}^\Lambda$, and the *twist* $d\Gamma(Q)$ denotes its second quantization. Clearly, $d_{H,Q}^2 = 0$. Moreover, $d_{H,Q} \, e^{-\beta H} = 0$ because $d\Gamma(Q)$ restricted to 0-forms vanishes. Denoting by $d_{H,Q}^*$ the adjoint of $d_{H,Q}$ with respect to Lebesgue measure, we introduce a *twisted Dirac operator* by $D_{H,Q} := d_{H,Q}^* + d_{H,Q}$. The twisted Witten Laplacian (or Hodge Laplacian with respect to the complex $d_{H,Q}$) is then defined as the square of $D_{H,Q}$,

$$\Delta_{H,Q} \; := \; D_{H,Q}^2 \; = \; d_{H,Q}^* \, d_{H,Q} \; + \; d_{H,Q} \, d_{H,Q}^* \, . \tag{I.3}$$

By construction, $\Delta_{H,Q}$ is a positive operator. Its kernel is one-dimensional and spanned by the 0-form $e^{-\beta H}$, as we show below.

The (untwisted, $Q = 0$) Witten Laplacian $\Delta_{H,0}$ was originally introduced by Witten [22] in the context of differential geometry. Its importance for the study of (I.1) lies in the identity

$$\begin{aligned} \mathbb{E}_\beta^T(u\,;\,v) \;&:=\; \mathbb{E}_\beta(u \cdot v) \;-\; \mathbb{E}_\beta(u)\cdot\mathbb{E}_\beta(v) \\ &=\; \frac{1}{\beta^2\,\mathcal{Z}_\beta}\big\langle e^{-\beta H}\,du \,\big|\, \big(\Delta_{H,0}^{(1)}\big)^{-1} e^{-\beta H}\,dv\big\rangle\,, \end{aligned} \tag{I.4}$$

for the truncated correlation function of the probability measure $d\mu(x)$, where $\mathbb{E}_\beta(u) := \int u(x)\,d\mu(x)$ denotes the expectation value of a continuously differentiable observable $u \in C^1(\mathbb{R}^\Lambda;\mathbb{R})$, and $\Delta_{H,0}^{(1)} > 0$ denotes the (strictly positive) restriction of $\Delta_{H,0}$ to the space of 1-forms. Identity (I.4) shows that the decay (mixing property) of $d\mu(x)$ corresponds to the decay of the resolvent of $\Delta_{H,0}$. This was realized by Helffer and Sjöstrand [15] and, more explicitly, by Sjöstrand in [18]. In particular, the truncated two-point correlation function has an explicit representation as a matrix element of the resolvent of $\Delta_{H,0}$,

$$\mathbb{E}_\beta^T(x_i\,;\,x_j) \;=\; \frac{1}{\beta^2\,\mathcal{Z}_\beta}\big\langle e^{-\beta H}\,dx_i \,\big|\, \big(\Delta_{H,0}^{(1)}\big)^{-1} e^{-\beta H}\,dx_j\big\rangle\,. \tag{I.5}$$

Here, $i, j \in \Lambda$ are two lattice points. The asymptotics of the decay of this truncated correlation function $\mathbb{E}_\beta^T(x_i\,;\,x_j)$, as the distance of i and j gets large, is of special interest. Eq. (I.5) links the asymptotics, in the low-temperature regime, to matrix elements of the inverse of $H''(0)$. To realize this connection, we need to analyze the low-lying spectrum of $\Delta_{H,0}^{(1)}$. In this paper we prove an upper bound on the correlations and in a future work we will derive the asymptotics of the correlations, following ideas of [2].

A certain class of Hamiltonian functions were studied by means of $\Delta_{H,0}^{(1)}$ in the above mentioned paper by Sjöstrand [18]. He derived the asymptotics of the truncated correlation function in leading order in β^{-1} under the assumption that the Hamiltonian function H is uniformly strictly convex. In [20], he considerably refined his analysis and derived the correlation asymptotics to all orders in β^{-1}. Sjöstrand's paper [18] was also the starting point for Jecko, Sjöstrand, and the first author, who derived in [2] the (same) correlation asymptotics, to leading order in β^{-1}, under weaker assumptions. Yet, the the requirements in [2] still imply that H has no critical point except for its (unique) minimum. While heuristic arguments suggest that uniqueness of the global minimum of H is actually a necessary condition for exponential decay of the correlations, the exclusion of other critical points in [2] appeared to be purely technical and not essential. This is precisely the point to be established in the present paper. Loosely speaking, the introduction of the twist $d\Gamma(Q)$ convexifies the potential in $\Delta_{H,Q}^{(1)}$ such that it has all properties required in [2], and all the methods developed therein apply again. Apart from this (desired) convexification, $d\Gamma(Q)$ also generates (undesired) exponentially large weight factors. These large weights lead to technical complications in our analysis because we have to show that they do not affect the main relative form bound implying the expected spectral properties of $\Delta_{H,Q}^{(1)}$ (see Sect. II).

The twist with $d\Gamma(Q)$ leads us to analyze the stability at low temperatures of the partition function $\mathcal{Z}_\beta$, under perturbations of H in a single variable, more precisely of the type $H \to H - q_i$, where $q_i(x) = q_i(x_i)$ and q_i vanishes around 0. We prove that pertubations of this type only change $\mathcal{Z}_\beta$ by an error which is exponentially small in β. We are inspired by techniques of Sjöstrand [19], cf. also [10, 16].

A variety of Hamiltonian functions, especially those of ϕ^4-type, were analyzed by Helffer [11, 13, 12, 14] by means of the Witten Laplacian $\Delta_{H,0}^{(1)}$. He also uses a "twist", which is, however, different from ours. Nevertheless, Helffer's approach and the models he studied do overlap with the models treated in the present paper. Therefore, a comparison is appropriate and will be the subject of future research.

It turns out that, through (I.3), the spectral properties of $\Delta_{H,0}^{(1)}$ are also intimately connected with the validity of a Log-Sobolev inequality for the measure $d\mu$. Roughly speaking, a Log-Sobolev inequality implies the uniform (w.r.t. $|\Lambda|$) positivity of $\Delta_{H,0}^{(1)}$, and the converse is true, as well, provided the positivity of $\Delta_{H,0}^{(1)}$ is not only uniform w.r.t. $|\Lambda|$, but also with respect to all possible boundary conditions (in the sense of the DLR equations). This has been established by Zegarlinski [25], Yoshida [23], Bodineau and Helffer [3] and Procacci and Scoppola [17]. These works, in particular, prove exponential decay of correlations for weakly coupled systems at an arbitrary, but fixed inverse temperature $0 < \beta < \infty$. In contrast to our paper, however, the methods employed in these papers do not apply in the low-temperature limit $\beta \to \infty$. The paper [3] is close to ours in spirit whereas [17] uses cluster expansion techniques.

One of the interesting aspects about the Witten Laplacian is that it appears in various branches of mathematics and mathematical physics as an object of fundamental importance. Aside from the original application to problems in differential geometry, now sometimes referred to as *Witten-Helffer-Sjöstrand* theory [5], it coincides (on a formal level) with operators used in supersymmetry, cf. [7]. In stochastic analysis, the Witten Laplacian can be viewed as the generator of a parabolic, stochastic PDE that determine the correlation functions of $d\mu$ and the rate of convergence of initial distributions to a stationary one [1]. In the context of spin glasses, especially the Hopfield-Kac model, the Witten Laplacian has been used in [4], to estimate the tunneling rates between local minima of a spin glass Hamiltonian.

We remark that continuous spin systems with multiple phases have been successfully studied by other methods, e.g., the Pirogov-Sinai theory and contour methods [8, 9, 24]. It would be interesting to link the basic notions of Pirogov-Sinai theory to corresponding spectral properties of the Hamiltonian. We anticipate this link to go through the minima of the Hamiltonian function which, on one hand, counts the number of phases in Pirogov-Sinai theory, and which, on the other hand should correspond to the elements of the kernel of a Witten Laplacian which degenerates in the thermodynamic limit. This is a subject of future research, as well.

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I.1 Organization of the Paper

The present section is devoted to a presentation of all the objects used in this paper. We furthermore state the main result, Theorem I.1, and reduce its proof to two statememts. One is a semi-classical localization result, Theorem I.6, and the other is a form bound given in Theorem II.1. In Subsect. I.2 we formulate the assumptions we impose on the self-energies (or single-spin potentials) and the interaction between lattice sites. We furthermore present some properties of the self-energies and the interaction which follow from the basic assumptions (to be derived in Appendix A). In Subsect. I.3 we present the central object around which the analysis of this paper revolves, namely the twisted Witten Laplacian (I.3). In Subsect. I.4 we present an extension to non-zero Q of the identity (I.4) expressing the truncated two-point correlation function in terms of matrix elements of the inverse Witten Laplacian (restricted to 1-forms). We furthermore give a Combes-Thomas twisted version of this formula which will be used to prove exponential decay of correlations. In Subsect. I.5 we introduce a comparison operator which approximate the twisted Witten Laplacian in the low-temperature limit (in a semi-classical sense). In Subsect. I.6 we state and prove the main result of this paper, namely exponential decay of correlations.

In Sect. II we state and prove Theorem II.1, the main technical result of this paper. This theorem is a form bound which states that the twisted Witten Laplacian (with Combes-Thomas weights), in the low-temperature limit, is a small pertubation of the comparison operator to be introduced in Subsect. I.5. We start the section by breaking the proof up into a series of estimates, which are subsequently proven in a series of subsections. In Subsect. II.1 and II.2 we estimate the perturbation (which is a matrix) in terms of a diagonal matrix. In Subsects. II.3–II.5 this diagonal matrix is in turn estimated by other diagonal matrices which in the final step is identified with the diagonal part of the comparison operator.

In Sect. III we prove a semi-classical localization estimate for the partition function. The proof is divided into two steps. In Subsect. III.1 we analyze the interaction part of the Hamiltonian and in Subsect. III.2 we analyse the self-energy part of the Hamiltonian and prove the localization estimate.

In Appendix A we derive some properties of the self-energy and the interaction which follow directly from the basic assumptions.

Finally, in Appendix B we establishe self-adjointness of the Dirac operator and determine the kernel of the Witten Laplacian.

I.2 Assumptions and the main result

We consider a system of real-valued spins on a finite set Λ, i.e., $|\Lambda| < \infty$, which we henceforth refer to as *the lattice*. We assume Λ to be a metric space with respect to a metric $\rho : \Lambda \times \Lambda \to \mathbb{R}_0^+$. Typical examples for Λ we have in mind are finite subsets of $\mathbb{Z}^d$ or finite discrete tori $\mathbb{Z}_L^d = (\mathbb{Z}/L\mathbb{Z})^d$, with $d, L \in \mathbb{N}$. While $\mathbb{Z}_L^d$ is an abelian group, we shall not assume Λ to possess any additional structure at this point.

In order to allow for taking the thermodynamic limit, we shall think of the lattice Λ as being an element of a countable collection Γ of finite lattices, $\Lambda \in \Gamma$. For the existence of the thermodynamic limit and its analysis, we then require that all constants are uniform with respect to $\Lambda \in \Gamma$. Below, we call a quantity *universal* if it neither depends on $j \in \Lambda$ nor on $\Lambda \in \Gamma$. For example, we shall assume below that

$$\max_{j \in \Lambda} \left\{ \sum_{k \in \Lambda} e^{-\rho(j,k)} \right\} \;\le\; C_\rho \, , \tag{I.6}$$

for some universal constant $C_\rho < \infty$.

Given the lattice Λ, the corresponding spin configuration space is $\mathbb{R}^\Lambda$, and the energy of a spin configuration is determined by a pair interaction Hamiltonian function $H \,:\, \mathbb{R}^\Lambda \to \mathbb{R}$ given by

$$H(x) \;=\; \sum_{j \in \Lambda} f_j(x_j) \;+\; \alpha \sum_{i,j \in \Lambda} w_{i,j}(x_i, x_j) \, , \tag{I.7}$$

for $x = (x_j)_{j \in \Lambda} \in \mathbb{R}^\Lambda$. Here, $f_j \,:\, \mathbb{R} \to \mathbb{R}$ are the on-site energy functions at the lattice point $j \in \Lambda$, and $w_{i,j} \,:\, \mathbb{R}^2 \to \mathbb{R}$ represent the pair interaction between the lattice sites $i, j \in \Lambda$. Furthermore, $0 \le \alpha \le \alpha_0 < 1$ is a coupling constant which is assumed to be sufficiently small, although we do not consider the limit $\alpha \to 0$. For the derivation of our main results, we require the following hypotheses.

Hypothesis 1. *For any $j \in \Lambda$, zero is the unique, nondegenerate minimum of $f_j \in C^2(\mathbb{R};\mathbb{R})$, attained at $t = 0$, i.e., $f_j(0) = 0$, $f_j''(0) > 0$, and $f_j(t) > 0$, whenever $t \neq 0$. Moreover, there exist universal constants $0 < c_f \le 1 \le C_f < \infty$, $R_f < \infty$, such that*

$$c_f \;\le\; f_j''(0) \;\le\; C_f \, , \tag{I.8}$$

$$\forall t \neq 0 : \qquad \big\{ f_j'(t) = 0 \big\} \;\Rightarrow\; \big\{ f_j(t) \ge c_f \big\} \, , \tag{I.9}$$

$$\forall t \in \mathbb{R} : \quad |f_j''(t) - f_j''(0)| \;\le\; C_f \big(|f_j'(t)| \,+\, \min\{1, |t|\} \big) \, , \tag{I.10}$$

$$\forall |t| \ge R_f : \quad |f_j'(t)| \;\ge\; c_f \, \max \big\{ |f_k'(s)| \;\big|\; |s| \le |t| \, , \; k \in \Lambda \big\} \tag{I.11}$$

for all $j \in \Lambda$.

Hypothesis 1 roughly coincides with what is required in [2, Example I.1], except for Eq. (I.10), which is much weaker than the corresponding assumption

$$|f''(t) - f''(0)| \;\le\; \text{const} \, \big(|f'(t)| + f'(t)^2 \big) \tag{I.12}$$

in [2, Example I.1]. Note that Eq. (I.12) yields $f''(t_*) = f''(0) > 0$, for any critical point t_*, which implies that $t = 0$ is the only critical point of f. This does not only suggest that Eq. (I.12) is merely technical, but it also illustrates that the requirement (I.12) excludes one of the main examples for an application of our method: an Ising-type or ϕ^4-model in a magnetic field, in which case $f_j(x_j) = \text{const} + (x_j^2 - 1)^2 + \lambda_j x_j$, with $\lambda_j \in \mathbb{R}$ being the magnetic field at site j. In fact, the main goal of the present work is to weaken Assumption (I.12).

To formulate our hypothesis on $w_{i,j}$, we introduce functions

$$h_j(s) \;:=\; \min\{|f_j(s)|, |f_j(s)|^{\frac{1}{2}}\} \;+\; \min\{1, |s|\}\,, \tag{I.13}$$

for all $j \in \Lambda$.

Hypothesis 2. *For all $i, j \in \Lambda$, the pair interaction functions $\omega_{i,j} \in C^2(\mathbb{R}\times\mathbb{R};\mathbb{R})$ vanish on-site and at the origin, $w_{i,i}(s,t) \equiv 0$ and $w_{i,j}(0,0) = 0$. Furthermore, for $i, j \in \Lambda$, there exists a symmetric matrix $\underline{a} = (a_{ij})_{i,j\in\Lambda}$ of nonnegative weights (with $a_{i,i} = 0$ for convenience), such that*

$$\sum_{\tau\in\{i,j\}} \left| \frac{\partial w_{i,j}}{\partial x_\tau}(x_i, x_j) \right| \;+\; \sum_{\sigma,\tau\in\{i,j\}} \left| \frac{\partial^2 w_{i,j}}{\partial x_\sigma\, \partial x_\tau}(x_i, x_j) \;-\; \frac{\partial^2 w_{i,j}}{\partial x_\sigma\, \partial x_\tau}(0,0) \right|$$
$$\leq\; a_{i,j}\left(h_i(x_i) + h_j(x_j)\right)\,, \tag{I.14}$$

and

$$\sum_{\sigma,\tau\in\{i,j\}} \left| \frac{\partial^2 w_{i,j}}{\partial x_\sigma\, \partial x_\tau}(0,0) \right| \;\leq\; a_{i,j}\,. \tag{I.15}$$

The weights are assumed to have exponential decay with respect to the metric ρ on Λ,

$$\max_{i\in\Lambda} \sum_{j\in\Lambda} e^{\rho(i,j)}\, a_{i,j} \;\leq\; C_a\,, \tag{I.16}$$

for some universal constant $C_a < \infty$.

Similar to the example above, Hypothesis 2 also accommodates for the important example of an Ising or spin glass type interaction, for which $w_{i,j}(x_i, x_j) = x_i\, x_j$ (except that C_a may not be universal for spin glass models). Further note that in the case $\Lambda \in \mathbb{Z}^d$ and ρ the euclidean metric, the summability condition is easily fulfilled for interactions of finite range $D > 0$, i.e., in case that $w_{i,j} \equiv 0$ whenever $\rho(i,j) \geq D$.

Hypotheses 1 and 2 insure that H is growing at least linearly in x at infinity, provided $\alpha \geq 0$ is sufficiently small, cf. Lemma B.1. Thus $u \cdot \exp[-\beta H] \in L^2(\mathbb{R}^\Lambda)$, for any $\beta > 0$ and any polynomially bounded measurable function u :

$\mathbb{R}^\Lambda \to \mathbb{C}$. In particular, the expectation value $\mathbb{E}_\beta(u)$ in Eq. (I.1) exists for any such function (which represents an observable).

The main result of this paper is the following.

Theorem I.1 (Exponential Decay). *Assume Hypothesises 1 and 2. There exist universal constants $\beta_0 < \infty$ and $\alpha_1 > 0$ such that, for all $0 < \varepsilon \le 1/2$, $0 \le \kappa < 1$, $\beta > \beta_0 \varepsilon^{-1/2}$, and $0 \le \alpha < \alpha_0 := \varepsilon(1-\kappa)\alpha_1$, we have*

$$\left|\mathbb{E}_\beta^T(x_i\,;x_j)\right| \;\le\; (1+\varepsilon)\,\frac{1}{2\,\beta\,\lambda_{\min}}\,e^{-\kappa\rho(i,j)}\,. \tag{I.17}$$

Here $\lambda_{\min} := \min \sigma(H''(0))$ which, for $\alpha < \alpha_0$ small enough, satisfies $\lambda_{\min} > c_f/2$. We remark that the effect of boundary conditions (in a given example) on the decay estimate (I.17) is encoded in $\lambda_{\min}$ and in C_a, and hence, is easy to analyze. Control with respect to boundary conditions were considered in [25, 23, 3, 17], however, not with uniform control at low temperature.

An important point in our analysis is the observation made in Lemma I.2, below, that our hypothesis on f_j suffices to construct an auxilliary function g_j which obeys (I.12) and, yet, differs from f_j only away from its global minimum at $x_j = 0$, i.e., in a region which should be irrelevant (and, indeed, is irrelevant, as we prove in a subsequent paper) for the correlation asymptotics at low temperature.

Lemma I.2. *Assume Hypothesis 1. There exist universal numbers $q_{\max} > 0$,*

$$0 \;<\; \hat{R}_0 \;<\; \hat{R}_1 \;<\; \check{R}_0 \;<\; \check{R}_1 \;<\; \infty\,, \tag{I.18}$$

to which we associate the unions of intervals

$$\begin{aligned} I_0 &:= [-\hat{R}_0\,,\,\hat{R}_0]\,, && \text{(I.19)}\\ \hat{I} &:= (-\hat{R}_1, -\hat{R}_0) \cup (\hat{R}_0, \hat{R}_1)\,, && \text{(I.20)}\\ \check{I} &:= (-\infty, -\check{R}_0) \cup (\check{R}_0, \infty)\,, && \text{(I.21)}\\ I_\infty &:= (-\infty, -\check{R}_1) \cup (\check{R}_1, \infty)\,, && \text{(I.22)} \end{aligned}$$

and functions

$$\begin{aligned} \hat{q}_j \;\in\; C^2\big(\hat{I}\,,\,[0, q_{\max}]\big) \quad , \quad \check{q}_j \;\in\; C^2\big(\check{I}\,,\,[0, q_{\max}]\big)\,, && \text{(I.23)}\\ \text{and} \qquad q_j \;:=\; \hat{q}_j + \check{q}_j \;\in\; C^2\big(\mathbb{R}\,,\,[0, q_{\max}]\big)\,, && \text{(I.24)} \end{aligned}$$

possessing the following properties:

(i) On I_∞, we have $\check{q}_j \equiv q_{\max}$.

(ii) The functions

$$g_j \;:=\; f_j - q_j \;\; \text{and} \;\; \hat{g}_j \;:=\; f_j - \hat{q}_j \tag{I.25}$$

are nonnegative and have a unique critical point at $t = 0$.

(iii) There exist universal constants $0 < c_g \leq 1 \leq C_g < \infty$, such that the following estimates hold true:

$$c_f \;\leq\; g_j''(0) \;=\; f_j''(0) \;\leq\; C_f \,, \tag{I.26}$$

$$\forall t \in \mathbb{R}: \qquad |g_j'(t)| \;\geq\; c_g \, \min\{1, |t|\} \,, \tag{I.27}$$

$$\forall t \in \mathbb{R}: \quad |g_j''(t) - g_j''(0)| \;+\; |\hat{g}_j''(t) - \hat{g}_j''(0)| \;\leq\; C_g \, |g_j'(t)| \,, \tag{I.28}$$

$$\forall t \in \mathbb{R}: \quad |g_j'(t)| \;\geq\; c_g \, \max\Big\{ |g_k'(s)| \;\Big|\; |s| \leq |t| \,,\; k \in \Lambda \Big\} \,, \tag{I.29}$$

$$\forall t \in \mathbb{R}: \qquad c_g \, |g_j'(t)| \;\leq\; |\hat{g}_j'(t)| \;\leq\; C_g \, |g_j'(t)| \,, \tag{I.30}$$

$$\forall \tau > 0: \qquad \max_{|t| \leq \tau} \big\{ |g_j'(t)| \big\} \;\leq\; C_g \, e^{C_g \tau} \,, \tag{I.31}$$

$$\forall \tau > 0: \qquad \min_{|t| \geq \tau} \big\{ |g_j'(t)| \big\} \;\geq\; c_g \, \min\{1, \tau\} \,. \tag{I.32}$$

Although in Appendix A, we give a detailed proof of Lemma I.2, the reader might find our graphical illustration in Fig. 1 of its assertion for a typical function f_j helpful.

Note that an application of Lemma I.2 to Hypothesis 2 easily yields the following consequence.

Lemma I.3. *Assume Hypotheses 1 and 2. Then, there exists a universal constant $C_w < \infty$, such that, for all $i, j \in \Lambda$, the pair interaction functions $w_{i,j}$ obey*

$$\sum_{\tau \in \{i,j\}} \left| \frac{\partial w_{i,j}}{\partial x_\tau}(x_i, x_j) \right| \;+\; \sum_{\sigma, \tau \in \{i,j\}} \left| \frac{\partial^2 w_{i,j}}{\partial x_\sigma \, \partial x_\tau}(x_i, x_j) \;-\; \frac{\partial^2 w_{i,j}}{\partial x_\sigma \, \partial x_\tau}(0,0) \right| \tag{I.33}$$

$$\leq\; C_w \, a_{i,j} \Big(\min\big\{ |g_i'(x_i)| \,,\; |g_i'(x_i)|^{1/2} \big\} \;+\; \min\big\{ |g_j'(x_j)| \,,\; |g_j'(x_j)|^{1/2} \big\} \Big) \,.$$

I.3 Twisted Witten Complex

In this section we present most of the objects to which we reduce the analysis of the truncated two-point correlation functions (I.2). We introduce the fermionic Fock space over $\mathbb{C}^\Lambda$,

$$\mathcal{F} \;\equiv\; \mathcal{F}(\mathbb{C}^\Lambda) \;:=\; \oplus_{n=0}^{\infty} \mathcal{F}^{(n)} \,, \qquad \mathcal{F}^{(n)} \;:=\; \big(\mathbb{C}^\Lambda \big)^{\otimes_a n} \,, \tag{I.34}$$

where $\otimes_a n$ denotes the n-fold antisymmetric tensor product, and $\mathcal{F}^{(0)} \simeq \mathbb{C}\Omega$ is a one-dimensional subspace spanned by the normalized vacuum vector Ω. The standard annihilation and creation operators $\{a_i, a_i^*\}_{i \in \Lambda}$ represent the canonical anticommutation relations (CAR)

$$\forall i, j \in \Lambda: \;\; \{a_i, a_j\} \;=\; \{a_i^*, a_j^*\} \;=\; 0 \,, \quad \{a_i^*, a_j\} \;=\; \delta_{ij} \,, \;\text{ and }\; a_i \Omega \;=\; 0 \,, \tag{I.35}$$

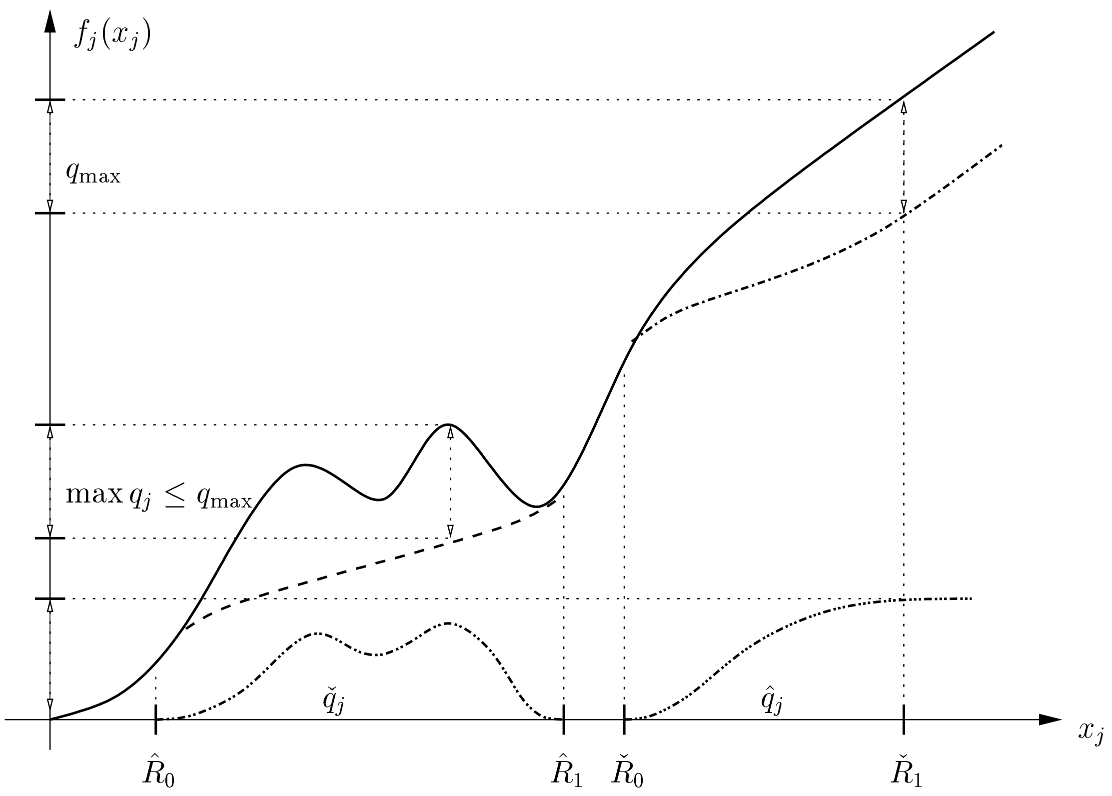

Figure 1: The solid curve is the graph of f_j. The graph of $\tilde{g}_j$ is the dashed curve on $(\hat{R}_0, \hat{R}_1)$ and agrees with f_j on $(0, \hat{R}_0)$ and on $(\hat{R}_1, \infty)$. The graph of g_j is the dashed curve on $(\hat{R}_0, \hat{R}_1)$, the dash-dotted curve on $(\check{R}_0, \infty)$ and agrees with f_j on $(0, r)$ and on $\hat{R}_1, \check{R}_0)$. The dash-dot-dotted curve depicts $\hat{q}_j$, and the dash-dot-dot-dotted curve depicts $\check{q}_j$

on $\mathcal{F}$, where $\{A, B\} = AB + BA$, and δ_{ij} is the Kronecker delta. By (I.35), an orthonormal basis of $\mathcal{F}^{(1)}$ is given by $\{a_i^* \Omega\}_{i \in \Lambda}$. Thus, $\mathcal{F}^{(0)}$ and $\mathcal{F}^{(1)}$ can be identified with $\mathbb{C}$ and $\mathbb{C}^\Lambda$, respectively. Due to (I.35), the products $a_i^* a_j$ and $a_j a_i^*$ leave $\mathcal{F}^{(n)}$ invariant. Thus, we can represent their action on $\mathcal{F}^{(0)}$ by a complex number,

$$\forall i, j \in \Lambda : \quad a_i^* \, a_j \;=\; 0 \quad \text{and} \quad a_j \, a_i^* \;=\; \delta_{ij} \quad \text{on } \mathcal{F}^{(0)} . \tag{I.36}$$

Likewise their action on $\mathcal{F}^{(1)}$ can be represented by a complex $\Lambda \times \Lambda$-matrix. Denoting the standard matrix units on $\mathcal{F}^{(1)} \equiv \mathbb{C}^\Lambda$ by E_{ij}, i.e., $\langle a_k^* \Omega | \; E_{ij} \, a_l^* \Omega \rangle := \delta_{ik} \, \delta_{jl}$, one easily checks that

$$\forall i, j \in \Lambda : \quad a_i^* \, a_j \;=\; E_{ij} \quad \text{and} \quad a_j \, a_i^* \;=\; \delta_{ij} \cdot \mathbf{1} - E_{ij} \quad \text{on } \mathcal{F}^{(1)} . \tag{I.37}$$

Note, in particular, that

$$\forall j \in \Lambda : \quad a_j^* \, a_j \;=\; E_{jj} \quad \text{and} \quad a_j \, a_j^* \;=\; \sum_{k(\neq j)} E_{kk} \quad \text{on } \mathcal{F}^{(1)} . \tag{I.38}$$

Here, $\sum_{k(\neq j)}$ denotes $\sum_{k \in \Lambda \setminus \{j\}}$. This is part of the summation convention that $\sum_i (\cdot) := \sum_{i \in \Lambda} (\cdot)$, $\sum_{i,j} (\cdot) := \sum_{(i,j) \in \Lambda^2} (\cdot)$, $\sum_{i \neq j} (\cdot) := \sum_{(i,j) \in \Lambda^2 \setminus \{(i,i) | i \in \Lambda\}} (\cdot)$, and $\sum_{i(\neq j)} (\cdot) := \sum_{i \in \Lambda \setminus \{j\}} (\cdot)$, which turns out to be convenient and is used throughout the paper.

The Hilbert space of forms over $\mathbb{R}^\Lambda$ is the tensor product

$$\mathcal{H} \;:=\; L^2(\mathbb{R}^\Lambda) \otimes \mathcal{F} \;=\; \oplus_{n=0}^{\infty} \mathcal{H}^{(n)} , \quad \mathcal{H}^{(n)} \;=\; L^2(\mathbb{R}^\Lambda) \otimes \mathcal{F}^{(n)} . \tag{I.39}$$

We introduce a (multiple of the) standard exterior derivative on $\mathcal{H}$, for $\beta > 0$, and its adjoint

$$d \;:=\; \sum_i \frac{1}{\beta} \partial_i \otimes a_i^* \quad \text{and} \quad d^* \;=\; - \sum_i \frac{1}{\beta} \partial_i \otimes a_i , \tag{I.40}$$

where ∂_i is shorthand for $\frac{\partial}{\partial x_i}$. Note that $d^2 = (d^*)^2 = 0$. The Hodge Laplacian associated to this exterior derivative is $dd^* + d^* d = (d + d^*)^2$.

To a bounded operator $T \in \mathcal{B}[\mathcal{H}^{(1)}]$ we associate its second quantization $d\Gamma(T) \in \mathcal{B}[\mathcal{H}]$ by the standard formula, i.e., if T is represented by a matrix $(T_{i,j})_{i,j \in \Lambda}$ whose entries take their values in the operators on $L^2(\mathbb{R}^\Lambda)$ then

$$d\Gamma(T) \;:=\; \sum_{i,j} T_{i,j} \otimes a_i^* \, a_j \, . \tag{I.41}$$

In the present paper we second-quantize only operators whose entries are semi-bounded self-adjoint operators on $L^2(\mathbb{R}^\Lambda)$. In particular, let Q, $\hat{Q}$, and $\check{Q}$ denote the matrix-valued functions with entries

$$\hat{Q}_{i,j}(x) \;=\; \delta_{i,j}\,\hat{q}_i(x_i)\;, \quad \check{Q}_{i,j}(x) \;=\; \delta_{i,j}\,\check{q}_i(x_i) \tag{I.42}$$

$$\text{and} \quad Q_{i,j}(x) \;=\; \hat{Q}_{i,j}(x) \;+\; \check{Q}_{i,j}(x)\;. \tag{I.43}$$

Here, $x \equiv (x_i)_{i\in\Lambda} \in \mathbb{R}^\Lambda$, and $\hat{q}_i$ and $\check{q}_i$ are introduced in (I.23)–(I.24). We frequently omit the argument and write $\hat{q}_i := \hat{q}_i(x_i)$ and $\check{q}_i := \check{q}_i(x_i)$, et cetera. Then their second quantization is given by

$$d\Gamma(Q^\#) \;=\; \sum_j q_j^\# \otimes\, a_j^*\, a_j\;, \tag{I.44}$$

where $Q^\#$ denotes Q, $\hat{Q}$, or $\check{Q}$, and $q_j^\#$ denotes q_j, $\hat{q}_j$, or $\check{q}_j$, respectively.

Now, we introduce the twisted exterior derivative

$$d_{H,Q} \;:=\; e^{-\beta(H-d\Gamma(Q))}\, d\, e^{\beta(H-d\Gamma(Q))} \tag{I.45}$$

on $C_0^\infty(\mathbb{R}^\Lambda)\otimes\mathcal{F}$. Here, $H \equiv H\otimes\mathbf{1}$ is considered a multiplication operator on $\mathcal{H}$. The *twisted Dirac operator* is defined as the sum of the twisted exterior derivative and its adjoint,

$$D_{H,Q} \;:=\; d_{H,Q} \;+\; d_{H,Q}^*\;. \tag{I.46}$$

By construction it is clear that $d_{H,Q}^2 = (d_{H,Q}^*)^2 = 0$. Thus the square of the twisted Dirac operator is the associated Hodge Laplacian

$$\Delta_{H,Q} \;:=\; D_{H,Q}^2 \;=\; d_{H,Q}\,d_{H,Q}^* \;+\; d_{H,Q}^*\,d_{H,Q}\;, \tag{I.47}$$

which we will call the *twisted Witten Laplacian*. It was introduced by Witten (in the case $Q=0$, H a Morse function, and $\mathbb{R}^\Lambda$ replaced by a compact manifold) in [22] to give a semiclassical proof of the Morse inequalities (see also [7]).

The twisted Witten Laplacian preserves the space $\mathcal{H}^{(n)}$ of n-forms on $\mathbb{R}^\Lambda$, so, writing $\Delta_{H,Q}^{(n)}$ for the restriction of $\Delta_{H,Q}$ to $\mathcal{H}^{(n)}$, we have that

$$\Delta_{H,Q} \;=\; \bigoplus_{n=0}^{|\Lambda|} \Delta_{H,Q}^{(n)}\;. \tag{I.48}$$

In Appendix B, we discuss domain properties of $d_{H,Q}$, $d_{H,Q}^*$, $D_{H,Q}$, and $\Delta_{H,Q}$, as well as, the question of self-adjointness of $D_{H,Q}$ and $\Delta_{H,Q}$. One of the results proved in Appendix B is the following.

Theorem I.4. *Assume Hypothesis 1 and Hypothesis 2.*

(i) There exist $\alpha_0 > 0$ and $\beta_0 < \infty$ such that, for $0 \leq \alpha < \alpha_0$ and $\beta > \beta_0$,

$$\mathrm{Ker}\{\Delta_{H,Q}^{(0)}\} \;=\; \mathbb{C}\, e^{-\beta H} \quad \textit{and} \quad \mathrm{Ker}\{\Delta_{H,Q}^{(1)}\} \;=\; \{0\}\,. \tag{I.49}$$

(ii) If all the f_i's and ω_{ij}'s are smooth in (not necessarily universal) neighbourhoods of 0, then there exists a universal constant $\alpha_0 > 0$ such that for $0 \leq \alpha < \alpha_0$ and all $\beta > 0$,

$$\mathrm{Ker}\,\Delta_{H,Q} \;=\; \mathbb{C}\, e^{-\beta H} \,\otimes\, \Omega\,. \tag{I.50}$$

We remark that our main result Theorem I.1 only requires (I.49). We have however included a proof in Appendix B of the more general (I.50).

In the remaining part of this section we give more explicit formulas for the objects introduced above, and we compare these with [2]. First, we introduce some exponential weights,

$$\hat{\Theta}_i \;:=\; e^{\beta \hat{q}_i}\,, \quad \check{\Theta}_i \;:=\; e^{\beta \check{q}_i} \quad \text{and} \quad \Theta_i \;:=\; e^{\beta q_i} \;=\; \hat{\Theta}_i\, \check{\Theta}_i\,, \tag{I.51}$$

and twisted derivatives, together with their adjoints,

$$Z_i(H) \;:=\; \frac{1}{\beta}\,\partial_i \;+\; H_i' \quad \text{and} \quad Z_i^*(H) \;=\; -\frac{1}{\beta}\,\partial_i \;+\; H_i'\,. \tag{I.52}$$

We compute, using (I.35) and (I.40)–(I.45),

$$d_{H,Q} \;=\; \sum_i \Theta_i\, Z_i(H) \,\otimes\, a_i^* \quad \text{and} \quad d_{H,Q}^* \;=\; \sum_i Z_i^*(H)\Theta_i \otimes a_i\,. \tag{I.53}$$

Note the intertwining relations

$$\Theta_i\, Z_i(H) \;=\; Z_i(G)\,\Theta_i \qquad \text{and} \qquad \widehat{\Theta}_i\, Z_i(H) \;=\; Z_i(\widehat{G})\,\widehat{\Theta}_i\,, \tag{I.54}$$

where

$$G \;:=\; H \,-\, \sum_i q_i \;=\; \sum_i g_i \,+\, \alpha \sum_{i,j} w_{i,j}\,, \tag{I.55}$$

$$\widehat{G} \;:=\; H \,-\, \sum_i \hat{q}_i \;=\; \sum_i \hat{g}_i \,+\, \alpha \sum_{i,j} w_{i,j}\,, \tag{I.56}$$

according to (I.7). In the derivation of (I.54) we use that $\partial_i(H - q_i) = \partial_i G$ and $\partial_i(H - \hat{q}_i) = \partial_i \widehat{G}$. The intertwining relations (I.54) are of key importance to our analysis, as it allows us to pass from H to a new Hamiltonian function G which agrees with H at zero, but yet has no critical points other than zero.

Note that $\mathcal{H}^{(0)} = L^2(\mathbb{R}^\Lambda) \otimes \mathcal{F}^{(0)}$ can be identified with $L^2(\mathbb{R}^\Lambda)$, and $\mathcal{H}^{(1)}$ with $L^2(\mathbb{R}^\Lambda) \otimes \mathbb{C}^\Lambda$. We will use the same notation, $\Delta^{(0)}_{H,Q}$ and $\Delta^{(1)}_{H,Q}$, for both realizations.

We first compute the restriction of the twisted Witten Laplacian to 0-forms, $\Delta^{(0)}_{H,Q}$, using (I.36) which implies that $d^*_{H,Q} = 0$ on $\mathcal{H}^{(0)}$. We obtain

$$\begin{aligned}\Delta^{(0)}_{H,Q} &= d^*_{H,Q}\, d_{H,Q} = \sum_{i,j} Z_i^*(H)\,\Theta_i\,\Theta_j\,Z_j(H) \otimes a_i\,a_j^* \\ &= \sum_i \Theta_i\,Z_i^*(G)\,Z_i(G)\,\Theta_i \otimes \mathbf{1} \equiv \sum_i \Theta_i\,Z_i^*(G)\,Z_i(G)\,\Theta_i\,,\end{aligned} \tag{I.57}$$

where in the last line, we have given its representation on $L^2(\mathbb{R}^\Lambda)$. In the special case $Q = 0$, we have $\Theta_i \equiv 1$, for all $i \in \Lambda$, and hence

$$\Delta^{(0)}_{H,0} = \sum_i Z_i^*(H)\,Z_i(H) = -\beta^{-2}\Delta + |\nabla H(x)|^2 - \beta^{-1}\Delta H(x)\,, \tag{I.58}$$

where $\Delta = \sum_{i\in\Lambda} \partial_i^2$. For $Q = 0$, the twisted Witten Laplacian restricted to 0-forms thus becomes a standard semiclassical Schrödinger operator, and the critical points of H become global minima of the potential (if we disregard ΔH, which enters at lower order in β). Comparing to (I.57), we see that the introduction of Q removes the nonzero local minima, which are irrelevant in the low-temperature limit. The price we pay is the presence of the exponential weights Θ_i. In [2] the assumptions were such that the low-temperature study of the Witten Laplacian, with $Q = 0$, became essentially a one-well semiclassical problem. Here we have relaxed the assumptions and introduced the extra twist by $d\Gamma(Q)$ to restore the essential features of a one-well problem.

The full twisted Witten Laplacian is of the form, cf. (I.35), (I.47), (I.53) and (I.54),

$$\begin{aligned}\Delta_{H,Q} &= \sum_{i,j} \Theta_i\,Z_i(H)\,Z_j^*(H)\,\Theta_j \otimes a_i^*\,a_j + Z_j^*(H)\,\Theta_j\,\Theta_i\,Z_i(H) \otimes a_j\,a_i^* \\ &= \sum_j \Big\{\Theta_j\,Z_j(H)\,Z_j^*(H)\,\Theta_j \otimes a_j^*\,a_j + \Theta_j\,Z_j^*(G)\,Z_j(G)\,\Theta_j \otimes a_j\,a_j^*\Big\} \\ &\qquad + \sum_{i\neq j} \big[\Theta_i\,Z_i(H)\,,\, Z_j^*(H)\,\Theta_j\big] \otimes a_i^*\,a_j\,, \\ &= \sum_j \Big\{\Theta_j\,Z_j(H)\,Z_j^*(H)\,\Theta_j \otimes a_j^*\,a_j + \Theta_j\,Z_j^*(G)\,Z_j(G)\,\Theta_j \otimes a_j\,a_j^*\Big\} \\ &\qquad + \frac{2}{\beta}\sum_{i\neq j} \Theta_i\,\Theta_j\,H''_{ij}(x) \otimes a_i^*\,a_j\,,\end{aligned} \tag{I.59}$$

where we use $[Z_i(H), \Theta_j] = 0$ and $[Z_i(H), Z_j^*(H)] = 2\beta^{-1} H''_{i,j}$, for $i \neq j$, and we denote $H''_{ij} := \partial_i \partial_j H$.

Inserting (I.38) into (I.59), we arrive at the twisted Witten Laplacian on 1-forms,

$$\begin{aligned} \Delta^{(1)}_{H,Q} \;&=\; \sum_j \Big\{ \Theta_j \, Z_j(H) \, Z_j^*(H) \, \Theta_j \;+\; \sum_{k(\neq j)} \Theta_k \, Z_k^*(G) \, Z_k(G) \, \Theta_k \Big\} \otimes E_{jj} \\ &\quad + \frac{2}{\beta} \sum_{j \neq k} \Theta_j \, \Theta_k \, H''_{jk} \otimes E_{jk} \, . \end{aligned} \tag{I.60}$$

I.4 Correlation Identity and Exponential Weights

A remarkable consequence of Theorem I.4 is that it allows us to write the truncated correlation (I.2) of two observables $u, v \in C^1(\mathbb{R}^\Lambda, \mathbb{R})$ as a matrix element of the resolvent of $\Delta^{(1)}_{H,Q}$, as follows.

Theorem I.5. *Assume (I.49) and let $u, v \in C^1(\mathbb{R}^\Lambda, \mathbb{R})$ such that $|\nabla u|$ and $|\nabla v|$ are polynomially bounded in x. Then*

$$\begin{aligned} &\mathbb{E}_\beta^T(u\,;\,v) \;=\; \beta^{-2} \, \mathcal{Z}_\beta^{-1} \\ &\quad \cdot \Big\langle \sum_k (\partial_k u) \, e^{-\beta(H - q_k)} \otimes a_k^* \Omega \,\Big|\, \Big(\Delta^{(1)}_{H,Q} \Big)^{-1} \Big(\sum_\ell (\partial_\ell v) \, e^{-\beta(H - q_\ell)} \otimes a_\ell^* \Omega \Big) \Big\rangle , \end{aligned} \tag{I.61}$$

where

$$\mathcal{Z}_\beta \;:=\; \int_{\mathbb{R}^\Lambda} e^{-2\beta H(x)} \, dx \;=\; \big\| e^{-\beta H} \otimes \Omega \big\|^2 \, . \tag{I.62}$$

Proof: We first observe that

$$\mathbb{E}_\beta(u) \;=\; \mathcal{Z}_\beta^{-1} \int_{\mathbb{R}^\Lambda} u(x) \, e^{-2\beta H(x)} \, dx \;=\; \mathcal{Z}_\beta^{-1} \big\langle e^{-\beta H} \otimes \Omega \,\big|\, u \, e^{-\beta H} \otimes \Omega \big\rangle , \tag{I.63}$$

and hence

$$\begin{aligned} \mathbb{E}_\beta^T(u\,;\,v) \;&=\; \mathbb{E}_\beta(u \cdot v) \;-\; \mathbb{E}_\beta(u) \, \mathbb{E}_\beta(v) \\ &=\; \mathbb{E}_\beta\big((u - \mathbb{E}_\beta(u)) \cdot (v - \mathbb{E}_\beta(v)) \big) \\ &=\; \mathcal{Z}_\beta^{-1} \big\langle (u - \mathbb{E}_\beta(u)) \, e^{-\beta H} \otimes \Omega \,\big|\, P_0 \, (v - \mathbb{E}_\beta(v)) \, e^{-\beta H} \otimes \Omega \big\rangle , \end{aligned} \tag{I.64}$$

where we use that v is real-valued and

$$P_0 \;:=\; \Big(\mathbf{1} \;-\; \mathcal{Z}_\beta^{-1} \, |e^{-\beta H}\rangle\langle e^{-\beta H}| \Big) \;\otimes\; P_\Omega \, , \tag{I.65}$$

denotes the orthogonal projection onto the orthogonal complement to $\mathbb{C} \cdot e^{-\beta H}$ times the projection onto the Fock vacuum, $P_\Omega := |\Omega\rangle\langle\Omega|$. We used above that $(u - \mathbb{E}_\beta(u)) \exp(-\beta H) \otimes \Omega \in \mathrm{Ran}\, P_0$.

Next, Eq. (I.49) implies that $\mathrm{Ran}\{D_{H,Q}\, P_0\} \perp \mathrm{Ker}\{\Delta_{H,Q}\}$, which allows us to write

$$\begin{aligned} P_0 &= P_0 D_{H,Q}\, \Delta_{H,Q}^{-1}\, D_{H,Q}\, P_0 \\ &= P_0\, d_{H,Q}^*\, \Delta_{H,Q}^{-1}\, d_{H,Q}\, P_0 \;=\; P_0\, d_{H,Q}^* \big(\Delta_{H,Q}^{(1)}\big)^{-1} d_{H,Q}\, P_0 \,. \end{aligned} \tag{I.66}$$

Using furthermore that $d_{H,Q}\mathbb{E}_\beta(u) \exp(-\beta H) \otimes \Omega = 0$, we obtain from Eqs. (I.64) and (I.66) that

$$\mathbb{E}_\beta^T(u\,;\,v) \;=\; \mathcal{Z}_\beta^{-1} \Big\langle d_{H,Q}\big(u\, e^{-\beta H} \otimes \Omega\big) \Big|\, \big(\Delta_{H,Q}^{(1)}\big)^{-1} d_{H,Q}\big(v\, e^{-\beta H} \otimes \Omega\big) \Big\rangle \,. \tag{I.67}$$

A simple computation gives

$$d_{H,Q}\big(v\, e^{-\beta H} \otimes \Omega\big) \;=\; \frac{1}{\beta} \sum_\ell (\partial_\ell v)\, e^{-\beta(H - q_\ell)} \;\otimes\; a_\ell^* \Omega \,. \tag{I.68}$$

Inserting (I.68) into (I.64), we arrive at the desired identity (I.61) for the truncated correlation. □

As an immediate consequence of (I.61), we have the identity

$$\mathbb{E}_\beta^T(x_i\,;\,x_j) \;=\; \beta^{-2}\, \mathcal{Z}_\beta^{-1} \Big\langle e^{-\beta(H - q_i)} \;\otimes\; e_i \,\Big|\, \big(\Delta_{H,Q}^{(1)}\big)^{-1} \big(e^{-\beta(H - q_j)} \;\otimes\; e_j\big) \Big\rangle \tag{I.69}$$

for the two-point correlation function, where $e_i \equiv a_i^* \Omega$ denotes the i^{th} standard basis vector in $\mathbb{C}^\Lambda$. The presence of the modifiers q_i and q_j in the exponents leads us to study the stability of the partition function under perturbations in a single spin variable. We have the following theorem, which is proved in Section III

Theorem I.6. *Assume Hypotheses 1 and 2. There exists a universal constant $c > 0$ such that, for all $o \in \Lambda$, $\xi > 0$, $\beta > c^{-1}$, and $0 \le \alpha \le \alpha_0 := c\, \min\{1, \xi\}^4$, the following localization estimates hold true,*

$$\mathbb{E}_\beta\big(\mathbf{1}[|x_o| \ge \xi]\big) \;=\; \mathcal{Z}_\beta^{-1} \int_{\{x : |x_o| \ge \xi\}} e^{-2\beta H(x)}\, d^\Lambda x \;\le\; c^{-1}\, e^{-c\beta\, \min\{\xi, \xi^2\}} \,. \tag{I.70}$$

This result states that perturbing the Hamiltonian function in one spin-variable only changes the partition function by an exponentially small error in the low-temperature regime, provided the perturbation is supported away from the global minimum.

Exponential decay estimates for the measure $\exp[-2\beta H]\, d^\Lambda x$, as a single spin variable is away from the global minimum are complemented by the decay properties of $\mathbb{E}_\beta^T(x_i\,;\,x_j)$ in (I.69), as the distance between i and j in Λ tends to be large. For the analysis of the latter, we introduce exponential weights $\exp[\kappa\rho(i,j)]$ and observe the following generalization of (I.69).

Lemma I.7. *Let $\kappa > 0$ and $i, j \in \Lambda$. Then*

$$e^{\kappa\rho(i,j)}\, \mathbb{E}_\beta^T(x_i\,;\,x_j) \tag{I.71}$$
$$= \beta^{-2}\, \mathcal{Z}_\beta^{-1} \Big\langle e^{-\beta(H-q_i)} \otimes a_i^*\Omega \,\Big|\, \big(\Delta_{H,Q}^{(1)}(\kappa,j)\big)^{-1} \big(e^{-\beta(H-q_j)} \otimes a_j^*\Omega\big) \Big\rangle\,,$$

where $\Delta_{H,Q}^{(1)}(\kappa,j)$ denotes the restriction to $\mathcal{H}^{(1)}$ of

$$\Delta_{H,Q}(\kappa,j)$$
$$:= \sum_k \Big\{ \Theta_k\, Z_k(H)\, Z_k^*(H)\, \Theta_k \otimes a_k^* a_k \;+\; \Theta_k\, Z_k^*(G)\, Z_k(G)\, \Theta_k \otimes a_k a_k^* \Big\}$$
$$+ \frac{2}{\beta} \sum_{k\neq\ell} e^{\kappa[\rho(k,j)-\rho(\ell,j)]}\, \Theta_k\, \Theta_\ell\, H_{k\ell}'' \otimes a_k^* a_\ell\,, \tag{I.72}$$

and $\rho : \Lambda \times \Lambda \to \mathbb{R}_0^+$ is the metric on Λ.

Before we turn to the proof of Lemma I.7, we remark that the exponential decay of the two-point correlation function is analyzed in detail in Subsect. I.6. Loosely speaking, the decay rate of the two-point correlation function is given by the limes superior of all $\kappa > 0$, for which (I.61) stays bounded, as $\rho(i,j)$ gets large.

Another remark to be made is that $\operatorname{Ker} \Delta_{H,Q}^{(0)}(\kappa,j) = \mathbb{C}\cdot e^{-\beta H}$ and $\operatorname{Ker} \Delta_{H,Q}^{(1)}(\kappa,j) = \{0\}$, for all $0 \le \kappa \le 1$ and $j \in \Lambda$, due to the fact that $\Delta_{H,Q}(\kappa,j)$ derives from $\Delta_{H,Q} \equiv \Delta_{H,Q}(0,j)$ by conjugation with the bounded and invertible operator $\exp[d\Gamma(M_j)]$, see below.

Proof: Define a diagonal $\Lambda \times \Lambda$-matrix M_j by $(M_j)_{k\ell} := \kappa\,\rho(k,j)\,\delta_{k\ell}$. Observe that $\pm d\Gamma(M_j)\, a_i^*\Omega = \pm\kappa\,\rho(i,j)\, a_i^*\Omega$ and that $d\Gamma(M_j)$ leaves $\mathcal{H}^{(n)}$ invariant, cf. (I.41). Hence

$$\beta^2\, \mathcal{Z}_\beta\, e^{\kappa\rho(i,j)}\, \mathbb{E}_\beta^T(x_i\,;\,x_j) \tag{I.73}$$
$$= \Big\langle e^{d\Gamma(M_j)} \big(e^{-\beta(H-q_i)} \otimes a_i^*\Omega\big) \,\Big|\, \Delta_{H,Q}^{-1}\, e^{-d\Gamma(M_j)} \big(e^{-\beta(H-q_j)} \otimes a_j^*\Omega\big) \Big\rangle$$
$$= \Big\langle \big(e^{-\beta(H-q_i)} \otimes a_i^*\Omega\big) \,\Big|\, \big(e^{d\Gamma(M_o)}\, \Delta_{H,Q}\, e^{-d\Gamma(M_o)}\big)^{-1} \big(e^{-\beta(H-q_j)} \otimes a_j^*\Omega\big) \Big\rangle\,.$$

Since

$$e^{d\Gamma(M_j)}\, a_k^*\, a_\ell\, e^{-d\Gamma(M_l)} \;=\; e^{\kappa[\rho(k,j)-\rho(\ell,j)]}\, a_k^*\, a_\ell\,, \tag{I.74}$$

we obtain Eq. (I.71) from the computation

$$e^{d\Gamma(M_j)}\, \Delta_{H,Q}\, e^{-d\Gamma(M_j)}$$
$$= \sum_k \Big\{ \Theta_k\, Z_k(H)\, Z_k^*(H)\, \Theta_k \otimes a_k^* a_k \;+\; \Theta_k\, Z_k^*(G)\, Z_k(G)\, \Theta_k \otimes a_k a_k^* \Big\}$$
$$+ \frac{2}{\beta} \sum_{k\neq\ell} e^{\kappa[\rho(k,j)-\rho(\ell,j)]}\, \Theta_k\, \Theta_\ell\, H_{k\ell}''(x) \otimes a_k^*\, a_\ell\,. \tag{I.75}$$

□

I.5 Comparison Operator

To motivate our next definitions, we observe that, by setting $Q = 0$ in (I.60), we obtain the Witten Laplacian that has been analyzed in [2],

$$\Delta_{H,0}^{(1)} \; = \; \Delta_{H,0}^{(0)} \otimes \mathbf{1} \; + \; \frac{2}{\beta} \, H''(x) \, . \tag{I.76}$$

In [2], the semiclassical confining properties of the one-well problem were used to show that $\Delta_{H,0}^{(1)}$ is close to the comparison operator

$$A_{H,0}^{(1)} \; := \; \Delta_{H,0}^{(0)} \otimes \mathbf{1} \; + \; \frac{2}{\beta} \, H''(0) \tag{I.77}$$

in the sense that their difference $W_{H,0}^{(1)} := \Delta_{H,0}^{(1)} - A_{H,0}^{(1)}$ is a small perturbation of $A_{H,0}^{(1)}$, as a quadratic form. More specifically, it was shown in [2] that if $\alpha \geq 0$ is sufficiently small and $\beta < \infty$ is sufficiently large, then for any vector ψ in the form domain of $A_{H,0}^{(1)}$,

$$\big\langle \psi \, \big| \, W_{H,0}^{(1)} \, \psi \big\rangle \; \leq \; \frac{C}{\beta^{1/2}} \, \big\langle \psi \, \big| \, A_{H,0}^{(1)} \, \psi \big\rangle \, , \tag{I.78}$$

for some universal constant $C < \infty$. The derivation of Form Bound (I.78) in [2], however, uses Property (I.12), which we are lacking in the present context, in an essential way.

To treat Hamiltonian functions that merely obey the weaker Hypotheses 1 and 2, we suppose we have constructed q_j and $\hat{q}_j$ according to Lemma I.2 and formed $\Theta_j = \exp[\beta q_j]$ and $\widehat{\Theta}_j = \exp[\beta \hat{q}_j]$, respectively, for all $j \in \Lambda$. We then define

$$A_{H,Q} \; := \; \sum_j \Big\{ \check{A}_j \otimes a_j^* \, a_j \; + \; A_j \otimes a_j \, a_j^* \Big\} + \; \frac{2}{\beta} \sum_{i,j} H_{ij}''(0) \; \otimes \; a_i^* \, a_j \, , \tag{I.79}$$

where

$$\check{A}_j \;\; := \;\; \Theta_j \, Z_j(H) \, Z_j^*(H) \, \Theta_j \; - \; 2\beta^{-1} \, \check{\Theta}_j^2 \, \hat{G}_{jj}''(x) \, , \tag{I.80}$$

$$A_j \;\; := \;\; \Theta_j \, Z_j^*(G) \, Z_j(G) \, \Theta_j \, . \tag{I.81}$$

We remark that Eq. (I.79) coincides with (I.77), for $Q = 0$. (The notation $\check{A}_j$ is motivated by the estimate (I.92).) Furthermore, for $0 \leq \kappa \leq 1$ and $o \in \Lambda$, we define

$$
\begin{aligned}
W_{H,Q}(\kappa, o; x) &:= W_{\mathrm{diag}}(x) \;+\; W_{\mathrm{off-d}}(\kappa, o; x) && \text{(I.82)} \\
&:= W_{\mathrm{diag}}(x) \;+\; \Delta W_{\mathrm{off-d}}(\kappa, o; x) \;+\; W_{\mathrm{off-d}}(\kappa, o; 0)\,, \\
W_{\mathrm{diag}}(x) &:= \frac{2}{\beta} \sum_j \Big(\check{\Theta}_j^2 \, \hat{G}''_{jj}(x) \;-\; H''_{jj}(0) \Big) \otimes a_j^* a_j\,, && \text{(I.83)} \\
W_{\mathrm{off-d}}(\kappa, o; x) &:= \frac{2}{\beta} \sum_{i \neq j} \Big(e^{\kappa[\rho(i,o) - \rho(j,o)]} \Theta_i \Theta_j \, H''_{ij}(x) \;-\; H''_{ij}(0) \Big) \otimes a_i^* a_j \\
& && \text{(I.84)} \\
&= \frac{2\,\alpha}{\beta} \sum_{i \neq j} \Big(e^{\kappa[\rho(i,o) - \rho(j,o)]} \Theta_i \Theta_j \, w''_{ij}(x) \;-\; w''_{ij}(0) \Big) \otimes a_i^* a_j\,, \\
\Delta W_{\mathrm{off-d}}(\kappa, o; x) &:= W_{\mathrm{off-d}}(\kappa, o; x) \;-\; W_{\mathrm{off-d}}(\kappa, o; 0) && \text{(I.85)} \\
&= \frac{2\,\alpha}{\beta} \sum_{i \neq j} e^{\kappa[\rho(i,o) - \rho(j,o)]} \Big(\Theta_i \Theta_j \, w''_{ij}(x) \;-\; w''_{ij}(0) \Big) \otimes a_i^* a_j\,, \\
W_{\mathrm{off-d}}(\kappa, o; 0) &= \frac{2\,\alpha}{\beta} \sum_{i \neq j} \big(e^{\kappa[\rho(i,o) - \rho(j,o)]} \;-\; 1 \big) \, w''_{ij}(0) \otimes a_i^* a_j\,. && \text{(I.86)}
\end{aligned}
$$

where $w''_{ij} := \partial_i \partial_j w_{ij}$, and we observe the decomposition identity

$$
\Delta_{H,Q}(\kappa, o) \;=\; A_{H,Q} \;+\; W_{H,Q}(\kappa, o)\,. \tag{I.87}
$$

The argument "x" in (I.82)–(I.84) indicates that these operators act as matrix-valued multiplication operators and contain no differential operator. We frequently omit to display x. Note also that we have the identites $W_{\mathrm{off-d}}(0, o; 0) = 0$ and $\Delta W_{\mathrm{off-d}}(0, o; x) = W_{\mathrm{off-d}}(0, o; x)$.

For $n = 0, 1$, we compute the restrictions of $A_{H,Q}$ onto $\mathcal{H}^{(n)}$, which we denote by $A_{H,Q}^{(n)}$, respectively. From (I.36)–(I.38), we immediately obtain

$$
\begin{aligned}
A_{H,Q}^{(0)} &= \sum_j A_j \;=\; \sum_j \Theta_j \, Z_j^*(G) \, Z_j(G) \, \Theta_j\,, && \text{(I.88)} \\
A_{H,Q}^{(1)} &= \sum_j \Big\{ \Theta_j \, Z_j(H) \, Z_j^*(H) \, \Theta_j \;-\; 2\beta^{-1} \, \check{\Theta}_j^2 \, \hat{G}''_{jj}(x) && \text{(I.89)} \\
&\quad + \sum_{k(\neq j)} \Theta_k \, Z_k^*(G) \, Z_k(G) \, \Theta_k \Big\} \otimes E_{jj} \;+\; \frac{2}{\beta} \sum_{i \neq j} H''_{ij}(0) \otimes E_{ij}\,.
\end{aligned}
$$

We further observe that

$$
\begin{aligned}
d\Gamma\big(A_{H,Q}^{(1)}\big) &= A_{H,Q} \;+\; \sum_j A_j \otimes \Big\{ \sum_{k(\neq j)} a_k^* a_k \;-\; a_j a_j^* \Big\} \\
&= A_{H,Q} \;+\; \Big(\sum_j A_j \Big) \otimes \big(N_{\mathcal{F}} - \mathbf{1} \big)\,, && \text{(I.90)}
\end{aligned}
$$

where

$$N_{\mathcal{F}} \;:=\; d\Gamma(\mathbf{1}) \;=\; \sum_k a_k^* \, a_k \tag{I.91}$$

is the fermion number (= form degree) operator. Note that $N_{\mathcal{F}}$ is bounded, more precisely $\|N_{\mathcal{F}}\| = |\Lambda|$, thanks to the Pauli principle.

We will write $W_{H,Q}^{(k)}(\kappa, o)$ for the restriction of $W_{H,Q}(\kappa, o)$ to k-forms, and likewise for the derived quantities presented in the formula (I.82).

I.6 Exponential Decay of Correlations

In this subsection we prove our main result, Theorem I.1. Our proof relies on a form bound to be presented below, cf. (I.94). This form bound is contained in Theorem II.1, which is the technical centerpiece of the present work.

Proof of Theorem I.1: We begin with the remark that from (I.80), (I.54), and the facts that $\widehat{\Theta}_j \check{\Theta}_j = \Theta_j$ and $\widehat{\Theta}_j \geq 1$, we get

$$\begin{aligned} \check{A}_j \;&=\; \check{\Theta}_j \, \{ Z_j(\hat{G}) \, \widehat{\Theta}_j^2 \, Z_j(\hat{G})^* \;-\; 2\beta^{-1} \, \hat{G}_{jj} \} \, \check{\Theta}_j \\ &\geq\; \check{\Theta}_j \, \{ Z_j(\hat{G}) \, Z_j(\hat{G})^* \;-\; 2\beta^{-1} \, \hat{G}_{jj} \} \, \check{\Theta}_j \\ &=\; \check{\Theta}_j \, Z_j(\hat{G})^* \, Z_j(\hat{G}) \, \check{\Theta}_j \, . \end{aligned} \tag{I.92}$$

In particular, this implies that $\check{A}_j \geq 0$, and hence we find that

$$A_{H,Q}^{(1)} \;\geq\; \frac{2}{\beta} \, H''(0) \;\geq\; \frac{2}{\beta} \, \lambda_{\min} \, \mathbf{1} \, , \tag{I.93}$$

where $\lambda_{\min} := \min \sigma(H''(0))$ is the lowest eigenvalue of the Hessian of H at $x = 0$. Observe that, since $f_j''(0) \geq c_f > 0$ and $|\partial_\sigma \partial_\tau w_{ij}''(0,0)| \leq a_{ij}$, by Hypotheses 1 and 2, the eigenvalue $\lambda_{\min} \geq c_f/2 > 0$ is strictly positive (for $\alpha < \alpha_0$ small).

Let $\epsilon' >$ (to be chosen below). Theorem II.1 implies that there exists a universal constant $C < \infty$, such that, for all $0 \leq \kappa < 1$, $\beta > C\varepsilon'^{-1/2}$, $0 \leq \alpha < C^{-1}\varepsilon'(1-\kappa)$ and all $j \in \Lambda$, the following estimate holds true

$$\big| \langle \psi \,|\, W_{H,Q}^{(1)}(\kappa, j) \phi \rangle \big| \;\leq\; \varepsilon' \, \big\| (A_{H,Q}^{(1)})^{1/2} \psi \big\| \, \big\| (A_{H,Q}^{(1)})^{1/2} \phi \big\| \, . \tag{I.94}$$

Write

$$\begin{aligned} &\big(\Delta_{H,Q}^{(1)}(\kappa, j) \big)^{-1} \\ &=\; \big(A_{H,Q}^{(1)} \big)^{-1/2} \Big(\mathbf{1} \;+\; \big(A_{H,Q}^{(1)} \big)^{-1/2} \, W_{H,Q}^{(1)}(\kappa, j) \, \big(A_{H,Q}^{(1)} \big)^{-1/2} \Big)^{-1} \, \big(A_{H,Q}^{(1)} \big)^{-1/2} \, . \end{aligned} \tag{I.95}$$

The formbound (I.94), with $\epsilon' < 1$, allows us to expand the resolvent on the right-hand side of (I.95) in a Neumann series. This observation together with Eqs. (I.93) imply that for small enough ϵ' (depending on ϵ) we have

$$\big\| \big(\Delta_{H,Q}^{(1)}(\kappa, j)\big)^{-1} \big\| \;\le\; \frac{\beta}{2\,(1-\varepsilon/3)\,\lambda_{\min}}\,, \tag{I.96}$$

under the conditions on α and β mentioned above and for all $j \in \Lambda$.

Lemma I.7 and the Cauchy-Schwarz inequality yield

$$\big|\mathbb{E}_\beta^T(x_i;\, x_j)\big| \;\le\; e^{-\kappa\rho(i,j)}\, \frac{\|e^{-\beta(H-q_i)}\|\,\|e^{-\beta(H-q_j)}\|}{\beta^2\,\mathcal{Z}_\beta}\, \big\| \big(\Delta_{H,Q}^{(1)}(\kappa, j)\big)^{-1} \big\|\,. \tag{I.97}$$

Note that the Hypothesis 1 and 2 remain true if we replace one of the f_j's with the corresponding g_j. We proceed to apply Theorem I.6,

$$\begin{aligned}\int_{\mathbb{R}^\Lambda} e^{-2\beta(H-q_i)} d^\Lambda x \;&=\; \int_{|x_i|<\hat{R}_0} e^{-2\beta H} d^\Lambda x \;+\; \int_{|x_i|>\hat{R}_0} e^{-2\beta(H-q_i)} d^\Lambda x \\ &\le\; \int_{\mathbb{R}^\Lambda} e^{-2\beta H} d^\Lambda x \;+\; e^{-c\beta} \int_{\mathbb{R}^\Lambda} e^{-2\beta(H-q_i)} d^\Lambda x\,. \end{aligned} \tag{I.98}$$

This implies, for all $i \in \Lambda$,

$$\int_{\mathbb{R}^\Lambda} e^{-2\beta(H-q_i)} d^\Lambda x \;\le\; (1-e^{-c\beta})^{-1}\,\mathcal{Z}_\beta\,, \tag{I.99}$$

provided $\beta_0 < \infty$ is sufficiently large. Inserting (I.96) and (I.99) into (I.97), we arrive at the desired bound,

$$\big|\mathbb{E}_\beta^T(x_i;\, x_j)\big| \;\le\; \frac{1+\varepsilon}{2\,\beta\,\lambda_{\min}}\, e^{-\kappa\rho(i,j)}\,. \tag{I.100}$$

□

We note that the formbound (I.94) implies the lower bound

$$\min \sigma\big(\Delta_{H,Q}^{(1)}\big) \;\ge\; \frac{2\,(1-\varepsilon')}{\beta}\,\lambda_{\min}\,. \tag{I.101}$$

II Form Bound on the Perturbation

This section is the technical heart of our paper. Here we prove a form bound showing that $\Delta_{H,Q}^{(1)} = A_{H,Q}^{(1)} + W_{H,Q}^{(1)}$ decomposes the twisted Witten Laplacian on $\mathcal{H}^{(1)}$ into its main part $A_{H,Q}^{(1)}$, see (I.89), and a small perturbation $W_{H,Q}^{(1)}$. Being slightly more general, we show that $W_{H,Q}$ is a small compared to $d\Gamma(A_{H,Q}^{(1)})$, displayed in (I.90), with relative bound of order $\beta^{-1/2}$, which becomes small in the low-temperature limit $\beta \to 0$.

The main result of this section is

Theorem II.1 (Main Form Bound). *There exist universal constants $C < \infty$, $\alpha_0 > 0$, and $\beta_0 < \infty$ such that, for all $0 \le \alpha \le \alpha_0$, all $\beta > \beta_0$, all $0 \le \kappa < 1$, and all $o \in \Lambda$, the estimate*

$$\left\| d\Gamma(A_{H,Q}^{(1)})^{-1/2} \, W_{H,Q}(\kappa, o) \, d\Gamma(A_{H,Q}^{(1)})^{-1/2} \right\| \;\le\; C \left(\frac{1}{\beta^{1/2}} + \frac{\alpha \, \kappa}{1-\kappa} \right) \tag{II.1}$$

holds true on $\big(\mathcal{H}^{(0)}\big)^{\perp}$, and $W_{H,Q}(\kappa, o) = 0$ on $\mathcal{H}^{(0)}$. Especially, for $\kappa = 0$ and on $\mathcal{H}^{(1)}$, we have

$$\pm W_{H,Q}^{(1)}(0, o) \;\le\; \frac{C}{\beta^{1/2}} \, A_{H,Q}^{(1)} \, , \tag{II.2}$$

in the sense of quadratic forms.

Proof: We break up the proof of Theorem II.1 into a series of lemmata proved in Subsects. II.1–II.5, below. Each of these lemmata contain a bootstrap argument close in spirit to the strategy applied in [2, Sect. III].

We recall from Eq. (I.90) the definition $N_{\mathcal{F}} = \sum_k \mathbf{1} \otimes a_k^* a_k = \bigoplus_{n=0}^{|\Lambda|} n \cdot \mathbf{1}_{\mathcal{H}^{(n)}}$ of the fermion number operator on $\mathcal{H}$.

We may assume without loss of generality that all operators are restricted to $\big(\mathcal{H}^{(0)}\big)^{\perp}$. Recall the decomposition, see (I.82)–(I.86),

$$W_{H,Q}(\kappa, o; x) \;=\; W_{\mathrm{diag}}(x) + \Delta W_{\mathrm{off-d}}(\kappa, o; x) + W_{\mathrm{off-d}}(\kappa, o; 0) \, . \tag{II.3}$$

We estimate $W_{\mathrm{diag}}(x) + \Delta W_{\mathrm{off-d}}(\kappa, o; x)$ and $W_{\mathrm{off-d}}(\kappa, o; 0)$ separately. In Lemma II.3 in Subsect. II.1 we prove that

$$\left\| N_{\mathcal{F}}^{-1/2} \, W_{\mathrm{off-d}}(\kappa, o; 0) \, N_{\mathcal{F}}^{-1/2} \right\| \;\le\; \frac{3\, C_a \, \alpha \, \kappa}{(1-\kappa)\, \beta} \, , \tag{II.4}$$

and the main part of the proof is actually to show that

$$\left\| \big(M[\underline{b}] + \beta^{-1} N_{\mathcal{F}} \big)^{-1/2} \big\{ W_{\mathrm{diag}} + \Delta W_{\mathrm{off-d}}(\kappa, o) \big\} \big(M[\underline{b}] + \beta^{-1} N_{\mathcal{F}} \big)^{-1/2} \right\| \;\le\; \frac{C}{\beta^{1/2}}, \tag{II.5}$$

for some universal constant $C < \infty$. Here, $M[\underline{b}]$ belongs to the following chain of (possibly unbounded) operators,

$$J[\underline{a}] := \sum_j J_j[\underline{a}] \otimes a_j^* a_j \,, \qquad K[\underline{b}] := \sum_j K_j[\underline{b}] \otimes a_j^* a_j \,, \tag{II.6}$$

$$L[\underline{b}] := \sum_j L_j[\underline{b}] \otimes a_j^* a_j \,, \qquad M[\underline{b}] := \sum_j M_j[\underline{b}] \otimes a_j^* a_j \,, \tag{II.7}$$

where $J_j[\underline{a}]$, $K_j[\underline{b}]$, $L_j[\underline{b}]$ and $M_j[\underline{b}]$ are multiplication operators, multiplying by the following functions

$$J_j[\underline{a}] := \check{\Theta}_j^2 \, |\hat{g}_j'| + \sum_{k(\neq j)} a_{jk} \, \Theta_k^2 \, |g_k'| \,, \tag{II.8}$$

$$K_j[\underline{b}] := \check{\Theta}_j^2 \, |\widehat{G}_j'| + \sum_{k(\neq j)} b_{jk} \, \Theta_k^2 \, |G_k'| \,, \tag{II.9}$$

$$L_j[\underline{b}] := \check{\Theta}_j^2 \, |\widehat{G}_j'|^2 + \sum_{k(\neq j)} b_{jk} \, \Theta_k^2 \, |G_k'|^2 \,, \tag{II.10}$$

$$M_j[\underline{b}] := \check{\Theta}_j^2 \, \widehat{V}_j + \sum_{k(\neq j)} b_{jk} \, \Theta_k^2 \, V_k \,, \tag{II.11}$$

and

$$\widehat{V}_j(x) := \widehat{G}_j'(x)^2 - \frac{1}{\beta} \widehat{G}_{jj}''(x) \quad \text{and} \quad V_k(x) := G_k'(x)^2 - \frac{1}{\beta} G_{kk}''(x) \,. \tag{II.12}$$

Furthermore, we denote the weights by $\underline{a} := (a_{ij})_{i,j \in \Lambda}$, $\underline{a}(\kappa) := \big(a_{ij}(\kappa)\big)_{i,j \in \Lambda} = \big(e^{\kappa \rho(i,j)} a_{ij}\big)_{i,j \in \Lambda}$, and introduce new weights $\underline{b} := (b_{ij})_{i,j \in \Lambda}$, with the property that $b_{ij} \geq e^{\rho(i,j)} a_{ij}$, in Subsect. II.3.

To explain the general strategy of the estimates we note that $|g_j'| + |g_k'|$ controls the derivatives of w_{jk}, by Hypothesis 2. This enables us to show in Theorem II.6 in Subsect. II.3 that for some universal constant $C < \infty$,

$$\Big\| J[\underline{a}(\kappa)]^{-1/2} \big\{ W_{\text{diag}} + \Delta W_{\text{off-d}}(\kappa, o) \big\} \, J[\underline{a}(\kappa)]^{-1/2} \Big\| \leq \frac{C}{\beta} \,. \tag{II.13}$$

Next, we remark that the difference between g_k' and G_k' also contains only derivatives of w_{jk} which we control with $|g_j'| + |g_k'|$. Bootstrapping the corresponding estimate, we derive in Theorem II.8 in Subsect. II.3 that, for some universal constant $C < \infty$,

$$J[\underline{a}(\kappa)] \leq J[\underline{b}] \leq C \, K[\underline{b}] \,. \tag{II.14}$$

For the bootstrap argument to work out, we need to pass from the weights $a_{ij}(\kappa) = e^{\rho(i,j)} a_{ij}$ to bigger weights $b_{ij} \geq e^{\rho(i,j)} a_{ij}$. These new weights are then chosen as to fulfill $\sum_k b_{ik} a_{kj} \le C' b_{ij}$, where $C' < \infty$ is a universal constant. This construction and the bootstrap argument require $\alpha \geq 0$ to be small.

Similar arguments are then used in Theorem II.9 in Subsect. II.4 to prove that, for some universal constant $C < \infty$,

$$K[\underline{b}] \ \le \ \frac{1}{\varepsilon} \, L[\underline{b}] \ + \ C \, \varepsilon \, N_{\mathcal{F}} \, , \tag{II.15}$$

provided $\alpha \geq 0$ and $\varepsilon > 0$ are sufficiently small, and finally in Theorem II.10 in Subsect. II.5 to prove that, for some universal constant $C < \infty$,

$$L[\underline{b}] \ \le \ \Big(1 + \frac{C}{\beta} \Big) \, M[\underline{b}] \ + \ \frac{C}{\beta} \, N_{\mathcal{F}} \, , \tag{II.16}$$

provided that $\beta < \infty$ is sufficiently large.

Inserting (II.14)–(II.16) into (II.13), we conclude that, for some universal constants $C, C' < \infty$ and all ϕ, ψ in the form domain of $M[\underline{b}]$,

$$\begin{aligned} & \Big| \big\langle \phi \mid \big\{ \Delta W_{\mathrm{off-d}}(\kappa, o) + W_{\mathrm{diag}} \big\} \, \psi \big\rangle \Big| \qquad\qquad \text{(II.17)} \\ \le \ & \Big\| \Big\{ \frac{C}{\varepsilon \, \beta} \Big(1 + \frac{C}{\beta} \Big) \, M[\underline{b}] \ + \ \frac{C}{\varepsilon \, \beta^2} \, N_{\mathcal{F}} \ + \ \frac{C \, \varepsilon}{\beta} \, N_{\mathcal{F}} \Big\}^{1/2} \phi \Big\| \\ & \quad \cdot \Big\| \Big\{ \frac{C}{\varepsilon \, \beta} \Big(1 + \frac{C}{\beta} \Big) \, M[\underline{b}] \ + \ \frac{C}{\varepsilon \, \beta^2} \, N_{\mathcal{F}} \ + \ \frac{C \, \varepsilon}{\beta} \, N_{\mathcal{F}} \Big\}^{1/2} \psi \Big\| \\ \le \ & \frac{C'}{\beta^{1/2}} \Big\| \Big(M[\underline{b}] \ + \ \frac{1}{\beta} \, N_{\mathcal{F}} \Big)^{1/2} \phi \Big\| \, \Big\| \Big(M[\underline{b}] \ + \ \frac{1}{\beta} \, N_{\mathcal{F}} \Big)^{1/2} \psi \Big\| \, , \end{aligned}$$

by choosing $\varepsilon := \beta^{-1/2}$, provided that $\alpha \geq 0$ is sufficiently small, and $\beta < \infty$ is sufficiently large. Additionally taking into account (II.4), we hence obtain

$$\begin{aligned} \Big| \big\langle \phi \mid W_{H,Q}(\kappa, o) \, \psi \big\rangle \Big| \ \le \ & \frac{C''}{\beta^{1/2}} \Big\| \Big(M[\underline{b}] \ + \ \Big[\frac{1}{\beta} + \frac{\alpha \, \kappa}{\beta^{1/2} \, (1 - \kappa)} \Big] \, N_{\mathcal{F}} \Big)^{1/2} \phi \Big\| \\ & \Big\| \Big(M[\underline{b}] \ + \ \Big[\frac{1}{\beta} + \frac{\alpha \, \kappa}{\beta^{1/2} \, (1 - \kappa)} \Big] \, N_{\mathcal{F}} \Big)^{1/2} \psi \Big\| \, , \end{aligned} \tag{II.18}$$

with $C'' < \infty$ being a universal constant.

The next step is to use the positivity of $-\partial_j^2$, as a quadratic form. This gives, cf. (II.12),

$$\begin{aligned} \Theta_k^2 \, V_k \ & \le \ \Theta_k \Big(- \frac{1}{\beta^2} \partial_k^2 \ + \ (G_k')^2 \ - \ \frac{1}{\beta} G_{kk}'' \Big) \Theta_k \\ & = \ \Theta_k \, Z_k^*(G) \, Z_k(G) \, \Theta_k \, , \end{aligned} \tag{II.19}$$

and similarly

$$\begin{aligned}\check{\Theta}_j^2\,\widehat{V}_j \;&\le\; \check{\Theta}_j\Big(-\frac{1}{\beta^2}\partial_j \;+\; (\widehat{G}_j')^2 \;-\; \frac{1}{\beta}\widehat{G}_{jj}''\Big)\check{\Theta}_j \\ &=\; \check{\Theta}_j\, Z_j^*(\widehat{G})\, Z_j(\widehat{G})\,\check{\Theta}_j\,. \end{aligned} \tag{II.20}$$

The distinction between $\check{\Theta}_j$ and Θ_j becomes important now. Namely, inserting the commutator $[Z_j^*(\widehat{G}), Z_j(\widehat{G})] = -2\beta^{-1}\widehat{G}_{jj}''$, using the second intertwining relation $Z_j(\widehat{G})\widehat{\Theta}_j = \widehat{\Theta}_j Z_j(H)$ in (I.54), and taking into account that $\widehat{\Theta}_j^2 \ge 1$, $\widehat{\Theta}_j\check{\Theta}_j = \Theta_j$, we have (cf. (I.92))

$$\begin{aligned}\check{\Theta}_j^2\,\widehat{V}_j \;&\le\; \check{\Theta}_j\, Z_j^*(\widehat{G})\, Z_j(\widehat{G})\,\check{\Theta}_j \\ &=\; \check{\Theta}_j\, Z_j(\widehat{G})\, Z_j^*(\widehat{G})\,\check{\Theta}_j \;-\; \frac{2}{\beta}\,\check{\Theta}_j^2\,\widehat{G}_{jj}'' \\ &\le\; \check{\Theta}_j\, Z_j(\widehat{G})\,\widehat{\Theta}_j\,\widehat{\Theta}_j\, Z_j^*(\widehat{G})\,\check{\Theta}_j \;-\; \frac{2}{\beta}\,\check{\Theta}_j^2\,\widehat{G}_{jj}'' \\ &=\; \check{\Theta}_j\,\widehat{\Theta}_j\, Z_j(H)\, Z_j^*(H)\,\widehat{\Theta}_j\,\check{\Theta}_j \;-\; \frac{2}{\beta}\,\check{\Theta}_j^2\,\widehat{G}_{jj}'' \\ &=\; \Theta_j\, Z_j(H)\, Z_j^*(H)\,\Theta_j \;-\; \frac{2}{\beta}\,\check{\Theta}_j^2\,\widehat{G}_{jj}''\,. \end{aligned} \tag{II.21}$$

Inserting (II.19) and (II.21) into the definition (II.11) of $M_j[\underline{b}]$, we obtain that

$$\begin{aligned} M_j[\underline{b}] \;&=\; \check{\Theta}_j^2\,\widehat{V}_j \;+\; \sum_{k(\neq j)} b_{jk}\,\Theta_k^2\,V_k \\ &\le\; \Theta_j\, Z_j(H)\, Z_j^*(H)\,\Theta_j \;-\; \frac{2}{\beta}\,\check{\Theta}_j^2\,\widehat{G}_{jj}'' \;+\; \sum_{k(\neq j)} b_{jk}\,\Theta_k\, Z_k^*(G)\, Z_k(G)\,\Theta_k \\ &\le\; (1+C_b)\,\big(\check{A}_j \;+\; A_j\big)\,, \end{aligned} \tag{II.22}$$

where $C_b := \max_j \Big\{ \sum_k b_{jk} \Big\} < \infty$ is a universal constant, as proved in Lemma II.7 below. For all $\gamma \ge 1$, Eqs. (II.22) and (I.79) yield

$$\begin{aligned} M[\underline{b}] \;&\le\; (1+C_b)\,\gamma\,\Big(\sum_j \big\{\check{A}_j \;+\; A_j\big\}\otimes a_j^* a_j\Big) \\ &\le\; (1+C_b)\,\big(\gamma\, d\Gamma(A_{H,Q}^{(1)}) \;-\; \frac{c_f\,\gamma}{\beta}\,N_{\mathcal{F}}\big)\,. \end{aligned} \tag{II.23}$$

Here we used the following lower bound, cf. (I.8), (I.15) and (I.16): For any

$\psi \in \mathcal{F}$,

$$\Big\langle \psi \;\Big|\; \sum_{i,j} H''_{ij}(0) \otimes a_i^* a_j \; \psi \Big\rangle \;\geq\; c_f \sum_j \|a_j\,\psi\|^2 \;-\; \alpha \sum_{i,j} a_{ij}\, \|a_i\,\psi\|\,\|a_j\,\psi\|$$
$$\geq\; \big(c_f \;-\; C_a \alpha\big) \sum_j \|a_j\,\psi\|^2$$
$$\geq\; \frac{c_f}{2} \big\langle \psi \big|\; N_{\mathcal{F}}\,\psi \big\rangle\,, \tag{II.24}$$

provided $0 \leq \alpha \leq c_f/(2C_a)$. This establishes (II.23), for all $\gamma \geq 1$.

Finally, we insert (II.23) into (II.18) and obtain

$$\Big| \big\langle \phi \;|\; W_{H,Q}(\kappa, o)\,\psi \big\rangle \Big| \tag{II.25}$$
$$\leq\; \frac{C}{\beta^{1/2}} \Big\| \Big(\gamma\, d\Gamma(A^{(1)}_{H,Q}) \;+\; \frac{1}{\beta}\Big[1 + \frac{\beta^{1/2}\,\alpha\,\kappa}{(1-\kappa)} - c_f\,\gamma \Big]\, N_{\mathcal{F}} \Big)^{1/2} \phi \Big\|$$
$$\cdot \Big\| \Big(\gamma\, d\Gamma(A^{(1)}_{H,Q}) \;+\; \frac{1}{\beta}\Big[1 + \frac{\beta^{1/2}\,\alpha\,\kappa}{(1-\kappa)} - c_f\,\gamma \Big]\, N_{\mathcal{F}} \Big)^{1/2} \psi \Big\|\,,$$

for some universal constant $C < \infty$. We choose

$$\gamma \;:=\; \frac{1}{c_f} \Big(1 + \frac{\beta^{1/2}\,\alpha\,\kappa}{1-\kappa} \Big) \tag{II.26}$$

and observe that $\gamma \geq 1$, since $c_f \leq 1$. Moreover, this choice of γ insures that the terms in brackets in (II.25) vanish. Therefore,

$$\Big| \big\langle \phi \;|\; W_{H,Q}(\kappa, o)\,\psi \big\rangle \Big| \;\leq\; \frac{C\,\gamma}{\beta^{1/2}}\, \big\| d\Gamma(A^{(1)}_{H,Q})^{1/2}\,\phi \big\| \, \big\| d\Gamma(A^{(1)}_{H,Q})^{1/2}\,\psi \big\|$$
$$=\; \frac{C'}{\beta^{1/2}} \Big(1 + \frac{\beta^{1/2}\,\alpha\,\kappa}{1-\kappa} \Big)\, \big\| d\Gamma(A^{(1)}_{H,Q})^{1/2}\,\phi \big\| \, \big\| d\Gamma(A^{(1)}_{H,Q})^{1/2}\,\psi \big\|\,, \tag{II.27}$$

finishing the proof. □

II.1 Estimate on $W_{\mathrm{off-d}}(\kappa, o)$ in Terms of $J[\underline{\mathbf{a}}(\kappa)]$

We recall from (I.84) the definition of $W_{\mathrm{off-d}}(\kappa, o)$,

$$W_{\mathrm{off-d}}(\kappa, o; x) \;=\; \frac{2\,\alpha}{\beta} \sum_{i \neq j} \Big(e^{\kappa[\rho(i,o) - \rho(j,o)]} \Theta_i \Theta_j \, w''_{ij}(x) - w''_{ij}(0) \Big) \otimes a_i^* a_j\,, \tag{II.28}$$

and the definition of $\Theta_j := e^{\beta q_j}$. We decompose $W_{\mathrm{off-d}}(\kappa, o) = \Delta W_{\mathrm{off-d}}(\kappa, o) + W_{\mathrm{off-d}}(\kappa, o; 0)$, where

$$\Delta W_{\mathrm{off-d}}(\kappa, o; x) \;:=\; W_{\mathrm{off-d}}(\kappa, o; x) - W_{\mathrm{off-d}}(\kappa, o; 0) \tag{II.29}$$
$$= \frac{2\,\alpha}{\beta} \sum_{i \neq j} e^{\kappa[\rho(i,o)-\rho(j,o)]} \Big(\Theta_i \Theta_j\, w''_{ij}(x) \;-\; w''_{ij}(0) \Big) \;\otimes\; a_i^* a_j\;,$$
$$W_{\mathrm{off-d}}(\kappa, o; 0) \;=\; \frac{2\,\alpha}{\beta} \sum_{i \neq j} \Big(e^{\kappa[\rho(i,o)-\rho(j,o)]} \;-\; 1 \Big)\, w''_{ij}(0) \;\otimes\; a_i^* a_j\;. \tag{II.30}$$

Below, we prove the following estimate.

Lemma II.2. *For some universal constant $C < \infty$ and all $\kappa \in [0, 1]$,*

$$\Big\| J[\underline{a}(\kappa)]^{-1/2}\, \Delta W_{\mathrm{off-d}}(\kappa, o)\; J[\underline{a}(\kappa)]^{-1/2} \Big\| \;\leq\; \frac{C\,\alpha}{\beta}\,. \tag{II.31}$$

where $J[a]$ is defined in Eqs. (II.6) and (II.8), and $\underline{a}(\kappa) \;:=\; \big(a_{jk}(\kappa) \big)_{j,k \in \Lambda} \;=\; \big(e^{\kappa \rho(j,k)}\, a_{jk} \big)_{j,k \in \Lambda}$.

Proof: We use a variant of the Cauchy-Schwarz inequality. For $x \in \mathbb{R}^\Lambda$, we introduce the characteristic functions $\chi_j := \mathbf{1}[|x_j| < \hat{R}_0]$ and $\bar{\chi}_j := \mathbf{1}[|x_j| \geq \hat{R}_0]$, so that

$$1 \;=\; \chi_j + \bar{\chi}_j\;. \tag{II.32}$$

Note that if $|x_i| \leq \hat{R}_0$ then $\Theta_i \equiv 1$ and $|g_i'(x_i)| \leq C_g e^{C_g \hat{R}_0}$, by (I.31). Furthermore, $|g_j'(x_j)| \geq c_g \min\{1, \hat{R}_0\}$, for $|x_j| \geq \hat{R}_0$, by (I.32). Hence, for any $i, j \in \Lambda$,

$$\chi_i\, \bar{\chi}_j\, \big(|g_i'(x_i)| + |g_j'(x_j)| + 2 \big) \;\leq\; \chi_i\, \bar{\chi}_j\, \big(C_g e^{C_g \hat{R}_0} + |g_j'(x_j)| + 2 \big) \tag{II.33}$$
$$\leq\; C_1\, \chi_i\, \bar{\chi}_j\, |g_j'(x_j)|\,,$$

and

$$\bar{\chi}_i\, \bar{\chi}_j\, \big(|g_i'(x_i)|^{1/2} + |g_j'(x_j)|^{1/2} + 2 \big) \;\leq\; C_2\, \bar{\chi}_i\, \bar{\chi}_j\, |g_i'(x_i)|^{1/2}\, |g_j'(x_j)|^{1/2}\,, \tag{II.34}$$

where $C_1 := 1 + (C_g e^{C_g \hat{R}_0} + 2) c_g^{-1} \min\{1, \hat{R}_0\}^{-1}$ and $C_2 := 2 c_g^{-1/2} \min\{1, \hat{R}_0\}^{-1/2} + 2 c_g^{-1} \min\{1, \hat{R}_0\}^{-1}$. Additionally using that $\Theta_j \chi_j = \chi_j$ and $\Theta_j \geq 1$, we obtain

$$\big| \Theta_i \Theta_j\, w''_{ij}(x) \;-\; w''_{ij}(0) \big| \tag{II.35}$$
$$\leq\; \chi_i\, \chi_j\, \big| w''_{ij}(x) \;-\; w''_{ij}(0) \big| \;+\; \bar{\chi}_i\, \bar{\chi}_j\, \Theta_i\, \Theta_j \big(|w''_{ij}(x)| \;+\; |w''_{ij}(0)| \big)$$
$$+\; \chi_i\, \bar{\chi}_j\, \Theta_j \big(|w''_{ij}(x)| \;+\; |w''_{ij}(0)| \big) \;+\; \bar{\chi}_i\, \chi_j\, \Theta_i \big(|w''_{ij}(x)| \;+\; |w''_{ij}(0)| \big)$$
$$\leq\; C_w\, a_{ij}\, \chi_i\, \chi_j\, \big(|g_i'| + |g_j'| \big) \;+\; C_w\, C_2\, a_{ij}\, \bar{\chi}_i\, \bar{\chi}_j\, \Theta_i\, \Theta_j\, |g_i'|^{1/2}\, |g_j'|^{1/2}$$
$$+\; C_w\, C_1\, a_{ij}\, \big(\chi_i\, \bar{\chi}_j\, \Theta_j\, |g_j'| \;+\; \bar{\chi}_i\, \chi_j\, \Theta_i\, |g_i'| \big)$$
$$\leq\; C_3\, a_{ij} \Big(\Theta_i\, |g_i'| \;+\; \Theta_j\, |g_j'| \;+\; \Theta_i\, \Theta_j\, |g_i'|^{1/2} |g_j'|^{1/2} \Big)\,,$$

for some universal constant $C_3 < \infty$. Next, the triangle inequality implies that $\rho(i,o) - \rho(j,o) \leq \rho(i,j)$, and hence $e^{\kappa[\rho(i,o)-\rho(j,o)]}\, a_{ij} \leq a_{ij}(\kappa)$, uniformly in the reference point $o \in \Lambda$. Thus, Eq. (II.35) implies that

$$\begin{aligned} & e^{\kappa[\rho(i,o)-\rho(j,o)]} \left| \Theta_i \Theta_j \, w''_{ij}(x) \, - \, w''_{ij}(0) \right| \\ & \quad \leq \; C_3 \, a_{ij}(\kappa) \left(\Theta_i \, |g'_i| \, + \, \Theta_j \, |g'_j| \, + \, \Theta_i \, \Theta_j \, |g'_i|^{1/2} |g'_j|^{1/2} \right) . \end{aligned} \tag{II.36}$$

Now, let $\phi, \psi \in \mathcal{H}$ and observe that

$$\begin{aligned} & \sum_{i \neq j} a_{ij}(\kappa) \, \Theta_j \, |g'_j| \, \|a_i \phi(x)\| \, \|a_j \psi(x)\| \\ & \leq \; \left(\sum_{i \neq j} a_{ij}(\kappa) \, \Theta_j^2 \, |g'_j| \, \|a_i \phi(x)\|^2 \right)^{1/2} \left(\sum_{i \neq j} a_{ij}(\kappa) \, |g'_j| \, \|a_j \psi(x)\|^2 \right)^{1/2} \\ & \leq \; \left(\sum_j \left\{ C_a \, |g'_j| \, + \, \sum_{i(\neq j)} a_{ij}(\kappa) \, \Theta_i^2 \, |g'_i| \right\} \|a_j \phi(x)\|^2 \right)^{1/2} \\ & \qquad \cdot \left(\sum_j \left\{ C_a \, |g'_j| \, + \, \sum_{i(\neq j)} a_{ij}(\kappa) \, \Theta_i^2 \, |g'_i| \right\} \|a_j \psi(x)\|^2 \right)^{1/2} \end{aligned} \tag{II.37}$$

and

$$\begin{aligned} & \sum_{i \neq j} a_{ij}(\kappa) \, \Theta_i \, \Theta_j \, |g'_i|^{1/2} |g'_j|^{1/2} \, \|a_i \phi(x)\| \, \|a_j \psi(x)\| \; \leq \\ & \qquad \left(\sum_j \left\{ \sum_{i(\neq j)} a_{ij}(\kappa) \, \Theta_i^2 \, |g'_i| \right\} \|a_j \phi(x)\|^2 \right)^{1/2} \\ & \qquad\qquad \left(\sum_j \left\{ \sum_{i(\neq j)} a_{ij}(\kappa) \, \Theta_i^2 \, |g'_i| \right\} \|a_j \psi(x)\|^2 \right)^{1/2} , \end{aligned} \tag{II.38}$$

where $a_{ij}(\kappa) := e^{\kappa \rho(i,j)} \, a_{ij}$. Eqs. (II.35)–(II.38) and $\check{\Theta}_j \geq 1$ imply that

$$\left| \langle \phi \, | \, \Delta W_{\mathrm{off-d}}(\kappa, o) \, \psi \rangle \right| \; \leq \; \frac{3 C_3 \, \alpha}{\beta} \, \left\| J[\underline{a}(\kappa)]^{1/2} \, \phi \right\| \, \left\| J[\underline{a}(\kappa)]^{1/2} \, \psi \right\| , \tag{II.39}$$

and hence (II.31). □

Next we turn to estimating $W_{\mathrm{off-d}}(\kappa, o; 0)$ which is a constant matrix. Here we use the fact that $W_{\mathrm{off-d}}(0, o; 0) = 0$ to bound $W_{\mathrm{off-d}}(\kappa, o; 0)$ by a term of order κ.

Lemma II.3. *For all* $0 \leq \kappa < 1$,

$$\left\| N_{\mathcal{F}}^{-1/2} \, W_{\mathrm{off-d}}(\kappa, o; 0) \, N_{\mathcal{F}}^{-1/2} \right\| \; \leq \; \frac{3 \, C_a \, \alpha \, \kappa}{(1-\kappa) \, \beta} \, , \tag{II.40}$$

where $N_{\mathcal{F}} = \sum_j \mathbf{1} \otimes a_j^* a_j$.

Proof: We pick $\phi, \psi \in \mathcal{F}$ and apply the Cauchy-Schwarz inequality. Additionally using the triangle inequality, implying $e^{\kappa[\rho(i,o)-\rho(j,o)]} \le e^{\kappa\rho(i,j)}$, and taking (I.15) and the symmetry $a_{ij} = a_{ji}$ into account, we obtain

$$\begin{aligned} \big|\langle\phi|\, W_{\text{off}-\text{d}}(\kappa, o; 0)\, \psi\rangle\big| \ &\le\ \frac{2\,\alpha}{\beta} \sum_{i,j} \big(e^{\kappa\rho(i,j)} \,-\, 1\big)\, a_{ij}\, \|a_i\, \phi\|\, \|a_j\, \psi\| \qquad \text{(II.41)} \\ \le\ \frac{2\,\alpha}{\beta} \max_i \Big\{ \sum_j \big(e^{\kappa\rho(i,j)} \,-\, 1\big)\, a_{ij} \Big\} &\Big\{ \sum_k \|a_k\, \phi\|^2 \Big\}^{1/2} \Big\{ \sum_k \|a_k\, \psi\|^2 \Big\}^{1/2}, \end{aligned}$$

and hence (II.40) follows if we show that, for all $i \in \Lambda$,

$$\sum_j \big(e^{\kappa\rho(i,j)} \,-\, 1\big)\, a_{ij} \ \le\ \frac{3\, C_a\, \kappa}{1-\kappa}\, . \tag{II.42}$$

To this end, we observe that $e^r - 1 \le r\, e^r$, which implies

$$\begin{aligned} \sum_j \big(e^{\kappa\rho(i,j)} \,-\, 1\big)\, a_{ij} \ &\le\ \sum_j \kappa\, \rho(i,j)\, e^{\kappa\rho(i,j)}\, a_{ij} \qquad \text{(II.43)} \\ &=\ \frac{\kappa}{1-\kappa} \sum_j \Big((1-\kappa)\, \rho(i,j)\, e^{-(1-\kappa)\rho(i,j)}\Big) \Big(e^{\rho(i,j)}\, a_{ij}\Big) \\ &\le\ \frac{C_a\, \kappa}{1-\kappa} \max_{\rho \ge 0} \big\{\rho\, e^{-\rho}\big\} \ \le\ \frac{3\, C_a\, \kappa}{1-\kappa}\, , \end{aligned}$$

using (I.16). This gives (II.42). □

II.2 Estimate on $\Delta W_{\text{off}-\text{d}}(\kappa, o) + W_{\text{diag}}$ in Terms of $J[\underline{\mathbf{a}}(\kappa)]$

We begin by deriving two simple estimates which are of key importance to our approach.

Lemma II.4. *For some universal constant $C_\Theta < \infty$, all $j, k \in \Lambda$, and all $x \in \mathbb{R}^\Lambda$,*

$$\check{\Theta}_j^2\, |g_k'| \ \le\ \check{\Theta}_k^2\, |g_k'| \ +\ C_\Theta\, \check{\Theta}_j^2\, |g_j'|\, , \tag{II.44}$$
$$\Theta_j^2\, |g_k'| \ \le\ \check{\Theta}_k^2\, |g_k'| \ +\ C_\Theta\, \Theta_j^2\, |g_j'|\, . \tag{II.45}$$

Proof: First, Eqs. (II.44) and (II.45) hold true trivially, for $x_k \in I_\infty$, because on that set $\check{\Theta}_k \equiv e^{\beta q_{\max}} \ge \Theta_j \ge \check{\Theta}_j$.

Conversely, if $x_k \notin I_\infty$ then we distinguish the cases $x_j \in I_0$ and $x_j \notin I_0$. In the former case, $\check{\Theta}_j = \Theta_j \equiv 1 \le \check{\Theta}_k$, and there is nothing to prove.

It remains to demonstrate Eqs. (II.44) and (II.45) for $x_k \notin I_\infty$ and $x_j \notin I_0$, i.e., for $|x_k| \le \check{R}_1$ and $|x_j| \ge \hat{R}_0$. For this, we remark that Eqs. (I.31) and (I.32) imply

$$|g_k'(x_k)| \ \le\ \frac{C_g\, e^{C_g \check{R}_1}}{c_g\, \min\{1, \hat{R}_0\}}\, |g_j'(x_j)|\, , \tag{II.46}$$

which completes the proof upon the choice $C_\Theta := C_g c_g^{-1} e^{C_g \check{R}_1} \min\{1, \hat{R}_0\}^{-1}$. $\square$

Lemma II.4 is used to estimate

$$W_{\rm diag}(x) \;=\; \frac{2}{\beta} \sum_j \Big(\check{\Theta}_j^2 \, \hat{G}''_{jj}(x) \;-\; H''_{jj}(0) \Big) \;\otimes\; a_j^* a_j \tag{II.47}$$

in terms of $J[\underline{a}(\kappa)]$, where $J[a]$ is defined in Eqs. (II.6) and (II.8). Recall furthermore that $\underline{a}(\kappa) \;:=\; \big(a_{jk}(\kappa)\big)_{j,k\in\Lambda} \;=\; \big(e^{\kappa\rho(j,k)}\, a_{jk}\big)_{j,k\in\Lambda}$, and that $\check{\Theta}_j \;:=\; \exp[\beta \check{q}_j]$ and $\widehat{G} = H - \sum_k \hat{q}_k$. We have the following estimates.

Lemma II.5. *For some universal constant $C < \infty$, all $0 \le \alpha \le 1$, and all $x \in \mathbb{R}^\Lambda$,*

$$\check{\Theta}_j^2(x_j) \hat{G}''_{jj}(x) \;-\; H''_{jj}(0) \;\;\le\;\; C \; J_j[\underline{a}(\kappa)](x) \; , \tag{II.48}$$

$$\Theta_j^2(x_j) G''_{jj}(x) \;-\; H''_{jj}(0) \;\;\le\;\; C \; J_j[\underline{a}(\kappa)](x) \; . \tag{II.49}$$

Proof: We only give the proof of Eq. (II.48), the proof of Eq. (II.48) is similar. Furthermore we note that $a_{jk} \le a_{jk}(\kappa)$ and it hence suffices to prove Eq. (II.48) for a_{jk}.

First, we estimate in the region $\{|x_j| < \hat{R}_0\}$, where we have $\check{\Theta}_j \equiv 1$. This, Eq. (I.28), Eq. (I.30), and $H''_{jj}(0) = G''_{jj}(0)$ give

$$\begin{aligned}
\big|\check{\Theta}_j^2 \, \hat{G}''_{jj}(x) \;-\; H''_{jj}(0)\big| \;&=\; \big|\hat{G}''_{jj}(x) \;-\; \hat{G}''_{jj}(0)\big| \\
\le\;\; & \big|\hat{g}''_j(x_j) \;-\; \hat{g}''_j(0)\big| \;+ \\
& \alpha \sum_{k(\ne j)} \Big\{ \big|\partial_j^2 w_{kj}(x_k, x_j) \;-\; \partial_j^2 w_{kj}(0,0)\big| \;+\; \big|\partial_j^2 w_{jk}(x_j, x_k) \;-\; \partial_j^2 w_{jk}(0,0)\big| \Big\} \\
\le\;\; & C_g \, |\hat{g}'_j| \;+\; 2C_w \, \alpha \sum_{k(\ne j)} a_{kj} \, \big(|g'_k| + |g'_j|\big) \\
\le\;\; & \big(C_g + 2C_w C_a \alpha\big) \, |\hat{g}'_j| \;+\; 2C_w \, c_g^{-1} \, \alpha \sum_{k(\ne j)} a_{kj} \, \Theta_k^2 \, |\hat{g}'_k| \, ,
\end{aligned} \tag{II.50}$$

yielding Eq. (II.48) in the region $\{|x_j| < \hat{R}_0\}$.

Conversely, on $\{|x_j| \geq \hat{R}_0\}$, we have

$$\begin{aligned}
&\big|\check{\Theta}_j^2\, \hat{G}''_{jj}(x) \;-\; H''_{jj}(0)\big| \\
&\leq\; \check{\Theta}_j^2 \big|\widehat{G}''_{jj}(x) - \widehat{G}''_{jj}(0)\big| \;+\; \big(\check{\Theta}_j^2 + 1\big)\, \big|G''_{jj}(0)\big| \\
&\leq\; \check{\Theta}_j^2 \big|\hat{g}''_j(x_j) - \hat{g}''_j(0)\big| \;+\; \big(\check{\Theta}_j^2 + 1\big)\, \big|g''_j(0)\big| \\
&\qquad\qquad + 2C_w\, \alpha \sum_{k(\neq j)} a_{kj} \Big\{ \check{\Theta}_j^2\, \big(|g'_k| + |g'_j|\big) \;+\; \check{\Theta}_j^2 \;+\; 1 \Big\} \\
&\leq\; \big(1 + C_g + 4C_w C_a \alpha\big)\, \check{\Theta}_j^2 \Big[|\hat{g}'_j(x_j)| \;+\; 2\big(1 + |g''_j(0)|\big) \Big] \\
&\qquad\qquad + 2C_w\, C_\Theta\, c_g^{-1}\, \alpha \sum_{k(\neq j)} a_{kj}\, \Theta_k^2\, |g'_k|\, ,
\end{aligned} \tag{II.51}$$

where we use (I.30) again and $\check{\Theta}_j^2 |g'_k| \leq \check{\Theta}_k^2 |g'_k| + C_\Theta \check{\Theta}_j^2 |\hat{g}'_j|$, which we established in Lemma II.4. Observing that on $\{|x_j| \geq \hat{R}_0\}$, we have $|g'_j| \geq c_g \min\{1, \hat{R}_0\}$, and additionally using $|g''_j(0)| \leq C_f$, we further obtain

$$|g'_j(x_j)| \;+\; 2\big(1 + |g''_j(0)|\big) \;\leq\; C\, |\hat{g}'_j(x_j)|\, , \tag{II.52}$$

for the universal constant $C := 1 + 2c_g^{-2}(1 + C_f)\min\{1, \hat{R}_0\}^{-1} < \infty$. Inserting (II.52) into (II.51), we arrive at Eq. (II.48) in the region $\{|x_j| \geq \hat{R}_0\}$, as well. $\square$

Combining Lemma II.2 and Eq. (II.48) in Lemma II.5, we arrive at the following intermediate conclusion,

Theorem II.6. *For some universal constant $C < \infty$ and all $0 \leq \kappa \leq 1$,*

$$\big\| J[\underline{a}(\kappa)]^{-1/2} \left\{ \Delta W_{\mathrm{off-d}}(\kappa, o) + W_{\mathrm{diag}} \right\} J[\underline{a}(\kappa)]^{-1/2} \big\| \;\leq\; \frac{C}{\beta}\, , \tag{II.53}$$

where $J[\underline{a}]$ is defined in Eqs. (II.6) and (II.8).

II.3 Estimate on $J[\underline{\mathbf{a}}(\kappa)]$ in Terms of $K[\underline{\mathbf{b}}]$

In this subsection we replace $|\hat{g}'_j|$ and $|g'_k|$ in $J_j[\underline{a}]$ by $|\hat{G}'_j|$ and $|G'_k|$, respectively. It turns out that it is convenient to slighty increase the weights $\underline{a}(\kappa) := \big(a_{jk}(\kappa)\big)_{j,k\in\Lambda}$ by passing to $\underline{b} := (b_{jk})_{j,k\in\Lambda}$ defined by

$$b_{jk} \;:=\; \frac{1}{\alpha'} \sum_{N=1}^{\infty} \Big(\frac{\alpha'}{1 - C_a \alpha'} \Big)^N \, \big(\underline{a}(1)^N\big)_{jk}\, , \tag{II.54}$$

where $0 < \alpha' < (2C_a)^{-1}$ and $\underline{a}(1)^N$ denotes the N^{th} power of $\underline{a}(1)$ as a matrix with $\kappa = 1$, i.e.,

$$\big(\underline{a}(\kappa)^N\big)_{jk} \;:=\; \sum_{n_1,\ldots,n_{N-1}} e^{\rho(j,n_1)}\, a_{j\,n_1} \cdots e^{\rho(n_{N-1},k)}\, a_{n_{N-1}\,k} \;, \tag{II.55}$$

recalling that $a_{jk}(\kappa) = e^{\kappa\rho(i,j)}\, a_{jk}$. To prove the convergence of the series in Eq. (II.54) we view $\underline{a}(1)$ as a real, symmetric (and thus selfadjoint) matrix on $\mathbb{C}^\Lambda$ equipped with the norm induced by the standard scalar product $\langle\varphi|\psi\rangle = \sum_i \overline{\varphi_i}\psi_i$. The operator norm of $\underline{a}(1)$ is bounded by C_a, and hence $\|\eta\,\underline{a}\| < 1$, provided $0 < \eta := \alpha'(1-\alpha' C_a) < C_a^{-1}$ which is equivalent to $0 < \alpha' < C_a^{-1}/2$. We note that trivially

$$b_{jk} \;\geq\; \frac{e^{\rho(j,k)}\, a_{jk}}{1 - C_a\alpha'} \;\geq\; e^{\rho(j,k)}\, a_{jk} \;, \tag{II.56}$$

by ignoring all terms in the series (II.55) of order $N \geq 2$.

The following Proposition shows that $\underline{b}$ possesses similar summability properties as $\underline{a}(\kappa)$ and has the additional property (II.58), which $\underline{a}(\kappa)$ does not.

Lemma II.7. *For $0 < \alpha' < C_a^{-1}/2$, the weights $\underline{b} := (b_{jk})_{j,k\in\Lambda}$ defined in (II.54) exist, are symmetric, $b_{jk} = b_{kj}$, vanish on the diagonal, $b_{jj} = 0$, and obey the summability condition*

$$\max_j \Big\{ \sum_k b_{jk} \Big\} \;\leq\; \frac{C_a}{1 - 2C_a\alpha'} \;=:\; C_b \;. \tag{II.57}$$

Moreover they additionally obey the estimate

$$\frac{\alpha'}{1 - C_a\alpha'} \sum_m b_{jm}\, e^{\rho(j,k)}\, a_{mk} \;\leq\; b_{jk} \;, \tag{II.58}$$

for all $j, k \in \Lambda$.

Proof: We first apply the triangle inequality and (I.16) to derive

$$\max_j \Big\{ \sum_k \big(\underline{a}(1)^N\big)_{jk} \Big\} \tag{II.59}$$
$$\leq\; \max_j \Big\{ \sum_{n_1,\ldots,n_{N-1},k} e^{\rho(j,n_1)}\, a_{j\,n_1} \cdots e^{\rho(n_{N-1},k)}\, a_{n_{N-1}\,k} \Big\} \;\leq\; C_a^N \;,$$

which yields (II.57),

$$\max_j \Big\{ \sum_k e^{\rho(j,k)}\, b_{jk} \Big\} \;\leq\; \frac{1}{\alpha'} \sum_{N=1}^{\infty} \Big(\frac{C_a\alpha'}{1 - C_a\alpha'}\Big)^N \;=\; \frac{C_a}{1 - 2C_a\alpha'} \,. \tag{II.60}$$

Next, setting $\eta := \alpha'(1-\alpha' C_a)$, we note that $0 < \eta < C_a^{-1}$. Hence $\mathbf{1} - \eta \underline{a}(1)$ is invertible, and its inverse can be expanded in a norm-convergent Neumann series,

$$\big(\mathbf{1} \; - \; \eta \, \underline{a}(1)\big)^{-1} \; = \; \sum_{N=0}^{\infty} \big(\eta \, \underline{a}(1)\big)^N \; = \; \mathbf{1} \; + \; \sum_{N=1}^{\infty} \big(\eta \, \underline{a}(1)\big)^N \, , \tag{II.61}$$

and hence we have

$$\big(\mathbf{1} \; - \; \eta \, \underline{a}(1)\big)^{-1} \; = \; \mathbf{1} \; + \; \alpha' \, \underline{b} \, . \tag{II.62}$$

For the derivation of (II.58), we multiply (II.62) by $\mathbf{1} - \eta \underline{a}(1)$ and obtain

$$\mathbf{1} \; = \; \mathbf{1} \; - \; \eta \, \underline{a}(1) \; + \; \alpha' \underline{b} \; - \; \alpha' \eta \, \underline{b} \, \underline{a}(1) \, . \tag{II.63}$$

Since the matrix elements of $\eta \underline{a}$ are nonnegative, this implies

$$\eta \big(\underline{b} \, \underline{a}(1) \big)_{jk} \; \leq \; b_{jk} \, , \tag{II.64}$$

which is equivalent to (II.58). $\square$

Next we use the new weights $\underline{b} = (b_{jk})_{j,k \in \Lambda}$ to implement the change from $|g'_k|$ to $|G'_k|$. The main result of this subsection is a form bound on $J[\underline{a}(\kappa)]$ in terms of $K[\underline{b}]$, where $J[\underline{a}]$ and $K[\underline{a}]$ are defined in Eqs. (II.6), (II.8), and (II.9), and we recall that $\underline{a}(\kappa) := \big(a_{jk}(\kappa)\big)_{j,k\in\Lambda} = \big(e^{\kappa \rho(j,k)} \, a_{jk}\big)_{j,k\in\Lambda}$.

Theorem II.8. *There exists a universal constant* $1 \leq C < \infty$ *such that, for all* $0 \leq \alpha < \alpha' \leq C^{-1}$ *and all* $0 \leq \kappa \leq 1$,

$$J[\underline{a}(\kappa)] \; \leq \; J[\underline{b}] \; \leq \; \Big(1 + C \frac{\alpha}{\alpha'}\Big) \, K[\underline{b}] \, . \tag{II.65}$$

Proof: We first remark that (II.65) is equivalent to

$$\forall j \in \Lambda \; : \qquad J_j[\underline{a}(\kappa)] \; \leq \; J_j[\underline{b}] \; \leq \; \Big(1 + C \frac{\alpha}{\alpha'}\Big) \, K_j[\underline{b}] \, . \tag{II.66}$$

Secondly, the first inequality in (II.66) trivially follows from $a_{jk} \leq e^{\rho(j,k)} \, a_{jk} \leq b_{jk}$. Hence it remains to prove the second inequality in (II.66), and for this it is sufficient to show the existence of a universal constant $C' < \infty$ such that

$$\forall j \in \Lambda \; : \qquad \big| J_j[\underline{b}] \; - \; K_j[\underline{b}] \big| \; \leq \; C' \left(\frac{\alpha}{\alpha'}\right) J_j[\underline{b}] \, . \tag{II.67}$$

To this end, we proceed as in the proof of Lemma II.5 and observe that (I.33) implies

$$\begin{aligned} & \big| J_j[\underline{b}] \; - \; K_j[\underline{b}] \big| \\ & \leq \; \check{\Theta}_j^2 \, |\widehat{G}'_j - \hat{g}'_j| \; + \; \sum_k b_{jk} \, \Theta_k^2 \, |G'_k - g'_k| \\ & \leq \; \alpha \sum_m a_{jm} \big(\check{\Theta}_j^2 |g'_j| \; + \; \check{\Theta}_j^2 |g'_m| \big) \; + \; \alpha \sum_{k,m} b_{jk} \, a_{km} \big(\Theta_k^2 |g'_k| \; + \; \Theta_k^2 |g'_m| \big) \, . \end{aligned} \tag{II.68}$$

Now we apply Lemma II.4 and (II.58),

$$
\begin{aligned}
\big|J_j[\underline{b}] - K_j[\underline{b}]\big| \;\leq\; & \alpha \sum_m a_{jm} \big((1+C_\Theta)\check{\Theta}_j^2 |g_j'| + \check{\Theta}_m^2 |g_m'| \big) \\
& \qquad + \alpha \sum_{k,m} b_{jk}\, a_{km} \big((1+C_\Theta)\Theta_k^2 |g_k'| + \check{\Theta}_m^2 |g_m'| \big) \\
\leq\; & (1+C_\Theta)\, C_a\, \alpha\, \check{\Theta}_j^2 |g_j'| + \alpha \sum_m a_{jm}\, \check{\Theta}_m^2 |g_m'| \\
& \qquad + \alpha\, C_a\, (1+C_\Theta) \sum_k b_{jk}\, \Theta_k^2 |g_k'| + \left(\frac{\alpha}{\alpha'}\right) \sum_m b_{jm}\, \check{\Theta}_m^2 |g_m'| \\
\leq\; & (1+C_\Theta)\, C_a\, \alpha\, \check{\Theta}_j^2 |g_j'| + C'' \left(\frac{\alpha}{\alpha'}\right) \sum_k b_{jk}\, \check{\Theta}_k^2 |g_k'| \,, \qquad \text{(II.69)}
\end{aligned}
$$

for some universal constant $C'' < \infty$, where we additionally use $a_{jk} \leq b_{jk}$, $\check{\Theta}_k \leq \Theta_k$, $\alpha' \leq 1$, and $|g_j'| \leq c_g^{-1} |\hat{g}_j'|$ to derive the last inequality. This clearly implies (II.67). □

II.4 Estimate on $K[\underline{\mathbf{b}}]$ in Terms of $L[\underline{\mathbf{b}}]$

The purpose of this section is to replace $|G_k'|$ and $|\widehat{G}_k'|$ in $K[\underline{b}]$ by $\varepsilon (G_k')^2 + 1/\varepsilon$ and $\varepsilon (\widehat{G}_k')^2 + 1/\varepsilon$, respectively, where $\varepsilon > 0$ is a small parameter which we later choose to be of order $\beta^{-1/2}$. This replacement requires the introduction of $L[\underline{b}]$, defined in Eqs. (II.7) and (II.10). The main result of this subsection is the following theorem.

Theorem II.9. *There exists a universal constant $1 \leq C < \infty$ such that, for all $0 \leq \alpha < \alpha' \leq C^{-1}$ and all $0 < \varepsilon \leq C^{-1}$,*

$$
K[\underline{b}] \;\leq\; \frac{1}{\varepsilon}\, L[\underline{b}] \;+\; C\, \varepsilon\, N_{\mathcal{F}} \,, \qquad \text{(II.70)}
$$

where $N_{\mathcal{F}} = \sum_k \mathbf{1} \otimes a_k^* a_k$ is the number operator on $\mathcal{H}$.

Proof: We start with the observation that if $|t| \geq \hat{R}_0$ then $|g_k'(t)|$ and $|\hat{g}_j'(t)|$ are bounded below by a universal, positive constant $C^{-1} > 0$. Conversely, if $|t| < \hat{R}_0$ then $\check{\Theta}_j(t) = \Theta(t) = 1$. Thus we have the following estimates,

$$
\begin{aligned}
\Theta_k^2(x_k) &= \mathbf{1}\big[|x_k| < \hat{R}_0\big] + \mathbf{1}\big[|x_k| \geq \hat{R}_0\big]\, \Theta_k^2(x_k) \\
&\leq 1 + C\, \Theta_k^2(x_k) |g_k'(x_k)| \,, \qquad \text{(II.71)} \\
\check{\Theta}_j^2(x_j) &= \mathbf{1}\big[|x_j| < \hat{R}_0\big] + \mathbf{1}\big[|x_j| \geq \hat{R}_0\big]\, \check{\Theta}_j^2(x_j) \\
&\leq 1 + C\, \check{\Theta}_j^2(x_j) |\hat{g}_j'(x_j)| \,, \qquad \text{(II.72)}
\end{aligned}
$$

and, as an immediate consequence,

$$\Theta_k^2 |G_k'| \;\leq\; \frac{1}{2\varepsilon}\, \Theta_k^2 |G_k'|^2 \;+\; \frac{\varepsilon}{2}\big(1 \;+\; C\,\Theta_k^2 |g_k'|\big)\,, \tag{II.73}$$

$$\check{\Theta}_j^2 |\widehat{G}_j'| \;\leq\; \frac{1}{2\varepsilon}\, \check{\Theta}_j^2 |\widehat{G}_j'|^2 \;+\; \frac{\varepsilon}{2}\big(1 \;+\; C\,\check{\Theta}_j^2 |\hat{g}_j'|\big)\,. \tag{II.74}$$

Inserting (II.73)–(II.74) into the definition (II.9) of $K_j[\underline{b}]$ and observing the definition (II.8) of $J_j[\underline{b}]$, we obtain

$$\begin{aligned} K_j[\underline{b}] \;&=\; \check{\Theta}_j^2\, |\widehat{G}_j'| \;+\; \sum_k b_{jk}\, \Theta_k^2\, |G_k'| \qquad \text{(II.75)} \\ &\leq\; \frac{1}{2\varepsilon}\, L_j[\underline{b}] \;+\; \frac{\varepsilon}{2}\Big\{\check{\Theta}_j^2 \;+\; \sum_k b_{jk}\Big\} \;+\; \frac{C\,\varepsilon}{2}\Big\{\check{\Theta}_j^2 |\hat{g}_j'| \;+\; \sum_k b_{jk}\Theta_k^2 |g_k'|\Big\} \\ &\leq\; \frac{1}{2\varepsilon}\, L_j[\underline{b}] \;+\; \frac{(1+C_b)\,\varepsilon}{2} \;+\; \frac{C\,\varepsilon}{2}\, J_j[\underline{b}]\,. \\ &\leq\; \frac{1}{2\varepsilon}\, L_j[\underline{b}] \;+\; \frac{(1+C_b)\,\varepsilon}{2} \;+\; \frac{C\,C'\,\varepsilon}{2}\Big(1+\frac{\alpha}{\alpha'}\Big)\, K_j[\underline{b}]\,, \end{aligned}$$

where the last inequality results from Theorem II.8, which ensures the existence of a universal constant $C' < \infty$ such that $J_j[\underline{b}] \leq (1 + C'\alpha/\alpha')\, K_j[\underline{b}]$, provided α/α' is sufficiently small. Thus, if $\alpha \leq \alpha'$ and $\varepsilon \leq (2CC')^{-1}$ then we can solve in Eq. (II.75) for $K_j[\underline{b}]$ and obtain

$$K_j[\underline{b}] \;\leq\; \frac{1}{\varepsilon}\, L_j[\underline{b}] \;+\; (1+C_b)\,\varepsilon\,, \tag{II.76}$$

proving the assertion. □

II.5 Estimate on $L[\underline{\mathbf{b}}]$ in Terms of $M[\underline{\mathbf{b}}]$

In this subsection we show that subtracting $\beta^{-1} G_{kk}''$ from $(G_k')^2$ in $L[\underline{b}]$ does not cause a significant error. This is our last preparatory step for the derivation of the main form bound (II.1). Again, we first make the form of the desired upper bound precise by using the operator $M[\underline{b}]$, defined in Eq. (II.7) and Eqs. (II.11)–(II.12). The following theorem states the main result of this subsection.

Theorem II.10. *There exists a universal constant $1 \leq C < \infty$ such that, for all $\beta \geq C$,*

$$L[\underline{b}] \;\leq\; \Big(1+\frac{C}{\beta}\Big)\, M[\underline{b}] \;+\; \frac{C}{\beta}\, N_{\mathcal{F}}\,, \tag{II.77}$$

where $N_{\mathcal{F}} = \sum_k \mathbf{1} \otimes a_k^* a_k$ is the number operator on $\mathcal{H}$.

Proof: First we note that the difference between $M_j[\underline{b}]$ and $L_j[\underline{b}]$ is bounded by

$$\begin{aligned} \big|M_j[\underline{b}] - L_j[\underline{b}]\big| \; &\leq \; \frac{1}{\beta}\Big(\check{\Theta}_j^2|\widehat{G}''_{jj}(x)| \; + \; \sum_k b_{jk}\,\Theta_k^2|G''_{kk}(x)|\Big) \\ &\leq \; \frac{1}{\beta}\Big(|H''_{jj}(0)| \; + \; \sum_k b_{jk}\,|H''_{kk}(0)|\Big) \; + \\ &\quad\; \frac{1}{\beta}\Big(\big|\check{\Theta}_j^2\widehat{G}''_{jj}(x) \; - \; H''_{jj}(0)\big| \; + \; \sum_k b_{jk}\,\big|\Theta_k^2 G''_{kk}(x) \; - \; H''_{kk}(0)\big|\Big). \end{aligned} \tag{II.78}$$

Due to Eqs. (I.8) and (I.15), we have that

$$|H''_{jj}(0)| \; + \; \sum_k b_{jk}\,|H''_{kk}(0)| \; \leq \; (C_f + \alpha)\,(1 + C_b)\,. \tag{II.79}$$

Next, it follows from Eqs. (II.48)–(II.49) and (I.30) that

$$\big|\check{\Theta}_j^2\,\hat{G}''_{jj}(x) \; - \; H''_{jj}(0)\big| \; \leq \; C\Big(\check{\Theta}_j^2\,|\hat{g}'_j| \; + \; \alpha\sum_k a_{jk}\,\Theta_k^2\,|g'_k|\Big) \tag{II.80}$$

and, for $k \in \Lambda \setminus \{j\}$,

$$\begin{aligned} \big|{\Theta_k}^2\,{G_{kk}}''(x) \; - \; H''_{kk}(0)\big| \; &\leq \; C\Big(\Theta_k^2\,|g'_k| \; + \; \alpha\sum_\ell a_{k\ell}\,\check{\Theta}_\ell^2\,|g'_\ell|\Big) \\ &\leq \; C\Big(a_{jk}\check{\Theta}_j^2\,|\hat{g}'_j| \; + \; \Theta_k^2\,|g'_k| \; + \; \alpha\sum_{\ell(\neq j)} a_{k\ell}\,\Theta_\ell^2\,|g'_\ell|\Big)\,, \end{aligned} \tag{II.81}$$

for some universal constant $C < \infty$, since $\check{\Theta}_\ell \leq \Theta_\ell$. Inserting these estimates into the last term on the right side of (II.78), we obtain

$$\begin{aligned} &\big|\check{\Theta}_j^2\widehat{G}''_{jj}(x) \; - \; H''_{jj}(0)\big| \; + \; \sum_k b_{jk}\,\big|\Theta_k^2 G''_{kk}(x) \; - \; H''_{kk}(0)\big| \\ &\leq \; C\Big\{\check{\Theta}_j^2|\hat{g}'_j| \; + \; \alpha\sum_k a_{jk}\,\Theta_k^2|g'_k| \\ &\qquad\qquad + \; \sum_k b_{jk}\Big(a_{jk}\check{\Theta}_j^2|\hat{g}'_j| \; + \; \Theta_k^2|g'_k| \; + \; \alpha\sum_{\ell(\neq j)} a_{k\ell}\,\Theta_\ell^2|g'_\ell|\Big)\Big\} \\ &\leq \; C\Big\{(1 + C_a)\,\check{\Theta}_j^2|\hat{g}'_j| \; + \; \sum_k (a_{jk} + b_{jk})\,\Theta_k^2|g'_k| \; + \; \alpha\sum_{k,\ell} b_{jk}\,a_{k\ell}\,\Theta_\ell^2|g'_\ell|\Big\} \\ &\leq \; C\Big\{(1 + C_a)\,\check{\Theta}_j^2|\hat{g}'_j| \; + \; \sum_k 3\,b_{jk}\,\Theta_k^2|g'_k|\Big\}\,, \end{aligned} \tag{II.82}$$

where we make use of (II.56), (II.58), and assumed $\alpha/\alpha' < 1$ to be sufficiently small to derive the last inequality. Observe that the last line in (II.82) is bounded by a multiple of $J_j[\underline{b}]$. Thus, putting together (II.78), (II.79), and (II.82), we have

$$\big|M_j[\underline{b}] - L_j[\underline{b}]\big| \;\le\; \frac{C}{\beta} \;+\; \frac{C}{\beta}\, J_j[\underline{b}]\,, \tag{II.83}$$

for some universal constant $C < \infty$. Applications of Theorem II.8 and of Theorem II.9, choosing $\varepsilon > 0$ to be a small, but universal constant, yield

$$J_j[\underline{b}] \;\le\; C'\, K_j[\underline{b}] \;\le\; C''\big(L_j[\underline{b}] + 1\big)\,, \tag{II.84}$$

for some universal constants $C', C'' < \infty$. Thus, if $\beta < \infty$ is sufficiently large, we can solve in Eq. (II.82) for $L_j[\underline{b}]$ and get

$$L_j[\underline{b}] \;\le\; \Big(1 + \frac{C'''}{\beta}\Big)\, M_j[\underline{b}] \;+\; \frac{C'''}{\beta}\,, \tag{II.85}$$

for some universal constant $C''' < \infty$. $\square$

III Semiclassical Localization of the Partition Function

This section is devoted to the proof of Theorem I.6. The method we use here is inspired by a related method which appeared first in a paper by Sjöstrand [19]. The method was used to study the first eigenvalue of a Schrödinger operator divided by the dimension. He showed that this object is in the semiclassical limit given, up to an exponentially small error, by using the first eigenvalue of the operator localized to an l^∞ neighbourhood of a global minimum. The exponential error is uniform in dimension (i.e., universal). This method was later applied in [10] to Laplace integrals (partition functions) and in [16] to Laplace integrals and transfer operators. The results mentioned above are for localization in an l^∞ neighbourhood, whereas our result is for localization in a slab of the form $\{x : |x_i| \geq \delta\}$, where $i \in \Lambda$ and $\delta > 0$. The two methods apply to both localization problems, but under different sets of conditions. The method presented here seems particularly well adapted for the model considered in this paper. We note that the α_0 given by Theorem I.6 is not uniform in κ. In the related results mentioned above, α_0 can be chosen uniformly in the size of the neighbourhood.

III.1 The Dilated Interaction

Throughout this section we impose Hypotheses 1 and 2. We prove Theorem I.6 through a sequence of lemmata.

Lemma III.1. *Let* $\underline{\varepsilon} := (\varepsilon_k)_{k\in\Lambda} \subseteq (0,1)$ *and* $e_0 = 1 - \max_{k\in\Lambda}\{\varepsilon_k\}$. *There exists a universal constant* $C > 0$*, independent of* $\underline{\varepsilon}$*, such that*

$$\left| \sum_{j,k} \Big\{ w_{jk}\big(x_j, x_k\big) \; - \; w_{jk}\big((1-\varepsilon_j)x_j, (1-\varepsilon_k)x_k\big) \Big\} \right| \tag{III.1}$$
$$\leq \; \frac{C}{e_0} \sum_j \Big\{ \sum_k a_{jk} \Big(1 \; + \; \frac{\varepsilon_k}{\varepsilon_j} \Big) \Big\} \Big\{ g_j(x_j) \; - \; g_j\big((1-\varepsilon_j)x_j\big) \Big\} \, ,$$

for all $x \in \mathbb{R}^\Lambda$.

Proof: We abbreviate

$$y_j \; := \; (1-\varepsilon_j)\, x_j \quad \text{and} \quad z_j(t) \; := \; t x_j \; + \; (1-t) y_j \, . \tag{III.2}$$

The Fundamental Theorem of Calculus yields, cf. also the estimate (B.2),

$$\begin{aligned}
\big|w_{jk}(x_j, x_k) \;-\; w_{jk}(y_j, y_k)\big| \;&=\; \left| \int_0^1 \frac{d}{dt}\big\{w_{jk}(z_j(t), z_k(t))\big\}\, dt \right| \\
=\; &\left| \int_0^1 \Big\{ (x_j - y_j)\, \partial_j w_{jk}\big(z_j(t), z_k(t)\big) \;+\; (x_k - y_k)\, \partial_k w_{jk}\big(z_j(t), z_k(t)\big) \Big\}\, dt \right| \\
\leq\; &\int_0^1 a_{jk}\, \big(\varepsilon_j |x_j| \,+\, \varepsilon_k |x_k|\big)\, \big(|g_j'(z_j(t))| \,+\, |g_k'(z_k(t))|\big)\, dt\,. \qquad \text{(III.3)}
\end{aligned}$$

Using that $|g_j'(z_j(t))| = \mathrm{sgn}(x_j) g_j'(z_j(t))$ and $g_j'(z_j(t)) = \varepsilon_j^{-1} x_j^{-1} \frac{d}{dt}\big[g_j(z_j(t))\big]$, we rewrite the estimate above as follows,

$$\begin{aligned}
\big|w_{jk}(x_j, x_k) \;-\; w_{jk}(y_j, y_k)\big| \;&\leq\; a_{jk}\, \big(\varepsilon_j |x_j| \,+\, \varepsilon_k |x_k|\big) \\
&\int_0^1 \left\{ \frac{1}{\varepsilon_j |x_j|}\, \frac{d}{dt}\Big[g_j(z_j(t))\Big] \;+\; \frac{1}{\varepsilon_k |x_k|}\, \frac{d}{dt}\Big[g_k(z_k(t))\Big] \right\} dt \\
=\; a_{jk}\, \big(\varepsilon_j |x_j| \,+\, \varepsilon_k |x_k|\big) &\left\{ \frac{g_j(x_j) \,-\, g_j(y_j)}{\varepsilon_j |x_j|} \;+\; \frac{g_k(x_k) \,-\, g_k(y_k)}{\varepsilon_k |x_k|} \right\} \qquad \text{(III.4)}
\end{aligned}$$

Hence, by the symmetry in j and k, we get

$$\begin{aligned}
&\sum_{j,k} \big|w_{jk}(x_j, x_k) \;-\; w_{jk}(y_j, y_k)\big| \qquad \text{(III.5)} \\
&\quad \leq\; 2 \sum_{j,k} a_{jk} \left(1 \,+\, \frac{\varepsilon_k |x_k|}{\varepsilon_j |x_j|} \right) \Big(g_j(x_j) \;-\; g_j(y_j) \Big)\,.
\end{aligned}$$

Now we temporarily fix $j, k \in \Lambda$ and decompose the configuration space $\mathbb{R}^\Lambda$ into the two disjoint regions

$$\Big\{ x \in \mathbb{R}^\Lambda \;\Big|\; |x_j| \geq |y_k| \Big\} \quad \text{and} \quad \Big\{ x \in \mathbb{R}^\Lambda \;\Big|\; |x_j| < |y_k| \Big\}\,. \qquad \text{(III.6)}$$

For $|x_j| \geq |y_k| = (1 - \varepsilon_k)|x_k|$, we have

$$\frac{\varepsilon_k |x_k|}{\varepsilon_j |x_j|} \;\leq\; \frac{\varepsilon_k}{(1 - \varepsilon_k)\, \varepsilon_j} \;\leq\; \frac{\varepsilon_k}{e_0\, \varepsilon_j}\,, \qquad \text{(III.7)}$$

where we used that $1 - \varepsilon_k \geq e_0$. Conversely, if $|x_j| < |y_k| = (1 - \varepsilon_k)|x_k|$ then

(I.29) implies

$$
\begin{aligned}
\frac{\varepsilon_k |x_k|}{\varepsilon_j |x_j|} \big(g_j(x_j) \; - \; g_j(y_j) \big) \;&\leq\; \varepsilon_k \, |x_k| \, \max \Big\{ |g_j'(s)| \; \Big| \; 0 \,\leq\, \mathrm{sgn}(x_j) s \,\leq\, |x_j| \Big\} \\
&=\; \int_{|y_k|}^{|x_k|} \max \Big\{ |g_j'(s)| \; \Big| \; 0 \,\leq\, \mathrm{sgn}(x_k) s \,\leq\, |x_k| \Big\} \, dt \\
&\leq\; c_g^{-1} \int_{|y_k|}^{|x_k|} \mathrm{sgn}(x_k) \, g_k'\big(\mathrm{sgn}(x_k) t \big) \, dt \\
&=\; c_g^{-1} \big(g_k(x_k) \; - \; g_k(y_k) \big) \, . \qquad \text{(III.8)}
\end{aligned}
$$

Combining the estimates (III.7)–(III.8) and inserting them into (III.5) yields the lemma. □

We now specify $\underline{\varepsilon}$ in terms of the metric ρ on the lattice Λ and a fixed reference point $o \in \Lambda$.

Lemma III.2. *Fix $o \in \Lambda$ and $0 < \tau \leq \varepsilon_o < 1$, and set $\varepsilon_k := \varepsilon_o \delta_{k,o} + (1 - \delta_{k,o}) \tau e^{-\rho(k,o)}$. Then there exists a universal constant $C' > 0$ such that, for all $x \in \mathbb{R}^\Lambda$,*

$$
\begin{aligned}
&\left| \sum_{j,k} \Big\{ w_{jk}(x_j, x_k) \; - \; w_{jk}((1-\varepsilon_j) x_j, (1-\varepsilon_k) x_k) \Big\} \right| \\
&\quad \leq\; \frac{C'}{(1-\varepsilon_o)\,\tau} \sum_j \Big\{ g_j\big(x_j\big) \; - \; g_j\big((1-\varepsilon_j) x_j\big) \Big\} \, . \qquad \text{(III.9)}
\end{aligned}
$$

Proof: In view of Lemma III.1 and the fact that $e_0 \geq 1 - \varepsilon_o$, it is sufficient to show that

$$
\max_j \left\{ \sum_k a_{jk} \Big(1 + \frac{\varepsilon_k}{\varepsilon_j} \Big) \right\} \;\leq\; \frac{C''}{\tau} \, , \qquad \text{(III.10)}
$$

for some universal constant $C'' < \infty$. For $j \neq o$, we observe that the triangle inequality for the metric ρ and Eq. (I.16) implies

$$
\begin{aligned}
\sum_k a_{jk} \Big(1 + \frac{\varepsilon_k}{\varepsilon_j} \Big) \;&\leq\; e^{\rho(o,j)} \, a_{jo} \left(1 + \frac{\varepsilon_o}{\tau} \right) \;+\; \sum_{k(\neq o)} a_{jk} \left(1 \;+\; e^{\rho(o,j) - \rho(o,k)} \right) \\
&\leq\; \left(1 + \frac{\varepsilon_o}{\tau} \right) \max_j \left\{ \sum_k e^{\rho(j,k)} \, a_{jk} \right\} \;\leq\; C_a \left(1 + \frac{\varepsilon_o}{\tau} \right) \;\leq\; \frac{2\, C_a}{\tau} \, . \qquad \text{(III.11)}
\end{aligned}
$$

Conversely, for $j = o$, we estimate

$$
\sum_k a_{ok} \Big(1 + \frac{\varepsilon_k}{\varepsilon_o} \Big) \;\leq\; \left(1 + \frac{\tau}{\varepsilon_o} \right) \sum_{k(\neq o)} a_{ok} \;\leq\; 2\, C_a \, , \qquad \text{(III.12)}
$$

using (I.16). This completes the proof since $\tau < 1$. □

III.2 The Dilated Hamiltonian Function

In the following we fix $o \in \Lambda$ and $\varepsilon_o, \tau \in (0,1)$ and set $\varepsilon_k := \varepsilon_o \delta_{k,o} + (1 - \delta_{k,o})\tau e^{-\rho(k,o)}$. Furthermore, for $x = (x_k)_{k\in\Lambda} \in \mathbb{R}^\Lambda$, we define $y = (y_k)_{k\in\Lambda} \equiv y(x, \underline{\varepsilon}) \in \mathbb{R}^\Lambda$ by $y_k = (1 - \varepsilon_k)x_k$.

Proposition III.3. *Let $\xi > 0$ and define $\varepsilon_o := \max\left\{1/2, 1 - \hat{R}_0 \check{R}_1^{-1}\right\}$. There exists a universal constant $c > 0$ such that if $\tau := c\ \min\{1, \xi\}^2$ and $0 \le \alpha \le \alpha_0 := c\,\tau$ then*

$$H(x) \ - \ H(y) \ \ge \ c\xi\ \min\{1,\xi\}\,, \tag{III.13}$$

for all $x, y \in \mathbb{R}^\Lambda$ with $y_k = (1 - \varepsilon_k)x_k$, $k \in \Lambda$ and such that $|x_o| \ge \xi$.

Proof: We first observe that

$$\begin{aligned} H(x) \ - \ H(y) \ &= \ \Big\{ f_o(x_o) - f_o(y_o) \Big\} \ + \sum_{k(\neq o)} \big\{ f_k(x_k) - f_k(y_k) \big\} \\ &\quad + \alpha \sum_{j \neq k} \big\{ w_{jk}(x_j, x_k) \ - \ w_{jk}(y_j, y_k) \big\}\,. \end{aligned} \tag{III.14}$$

According to Lemma III.2, there exists a universal constant $C' < \infty$ such that

$$\alpha \sum_{j\neq k} \big\{ w_{jk}(x_j, x_k) \ - \ w_{jk}(y_j, y_k) \big\} \ \ge \ -\frac{C'\,\alpha}{(1-\varepsilon_o)\,\tau} \sum_j \big\{ g_j(x_j) \ - \ g_j(y_j) \big\}\,. \tag{III.15}$$

To bound the second term on the right side of (III.14) from below, we use that, for $|x_k| < \check{R}_1$, the boundedness of the derivative of q_k by a universal constant $C'' < \infty$ implies

$$q_k(x_k) - q_k(y_k) \ \ge \ -|x_k - y_k| \sup_{|s| \le \check{R}_1} \left| q_k'(s) \right| \ \le \ -C''\,\varepsilon_k\,\check{R}_1\,, \tag{III.16}$$

which also holds for $|x_k| \ge \check{R}_1$, since then $q_k(x_k) = q_{\max} \ge q_k(y_k)$. Thus (III.16) holds true for all x_k and $y_k = (1 - \varepsilon_k)x_k$ which yields

$$\begin{aligned} \sum_{k(\neq o)} \big\{ f_k(x_k) - f_k(y_k) \big\} \ &= \ \sum_{k(\neq o)} \big\{ g_k(x_k) - g_k(y_k) \big\} + \sum_{k(\neq o)} \big\{ q_k(x_k) - q_k(y_k) \big\} \\ &\ge \ \sum_{k(\neq o)} \big\{ g_k(x_k) - g_k(y_k) \big\} \ - \ C''\,\check{R}_1 \sum_{k(\neq o)} \varepsilon_k \\ &\ge \ \sum_{k(\neq o)} \big\{ g_k(x_k) - g_k(y_k) \big\} \ - \ C''\,\check{R}_1\,C_\rho\,\tau\,, \end{aligned} \tag{III.17}$$

where $C_\rho \ge \sum_{k(\neq o)} e^{-\rho(o,k)}$ is the universal constant introduced in Eq. (I.6).
To estimate the first term on the right side of (III.14), we observe that if $|x_o| < \check{R}_1$

and $0 < (1-\varepsilon_o) \le \hat{R}_0 \check{R}_1^{-1}$ then $y_o = (1-\varepsilon_o)x_o \in I_0$ and $q_o(y_o) = 0$. Since $q_o \ge 0$ everywhere, we therefore have

$$f_o(x_o) - f_o(y_o) \;=\; g_o(x_o) - g_o(y_o) \,+\, q_o(x_o) \;\ge\; g_o(x_o) - g_o(y_o)\,, \qquad \text{(III.18)}$$

which also holds for $|x_o| \ge \check{R}_1$, again because then $q_o(x_o) = q_{\max} \ge q_o(y_o)$. Thus, also (III.18) holds true, for all x_o and $y_o = (1-\varepsilon_o)x_o$.

Inserting (III.15), (III.17), and (III.18) into (III.14), we obtain

$$\begin{aligned} H(x) \;-\; H(y) & \\ \ge\; & \left(1 \,-\, \frac{C'\,\alpha}{(1-\varepsilon_o)\,\tau}\right) \sum_j \big\{g_j(x_j) \,-\, g_j(y_j)\big\} \;-\; C''\,\check{R}_1\,C_\rho\,\tau \\ \ge\; & \frac{1}{2} \sum_j \big\{g_j(x_j) \,-\, g_j(y_j)\big\} \;-\; C''\,\check{R}_1\,C_\rho\,\tau \\ \ge\; & \frac{1}{2} \big\{g_o(x_o) \,-\, g_o(y_o)\big\} \;-\; C''\,\check{R}_1\,C_\rho\,\tau\,, \end{aligned} \qquad \text{(III.19)}$$

provided $0 \le \alpha \le (1-\varepsilon_o)\,\tau\,(2C')^{-1}$.

By assumption, we have $|x_o| \ge \xi$ and $\varepsilon_o \ge 1/2$. Hence (I.29) and (I.32) give

$$\begin{aligned} g_o(x_o) - g_o(y_o) \;&\ge\; g_o(x_o) - g_o(x_o/2) \;\ge\; \frac{c_g\,|x_o|}{2}|g_o'(x_o/2)| \\ &\ge\; \frac{c_g^2\,\xi}{2}\,\min\{1, \xi/2\} \;\ge\; \frac{c_g^2}{4}\,\xi\,\min\{1,\xi\}\,. \end{aligned} \qquad \text{(III.20)}$$

Thus, choosing $\varepsilon_o := \max\big\{1/2\,,\, 1 - \hat{R}_0 \check{R}_1^{-1}\big\}$,

$$\tau \;:=\; \frac{c_g^2}{c_g^2 \,+\, 16\,C''\,\check{R}_1\,C_\rho}\,\min\{1, \xi^2\}\,, \qquad \text{and} \qquad \alpha_0 \;:=\; \frac{(1-\varepsilon_o)\,\tau}{2C'}\,, \qquad \text{(III.21)}$$

we have $\varepsilon_o, \tau \in (0,1)$, and we obtain

$$H(x) \;-\; H(y) \;\ge\; \frac{c_g^2}{16}\,\xi\,\min\{1,\xi\}\,, \qquad \text{(III.22)}$$

for all $0 \le \alpha \le \alpha_0$ and $x, y \in \mathbb{R}^\Lambda$ with $y_k = (1-\varepsilon_k)x_k$ and such that $|x_o| \ge \xi$. $\square$

We are now in a position to prove the main result of this section.

Proof of Theorem I.6: Fix $\xi > 0$, $o \in \Lambda$, and define $\varepsilon_o := \max\big\{1/2, 1 - \hat{R}_0 \check{R}_1^{-1}\big\}$, $\tau := c\,\min\{1,\xi\}^2$, and $0 \le \alpha \le \alpha_0 := c\,\tau^2$, where $c > 0$ is the universal constant in Proposition III.3. Set $\varepsilon_k := \varepsilon_o \delta_{k,o} + (1-\delta_{k,o})\tau e^{-\rho(k,o)}$.

Denoting $(\mathbf{1}-\underline{\varepsilon})x := \big((1-\varepsilon_k)x_k\big)_{k\in\Lambda} \in \mathbb{R}^\Lambda$ and $m(\xi) := 2c\xi\min\{1,\xi\}$, we use Proposition III.3 and a change of integration variables $y := (\mathbf{1}-\underline{\varepsilon})x$ to estimate

$$\begin{aligned}
&\int_{\{x:|x_o|\geq\xi\}} \exp\big[-2\beta H(x)\big]\, d^\Lambda x \\
&\leq\ e^{-\beta m(\xi)} \int_{\mathbb{R}^\Lambda} \exp\Big[-2\beta H\big((\mathbf{1}-\underline{\varepsilon})x\big)\Big]\, d^\Lambda x \\
&=\ e^{-\beta m(\xi)} \left(\prod_k \frac{1}{1-\varepsilon_k}\right) \int_{\mathbb{R}^\Lambda} \exp\big[-2\beta H(y)\big]\, d^\Lambda y \\
&\leq\ e^{-\beta m(\xi)} \left(\frac{1}{1-\varepsilon_o}\right) \exp\Big[\sum_{k(\neq 0)} \varepsilon_k\Big]\, \mathcal{Z}_\beta \\
&\leq\ C\, e^{C_\rho}\, e^{-\beta m(\xi)}\, \mathcal{Z}_\beta\,.
\end{aligned} \tag{III.23}$$

This proves Eq. (I.70). □

A Construction of $\hat{g}_j$ and g_j from f_j

In this appendix, we consider a family $\{f_j\}_{j\in\Lambda}$ of single-spin potentials fulfilling Hypothesis 1. Recall that Hypothesis 1 requires, for any $j \in \Lambda$, that zero is the unique, nondegenerate minimum of f_j, attained at $t = 0$, i.e., $f_j(0) = 0$, $f_j''(0) > 0$, and $f_j(t) > 0$, whenever $t \neq 0$. Moreover, there exist universal constants $0 < c_f \leq 1 \leq C_f < \infty$, $R_f < \infty$, such that

$$c_f \ \leq \ f_j''(0) \ \leq \ C_f \, , \tag{A.1}$$

$$\forall t \neq 0 : \qquad \big\{ f_j'(t) = 0 \big\} \ \Rightarrow \ \big\{ f_j(t) \geq c_f \big\} \, , \tag{A.2}$$

$$\forall t \in \mathbb{R} : \quad |f_j''(t) - f_j''(0)| \ \leq \ C_f \big(|f_j'(t)| \, + \, \min\{1, |t|\} \big) \, , \tag{A.3}$$

$$\forall |t| \geq R_f : \quad |f_j'(t)| \ \geq \ c_f \, \max \Big\{ |f_k'(s)| \ \Big| \ |s| \leq |t| \, , \, k \in \Lambda \Big\} \, , \tag{A.4}$$

for all $j \in \Lambda$.

Given $\{f_j\}_{j\in\Lambda}$ fulfilling Hypothesis 1, our purpose in this appendix is to prove Lemma I.2, i.e., to construct families $\{g_j\}_{j\in\Lambda}$ and $\{\hat{g}_j\}_{j\in\Lambda}$, which fulfill the stronger requirement (I.12), besides Hypothesis 1, and agree with $\{f_j\}_{j\in\Lambda}$ at zero and at infinity.

For the sake of readability, we shall only give the details of our construction on the positive half-axis, and we henceforth assume that $t, s \geq 0$ without further mentioning. We divide our construction into several steps, each casted into a lemma.

Lemma A.1. *Assume Hypothesis 1 and let $j \in \Lambda$. Then*

$$\forall t \leq C_f^{-1} : \quad \big| f_j'(t) \, - \, t f_j''(0) \big| \ \leq \ C_f^2 \, t^2 \, , \tag{A.5}$$

$$\forall t \in \mathbb{R}_0^+ : \quad \big| f_j'(t) \big| \ \leq \ 3 \, C_f \, \exp[2 \, C_f \, t] \, . \tag{A.6}$$

Proof: The asserted bounds (A.5) and (A.6) follow from Gronwall-type estimates. We fix $j \in \Lambda$ and define

$$\mu(t) \ := \ \max_{0 \leq s \leq t} \big| s^{-1} \, f_j'(s) \ - \ f_j''(0) \big| \, , \tag{A.7}$$

observing that $\lim_{s\to 0}\{s^{-1} f_j'(s)\} = f_j''(0)$ exists, since $f_j'(0) = 0$. Using this definition and (A.3), we estimate

$$\begin{aligned} |f_j'(s) \, - \, s f_j''(0)| \ &\leq \ \int_0^s \big| f_j''(r) - f_j''(0) \big| \, dr \\ &\leq \ C_f \int_0^s \big(|f_j'(r)| \, + \, \min\{1, r\} \big) \, dr \\ &\leq \ C_f \int_0^s \big(|f_j'(r) - r f_j''(0)| \, + \, [1 + f_j''(0)] \, r \big) \, dr \\ &\leq \ \frac{C_f \, s^2}{2} \big(\mu(s) + 1 + f_j''(0) \big) \, . \end{aligned} \tag{A.8}$$

Since μ is increasing, (A.8), $f_j''(0) \le C_f$, and $C_f \ge 1$ imply

$$\mu(t) \;\le\; \frac{C_f\, t}{2}\,\mu(t) \;+\; \frac{C_f + C_f^2}{2}\, t \;\le\; \frac{C_f\, t}{2}\,\mu(t) \;+\; C_f^2\, t\,, \tag{A.9}$$

which yields

$$\big|f_j'(t) \;-\; t f_j''(0)\big| \;\le\; t\,\mu(t) \;\le\; 2\,C_f^2\, t^2\,, \tag{A.10}$$

for all $0 \le t \le C_f^{-1}$.

Similarly, we define

$$\nu_\kappa(t) \;:=\; \max_{0\le s\le t} \big\{ e^{-\kappa s}\, |f_j'(s)| \big\}\,, \tag{A.11}$$

where $\kappa > 0$ is chosen later, and observe that, by (A.3),

$$\begin{aligned} |f_j'(s)| \;&\le\; C_f \int_0^s \big(|f_j'(r)| \,+\, |f_j''(0)| \,+\, \min\{1, r\} \big)\, dr \qquad\qquad (A.12)\\ &\le\; C_f \int_0^s e^{\kappa r}\, \nu_\kappa(r)\, dr \;+\; 2\,C_f^2\, s \\ &\le\; \frac{C_f}{\kappa}\, e^{\kappa s}\, \nu_\kappa(t) \;+\; 2\,C_f^2\, s\,. \end{aligned}$$

Since ν_κ is increasing, (A.12) implies,

$$\big(1 \;-\; C_f\, \kappa^{-1}\big)\, \nu_\kappa(t) \;\le\; 6\,C_f^2\, \kappa^{-1}\,, \tag{A.13}$$

Choosing $\kappa := 2C_f$, we arrive at (A.6). $\square$

For the next lemmata, we define

$$\tilde r_0 := \frac{c_f}{C_f^2} \quad \text{and} \quad \hat r_1 := \max\big\{ R_f\,,\; \tilde r_0 \big\}\,. \tag{A.14}$$

Lemma A.2. *Under the assumption of Hypothesis 1,*

$$\begin{aligned} &\forall\, t \le \tilde r_0 : \quad \tfrac14\, c_f\, t^2 \;\le\; f_j(t) \;\le\; \tfrac34\, C_f\, t^2\,, && (A.15)\\ &\forall\, t \le \tilde r_0 : \quad \tfrac12\, c_f\, t \;\le\; f_j'(t) \;\le\; \tfrac32\, C_f\, t\,, && (A.16)\\ &\forall\, t \ge \tilde r_0 : \qquad f_j(t) \;\ge\; \tfrac14\, c_f^3\, C_f^{-4}\,, && (A.17)\\ &\forall\, t \ge \hat r_1 : \qquad f_j'(t) \;\ge\; \tfrac12\, c_f^2\, C_f^{-2}\,, && (A.18) \end{aligned}$$

for all $j \in \Lambda$.

Proof: Estimate (A.16) follows from Lemma A.1, taking into account that $t \le \tilde r_0 = c_f\, C_f^{-2}$ implies $t \le C_f^{-1}$, since $c_f \le C_f$, and (A.15) is obtained from (A.16) by integration with respect to t.

Eq. (A.18) follows from (A.16) and (I.11),

$$f_j'(t) \;\geq\; c_f \max_{0 \leq s \leq t}\{f_j'(s)\} \;\geq\; c_f\, f_j'(\tilde r_0) \;\geq\; \frac{c_f^2}{2\, C_f^2}\,, \tag{A.19}$$

additionally using that $\tilde r_0 \leq \hat r_1 \leq t$.

For the derivation of (A.17), we remark that f_j is strict monotonically increasing on $[\hat r_1, \infty)$, by (A.18). Next, if $t_* \in (\tilde r_0, \hat r_1)$ is a critical point of f_j, then

$$f_j(t_*) \;\geq\; c_f \;\geq\; \frac{3c_f^2}{4C_f^3} \;\geq\; f_j(\tilde r_0)\,, \tag{A.20}$$

using (I.9), (A.15), and $C_f \geq 1 \geq c_f$. Thus

$$\min_{t \geq \tilde r_0}\{f_j(t)\} \;=\; f_j(\tilde r_0) \;\geq\; \frac{c_f^3}{4\, C_f^4}\,, \tag{A.21}$$

once more employing (A.15). □

For the construction of g_j and $\hat g_j$, we introduce the continuously differentiable function

$$\gamma(t) \;:=\; \begin{cases} \frac{1}{16}\, c_f\, t^2 & \text{if} \quad t \leq \tilde r_0^2\, (\hat r_1)^{-1}\,, \\ \frac{1}{8}\, c_f\, \tilde r_0^2\, (\hat r_1)^{-1}\, t \;-\; \frac{1}{16}\, c_f\, \tilde r_0^4\, (\hat r_1)^{-2} & \text{if} \quad t \geq \tilde r_0^2\, (\hat r_1)^{-1}\,. \end{cases} \tag{A.22}$$

Note that, by Lemma A.2,

$$\gamma(t) \;\leq\; \tfrac{c_f\, t^2}{8} \;\leq\; \tfrac{1}{2} f_j(t)\,, \tag{A.23}$$

$$\gamma'(t) \;\leq\; \tfrac{c_f\, t}{4} \;\leq\; \tfrac{1}{2} f_j'(t)\,, \tag{A.24}$$

for all $0 \leq t \leq \tilde r_0$,

$$f_j(t) - \gamma(t) \;\geq\; f_j(\tilde r_0) - \gamma(\hat r_1) \;\geq\; \frac{1}{8}\, c_f\, \tilde r_0^2\,, \tag{A.25}$$

for $\tilde r_0 \leq t \leq \hat r_1$, and

$$\gamma'(t) \;=\; \frac{c_f\, \tilde r_0^2}{4\, \hat r_1} \;\leq\; \frac{c_f\, \tilde r_0}{4} \;=\; \frac{c_f^2}{8\, C_f^2} \;\leq\; f_j'(t)\,, \tag{A.26}$$

provided $t \geq \hat r_1$. From (A.23)–(A.26) we deduce that

$$\forall\, t \in \mathbb{R}^+ : \quad 0 \;<\; \gamma(t) \;<\; f_j(t)\,, \tag{A.27}$$

$$\forall\, t \in \mathbb{R}^+ \setminus (\tilde r_0, \hat r_1) : \quad 0 \;<\; \gamma'(t) \;<\; f_j'(t)\,. \tag{A.28}$$

(A.29)

We define

$$\hat{\gamma}_j(t) \; := \; \min_{s \geq t} \big\{ f_j(s) - \gamma(s) \big\} \; + \; \gamma(t) \; , \tag{A.30}$$

which is reminiscent of the Legendre transform of f_j. We now show that $\hat{\gamma}_j$ has all the desired properties of $\hat{g}_j$ except for being twice continuously differentiable.

Lemma A.3. *Assume Hypothesis 1. Then $\hat{\gamma}_j$, defined as in (A.30), fulfills*

$$\gamma(t) \; \leq \; \hat{\gamma}_j(t) \; \leq \; f_j(t) \; , \tag{A.31}$$

for all $t \in \mathbb{R}_0^+$. Moreover, $\hat{\gamma}_j$ is left and right differentiable on $\mathbb{R}^+$, i.e., the limits $\partial_\pm \hat{\gamma}_j(t) := \lim_{h \to 0\pm} h^{-1}[\hat{\gamma}_j(t+h) - \hat{\gamma}_j(t)]$ exist, and its left and right derivatives obey

$$\gamma'(t) \; \leq \; \partial_\pm \hat{\gamma}_j(t) \; \leq \; \gamma'(t) \; + \; \max\{ f_j'(t) - \gamma'(t) \, , \, 0 \} \; . \tag{A.32}$$

for all $t \in \mathbb{R}^+$.

Proof: It suffices to prove that $n(t) := \min_{s \geq t}\{m(s)\} \leq m(t)$ is left and right differentiable and that $0 \leq \partial_\pm n(t) \leq m'(t)$, where $m := f_j - \gamma$. We distinguish several regions of $\mathbb{R}^+$.

1. Since m is continuous, so is n, and $\mathcal{A}_1 := \{t | n(t) < m(t)\}$ is open. Thus, if $t \in \mathcal{A}_1$, then n is constant in a neighborhood of t and, in particular, differentiable, with $\partial_+ n(t) = \partial_- n(t) = 0$.

2. Consider the complement, $\mathcal{A}_2 := \{t | n(t) = m(t)\}$. Let $t \in \mathcal{A}_2$ and $\delta > 0$. Then

$$\begin{aligned} n(t) - n(t-\delta) \;&=\; m(t) \;-\; \min\big[m(t) \, , \, \min_{t-\delta \leq s \leq t} \{m(s)\} \big] \\ &=\; \max\big[0 \, , \, \max_{t-\delta \leq s \leq t} \{ m(t) - m(s) \} \big] \\ &=\; \max\big[0 \, , \, \max_{t-\delta \leq s \leq t} \{ m'(t)\,(t-s) \;+\; o(t-s) \} \big] \\ &=\; \max\big[0 \, , \, \delta\, m'(t) \big] \;+\; o(\delta) \; . \end{aligned} \tag{A.33}$$

   Hence, $\partial_- n = \max\{m', 0\}$ on $\mathcal{A}_2$.

2.1 Observe that $\mathcal{A}_2 \cap \{t | m'(t) < 0\} = \emptyset$.

2.2 Let $t \in \mathcal{A}_2 \cap \{t | m'(t) = 0\}$ and $t' > t$. Then

$$0 \;\leq\; \frac{n(t') - n(t)}{t' - t} \;=\; \frac{\min_{s \geq t'}\{m(s)\} - m(t)}{t' - t} \;\leq\; \frac{m(t') - m(t)}{t' - t} \;\to\; 0 \, , \tag{A.34}$$

as $t' \searrow t$, and therefore $\partial_+ n(t) = 0$.

2.3.1 Let $t \in \mathcal{A}_2 \cap \{t | m'(t) > 0\}$ and suppose that $m(s) > m(t)$, for all $s > t$. Since $m'(t) > 0$ and $m' > 0$ at infinity, $m(t) < m_*$, where $m_* := \min\{m(s) \mid m'(s) = 0,\ s > t\}$, if such critical points exist, and $m_* := \infty$ otherwise. But then $n \equiv m$ in a neighborhood of t, and $\partial_+ n(t) = m'(t)$.

2.3.2 Let $t \in \mathcal{A}_2 \cap \{t | m'(t) > 0\}$ and suppose that $m(t_*) = m(t)$, for some $t_* > t$. Then $n \equiv m(t)$ on $[t, t_*]$, and $\partial_+ n(t) = 0$. □

Having established the left and right differentiability of $\hat{\gamma}_j$, we derive the following bounds on $\hat{\gamma}_j$.

Lemma A.4. *Assume Hypothesis 1, and set*

$$\hat{r}_0 \ := \ \frac{c_f^3}{3\, C_f^5} \qquad \textit{and} \qquad \hat{q}_{\max} \ := \ 2\, \exp\left[2\, C_f\, \hat{r}_1 \right] . \tag{A.35}$$

Then

$$\forall\, t \le \hat{r}_0 : \qquad \hat{\gamma}_j(t) \ = \ f_j(t)\,, \tag{A.36}$$

$$\forall\, \hat{r}_0 < t < \hat{r}_1 : \quad 0 \ \le \ f_j(t) - \hat{\gamma}_j(t) \ \le \ \hat{q}_{\max}\,, \tag{A.37}$$

$$\forall\, t \ge \hat{r}_1 : \qquad \hat{\gamma}_j(t) \ = \ f_j(t)\,. \tag{A.38}$$

Moreover, the function $\hat{\gamma}_j$ is left and right differentiable and Lipschitz continuous on $\mathbb{R}^+$, and there exists a universal constant $\hat{c}_g > 0$, such that

$$\forall t \in \mathbb{R}_0^+\ ,\ \ \sigma = \pm : \qquad \partial_\sigma \hat{\gamma}_j(t) \ \ge \ \hat{c}_g\, \min\{1, t\}\,, \tag{A.39}$$

$$\forall t \in \mathbb{R}_0^+\ ,\ \ \sigma = \pm : \ \ \partial_\sigma \hat{\gamma}_j(t) \ \ge \ \hat{c}_g\, \max\left\{ \partial_\tau \hat{\gamma}_j(s) \ : \ 0 \le s \le t\,,\ \tau = \pm \right\}. \tag{A.40}$$

Proof: We first observe that $f_j - \gamma$ is strict monotonically increasing on $[\hat{r}_1, \infty)$, by (A.26). Hence, if $t \in [\hat{r}_1, \infty)$, then $\min_{s \ge t}\{f_j(s) - \gamma(s)\} = f_j(t) - \gamma(t)$ and $\hat{\gamma}_j(t) = f_j(t)$, proving (A.38).

Next we remark that, by (A.31) and (A.38),

$$0 \ \le \ f_j(t) - \hat{\gamma}_j(t) \ \le \ \max_{0 \le s \le \hat{r}_1} \left\{ f_j(t) - \hat{\gamma}_j(t) \right\} \ \le \ \max_{0 \le s \le \hat{r}_1} \left\{ f_j(t) \right\}. \tag{A.41}$$

The right inequality in (A.37) then follows from integrating (A.6) on $[0, \hat{r}_1]$.

If $\hat{r}_0 \le t \le \hat{r}_1$, then (A.25), Lemma A.2, and (A.27) imply that

$$f_j(t) - \gamma(t) \ \ge \ \frac{1}{8}\, c_f\, \tilde{r}_0^2 \ \ge \ \frac{3}{4}\, C_f\, (\hat{r}_0)^2 \ \ge \ f_j(\hat{r}_0) \ \ge \ f_j(\hat{r}_0) - \gamma(\hat{r}_0)\,. \tag{A.42}$$

Moreover, $f_j - \gamma$ is strict monotonically increasing on $(0, \hat{r}_0) \subseteq (0, \tilde{r}_0)$, by (A.28). Thus, for $t \le \hat{r}_0$, we have $\min_{s \ge t}\{f_j(s) - \gamma(s)\} = f_j(t) - \gamma(t)$ and $\hat{\gamma}_j(t) = f_j(t)$, proving (A.36).

Estimate (A.39) is a direct consequence of (A.32),

$$\partial_\sigma \hat{\gamma}_j(t) \;\geq\; \gamma'(t) \;=\; \frac{1}{8}\, c_f \, \min\left\{ \tilde{r}_0^2 \, (\hat{r}_1)^{-1} \,,\; t \right\} . \tag{A.43}$$

For the derivation of (A.40) we distinguish the regions $(0, \hat{r}_0]$, $(\hat{r}_0, \hat{r}_1)$, and $[\hat{r}_1, \infty)$.
1) On $(0, \hat{r}_0] \subseteq (0, \tilde{r}_0]$, the function f_j is convex, according to Lemma A.1, and it agrees with $\hat{\gamma}_j$. Hence (A.40) holds true on $(0, \hat{r}_0]$ (with constant $\hat{c}_g = 1$).
2) On $(\hat{r}_0, \hat{r}_1)$, Eq. (A.40) follows from the fact that $\partial_\sigma \hat{\gamma}_j > 0$ is bounded away from zero, by (A.39), and bounded above, by (A.32).
3) On $[\hat{r}_1, \infty)$, we have $\hat{\gamma}_j \equiv f_j$, by (A.38). Then Eq. (A.40) follows from the fact that f_j obeys (A.4) and (A.32). □

We turn to the construction of a Lipshitz continuous function γ_j which has all the properties desired from g_j, except for being twice continuously differentiable. We first introduce two universal constants,

$$\check{r}_0 \;:=\; 2\, R_f \;\geq\; 2\, \hat{r}_1 \qquad \text{and} \qquad \check{r}_1 \;:=\; \check{r}_0 \;+\; \frac{4\, C_f^2 \, \hat{q}_{\max}}{c_f^2} \,, \tag{A.44}$$

and we define $\check{r}_{1,j} > \check{r}_0$ by requiring

$$f_j(\check{r}_{1,j}) - f_j(\check{r}_0) \;=\; \hat{q}_{\max} \;+\; \frac{c_f^2}{4\, C_f^2} \big(\check{r}_{1,j} - \check{r}_0 \big) \,. \tag{A.45}$$

Note that $\check{r}_{1,j}$ is uniquely determined by (A.45) and bounded above by $\check{r}_1$, due to (A.18). We now set

$$\gamma_j(t) \;:=\; \begin{cases} \hat{\gamma}_j(t) & \text{if} \quad 0 \leq t \leq \check{r}_0 \,, \\ f_j(\check{r}_0) \;+\; \frac{1}{4} c_f^2 \, C_f^{-2} \big(t - \check{r}_0 \big) & \text{if} \quad \check{r}_0 \leq t \leq \check{r}_{1,j} \,, \\ f_j(t) \;-\; \hat{q}_{\max} & \text{if} \quad \check{r}_{1,j} \leq t \,, \end{cases} \tag{A.46}$$

and observe that γ_j fulfills the assertion of Lemma A.4 if in its formulation $\hat{r}_1$ is replaced by $\check{r}_1$.

We finally come to the
Proof of Lemma I.2: We first remark that $\hat{\gamma}_j$ and γ_j, constructed as above, agree with f_j on $[0, \hat{r}_0]$, γ_j further agrees with f_j on $[\check{r}_1, \infty]$, and on $[\hat{r}_1, \infty]$, the derivatives of $\hat{\gamma}_j$ and f_j coincide. We choose smooth characteristic functions $\hat{\chi}, \check{\chi} \in C_0^\infty(\mathbb{R}^+, [0,1])$, such that

$$\hat{\chi} \equiv 1 \text{ on } [\hat{r}_0, \hat{r}_1] \,, \qquad \operatorname{supp} \hat{\chi} \subseteq (\hat{r}_0 - \varepsilon, \hat{r}_1 + \varepsilon) \,, \tag{A.47}$$

$$\check{\chi} \equiv 1 \text{ on } [\check{r}_0, \check{r}_1] \,, \qquad \operatorname{supp} \check{\chi} \subseteq (\check{r}_0 - \varepsilon, \check{r}_1 + \varepsilon) \,, \tag{A.48}$$

$$\sup_{t \in \mathbb{R}^+} \left\{ \frac{d^n}{dt^n} \hat{\chi}(t) \right\} \;=\; \mathcal{O}(\varepsilon^n) \,, \qquad \sup_{t \in \mathbb{R}^+} \left\{ \frac{d^n}{dt^n} \check{\chi}(t) \right\} \;=\; \mathcal{O}(\varepsilon^n) \,, \tag{A.49}$$

where $\varepsilon > 0$ is sufficiently small universal number. Furthermore, we pick $m \in C_0^\infty((-1,0), \mathbb{R}_0^+)$ with $\int_{\mathbb{R}} m = 1$ and define a mollifier by $m_\varepsilon(t) := \varepsilon^{-2} m(t/\varepsilon^2)$.

Now the functions $\hat{g}_j$ and g_j are obtained from $\hat{\gamma}_j$ and γ_j by smearing out in small neighborhoods of the appropriate regions $(\hat{r}_0, \hat{r}_1)$ and $(\check{r}_0, \check{r}_1)$. More precisely, we set

$$\hat{g}_j \;:=\; \big(1 - \hat{\chi}\big)\, \hat{\gamma}_j \;+\; \hat{\chi}\big(\hat{\gamma}_j * m_\varepsilon\big) \,, \tag{A.50}$$

$$g_j \;:=\; \big(1 - \hat{\chi} - \check{\chi}\big)\, \gamma_j \;+\; \big(\hat{\chi} + \check{\chi}\big)\, \big(\gamma_j * m_\varepsilon\big) \,. \tag{A.51}$$

Obviously, $\hat{g}_j$ and g_j are twice continuously differentiable. Since $\operatorname{supp} m_\varepsilon \subseteq \mathbb{R}^-$ and $\hat{\gamma}_j$ and γ_j are monotonically increasing, we have that $\hat{g}_j \le \hat{\gamma}_j$ and $g_j \le \gamma_j$.

Moreover, $\hat{g}_j$ differs from $\hat{\gamma}_j$ only in $(\hat{r}_0 - \varepsilon, \hat{r}_1 + \varepsilon)$, and on this interval we estimate

$$\big|\hat{\gamma}_j \;-\; (\hat{\gamma}_j * m_\varepsilon)\big| \;\le\; 3\, C_f \;\exp[2\, C_f\, (\check{r}_1 + \varepsilon)]\; \varepsilon^2 \,, \tag{A.52}$$

using (A.32) and (A.6). Hence, on $(\hat{r}_0 - \varepsilon, \hat{r}_1 + \varepsilon)$, we have

$$\begin{aligned} (\hat{g}_j)' \;&=\; -(\hat{\chi})' \left[\hat{\gamma}_j - (\hat{\gamma}_j * m_\varepsilon)\right] \;+\; \big(1 - \hat{\chi}\big)\, (\hat{\gamma}_j)' \;+\; \hat{\chi}\big((\hat{\gamma}_j)' * m_\varepsilon\big) \\ &=\; \big(1 - \hat{\chi}\big)\, (\hat{\gamma}_j)' \;+\; \hat{\chi}\big((\hat{\gamma}_j)' * m_\varepsilon\big) \;-\; \mathcal{O}(\varepsilon) \,, \end{aligned} \tag{A.53}$$

and $(\hat{g}_j)'$ is bounded above and below on $(\hat{r}_0 - \varepsilon, \hat{r}_1 + \varepsilon)$ by universal positive constants. A similar argument applies to g_j. It is hence clear that all bounds asserted in Lemma I.2 hold, including those that involve the second derivatives, if we set

$$\hat{R}_0 \;:=\; \hat{r}_0 - \varepsilon \,, \qquad \hat{R}_1 \;:=\; \hat{r}_1 + \varepsilon \,, \tag{A.54}$$

$$\check{R}_0 \;:=\; \check{r}_0 - \varepsilon \,, \qquad \check{R}_1 \;:=\; \check{r}_1 + \varepsilon \,. \tag{A.55}$$

We leaves the details of their verification to the reader. □

B Some Basic Properties

We begin with an estimate which implies that the Gibbs measure is well-defined,

Lemma B.1. *Suppose Hypothesesis 1 and 2. There exist universal constants $\alpha_0 > 0$ and $C > 0$ such that for $0 \le \alpha < \alpha_0$,*

$$H(x) \;\ge\; C^{-1} \sum_j g_j(x_j) - C|\Lambda| \,. \tag{B.1}$$

Proof: Using (I.29) we get the following estimate, cf. also (III.3) and (III.4),

$$\begin{aligned} |w_{ij}(x_i, x_j)| &\le \int_0^1 \Big| x_i\, \partial_i w_{ij}(\sigma x_i, \sigma x_j) \;+\; x_j\, \partial_j w_{ij}(\sigma x_i, \sigma x_j) \Big|\, d\sigma \\ &\le a_{ij} \int_0^1 (|x_i| \,+\, |x_j|)\, (|g_i'| \,+\, |g_j'|)\, d\sigma \\ &\le 4\, c_g^{-1}\, a_{ij} \int_0^1 \big\{ |x_i|\, |g_i'| \,+\, |x_j|\, |g_j'| \big\}\, d\sigma \\ &= 4\, c_g^{-1}\, a_{ij}\, \big(g_i(x_i) \,+\, g_j(x_j) \big)\,. \end{aligned} \tag{B.2}$$

This implies the lemma since $g_j = f_j - q_j$ and q_j is bounded. □

Lemma B.1 together with the estimate $g_j(x_j) \ge C^{-1}|x_j| - C$, for some universal constant $C > 0$, implies that the measure (I.1) is normalizable and that polynomially bounded measurable functions are integrable.

We will in this appendix identify $\mathcal{H}$ with $L^2(\mathbb{R}^\Lambda; \mathcal{F})$. Let $\mathcal{D} := C_0^\infty(\mathbb{R}^\Lambda; \mathcal{F})$ be the space of test-functions. We write $\mathcal{D}'$ for the dual space of distributions (recall that $\mathcal{F}$ is a finite dimensional vector space). Note that $d_{H,Q}, d_{H,Q}^*$ maps $\mathcal{H}$ into $\mathcal{D}'$ and

$$\mathrm{dom}(d_{H,Q}) \;=\; \{\psi \in \mathcal{H} \;:\; d_{H,Q}\, \psi \;\in\; \mathcal{H}\,\}\,, \tag{B.3}$$

$$\mathrm{dom}(d_{H,Q}^*) \;=\; \{\psi \in \mathcal{H} \;:\; d_{H,Q}^*\, \psi \;\in\; \mathcal{H}\,\}\,. \tag{B.4}$$

We proceed to prove the following proposition.

Proposition B.2. *The Dirac operator $D_{H,Q}$ is essentially self-adjoint on $\mathcal{D}$ and self-adjoint on* $\mathrm{dom}(d_{H,Q}) \cap \mathrm{dom}(d_{H,Q}^*)$.

Proof: As for the self-adjointness part of the statement we refer the reader to the proof of [2, Lemma B.1]. In order to handle the presence of Q we need to replace the observation $(D_{0,0}^2 + \mathbf{1})^{-1} \;:\; H^k(\mathbb{R}^\Lambda; \mathcal{F}) \;\to\; H^{k+2}(\mathbb{R}^\Lambda; \mathcal{F})$, for *any* k, by the following statement

$$(D_{0,Q}^2 + \mathbf{1})^{-1} \;:\; H^k(\mathbb{R}^\Lambda; \mathcal{F}) \;\to\; H^{k+2}(\mathbb{R}^\Lambda; \mathcal{F})\,, \tag{B.5}$$

where k is restricted to $k = 0$ and $k = 1$. Here $H^k(\mathbb{R}^\Lambda; \mathcal{F})$ are the standard Sobolev spaces.

We proceed to prove that $D_{H,Q}$ is self-adjoint on $\mathcal{K} := \mathrm{dom}(d_{H,Q}) \cap \mathrm{dom}(d_{H,Q}^*)$. Clearly $D_{H,Q}$ is symmetric on $\mathcal{K}$, and it remains to show that it is closed. To this end, let $\{u_n, v_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathrm{Graph}(D_{H,Q}|_{\mathcal{K}})$, satisfying $u_n \to u$ and $v_n \to v$ in $\mathcal{H}$.

We have the following equality on $\mathcal{K}$,

$$\big\| D_{H,Q}\psi \big\|^2 \;=\; \big\| d_{H,Q}\psi \big\|^2 \;+\; \big\| d_{H,Q}^*\psi \big\|^2\,. \tag{B.6}$$

Since $D_{H,Q}\, u_n = v_n$ is bounded uniformly in n, we find that so are $d_{H,Q}\, u_n$ and $d^*_{H,Q}\, u_n$.

Clearly $d_{H,Q}\, u \in \mathcal{D}'$. Let $\psi \in \mathcal{D}$. We have

$$\begin{aligned} |\langle d_{H,Q}u|\ \psi\rangle| &= |\langle u|\ d^*_{H,Q}\psi\rangle| \ = \ \lim_{n\to\infty} |\langle u_n|\ d^*_{H,Q}\psi\rangle| \\ &= \lim_{n\to\infty} |\langle d_{H,Q}u_n, \psi\rangle| \ \leq \ \sup_n \|d_{H,Q}u_n\|\|\psi\| \,, \end{aligned} \tag{B.7}$$

which implies by (B.3) that $u \in \mathrm{dom}(d_{H,Q})$. Similar we find that $u \in \mathrm{dom}(d^*_{H,Q})$. Since $\mathcal{K}$ is contained in the domain of the closure of $D_{H,Q}|_{\mathcal{D}}$, we conclude from closedness of the latter that $D_{H,Q}u = v$ and hence $(u, v) \in \mathrm{Graph}(D_{H,Q}|_{\mathcal{K}})$. This completes the proof. (We note that this proof is slightly simpler than the corresponding proof in [2].) □

For $u_k \in \mathcal{H}^{(k)}$ we define the linear maps

$$(T_k\, u_k)(x) \ := \ e^{-\beta\{H(x) - d\Gamma(Q(x))\}} \int_0^1 t^{k-1}\, e^{\beta\{H(tx) - d\Gamma(Q(tx))\}} u_k(tx)\, dt \,, \tag{B.8}$$

$$S \ := \ \sum_\ell \beta x_\ell \otimes c_\ell \,, \tag{B.9}$$

$$\Psi_k \ := \ S \circ T_k \,. \tag{B.10}$$

We have the following result.

Lemma B.3. *Let $v_k \in \mathcal{H}^{(k)}$ be continuous in a (not necessarily universal) neighbourhood of 0. Then $T_k v_k \in \mathcal{H}^{(k)}$*

Proof: We begin by analyzing the difference $H(x) - H(tx)$. Assume first that $0 < t < 1/2$. From (I.29) we get the estimate

$$\begin{aligned} g_j(s) \ - \ g_j(s/2) &= 1/2 \int_0^1 s g_j'((\sigma s + s)/2)\, d\sigma \\ &\geq (2c_g)^{-1} \int_0^1 s g_j'(\sigma s)\, d\sigma \ = \ (2c_g)^{-1}\, g(s) \,. \end{aligned} \tag{B.11}$$

Then, by (B.2) and the boundedness of the q_j's, there exists a universal constant $C < \infty$ such that

$$\begin{aligned} H(x) \ - \ H(tx) &\geq \sum_j \{f_j(x_j) - f_j(tx_j)\} \ - \ \alpha C \sum_j g_j(x_j) \\ &\geq \sum_j \{g_j(x_j) - g_j(x_j/2) - C\} - \alpha C \sum_j g_j(x_j) \\ &\geq \sum_j \big((2c_g)^{-1} - \alpha C\big) g_j(x_j) - C|\Lambda| \,. \end{aligned} \tag{B.12}$$

Secondly, if $1/2 < t < 1$, we invoke Lemma III.1, applied with $\epsilon_k = 1 - t$, to obtain (for $\alpha < \alpha_0$)

$$H(x) - H(tx) \geq \sum_j (1 - \alpha C)\big(g_j(x) - g_j(tx)\big) - C|\Lambda| \geq -C|\Lambda|, \tag{B.13}$$

for some universal constant $C < \infty$.

Now, we estimate, using (B.12) and (B.13),

$$\begin{aligned} \int_{\mathbb{R}^\Lambda} |(T_k v_k)(x)|^2 \, d^\Lambda x \;&\leq\; \sup_{t,x} \left\| e^{\beta\{d\Gamma(Q(x)) - d\Gamma(Q(tx))\}} \right\|^2_{\mathcal{B}[\mathcal{F}]} \\ &\qquad \cdot \int_0^1 \int_{\mathbb{R}^\Lambda} t^{2k-2} e^{-2\beta(H(x)-H(tx))} |v_k(tx)|^2 \, d^\Lambda x \, dt \\ &\leq\; C_{\beta,\Lambda} \Big\{ \int_{1/2}^1 \int_{\mathbb{R}^\Lambda} t^{2k-2} |v_k(tx)|^2 \, d^\Lambda x \, dt \\ &\qquad + \int_0^{1/2} \int_{\mathbb{R}^\Lambda} t^{2k-2} e^{-C_\beta |x|_1} |v_k(tx)|^2 \, d^\Lambda x \, dt \Big\} \\ &=\; C_{\beta,\Lambda} \Big\{ \|v_k\|^2 \int_{1/2}^1 t^{2k-2-|\Lambda|} \, dt \\ &\qquad + \int_0^{1/2} \int_{\mathbb{R}^\Lambda} t^{2k-2-|\Lambda|} e^{-\frac{C_\beta |x|_1}{t}} |v_k(x)|^2 \, d^\Lambda x \, dt \Big\} \, . \end{aligned} \tag{B.14}$$

Here $|x|_1 := \sum_j |x_j|$. As for the last term on the right side, we note that in the case $2k \geq |\Lambda| + 2$, it can be estimated in the same way as the first term. For the remaining case, $2k < |\Lambda| + 2$, we fix a number $0 < \delta < 1/2$ and estimate

$$\begin{aligned} &\int_0^{1/2} \int_{\mathbb{R}^\Lambda} t^{2k-2-|\Lambda|} e^{-\frac{\beta C |x|_1}{t}} |v_k(x)|^2 \, d^\Lambda x \, dt \\ &\leq \sup_s \left\{ |s|^{|\Lambda|+2-2k-2\delta} e^{-\beta C|s|} \right\} \Big(\int_0^{1/2} t^{-2\delta} \, dt \Big) \left\| |x|_1^{k-1-|\Lambda|/2+\delta} v_k \right\|^2 \end{aligned} \tag{B.15}$$

Since v_k is assumed to be continuous at 0 and $|x|_1^{2k-2-|\Lambda|+2\delta}$ is locally integrable in $\mathbb{R}^\Lambda$, we find that $|x|_1^{k-1-|\Lambda|/2+\delta} v_k \in \mathcal{H}$ (provided $k \geq 1$). Combining this with (B.14) and (B.15), we obtain $T_k v_k \in \mathcal{H}$. □

We will need a Poincaré lemma, the proof of which is left to the reader, see for example [6, Proposition 2.13.2],

Lemma B.4 (Poincaré). *Let $v_k \in \mathcal{H}^{(k)} \cap \mathrm{dom}(d_{H,Q})$ be such that v_k and $d_{H,Q}\, v_k$ are continuous in a (not necessarily universal) neighbourhood of 0. The following identity holds between functions,*

$$(d_{H,Q}\, \Psi_k\, v_k)(x) + (\Psi_k\, d_{H,Q}\, v_k)(x) = v_k(x) \, . \tag{B.16}$$

Proof of Theorem I.4: As for the first part of the Theorem we refer the reader to the proof of [2, Lemma B.3]. We note that the pertubation argument of [2], which is based on the main form bound in Theorem II.1, breaks down on k-forms with $k \geq 2$, since we cannot bound $d\Gamma(A_{H,Q}^{(1)})$ by $A_{H,Q}$ in a universal way, cf. (I.90).

The proof presented below for the second part of the theorem is based on an approach from an earlier unpublished version of [2]. See also [18].

First, we note that $\mathrm{Ker}\, D_{H,Q} = \mathrm{Ker}\, \Delta_{H,Q}$, by self-adjointness of $D_{H,Q}$ (recall $\Delta_{H,Q} = D_{H,Q}^2$). Secondly, we observe that $\exp(-\beta H) \otimes \Omega \in \mathrm{Ker}\, D_{H,Q}$. It thus remains to prove that $\mathrm{Ker}\, \Delta_{H,Q} \subseteq \mathbb{C} \cdot \exp(-\beta H) \otimes \Omega$.

Let $u \in \mathrm{Ker}\, \Delta_{H,Q}$, and write $u = \sum_k u_k$, where $u_k \in \mathcal{H}^{(k)}$. Clearly,

$$u_k \;\in\; \mathrm{Ker}\, \Delta_{H,Q} \;\subseteq\; \mathrm{Ker}\, d_{H,Q} \,\cap\, \mathrm{Ker}\, d_{H,Q}^* \,, \tag{B.17}$$

for all k.

Since $u_k \in \mathrm{Ker}\, \Delta_{H,Q}$, it follows from elliptic regularity, cf. [21, Proposition 9.1], and the assumption of smoothness at 0 that u_k is continuous at 0. Hence Lemmata B.3 and B.4 apply.

Let $\chi \in C_0^\infty(\mathbb{R}^\Lambda)$ and write $\chi_n(x) := \chi(x/n)$. Then $\chi_n \Psi_k u_k \in \mathcal{H}$ and $\mathrm{w} - \lim_{n\to\infty}[d_{H,Q}, \chi_n^2]S = 0$, cf. (I.53), (B.9) and (B.10). We compute using these observations together with (B.16) and (B.17),

$$\begin{aligned} 0 &= \langle d_{H,Q}^*\, u_k, \chi_n^2\, \Psi_k\, u_k \rangle \\ &= \langle u_k, o_w(1)\, T_k\, u_k \rangle \;+\; \langle u_k, \chi_n^2\, d_{H,Q}\, \Psi_k\, u_k \rangle \\ &= o(1) \;+\; \|\chi_n\, u_k\|^2 \,. \end{aligned} \tag{B.18}$$

Here $\mathrm{w} - \lim_{n\to\infty} o_w(1) = \lim_{n\to\infty} o(1) = 0$. Taking the limit $n \to \infty$ in (B.18) yields $u_k = 0$ for $k \geq 1$. We have thus verified that $u = u_0 = f \otimes \Omega$, $f \in L^2(\mathbb{R}^\Lambda)$, and by assumption $Z_j(H)f = 0$ for all j. This implies $u = \exp(-\beta H) \otimes \Omega$ which concludes the proof. □

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